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The computation of the degree of the greatest common divisor of three Bernstein basis polynomials

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Abstract

This paper considers the application of the Sylvester resultant matrix to the computation of the degree of the greatest common divisor (GCD) of three Bernstein basis polynomials $f(y), g(y)$ and $h(y)$. It is shown that the governing equations can be written in two forms, which lead to different Sylvester matrices. The first form requires that the polynomials be considered in pairs, but different pairs of polynomials may yield different computational answers, for example, the solution of the computations $\operatorname{GCD}(f,g)$ and $\operatorname{GCD}(g,h)$ may differ from the solution of the computations $\operatorname{GCD}(f,g)$ and $\operatorname{GCD}(f,h)$, depending on $f(y), g(y)$ and $h(y)$. This problem does not arise when the second form is considered, which requires that the three polynomials be considered simultaneously. Complications arise in both forms because of the combinatorial terms in the Bernstein basis functions, which cause the entries of the matrices to span several orders of magnitude, even if the coefficients of the polynomials are of the same order of magnitude. It is shown that the adverse effects of this wide range of magnitudes can be mitigated by postmultiplying both forms of the Sylvester matrix by a diagonal matrix of combinatorial terms and preprocessing $f(y), g(y)$ and $h(y)$ by three operations. Results of GCD computations from the two forms of the Sylvester matrix when $f(y), g(y)$ and $h(y)$ are perturbed by noise, and with the omission and inclusion of the preprocessing operations, are shown.

Key words: Greatest common divisor; Sylvester resultant matrix; Bernstein basis polynomials

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1 Introduction

The computation of the greatest common divisor (GCD) of two polynomials arises in many applications, including control theory, image processing and the computation of multiple roots of a polynomial, and this has been the main motivation for the continued research into this problem [3–7,17–27]. More recent work has considered the computation of the GCD of several power basis polynomials [1,2,8,9,12,15], the GCD of two bivariate power basis polynomials [10,11] and the GCD of two multivariate power basis polynomials [14,16,28]. These GCD problems are extended in this paper by considering the computation of the degree of the GCD of three univariate Bernstein basis polynomials. These computations are important in computer-aided geometric design (CAGD), and the reason for restriction of the computation to the degree of the GCD, rather than consider the computation of its degree and coefficients, is discussed.

Bernstein basis polynomials are used extensively in CAGD because of their elegant geometric properties and superior numerical properties with respect to the power basis. These properties of the Bernstein basis functions are [13]:

(1) The recursive generation of the nth order Bernstein basis function from the $(n-1)$th order Bernstein basis function.
(2) The variation-diminishing property.
(3) The positivity and partition of unity.
(4) The degree elevation procedure.

Two of the most important computations in CAGD are the calculation of the points of intersection of curves, and the points and curves of intersection of surfaces. These computations have been considered for two intersecting curves and two intersecting surfaces, but the calculation of the points of intersection of three curves has not been addressed. This problem raises issues that are not present when two curves or two surfaces are considered. In particular, if the polynomials $f(x,y), g(x,y)$ and $h(x,y)$ are factorised as

$$
\begin{aligned}
f(x,y) &= d(x,y)u(x,y),\\
g(x,y) &= d(x,y)v(x,y),\\
h(x,y) &= d(x,y)w(x,y),
\end{aligned}
$$

where $u(x,y), v(x,y)$ and $w(x,y)$ are coprime polynomials and $d(x,y)$ is the GCD of $f(x,y), g(x,y)$ and $h(x,y)$, then the curves

$$
f(x,y)=0, \qquad g(x,y)=0, \qquad h(x,y)=0,
$$

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intersect at the points that satisfy $d(x, y) = 0$. It therefore follows that the determination of the points of intersection of two or three curves reduces to the calculation of the GCD of their polynomial forms. The computation of the GCD of two bivariate polynomials in the power basis has been considered, as noted above, but much less work has been devoted to the computation of the GCD of three bivariate polynomials.

The Sylvester matrix and its subresultant matrices [3] are often used for the computation of the GCD of two univariate polynomials, and it is desirable to extend their use to the computation of the GCD of three bivariate polynomials. This computation raises, however, several issues because the Sylvester matrices of three univariate or bivariate polynomials are not trivial extensions of their equivalent forms for two polynomials. For example, there are four forms of the Sylvester matrix and its subresultant matrices of three polynomials, and the Sylvester matrix of each form is rectangular but the Sylvester matrix of two polynomials is unique and square. It is therefore appropriate to consider initially the simpler problem of the computation of the degree of the GCD of three univariate polynomials, before the more difficult problem of the computation of the GCD of three bivariate polynomials is considered. This simpler problem is addressed in this paper in order to understand the properties of Sylvester matrices of three polynomials. This problem has not been addressed in the literature and there do not, therefore, exist results from other methods for comparison. The effectiveness of the Sylvester matrices for the computation of the degree of the GCD of three polynomials can, however, be quantified because its calculation reduces to the change from rank deficiency to full rank as rows and columns of a matrix are removed. A good (bad) result is therefore characterised by a large (small) change in the smallest singular value of each matrix as the rows and columns of the matrices are removed.

The discussion above has referred to the GCD of three polynomials, but these polynomials are corrupted by noise in practical problems and thus an approximate greatest common divisor (AGCD) they possess must be considered. An AGCD of three polynomials is discussed in Section 2, but it is convenient to derive the theory for the computation of the GCD of three polynomials and then show that if the polynomials are preprocessed, an AGCD they possess can be computed when their coefficients are corrupted by noise.

Four formulations of the GCD problem for three polynomials are considered in Section 3, and it is shown in Section 4 that each of them yields a Sylvester matrix, as noted above. Although these matrices are theoretically equivalent because the GCD and coprime polynomials can be computed from each of them, their numerical properties differ such that they may return different solutions to the same GCD computation. The adverse numerical effects of the combinatorial terms in the Bernstein basis functions are considered in Section 5, and it is shown in Section 6 that these adverse effects can be mitigated by processing the polynomials before computations are performed on their Sylvester matrices and subresultant matrices. Section 7 contains examples of these computations, and the paper is summarised in Section 8.

This paper considers only the computation of the degree of the GCD of three univariate Bernstein basis polynomials because the computation of the coefficients of the GCD involves considerable extra work. In particular, it requires the extension of the non-linear structure-preserving matrix described in [7] from the Sylvester matrix and its subresultant matrices of two polynomials to their equivalents for three polynomials. The structures of these matrices for three polynomials are more complex than the structures of their equivalents for two polynomials, and the theoretical development for the preservation of these structures is therefore more difficult and thus requires a separate study. The major contribution of this paper is therefore the extension of the Sylvester matrix from two univariate polynomials to three univariate polynomials, and investigations into its use for the computation of the degree of their GCD, and its use for the computation of an AGCD when the coefficients of the polynomials are perturbed by noise.

2 An AGCD of three polynomials

It was noted in Section 1 that practical problems require that an AGCD, rather than the GCD, of two or more polynomials be considered, and thus this section considers an AGCD of three polynomials.

The following definition of an AGCD of three polynomials is an extension of the definition of an AGCD of two polynomials [5,10]. It includes the properties of minimum distance and maximum degree of an AGCD [27].

Definition 2.1 (An AGCD) A polynomial $d(y)$ is an ε-divisor of $f(y), g(y)$ and $h(y)$, which are of degrees m, n and p respectively, if there exist polynomials $\tilde{f}(y), \tilde{g}(y)$ and $\tilde{h}(y)$ of degrees m, n and p respectively, such that

$$\begin{aligned}
\frac{\left\| f(y) - \tilde{f}(y) \right\|_l}{\| f(y) \|_l} &\leq \varepsilon \| f(y) \|_l , \\
\frac{\| g(y) - \tilde{g}(y) \|_l}{\| g(y) \|_l} &\leq \varepsilon \| g(y) \|_l , \\
\frac{\left\| h(y) - \tilde{h}(y) \right\|_l}{\| h(y) \|_l} &\leq \varepsilon \| h(y) \|_l ,
\end{aligned}$$

and $d(y)$ is a divisor of $\tilde{f}(y), \tilde{g}(y)$ and $\tilde{h}(y)$, where $l = 1, 2, \infty$. If $d(y)$ is the

ε-divisor, of maximum degree, of $f(y), g(y)$ and $h(y)$ for which the distance between the exact and perturbed polynomials is the minimum among the set of ε-divisors, then it is an ε-GCD or AGCD of $f(y), g(y)$ and $h(y)$. The polynomials $u(y) = \tilde{f}(y)/d(y)$, $v(y) = \tilde{g}(y)/d(y)$ and $w(y) = \tilde{h}(y)/d(y)$ are ε-cofactors.

