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Quasimonotone graphs[☆]

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ABSTRACT

For any class $\mathcal{C}$ of bipartite graphs, we define quasi-$\mathcal{C}$ to be the class of all graphs G such that every bipartition of G belongs to $\mathcal{C}$. This definition is motivated by a generalisation of the switch Markov chain on perfect matchings from bipartite graphs to nonbipartite graphs. The monotone graphs, also known as bipartite permutation graphs and proper interval bigraphs, are such a class of bipartite graphs. We investigate the structure of quasi-monotone graphs and hence construct a polynomial time recognition algorithm for graphs in this class.

1. Introduction

In [6] (with Jerrum) and [7] we considered the *switch* Markov chain on perfect matchings in bipartite and nonbipartite graphs. This chain repeatedly replaces two matching edges with two non-matching edges involving the same four vertices, if possible. (See [6,7] for details.) We considered the ergodicity and mixing properties of the chain.

In particular, we proved in [6] that the chain is *rapidly mixing* (i.e. converges in polynomial time) on the class of *monotone* graphs. This class of bipartite graphs was defined by Diaconis, Graham and Holmes in [5], motivated by statistical applications of perfect matchings. The biadjacency matrices of graphs in the class have a "staircase" structure. Diaconis et al. conjectured the rapid mixing property shown in [6]. We also showed in [6] that this class is, in fact, identical to the known class of *bipartite permutation* graphs [15], which is itself known to be identical to the class of *proper interval bigraphs* [10], see also [4].

In extending the work of [6] to nonbipartite graphs in [7], we showed that the rapid mixing proof for monotone graphs extends easily to a class of graphs which includes, besides the monotone graphs themselves, all *proper*, or *unit*, *interval graphs* [1]. In this class the bipartite graph given by the cut between any bipartition of the vertices of the graph must be a monotone graph. We called these graphs *quasimonotone*.

In fact, "quasi-" is an operator on bipartite graph classes, and can be applied more generally. In this view, quasimonotone graphs are quasi-monotone graphs, as formally defined in Section 2, and discussed in Section 2.1.

For any class of bipartite graphs that is recognisable in polynomial time, the definition of its quasi-class implies membership in co-$\mathbb{NP}$ and deterministically only an exponential time recognition algorithm. Thus an immediate question is whether we can recognise the quasi-class in polynomial time. The main contribution of this paper is a proof that quasimonotone graphs have a polynomial time recognition algorithm.

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1.1. Definitions and notation

If G is a graph, we will denote its vertex set by $V[G]$, and its edge set by $E[G]$. If $U \subseteq V[G]$, then $G[U]$ will denote the subgraph induced by U. To ease the notation we do not distinguish between U and the subgraph it induces in G where this does not cause ambiguity. So a cycle in G is either a subgraph or the set of its vertices. Similarly, we will write $G = H$ when G is isomorphic to H.

A subgraph of $G = (V, E)$ is a *cycle* in G if it is connected and 2-regular. The *length* or *size* of a cycle is the number of its edges (or vertices). A *chord* of a cycle (U, F) in (V, E) is an edge in $U^{(2)} \cap E \setminus F$. A chord in a cycle of even length is *odd* if the distance between its endpoints on the cycle is odd. That is, an *odd chord* splits an even cycle into two smaller cycles of even length. An *even chord* splits an even cycle into two smaller cycles of odd length.

A *hole* in a graph is a chordless cycle of length at least five. A cycle of length three is a *triangle*, and a cycle of length four a *quadrangle*. A hole is *odd* if it has an odd number of vertices, otherwise *even*. Let HoleFree be the class of graphs without a hole, and EvenHoleFree the class of graphs without even holes. For the purposes of this paper, a *long* hole will be defined as an odd hole of size at least 7.

A *bipartition* L, R of a set V is such that $L \subseteq V$ and $R = V \setminus L$. Then, if $G = (V, E)$ is any graph, the graph $G[L{:}R]$ is the bipartite graph with vertex bipartition L, R, and edge set the cut $L{:}R = \{xy \in E : x \in L, y \in R\}$. We refer to $G[L{:}R]$ as a bipartition of G.

The *distance* $\mathrm{dist}(u, v)$ between two vertices u and v is the length of a shortest $(u, \ldots, v)$ path in G. If H is an induced subgraph of G, and $x, y \in H$, we denote the distance from x to y in H by $\mathrm{dist}_H(x, y)$. If $v \in V$, $\mathrm{dist}(v, H)$ is the minimum distance $\mathrm{dist}(v, w)$ from v to any vertex $w \in H$. The maximum distance between two vertices in G is the *diameter* of G.

If $G = (V, E)$ and $v \in V$, we denote the neighbourhood of v by $N(v)$, and $N(v) \cup \{v\}$ by $N[v]$.

1.2. Structure of the paper

The focus of the paper, of which [8] is an extended abstract, is on the class of quasimonotone graphs. In Section 2 we discuss a generalisation of the construction of the class and some immediate properties. In Section 2.1 we give some examples.

Section 3 shows that quasimonotone graphs can be recognised in polynomial time. We begin, in Section 3.1, by proving some properties of quasimonotone graphs for later use, using the characterisation of monotone graphs by forbidden induced subgraphs. The anticipated recognition algorithm first looks for flaws (defined in 3.1) and then branches into different procedures depending on the length of a short hole (defined in 3.3) in the input graph. We describe how to find such a hole in 3.3. The remaining forbidden subgraphs are pre-holes, also defined in 3.1.

Sections 4 and 5 deal with graphs containing a long hole. Again we start with some technical lemmas showing that the long hole enforces an annular structure in the absence of flaws. The structure is determined by *splitting*, described in 5.1. Possible pre-holes must wind round this annulus once or twice. We complete the process by checking for pre-holes, using a procedure given in 5.2.

If there is no long hole we can list all the triangles and 5-holes in the input graph. In this case a shortest pre-hole consists of two of these odd cycles and two vertex-disjoint paths between them. We describe this in more detail in Section 6.

Section 7 summarises the algorithm with a formal description, and discusses its running time.

In Section 8, we give a short discussion of a central question raised in the paper, recognising a pre-hole in an arbitrary graph. Though we do not settle this question, we show that a related question is $\mathbb{NP}$-complete. That is, given a graph, is it a pre-hole?

Finally, Section 9 concludes the paper.

2. Quasi-classes and pre-graphs

A *hereditary* class of graphs is closed under taking induced subgraphs. Let Bipartite denote the class of bipartite graphs, and let $\mathcal{C} \subseteq$ Bipartite. Then we will say that the graph G is *quasi-*$\mathcal{C}$ if $G[L{:}R] \in \mathcal{C}$ for all bipartitions L, R of V.

Lemma 1. *If $\mathcal{C} \subseteq$ Bipartite is a hereditary class that is closed under disjoint union then $\mathcal{C} =$ Bipartite $\cap$ quasi-$\mathcal{C}$.*

Proof. First let $G = (L \cup R, E)$ be any bipartite graph that does not belong to $\mathcal{C}$. Since $G = G[L{:}R]$ the graph G does not belong to quasi-$\mathcal{C}$. Hence $\mathcal{C} \supseteq$ Bipartite $\cap$ quasi-$\mathcal{C}$.

Next we show $\mathcal{C} \subseteq$ Bipartite $\cap$ quasi-$\mathcal{C}$. Let $G = (X \cup Y, E)$ be a graph in $\mathcal{C}$ and let $L{:}R$ be a bipartition of $X \cup Y$. Now $G[L{:}R]$ is the disjoint union of $G_1 = G[(X \cap L) \cup (Y \cap R)]$ and $G_2 = G[(X \cap R) \cup (Y \cap L)]$. The graphs G_1 and G_2 belong to $\mathcal{C}$ since the class is hereditary, and hence $G[L{:}R]$ is in $\mathcal{C}$ because $\mathcal{C}$ is closed under disjoint union. Thus $G \in$ quasi-$\mathcal{C}$. □

A hereditary graph class can be characterised by a set $\mathcal{F}$ of forbidden subgraphs. The set $\mathcal{F}$ is minimal if no graph in $\mathcal{F}$ contains any other as an induced subgraph.

For a bipartite graph H, a graph $G = (V, E)$ is a *pre-*H if there is a bipartition L, R of V such that $G[L{:}R] = H$. In this case H is a spanning subgraph of G. Clearly any bipartite H is itself a pre-H.

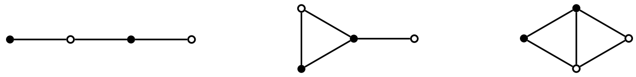


Fig. 1. The pre-P_4's: the path P_4, the paw and the diamond.