3 The GCD of three polynomials

The GCD of three polynomials, $f(y), g(y)$ and $h(y)$, can be obtained by two GCD computations, each of which is performed on two polynomials,

$$
\begin{aligned}
\text{GCD}\,(f, g, h) &= \text{GCD}\,\left(\text{GCD}\,(f, g), h\right) = \text{GCD}\,(d_{f,g}, h) \\
&= \text{GCD}\,\left(\text{GCD}\,(g, h), f\right) = \text{GCD}\,(d_{g,h}, f) \\
&= \text{GCD}\,\left(\text{GCD}\,(h, f), g\right) = \text{GCD}\,(d_{h,f}, g),
\end{aligned}
\tag{1}
$$

where $d_{p,q} = d_{p,q}(y) = \text{GCD}\,(p, q)$. The methods described in [3–7,18,21–24,26] can be used for these GCD computations on two polynomials, and it may occur that

$$
d_{f,g}(y) = \text{GCD}\,(d_{f,g}, h),
$$

in which case all the subresultant matrices of the Sylvester matrix $S(d_{f,g}, h)$ are rank deficient. It is therefore difficult to determine the degree of the GCD of $d_{f,g}(y)$ and $h(y)$ because there does not exist a change from rank deficiency to full rank as the index k of the Sylvester subresultant matrices $S_k(d_{f,g}, h)$, $k = 1, \ldots, \min(\deg d_{f,g}, \deg h)$, increases. Another problem arises when computations suggest that

$$
\text{GCD}\,(f, g, h) \neq \text{GCD}\,\left(\text{GCD}\,(f, g), h\right),
$$

and this situation may exist for one or more pairings of the polynomials, but other pairings may yield good results. It is, however, not possible to determine *a priori* the pairings that yield good results, which is a disadvantage of the reduction of the three-polynomial GCD problem to two GCD computations, each of which is performed on two polynomials (1). These problems that arise when two GCD computations are performed on two polynomials do not occur when $f(y), g(y)$ and $h(y)$ are considered simultaneously and only one GCD computation is therefore performed. In this case, it is necessary to extend the Sylvester matrix and its subresultant matrices from two polynomials to three polynomials. There are two forms of these matrices for three polynomials and they are now discussed briefly, and they are considered in detail in Section 4.

Let $d(y)$ be the GCD of $f(y), g(y)$ and $h(y)$, and let their coprime polynomials be $u(y), v(y)$ and $w(y)$, respectively,

$$f(y) = u(y)d(y), \qquad g(y) = v(y)d(y), \qquad h(y) = w(y)d(y). \tag{2}$$

The simultaneous consideration of these equations leads to one form of the Sylvester matrix, and the second form of the Sylvester matrix is obtained by grouping the three equations (2) pairwise:

$$\begin{aligned}
&\text{Group 1:} \quad f(y) = u(y)d(y), \quad g(y) = v(y)d(y),\\
&\text{Group 2:} \quad f(y) = u(y)d(y), \quad h(y) = w(y)d(y),\\
&\text{Group 3:} \quad g(y) = v(y)d(y), \quad h(y) = w(y)d(y).
\end{aligned}$$

It is clear that consideration of any two groups includes the three equations (2), and thus computations on any two groups is adequate to compute the GCD of $f(y), g(y)$ and $h(y)$.

This discussion leads to two variants of the Sylvester matrix of three polynomials:

Variant 1 The polynomials $f(y), g(y)$ and $h(y)$ are considered simultaneously and there is only one Sylvester matrix.

Variant 2 The polynomials $f(y), g(y)$ and $h(y)$ are considered pairwise in three sets,

$$((f, g), (f, h)), \qquad ((f, g), (g, h)), \qquad ((f, h), (g, h)), \tag{3}$$

which correspond to Groups 1 and 2, Groups 1 and 3, and Groups 2 and 3, respectively. Each set yields a different Sylvester matrix, but each Sylvester matrix can be used to compute the GCD of $f(y), g(y)$ and $h(y)$. These three matrices yield, theoretically, the same result, but they may return different computational results. The Sylvester matrix that yields the best result may not be known *a priori*, which is a disadvantage of this variant.

Section 4 considers the Sylvester matrices and subresultant matrices for Variants 1 and 2, and their application to the calculation of the degree of the GCD of $f(y), g(y)$ and $h(y)$. These matrices contain combinatorial terms, some of which are very large and may therefore lead to computational problems. It is shown in Section 6 that the adverse numerical effects of these problems can be mitigated by processing $f(y), g(y)$ and $h(y)$ before computations are performed on their Sylvester matrices and subresultant matrices.

4 The Sylvester matrices for three polynomials

The discussion in Section 1 considered three polynomials in an arbitrary basis, but the polynomials $f(y), g(y)$ and $h(y)$ in this section and all subsequent sections are expressed in the Bernstein basis. The polynomials are real and of degrees m, n and p respectively, and the degree of their GCD is t,

$$f(y) = \sum_{i=0}^{m} a_i \binom{m}{i} (1-y)^{m-i} y^i, \tag{4}$$

$$g(y) = \sum_{i=0}^{n} b_i \binom{n}{i} (1-y)^{n-i} y^i, \tag{5}$$

$$h(y) = \sum_{i=0}^{p} c_i \binom{p}{i} (1-y)^{p-i} y^i. \tag{6}$$

The polynomials have more than one common divisor $d_{(k)}(y)$ of degree $k = 1, \ldots, t-1$, and one common divisor (the GCD) of degree $k = t$. There therefore exist quotient polynomials $u_{(k)}(y), v_{(k)}(y)$ and $w_{(k)}(y)$ of degrees $m-k, n-k$ and $p-k$ respectively, such that

$$d_{(k)}(y) = \frac{f(y)}{u_{(k)}(y)} = \frac{g(y)}{v_{(k)}(y)} = \frac{h(y)}{w_{(k)}(y)}, \qquad k = 1, \ldots, t, \tag{7}$$

where

$$u_{(k)}(y) = \sum_{i=0}^{m-k} u_{k,i} \binom{m-k}{i} (1-y)^{m-k-i} y^i, \tag{8}$$

$$v_{(k)}(y) = \sum_{i=0}^{n-k} v_{k,i} \binom{n-k}{i} (1-y)^{n-k-i} y^i, \tag{9}$$

$$w_{(k)}(y) = \sum_{i=0}^{p-k} w_{k,i} \binom{p-k}{i} (1-y)^{p-k-i} y^i. \tag{10}$$

It follows from (7) that the polynomials $d_{(k)}(y)$ are common divisors of $f(y), g(y)$ and $h(y)$ if the equations

$$f(y)v_{(k)}(y) - g(y)u_{(k)}(y) = 0, \tag{11}$$

$$f(y)w_{(k)}(y) - h(y)u_{(k)}(y) = 0, \tag{12}$$

$$g(y)w_{(k)}(y) - h(y)v_{(k)}(y) = 0, \tag{13}$$

are satisfied for $k = 1, \ldots, t$, and each of these three equations defines the GCD problem for two Bernstein basis polynomials [6,7]. In particular, it is shown in these references that the kth Sylvester subresultant matrix $S_k(f,g)$ of $f(y)$ and $g(y)$ is

$$\begin{aligned} S_k(f,g) &= P_k^{-1} T_k(f,g) \\ &= \operatorname{diag}\left[\frac{1}{\binom{m+n-k}{0}} \; \frac{1}{\binom{m+n-k}{1}} \; \cdots \; \frac{1}{\binom{m+n-k}{m+n-k}} \right] \times \\ & \left[\begin{array}{cccccc} a_0\binom{m}{0} & & & b_0\binom{n}{0} & & \\ a_1\binom{m}{1} & \ddots & & b_1\binom{n}{1} & \ddots & \\ \vdots & \ddots & a_0\binom{m}{0} & \vdots & \ddots & b_0\binom{n}{0} \\ \vdots & \ddots & a_1\binom{m}{1} & \vdots & \ddots & b_1\binom{n}{1} \\ a_m\binom{m}{m} & \ddots & \vdots & b_n\binom{n}{n} & \ddots & \vdots \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_m\binom{m}{m} & & & b_n\binom{n}{n} \end{array} \right], \end{aligned}$$

where

$$P_k^{-1} = \operatorname{diag}\left[\frac{1}{\binom{m+n-k}{0}} \; \frac{1}{\binom{m+n-k}{1}} \; \cdots \; \frac{1}{\binom{m+n-k}{m+n-k}} \right],$$

$$T_k(f,g) = \left[F_k(f) \;\; G_k(g) \right] \in \mathbb{R}^{(m+n-k+1)\times(m+n-2k+2)},$$

and $F_k(f) \in \mathbb{R}^{(m+n-k+1)\times(n-k+1)}$ and $G_k(g) \in \mathbb{R}^{(m+n-k+1)\times(m-k+1)}$ are Tœplitz matrices that contain the coefficients of $f(y)$ and $g(y)$ scaled by their combinatorial terms,

$$F_k(f) = \left[\begin{array}{ccc} a_0\binom{m}{0} & & \\ a_1\binom{m}{1} & \ddots & \\ \vdots & \ddots & a_0\binom{m}{0} \\ \vdots & \ddots & a_1\binom{m}{1} \\ a_m\binom{m}{m} & \ddots & \vdots \\ & \ddots & \vdots \\ & & a_m\binom{m}{m} \end{array} \right], \quad G_k(g) = \left[\begin{array}{ccc} b_0\binom{n}{0} & & \\ b_1\binom{n}{1} & \ddots & \\ \vdots & \ddots & b_0\binom{n}{0} \\ \vdots & \ddots & b_1\binom{n}{1} \\ b_n\binom{n}{n} & \ddots & \vdots \\ & \ddots & \vdots \\ & & b_n\binom{n}{n} \end{array} \right]. \tag{14}$$

Theorem 4.1 shows that the degree of the GCD of two polynomials can be computed from their subresultant matrices.

Theorem 4.1 The degree $t_{f,g}$ of the GCD of $f(y)$ and $g(y)$ is equal to the largest integer k such that $S_k(f,g)$ is rank deficient,

$$\begin{aligned} \text{rank } S_k(f,g) &< m+n-2k+2, \quad k=1,\ldots,t_{f,g}, \\ \text{rank } S_k(f,g) &= m+n-2k+2, \quad k=t_{f,g}+1,\ldots,\min(m,n). \end{aligned} \tag{15}$$

It is shown in [6,7] that bad results may be obtained when the degree of the GCD of $f(y)$ and $g(y)$ is computed from $S_k(f,g)$, $k=1,\ldots,\min(m,n)$, because of the combinatorial terms in P_k^{-1} and $T_k(f,g)$. Better results are obtained when the modified Sylvester subresultant matrices

$$S_k(f,g)R_k = P_k^{-1}T_k(f,g)R_k, \qquad k=1,\ldots,\min(m,n),$$

are used, where

$$R_k = \begin{bmatrix} Q_{n-k} & \\ & Q_{m-k} \end{bmatrix}, \qquad Q_s = \text{diag}\left[\binom{s}{0} \; \binom{s}{1} \; \cdots \; \binom{s}{s} \right]. \tag{16}$$

Since P_k^{-1} and $R_k(f,g)$ have full rank, the four matrices

$$\left\{ T_k(f,g), \;\; P_k^{-1}T_k(f,g), \;\; T_k(f,g)R_k, \;\; P_k^{-1}T_k(f,g)R_k \right\}, \tag{17}$$

have the same rank for each value of $k=1,\ldots,\min(m,n)$, and thus the rank property (15) of the subresultant matrices is satisfied by all these matrices, and not only by $S_k(f,g) = P_k^{-1}T_k(f,g)$. Although these four matrices are theoretically equivalent for the calculation of $t_{f.g}$, it is shown in [6] that they may return different results, in particular in the presence of noise and for polynomials of high degree.

Theorem 4.1 and the four forms (17) can be extended to the computation of the GCD of three polynomials, and the differences, due to the combinatorial terms, in these forms are considered in Section 5. It is shown in [6,7] that, in general, the form $P_k^{-1}T_k(f,g)R_k$ yields the best result for the computation of the GCD of two polynomials, and the extension of this form from two polynomials to three polynomials will therefore be used in Sections 4.1 and 4.2 for consideration of the four forms of the Sylvester matrix of three polynomials. It is shown, however, that even with the best form, an incorrect result may be obtained and additional methods must therefore be used to minimise the adverse numerical effects of large combinatorial terms.

It is convenient to introduce a small change of notation between the two-polynomial GCD problem and the three-polynomial GCD problem. In particular, the kth Sylvester subresultant matrix for the polynomials $f(y)$ and $g(y)$ is $P_k^{-1}T_k(f,g)$, and (17) shows that four matrices must be considered for the computation of the degree of the GCD of $f(y)$ and $g(y)$. Since, as noted above, $P_k^{-1}T_k(f,g)R_k$ yields the best result for this problem, it is convenient, when considering the computation of the degree of the GCD of three polynomials, to redefine the kth Sylvester subresultant matrix as the extension of $P_k^{-1}T_k(f,g)R_k$ to its three-polynomial equivalent. This modified definition of the Sylvester matrix and its subresultant matrices is used for Variants 1 and 2 of the Sylvester matrix of three polynomials, and they are considered in Sections 4.1 and 4.2, respectively.

4.1 The Sylvester matrix for Variant 1

Variant 1 of the Sylvester matrix of three polynomials follows from the simultaneous consideration of (11), (12) and (13), and this leads to the Sylvester matrix and its subresultant matrices $\tilde{S}_k(f,g,h)$, $k=1,\ldots,q$, $q=\min(m,n,p)$. The forms of these matrices follow easily from their forms for two polynomials, and in particular, the extension of the proof of Theorem 4.1 shows that (11), (12) and (13) can be combined and written in matrix form,

$$\tilde{S}_k(f,g,h)\tilde{x}_k=0, \quad \tilde{S}_k(f,g,h)=\tilde{D}_k^{-1}\tilde{T}_k(f,g,h)\tilde{Q}_k, \quad k=1,\ldots,t, \tag{18}$$

where $\tilde{S}_k(f,g,h)$ is a 3×3 block matrix of order $(2(m+n+p)-3k+3)\times(m+n+p-3k+3)$, $\tilde{D}_k^{-1}$ is a square diagonal matrix of order $2(m+n+p)-3k+3$,

$$\begin{aligned}
\tilde{D}_k^{-1} &= \operatorname{diag}\left[\begin{array}{ccc} D_{m+n-k}^{-1} & D_{m+p-k}^{-1} & D_{n+p-k}^{-1}\end{array}\right],\\
D_s^{-1} &= \operatorname{diag}\left[\begin{array}{cccc} \frac{1}{\binom{s}{0}} & \frac{1}{\binom{s}{1}} & \cdots & \frac{1}{\binom{s}{s}}\end{array}\right],\\
\tilde{T}_k(f,g,h) &= \left[\begin{array}{ccc} T_{n-k}(f) & & T_{m-k}(g)\\ & T_{p-k}(f) & T_{m-k}(h)\\ T_{n-k}(h) & -T_{p-k}(g) & \end{array}\right], \quad \tilde{x}_k=\left[\begin{array}{c} v_{(k)}\\ w_{(k)}\\ -u_{(k)}\end{array}\right],
\end{aligned}$$

the matrices

$$T_r(f)\in\mathbb{R}^{(m+r+1)\times(r+1)},\ T_r(g)\in\mathbb{R}^{(n+r+1)\times(r+1)} \text{ and } T_r(h)\in\mathbb{R}^{(p+r+1)\times(r+1)}$$

are lower triangular and Tœplitz, and their non-zero entries contain the coefficients of $f(y), g(y)$ and $h(y)$, scaled by their combinatorial terms, respectively, as shown in (14), $u_{(k)}, v_{(k)}$ and $w_{(k)}$ are vectors of the coefficients of $u_{(k)}(y), v_{(k)}(y)$ and $w_{(k)}(y)$, which are defined in (8), (9) and (10), $\tilde{Q}_k$ is a square diagonal matrix of order $m+n+p-3k+3$,

$$\tilde{Q}_k = \text{diag}\begin{bmatrix} Q_{n-k} & Q_{p-k} & Q_{m-k} \end{bmatrix}, \tag{19}$$

and Q_s is defined in (16). These definitions of $\tilde{D}_k^{-1}, \tilde{T}_k(f,g,h)$ and $\tilde{Q}_k$ allow $\tilde{S}_k(f,g,h)$ to be written as

$$\tilde{S}_k(f,g,h) = \begin{bmatrix} C_{n-k}(f) & & C_{m-k}(g) \\ & C_{p-k}(f) & C_{m-k}(h) \\ C_{n-k}(h) & -C_{p-k}(g) & \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
C_{n-k}(f) &= D_{m+n-k}^{-1} T_{n-k}(f) Q_{n-k} \in \mathbb{R}^{(m+n-k+1)\times(n-k+1)}, \\
C_{m-k}(g) &= D_{m+n-k}^{-1} T_{m-k}(g) Q_{m-k} \in \mathbb{R}^{(m+n-k+1)\times(m-k+1)}, \\
C_{p-k}(f) &= D_{m+p-k}^{-1} T_{p-k}(f) Q_{p-k} \in \mathbb{R}^{(m+p-k+1)\times(p-k+1)}, \\
C_{m-k}(h) &= D_{m+p-k}^{-1} T_{m-k}(h) Q_{m-k} \in \mathbb{R}^{(m+p-k+1)\times(m-k+1)}, \\
C_{n-k}(h) &= D_{n+p-k}^{-1} T_{n-k}(h) Q_{n-k} \in \mathbb{R}^{(n+p-k+1)\times(n-k+1)}, \\
C_{p-k}(g) &= D_{n+p-k}^{-1} T_{p-k}(g) Q_{p-k} \in \mathbb{R}^{(n+p-k+1)\times(p-k+1)}.
\end{aligned} \tag{21}$$

The Sylvester matrix is defined by the condition $k=1$, $\tilde{S}(f,g,h) = \tilde{S}_1(f,g,h)$, and thus the Sylvester matrix of three polynomials is rectangular, which must be compared with the Sylvester matrix of two polynomials, which is square.

Theorem 4.2 shows that the degree t of the GCD of $f(y), g(y)$ and $h(y)$ can be computed from $\tilde{S}(f,g,h)$. The proof of the theorem is very similar to the proof of Theorem 4.1 and it is therefore omitted.

Theorem 4.2 The degree t of the GCD of $f(y), g(y)$ and $h(y)$ is equal to the largest integer k such that $\tilde{S}_k(f,g,h)$ is rank deficient,

$$\begin{aligned}
\text{rank } \tilde{S}_k(f,g,h) &< m+n+p-3k+3, \quad k = 1,\ldots,t, \\
\text{rank } \tilde{S}_k(f,g,h) &= m+n+p-3k+3, \quad k = t+1,\ldots,q,
\end{aligned}$$

where $q = \min(m,n,p)$.

The polynomials $u_{(k)}(y), v_{(k)}(y)$ and $w_{(k)}(y)$ are defined for $k = 1, \ldots, t$, and if this definition is extended to $k = 1, \ldots, q$, such that each polynomial is equal to the zero polynomial for $k = t+1, \ldots, q$,

$$u_{(k)}(y) = v_{(k)}(y) = w_{(k)}(y) \equiv 0, \qquad k = t+1, \ldots, q,$$

then it follows from (18) and the proof of Theorem 4.2 that $\tilde{S}_t(f, g, h)$ has unit rank loss, and

$$\tilde{S}_k(f, g, h)\tilde{x}_k = \left(\tilde{D}_k^{-1}\tilde{T}_k(f, g, h)\tilde{Q}_k\right)\tilde{x}_k = 0, \qquad k = 1, \ldots, q,$$

where $\tilde{x}_k$ is not unique for $k = 1, \ldots, t-1$, and $\tilde{x}_t$ contains the coefficients of $u_{(t)}(y), v_{(t)}(y)$ and $w_{(t)}(y)$ and is therefore unique up to a non-zero constant multiplier,

$$\tilde{x}_k = \begin{cases} \begin{bmatrix} v_{(k)}^T & w_{(k)}^T & -u_{(k)}^T \end{bmatrix}^T \neq 0, & k = 1, \ldots, t-1, \\ \begin{bmatrix} v_{(t)}^T & w_{(t)}^T & -u_{(t)}^T \end{bmatrix}^T \text{(coprime polynomials)}, & k = t, \\ \begin{bmatrix} v_{(k)}^T & w_{(k)}^T & -u_{(k)}^T \end{bmatrix}^T \equiv 0, & k = t+1, \ldots, q. \end{cases}$$

4.2 The Sylvester matrix for Variant 2

Variant 2 of the Sylvester matrix of three polynomials follows from the observation that any two of the three equations (11), (12) and (13) implies the third equation. This variant therefore leads to three Sylvester matrices and subresultant matrices, and their forms have the same structure as (20), but each form has two rows, not three rows, because the polynomials are considered in pairs, as shown by the three pairings in (3). It therefore follows that the polynomial pairs $((f, g), (f, h))$ yield the equation,

$$\bar{S}_k(f, g, h)\bar{x}_{k,1} = \begin{bmatrix} C_{n-k}(f) & & C_{m-k}(g) \\ & C_{p-k}(f) & C_{m-k}(h) \end{bmatrix} \begin{bmatrix} v_{(k)} \\ w_{(k)} \\ -u_{(k)} \end{bmatrix} = 0, \qquad (22)$$

the polynomial pairs $((f, g), (g, h))$ yield the equation,

$$\bar{S}_k(g,f,h)\bar{x}_{k,2} = \begin{bmatrix} C_{m-k}(g) & & C_{n-k}(f) \\ & C_{p-k}(g) & C_{n-k}(h) \end{bmatrix} \begin{bmatrix} u_{(k)} \\ w_{(k)} \\ -v_{(k)} \end{bmatrix} = 0, \tag{23}$$

and the polynomial pairs $((f,h),(g,h))$ yield the equation,

$$\bar{S}_k(h,g,f)\bar{x}_{k,3} = \begin{bmatrix} C_{n-k}(h) & & C_{p-k}(g) \\ & C_{m-k}(h) & C_{p-k}(f) \end{bmatrix} \begin{bmatrix} v_{(k)} \\ u_{(k)} \\ -w_{(k)} \end{bmatrix} = 0, \tag{24}$$

for $k = 1, \ldots, q$. The dimensions of the matrices, and the equations from which they are formed, are

$\bar{S}_k(f,g,h) \in \mathbb{R}^{(2m+n+p-2k+2)\times(m+n+p-3k+3)}$ is formed from (11) and (12),
$\bar{S}_k(g,f,h) \in \mathbb{R}^{(m+2n+p-2k+2)\times(m+n+p-3k+3)}$ is formed from (11) and (13),
$\bar{S}_k(h,g,f) \in \mathbb{R}^{(m+n+2p-2k+2)\times(m+n+p-3k+3)}$ is formed from (12) and (13),

and each matrix is a 2×3 block matrix, which must be compared with the Sylvester matrix of Variant 1, which is a 3×3 block matrix. Theorem 4.2 is also satisfied by the matrices $\bar{S}_k(f,g,h)$, $\bar{S}_k(g,f,h)$ and $\bar{S}_k(h,g,f)$, and its application to these matrices is stated in Theorem 4.3.

Theorem 4.3 The degree t of the GCD of $f(y), g(y)$ and $h(y)$ is equal to the largest integer k such that $\bar{S}_k(f,g,h)$, $\bar{S}_k(g,f,h)$ and $\bar{S}_k(h,g,f)$ are rank deficient,

$$\begin{aligned} \text{rank } \bar{S}_k(f,g,h) &< m+n+p-3k+3, \quad k = 1,\ldots,t, \\ \text{rank } \bar{S}_k(g,f,h) &< m+n+p-3k+3, \quad k = 1,\ldots,t, \\ \text{rank } \bar{S}_k(h,g,f) &< m+n+p-3k+3, \quad k = 1,\ldots,t, \end{aligned}$$

and

$$\begin{aligned} \text{rank } \bar{S}_k(f,g,h) &= m+n+p-3k+3, \quad k = t+1,\ldots,q, \\ \text{rank } \bar{S}_k(g,f,h) &= m+n+p-3k+3, \quad k = t+1,\ldots,q, \\ \text{rank } \bar{S}_k(h,g,f) &= m+n+p-3k+3, \quad k = t+1,\ldots,q. \end{aligned}$$

The matrices $\tilde{S}_k(f,g,h)$ and $(\bar{S}_k(f,g,h), \bar{S}_k(g,f,h), \bar{S}_k(h,g,f))$ have a partitioned structure and badly scaled polynomials $f(y), g(y)$ and $h(y)$ may therefore lead to incorrect results. These problems that arise from badly scaled polynomials also exist when the GCD of two polynomials is considered [6,24].

Example 4.1 Consider the Bernstein forms of the polynomials $f(y)$, $g(y)$ and $h(y)$ of degrees $m = 29$, $n = 19$ and $p = 18$ respectively, whose factored forms are

$$\begin{aligned}
f(y) = {} & (y - 9.2657984335)^2(y - 1.2657984335)^4(y - 0.41564897)^6 \times \\
& (y - 0.21657894)(y - 0.0654654561)^2(y + 0.7879734)^9 \times \\
& (y + 1.654987654)^2(y + 1.932654987)(y + 2.3549879)^2, \\
g(y) = {} & (y - 9.2657984335)^2(y - 1.75292)(y - 1.2657984335)^4 \times \\
& (y - 0.99851354877)^3(y - 0.21657894)(y - 0.0654654561)^2 \times \\
& (y + 0.1654988136)^4(y + 1.654987654)^2, \\
h(y) = {} & (y - 9.2657984335)^2(y - 1.2657984335)^4(y - 0.564987986958)^3 \times \\
& (y - 0.21657894)(y - 0.0654654561)^2(y + 0.778912324654)^2 \times \\
& (y + 1.654987654)^2(y + 1.75)^2,
\end{aligned}$$

and whose GCD $d_{f,g,h}(y)$ is of degree $t = 11$,

$$\begin{aligned}
d_{f,g,h}(y) = {} & (y - 9.2657984335)^2(y - 1.2657984335)^4(y - 0.21657894) \times \\
& (y - 0.0654654561)^2(y + 1.654987654)^2.
\end{aligned}$$

Noise was added to the coefficients of the polynomials, thereby forming the inexact polynomials $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$, such that the upper bound of the relative error in each coefficient was 10^{-9}. The coefficients of these polynomials are plotted in Figure 1 and it is seen they span about 10 orders of magnitude.

Figures 2, 3, 4 and 5 show the singular values of $\tilde{S}_k(\hat{f},\hat{g},\hat{h})$ and $(\bar{S}_k(\hat{f},\hat{g},\hat{h})$, $\bar{S}_k(\hat{g},\hat{f},\hat{h})$, $\bar{S}_k(\hat{h},\hat{g},\hat{f}))$, respectively. The degree of the GCD can be deduced from Figures 2, 4 and 5, with different levels of clarity, but Figure 3 yields a bad result because there does not exist a significant change in the smallest singular value between $k = 11$ and $k = 12$. $\square$

Example 4.1 shows that Variant 1 may return the correct value of t but it may be poorly defined, and one or more pairs of polynomials in Variant 2 may yield the correct result, but the other pair(s) of polynomials may yield incorrect results. The next section considers the combinatorial terms in the Sylvester matrices and their subresultant matrices, and it is shown that, even for moderate values of m, n and p, they span many orders of magnitude. This wide range of magnitudes is a cause of incorrect results when computations are performed on these matrices, and Section 6 considers operations on $f(y), g(y)$

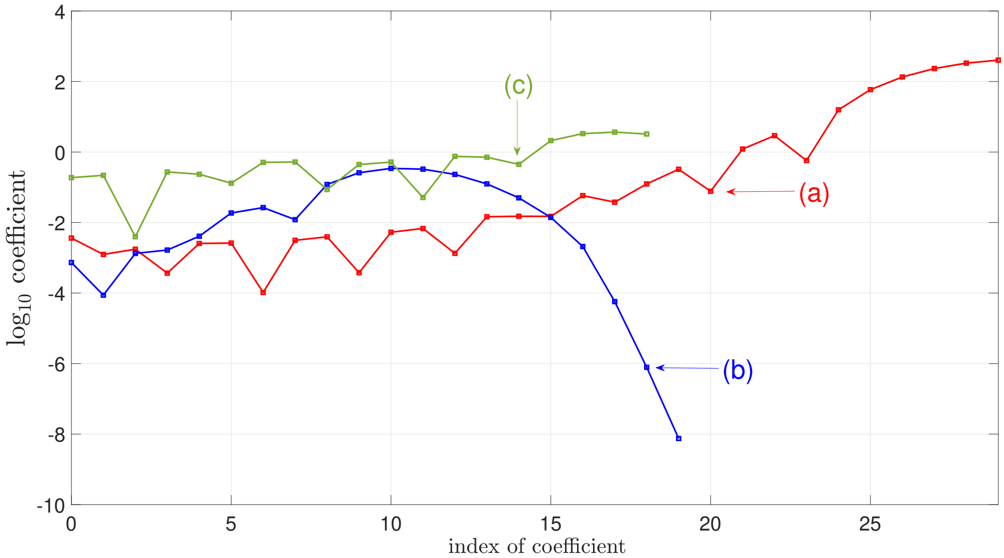

Fig. 1. The coefficients of (a) $\hat{f}(y)$, (b) $\hat{g}(y)$ and (c) $\hat{h}(y)$ for Example 4.1.

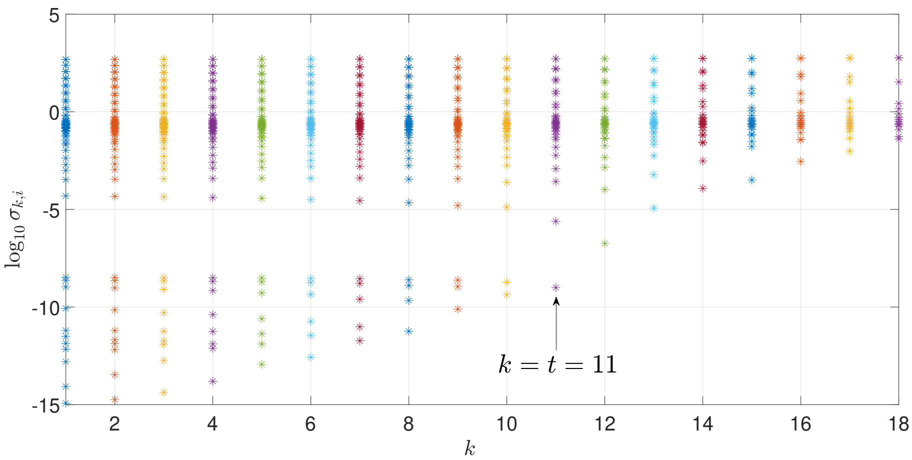

Fig. 2. The singular values $\sigma_{k,i}$ of $\tilde{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 4.1.

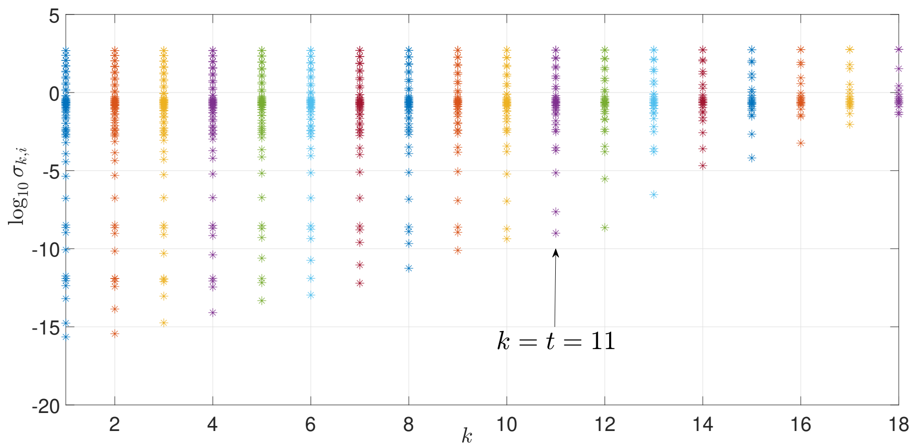

Fig. 3. The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 4.1.

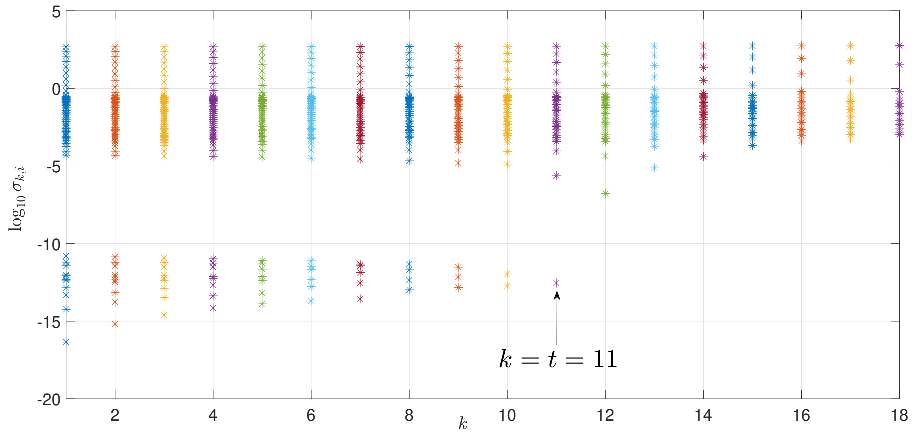

Fig. 4. The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{g}, \hat{f}, \hat{h})$ for Example 4.1.

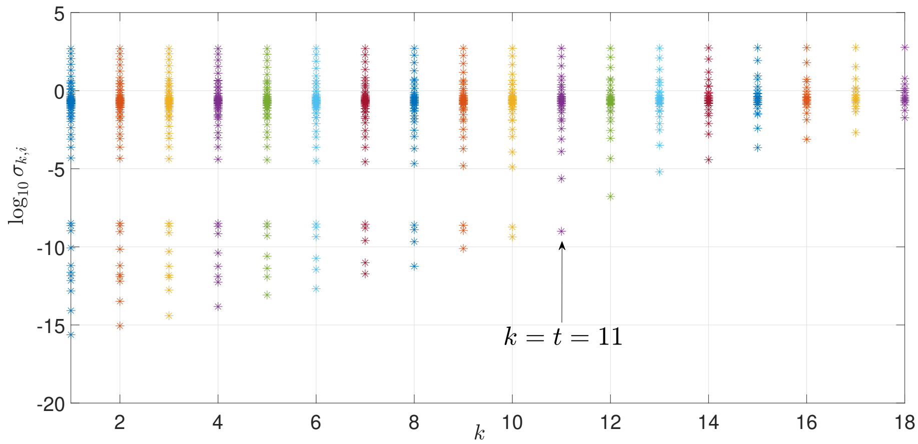

Fig. 5. The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{h}, \hat{g}, \hat{f})$ for Example 4.1.

and $h(y)$ that are implemented before computations are performed on their Sylvester matrices and subresultant matrices in order to minimise the adverse numerical effects of this wide range of magnitudes. The examples in Section 7 show that these preprocessing operations lead to significantly improved results.

5 The combinatorial terms in the Bernstein basis functions

It was shown in Sections 4.1 and 4.2 that the degree of the GCD of three Bernstein basis polynomials can be computed from Variants 1 and 2, respectively, of their Sylvester matrices and subresultant matrices. Furthermore, it follows from the form of $\tilde{S}_k(f, g, h)$ in (18) and the non-singular property of $\tilde{D}_k^{-1}$ and $\tilde{Q}_k$ that Theorem 4.2 can be applied to the matrices in the set

$$\left\{ \tilde{T}_k(f,g,h), \quad \tilde{D}_k^{-1}\tilde{T}_k(f,g,h), \quad \tilde{T}_k(f,g,h)\tilde{Q}_k, \quad \tilde{D}_k^{-1}\tilde{T}_k(f,g,h)\tilde{Q}_k \right\}. \quad (25)$$

It is clear that each polynomial pair in Variant 2 can also be expressed in the forms (25). For example, it follows from (21) and (22) that

$$\bar{S}_k(f,g,h) = \bar{D}_k^{-1}\bar{T}_k(f,g,h)\bar{Q}_k,$$

where

$$\bar{D}_k^{-1} = \begin{bmatrix} D_{m+n-k}^{-1} & \\ & D_{m+p-k}^{-1} \end{bmatrix},$$

$$\bar{T}_k(f,g,h) = \begin{bmatrix} T_{n-k}(f) & & T_{m-k}(g) \\ & T_{p-k}(f) & T_{m-k}(h) \end{bmatrix},$$

and $\bar{Q}_k = \tilde{Q}_k$ where $\tilde{Q}_k$ is defined in (19). The forms (25) of $\bar{S}_k(f,g,h)$ are therefore

$$\left\{ \bar{T}_k(f,g,h),\ \ \bar{D}_k^{-1}\bar{T}_k(f,g,h),\ \ \bar{T}_k(f,g,h)\bar{Q}_k,\ \ \bar{D}_k^{-1}\bar{T}_k(f,g,h)\bar{Q}_k \right\}, \tag{26}$$

and each of these matrices can, in theory, be used to compute the degree of the GCD of $f(y), g(y)$ and $h(y)$. The forms (26) of $\bar{S}_k(g,f,h)$ and $\bar{S}_k(h,g,f)$, which are defined in (23) and (24) respectively, for this computation follow similarly.

The combinatorial terms in the four forms (25) of $\tilde{S}_k(f,g,h)$ (Variant 1), and the four forms (26) of $\bar{S}_k(f,g,h)$ (Variant 2) and their extensions to $\bar{S}_k(g,f,h)$ and $\bar{S}_k(h,g,f)$, are different, but it is adequate to consider one set of matrices because the results for the other set of matrices follow identically. In particular, it follows from (20) and (21) that a typical block matrix in $\tilde{S}_k(f,g,h)$ is $C_{n-k}(f) = D_{m+n-k}^{-1}T_{n-k}(f)Q_{n-k}$, and the combinatorial terms in the matrix products in $C_{n-k}(f)$, that is, the matrices,

$$\left\{ T_{n-k}(f),\ \ D_{m+n-k}^{-1}T_{n-k}(f),\ \ T_{n-k}(f)Q_{n-k},\ \ D_{m+n-k}^{-1}T_{n-k}(f)Q_{n-k} \right\},$$

are, respectively,

$$\left\{ \binom{m}{i},\quad \frac{\binom{m}{i}}{\binom{m+n-k}{i+j}},\quad \binom{m}{i}\binom{n-k}{j},\quad \frac{\binom{m}{i}\binom{n-k}{j}}{\binom{m+n-k}{i+j}} \right\},$$

for $i = 0, \dots, m$, and $j = 0, \dots, n-k$. Consideration of each term shows that the ratio of the maximum value to the minimum value attains its minimum value for the term $\binom{m}{i}\binom{n-k}{j}/\binom{m+n-k}{i+j}$, that is, the matrix $D_{m+n-k}^{-1}T_{n-k}(f)Q_{n-k}$.

It follows that the adverse numerical effects of the combinatorial terms are minimised when the forms

$$\left\{ \tilde{S}_k(f,g,h),\ \ \bar{S}_k(f,g,h),\ \ \bar{S}_k(g,f,h),\ \ \bar{S}_k(h,g,f) \right\},$$

of the Sylvester matrix and its subresultant matrices are used. Improved results are obtained when $f(y), g(y)$ and $h(y)$ are processed before these matrices are formed, and these preprocessing operations are discussed in Section 6.

6 Preprocessing operations

Example 4.1 shows that the direct use of Variants 1 and 2 for the computation of the degree of the GCD of three Bernstein basis polynomials may lead to incorrect results, even in the absence of noise. Similar bad results are obtained when the GCD of two Bernstein basis polynomials is considered, and it is shown in [6,7] that processing the polynomials by three operations before computations are performed on their Sylvester matrix and its subresultant matrices yields a significant improvement in the results. This section considers these operations for the computation of the GCD of three polynomials.

Operation 1 The non-zero entries associated with $f(y), g(y)$ and $h(y)$ in each Sylvester matrix and subresultant matrix are normalised by their geometric mean in order to balance each block matrix (21) in the larger matrices $\tilde{S}_k(f,g,h)$ and $(\bar{S}_k(f,g,h), \bar{S}_k(g,f,h), \bar{S}_k(h,g,f))$.

Operation 2 The GCD of two or more polynomials is defined to within an arbitrary non-zero scalar multiplier, and thus two of the polynomials, $f(y)$ and $h(y)$, are multiplied by constants λ_k and ρ_k respectively,

$$\text{GCD}\,(f,g,h) \sim \text{GCD}\,(\lambda_k f, g, \rho_k h), \qquad k = 1, \ldots, q,$$

where $\sim$ denotes equivalence to within an arbitrary non-zero constant, and $f(y), g(y)$ and $h(y)$ are normalised by their geometric means. The subscript k is included in the constants λ_k and ρ_k because their optimal values must be calculated for each value of k, that is, for each subresultant matrix.

Operation 3 The substitution,

$$y = \theta_k \omega, \qquad k = 1, \ldots, q, \tag{27}$$

is made, where ω is the new independent variable and θ_k is a constant to be determined. The subscript k is included in θ_k because, like λ_k and ρ_k, its optimal value must be calculated for each subresultant matrix. The substitution (27) implies that all computations are performed in a basis that is closely related to, but distinct from, the Bernstein basis. These basis functions are, for a polynomial of degree m,

$$\left\{ \theta_k^i \binom{m}{i} (1-\theta_k \omega)^{m-i} \omega^i \right\}_{i=0}^{m}, \qquad k = 1, \ldots, q,$$

and thus a different basis is used for the Sylvester matrix and each subresultant matrix.

The first and second operations yield the polynomials $\lambda_k \dot{f}(y), \dot{g}(y)$ and $\rho_k \dot{h}(y)$,

$$\lambda_k \dot{f}(y) = \frac{\lambda_k f(y)}{G_k(f)}, \quad \dot{g}(y) = \frac{g(y)}{G_k(g)}, \quad \rho_k \dot{h}(y) = \frac{\rho_k h(y)}{G_k(h)}, \qquad k = 1, \ldots, q,$$

where $G_k(s)$ is the geometric mean of the non-zero entries in the partition of the kth subresultant matrix that contains the coefficients of the polynomial $s(y)$. For example, it follows from (20) and (21) that $f(y)$ occurs in the matrices $C_{n-k}(f)$ and $C_{p-k}(f)$ in $\tilde{S}_k(f, g, h)$, and thus

$$G_k(f) = \left(\prod_{j=0}^{n-k} \prod_{i=0}^{m} \frac{a_i \binom{m}{i} \binom{n-k}{j}}{\binom{m+n-k}{i+j}} \times \prod_{j=0}^{p-k} \prod_{i=0}^{m} \frac{a_i \binom{m}{i} \binom{p-k}{j}}{\binom{m+p-k}{i+j}} \right)^{\frac{1}{(m+1)(n+p-2k+2)}}, \tag{28}$$

is the geometric mean of the entries associated with $f(y)$ in the kth Sylvester subresultant matrix for $\tilde{S}_k(f, g, h)$ (Variant 1). The geometric means associated with $g(y)$ and $h(y)$, and these means for $f(y), g(y)$ and $h(y)$ for each of the forms (25) for this variant, are calculated in a similar manner. These computations for the different forms of the Sylvester matrix and its subresultant matrices for Variant 2, ($\bar{S}(f, g, h)$, $\bar{S}(g, f, h)$, $\bar{S}(h, g, f)$), and for each matrix in the set (26), follow identically.

The optimal values λ_k^*, ρ_k^* and θ_k^* of, respectively, λ_k, ρ_k and θ_k are obtained from the solution of a linear programming that is very similar to its form for the preprocessing operations for the two-polynomial GCD problem [6,7]. The polynomials that arise from the preprocessing operations are, for $k = 1, \ldots, q$,

$$\begin{aligned}
\lambda_k^* \ddot{f}(\omega, \theta_k^*) &= \lambda_k^* \sum_{i=0}^{m} \bar{a}_{k,i} \, (\theta_k^*)^i \binom{m}{i} (1-\theta_k^* \omega)^{m-i} \, \omega^i, \\
\ddot{g}(\omega, \theta_k^*) &= \sum_{i=0}^{n} \bar{b}_{k,i} \, (\theta_k^*)^i \binom{n}{i} (1-\theta_k^* \omega)^{n-i} \, \omega^i, \\
\rho_k^* \ddot{h}(\omega, \theta_k^*) &= \rho_k^* \sum_{i=0}^{p} \bar{c}_{k,i} \, (\theta_k^*)^i \binom{p}{i} (1-\theta_k^* \omega)^{p-i} \, \omega^i,
\end{aligned} \tag{29}$$

where

$$\bar{a}_{k,i} = \frac{a_i}{G_k(f)}, \qquad i = 0, \dots, m,$$

$$\bar{b}_{k,i} = \frac{b_i}{G_k(g)}, \qquad i = 0, \dots, n,$$

$$\bar{c}_{k,i} = \frac{c_i}{G_k(h)}, \qquad i = 0, \dots, p,$$

a_i, b_i and c_i are defined in (4), (5) and (6), respectively, the geometric mean $G_k(f)$ is given in (28), and the geometric means $G_k(g)$ and $G_k(h)$ are calculated similarly. All computations are performed on the Sylvester matrices and subresultant matrices formed from the polynomials (29).

Algorithm 1 shows the algorithm for the computation of the degree of the GCD of three polynomials.

Algorithm 1 Degree of the GCD of three polynomials

Input
(1) Polynomials $f(y), g(y)$ and $h(y)$ of degrees m, n and p respectively
(2) The variant $Y(f, g, h)$ of the four variants of the Sylvester matrix and its subresultants used for the GCD computation
Output The degree of the GCD of $f(y), g(y)$ and $h(y)$

$q \leftarrow \min(m, n, p)$
% Loop over the q subresultant matrices
for $k \leftarrow 1, q$ **do**
  (i) Compute λ_k^*, ρ_k^* and θ_k^*, the optimal values of λ_k, ρ_k and θ_k
  (ii) Form the polynomials $\lambda_k^* \ddot{f}(\omega, \theta_k^*)$, $\ddot{g}(\omega, \theta_k^*)$ and $\rho_k^* \ddot{h}(\omega, \theta_k^*)$
  (iii) Form the matrix $Y_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$ and calculate the number of columns of this matrix, $c = m + n + p - 3k + 3$
  (iv) Calculate the singular values $\sigma_{k,i}, i = 1, \dots, c$, of $Y_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$
end for
% Calculate $t = \deg \text{GCD}\,(f, g, h)$ from the singular values $\sigma_{k,i}$
$sv(i) \leftarrow 0,\ i = 1, \dots, q$
for $k \leftarrow 1, q$ **do**
  $sv(k) \leftarrow \frac{\max_i \{\sigma_{k,i}\}}{\min_i \{\sigma_{k,i}\}}$
end for
% Calculate the ratio of successive entries of sv
for $k \leftarrow 1, q - 1$ **do**
  $ratiosv(k) \leftarrow \frac{sv(k)}{sv(k+1)}$
end for
% Calculate the degree of the GCD of $f(y), g(y)$ and $h(y)$
$t \leftarrow \arg\max_k \{ratiosv(k)\}$

---

7 Results

This section contains two examples that show the results from Variants 1 and 2 for the computation of the degree of the GCD of three Bernstein basis polynomials. Results that show the effects of the preprocessing operations and the addition of noise to the polynomials are included.

Example 7.1 Consider the Bernstein forms of the polynomials $f(y)$, $g(y)$ and $h(y)$, of degrees $m = 12, n = 36$ and $p = 15$, whose factored forms are

$$\begin{aligned} f(y) = &(y - 0.5654654561)^5(y - 0.21657894)(y - 0.01564897)^2 \times \\ &(y + 0.2468796514)^3(y + 0.7879734) \\ g(y) = &(y - 0.99851354877)^7(y - 0.75292)^{20}(y - 0.5654654561)^5 \times \\ &(y - 0.21657894)(y + 0.2468796514)^3 \\ h(y) = &(y - 0.5654654561)^5(y - 0.21657894)(y + 0.2468796514)^3 \times \\ &(y + 0.778912324654)^4(y + 1.75)^2, \end{aligned}$$

for which $q = \min(12, 36, 15) = 12$ and whose GCD $d_{f,g,h}(y)$ is of degree $t = 9$,

$$d_{f,g,h}(y) = (y - 0.5654654561)^5(y - 0.21657894)(y + 0.2468796514)^3.$$

It is noted that $g(y)$ has four distinct roots in the unit interval, and one of these roots is of multiplicity 20. Noise was added to the coefficients of the polynomials $f(y), g(y)$ and $h(y)$, thereby forming the polynomials $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$ whose coefficients are

$$\begin{aligned} \hat{a}_i &= a_i + a_i\varepsilon_i r_i, \qquad i = 0, \dots, m, \\ \hat{b}_j &= b_j + b_j\varepsilon_j r_j, \qquad j = 0, \dots, n, \\ \hat{c}_l &= c_l + c_l\varepsilon_l r_l, \qquad l = 0, \dots, p, \end{aligned} \tag{30}$$

where $\varepsilon_i, \varepsilon_j$ and ε_l are uniformly distributed random variables in the interval $\mathcal{I} = [10^{-7}, 10^{-4}]$, and r_i, r_j and r_l are uniformly distributed random variables in the interval $[-1, 1]$. The inclusion of the interval $\mathcal{I}$ for the upper bound of the relative errors provides a stringent test for the computation of the degree of the GCD of $f(y), g(y)$ and $h(y)$ since it implies that a threshold for distinguishing between the non-zero and zero singular values of the subresultant matrices cannot be applied.

The polynomials $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$ were processed, thus forming the polynomials $\lambda_k^* \ddot{f}(\omega, \theta_k^*), \ddot{g}(\omega, \theta_k^*)$ and $\rho_k^* \ddot{h}(\omega, \theta_k^*)$, $k = 1, \dots, 12$, and the coefficients of these polynomials span a much smaller range, by several orders of magnitude, than the coefficients of the unprocessed polynomials. Figures 6 and 7 show, respectively, the singular values of $\bar{S}_k(\hat{f},\hat{g},\hat{h})$ and $\bar{S}_k(\lambda_k^*\ddot{f},\ddot{g},\rho_k^*\ddot{h})$, and it is clear that the unprocessed polynomials return an incorrect result, but the correct result ($t=9$) is obtained with the processed polynomials, and that the result is clearly defined. □

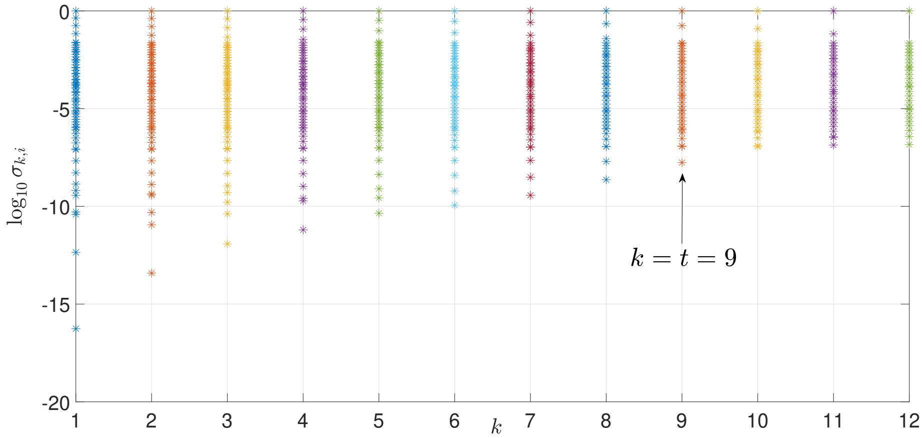

Fig. 6. The singular values of $\bar{S}_k(\hat{f},\hat{g},\hat{h})$ for Example 7.1.

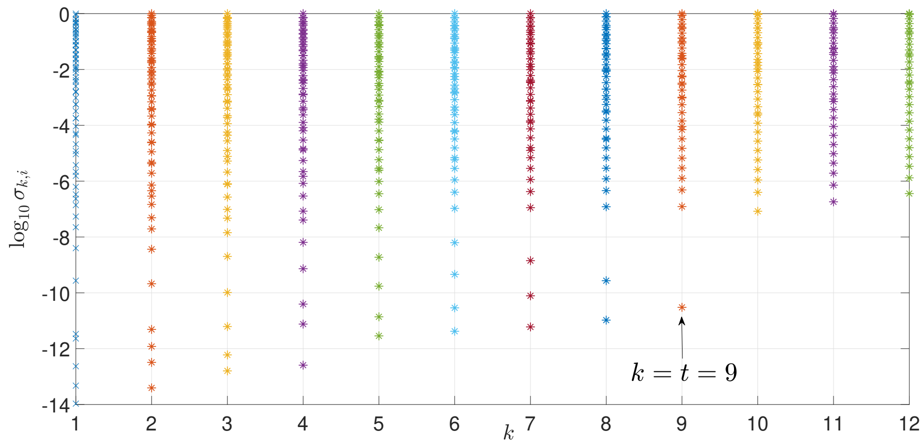

Fig. 7. The singular values of $\bar{S}_k(\lambda_k^*\ddot{f},\ddot{g},\rho_k^*\ddot{h})$ for Example 7.1.

Example 7.2 Consider the Bernstein forms of the exact polynomials $f(y)$, $g(y)$ and $h(y)$, of degrees $m=24, n=25$ and $p=24$, whose factored forms are

$$\begin{aligned} f(y) &= (y-1.46)^2(y-1.37)^3(y-1.20)(y-0.82)^3(y-0.75)^5 \times \\ &\quad (y-0.56)^4(y-0.10)^2(y+0.27)^4, \\ g(y) &= (y-0.99)^4(y-0.12)^4(y+0.20)^3(y-0.10)^2(y-0.56)^4 \times \\ &\quad (y-0.75)^5(y-1.37)^3, \\ h(y) &= (y-1.37)^3(y-0.75)^5(y-0.72)^8(y-0.56)^4(y-0.10)^2 \times \\ &\quad (y+0.75)^2, \end{aligned}$$

and whose GCD $d_{f,g,h}(y)$ is of degree $t = 14$,

$$d_{f,g,h}(y) = (y - 0.10)^2(y - 0.56)^4(y - 0.75)^5(y - 1.37)^3.$$

Noise was added to the coefficients of $f(y), g(y)$ and $h(y)$, as shown in (30), thus yielding the polynomials $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$, where $\varepsilon_i, \varepsilon_j$ and ε_l are uniformly distributed random variables in the interval $\mathcal{I} = [10^{-6}, 10^{-4}]$. As in Example 7.1, this interval for the upper bound of the relative errors of the coefficients provides a stringent test for the computation of the degree of the GCD of $f(y), g(y)$ and $h(y)$. The polynomials $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$ were processed, thereby forming the polynomials $\lambda_k^* \ddot{f}(\omega, \theta_k^*), \ddot{g}(\omega, \theta_k^*)$ and $\rho_k^* \ddot{h}(\omega, \theta_k^*)$, $k = 1, \ldots, 24$. Figures 8 and 9 show the singular values of $\bar{S}_k(\hat{f}, \hat{g}, \hat{h})$ and $\bar{S}_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$, and as for Example 7.1, the importance of the preprocessing operations is clear because the degree of the GCD cannot be deduced from the subresultant matrices of the unprocessed polynomials, but the correct result $(t = 14)$ is obtained when $\hat{f}(y), \hat{g}(y)$ and $\hat{h}(y)$ are processed. Also, the rank loss of $\bar{S}_1(\hat{f}, \hat{g}, \hat{h})$ is two, which suggests that the degree of the GCD is two, which is incorrect, but the rank loss of $\bar{S}_1(\lambda_1^* \ddot{f}, \ddot{g}, \rho_1^* \ddot{h})$ is 14, which is correct. The results for $\tilde{S}_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$ (Variant 1), and the matrices $\bar{S}_k(\ddot{g}, \lambda_k^* \ddot{f}, \rho_k^* \ddot{h})$ and $\bar{S}_k(\rho_k^* \ddot{h}, \ddot{g}, \lambda_k^* \ddot{f})$ (Variant 2), were very similar to the result for $\bar{S}_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$, which is consistent with the results in Example 7.1.

Figure 8 shows that the coefficients of the coprime polynomials and GCD cannot be computed from the unprocessed polynomials because the degree of the GCD cannot be determined from these polynomials. Figure 9 shows, however, that these coefficients can be computed from the processed polynomials $\lambda_t^* \ddot{f}(\omega, \theta_t^*), \ddot{g}(\omega, \theta_t^*)$ and $\rho_t^* \ddot{h}(\omega, \theta_t^*)$, $t = 14$, and thus the subresultant matrix $\bar{S}_t(\lambda_t^* \ddot{f}, \ddot{g}, \rho_t^* \ddot{h})$ was used to calculate these coefficients, after the polynomials were transformed, using (27) with $\theta_k = \theta_t^*$, from the independent variable ω to the independent variable y. This calculation reduced to the solution of a least squares problem, and the relative errors, defined in the 2-norm, in the coefficient vector of these polynomials are shown in Table 1. It is seen that the errors lie in the interval $\mathcal{I}$ of the noise levels $\varepsilon_i, \varepsilon_j$ and ε_l. $\square$

Polynomial	Error
$d_{f,g,h}(y)$	8.504447e-06
$u_{(t)}(y)$	1.744320e-05
$v_{(t)}(y)$	8.334257e-05
$w_{(t)}(y)$	3.447570e-05

Table 1
The relative errors in the GCD $d_{f,g,h}(y)$ and the coprime polynomials, $u_{(t)}(y), v_{(t)}(y)$ and $w_{(t)}(y)$ with $\varepsilon_i, \varepsilon_j$ and ε_l in the interval $\left[10^{-6}, 10^{-4}\right]$, for Example 7.2.

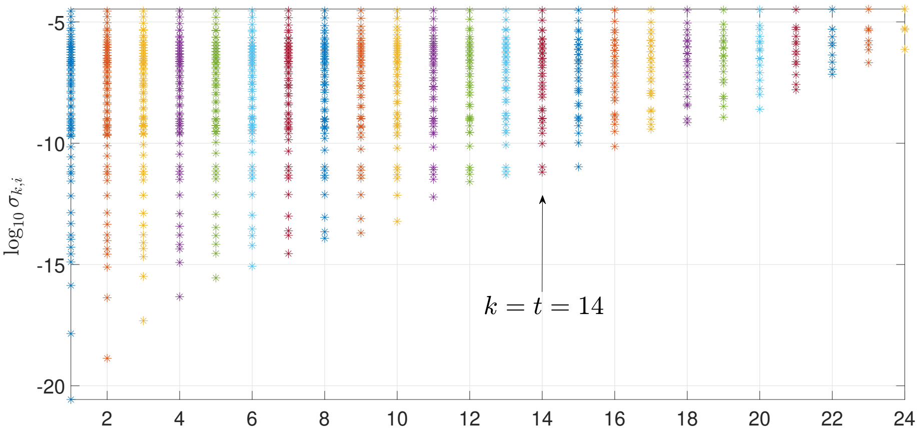

Fig. 8. The singular values of $\bar{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 7.2.

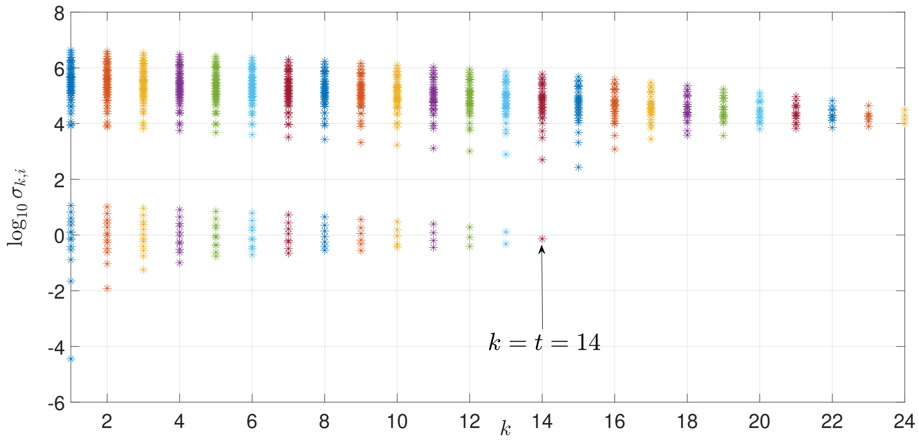

Fig. 9. The singular values of $\bar{S}_k(\lambda_k^* \ddot{f}, \ddot{g}, \rho_k^* \ddot{h})$ for Example 7.2.

These examples are typical of many other examples because they show the significant improvement in the results when the polynomials are processed before computations are performed on their Sylvester matrices and subresultant matrices. This improvement manifests itself in a large gap between the non-zero and zero singular values of all forms of these matrices, such that the degree of the GCD is clearly defined, including in the presence of noise. This good result for the degree of the GCD yielded good results for the coefficients of the GCD and coprime polynomials, even when a least squares solution of a linear algebraic equation is used and the structure of the coefficient matrix in the equation is not preserved. It is, however, expected that better results will be obtained when this structure is preserved, as shown in [7] for the computation of the coefficients of the GCD of two polynomials.

8 Summary

This paper has considered the computation of the degree of the GCD of three Bernstein basis polynomials by their Sylvester matrices and subresultant matrices. These matrices take two forms, denoted Variants 1 and 2, and the best form of each variant is obtained by postmultiplying the standard form of the Sylvester matrix by a diagonal matrix of combinatorial terms. One variant yields a 3×3 block matrix and the other variant yields three 2×3 block matrices.

The Sylvester matrix and its subresultant matrices for both variants may return different values of the degree of the GCD of three polynomials, or they may return an indeterminate result, particularly when the polynomials are badly scaled. Significantly improved results are obtained when the polynomials are processed by three operations before computations are performed on their Sylvester matrices and subresultant matrices because the degree of the GCD is then clearly defined.

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