Lemma 2. *If $\mathcal{C} \subseteq$ Bipartite is characterised by a set $\mathcal{F}$ of forbidden induced subgraphs, let pre-$\mathcal{F} = \{$pre-$H \mid H \in \mathcal{F}\}$. Then quasi-$\mathcal{C}$ is characterised by the set of forbidden induced subgraphs pre-$\mathcal{F}$.*

Proof. Suppose $G = (V, E)$ contains $H' = (V', E')$, a pre-H for some $H \in \mathcal{F}$. Then V' has a bipartition L', R' such that $H'[L'\colon R'] = H$. Extending L', R' to a bipartition L, R of V, $G[L{:}R]$ contains H. Then $G[L{:}R] \notin \mathcal{C}$, so $G \notin$ quasi-$\mathcal{C}$. Conversely, if $G \in$ quasi-$\mathcal{C}$, every $G[L{:}R] \in \mathcal{C}$, so no $G[L{:}R]$ contains H, for any $H \in \mathcal{F}$. Thus G contains no pre-H, for any $H \in \mathcal{F}$, that is, no $H' \in$ pre-$\mathcal{F}$. □

Note, however, that pre-$\mathcal{F}$ may not be minimal for quasi-$\mathcal{C}$ when $\mathcal{F}$ is minimal for $\mathcal{C}$.

2.1. Examples

The class quasi-Bipartite is clearly the set of all graphs.

If $\mathcal{C}$ is the class of complete bipartite graphs, it is easy to see that quasi-$\mathcal{C}$ is the class of complete graphs. Note however, that this class is not closed under disjoint union. Now, if $\mathcal{C}$ becomes the class of graphs for which every component is complete bipartite, then quasi-$\mathcal{C}$ is the class of graphs without P_4, paw or diamond. These three graphs are the pre-P_4's, see Fig. 1. To see this we observe the following:

- If a graph G contains a pre-P_4 then there is a bipartition $G[L{:}R]$ that contains a P_4 as induced subgraph. A connected component of $G[L{:}R]$ containing such a P_4 is not complete bipartite.
- Now G does not contain a pre-P_4. If a connected component H of a bipartition of G is not complete bipartite, then H contains a P_4, contradicting the fact that G does not contain a pre-P_4.

Bounded-degree graphs give another example. If $\mathcal{C}_d$ is the class of bipartite graphs with degree at most d, for a fixed integer $d > 0$, then quasi-$\mathcal{C}_d$ is the class of all graphs with degree at most d. To see this note that, if v has degree at most d in G, then v has degree at most d in any $G[L{:}R]$. Conversely, if v has degree $d' > d$, then v also has degree d' in any $G[L{:}R]$ such that $v \in L$, $N(v) \subseteq R$. The unique forbidden subgraph for $\mathcal{C}_d$ is clearly the star $K_{1,d+1}$. Therefore, the class quasi-$\mathcal{C}_d$ is characterised by forbidding pre-$K_{1,d+1}$'s, a set with size $O(d^2)$. Hence quasi-$\mathcal{C}_d$ can be recognised in polynomial time, for fixed d.

A less obvious example is for the class $\mathcal{C}$ of *linear forests*, which are disjoint unions of paths. Its quasi-class contains all graphs with connected components that are either a path or an odd cycle.

ChordalBipartite is the class of bipartite graphs in which every cycle of length at least six has a chord. OddChordal is the class of graphs in which every even cycle of length six or more has an odd chord. We show in [7] that quasi-ChordalBipartite = OddChordal. However, the complexity of the recognition problem for the class OddChordal is open, even though the class ChordalBipartite can be recognised in almost linear time [13]. More generally, polynomial time recognition of $\mathcal{C}$ does not directly imply the same property for quasi-$\mathcal{C}$. All we can assert is membership in co-$\mathbb{NP}$, by guessing a bipartition L, R and showing in polynomial time that $G[L{:}R] \notin \mathcal{C}$.

Finally, as remarked above, if Monotone is the class of monotone graphs and Quasimonotone is the class of quasimonotone graphs, then quasi-Monotone = Quasimonotone, by definition. Most of this paper examines the structure and polynomial time recognition of graphs in this class. As we have remarked, the linear-time recognition of Monotone has little relevance to this issue.

A further example, Quasichains, is discussed in [7]. It is the quasi-class arising from unions of *chain graphs* (defined in 5.1), so Quasichains $\subset$ Quasimonotone.

Note that quasi-$\mathcal{C}$ does not necessarily inherit the properties as $\mathcal{C}$. For example, consider *perfection*. Since $\mathcal{C} \subseteq$ Bipartite, $\mathcal{C} \subset$ Perfect, the class of perfect graphs. But every bipartition of an odd hole is a linear forest. Thus, for any bipartite superclass of linear forests, the quasi-class contains odd holes, which are imperfect. In particular, this holds for the class Quasimonotone. However, if $\mathcal{C}$ is hereditary, closed under disjoint union, since $\mathcal{C} \subseteq$ quasi-$\mathcal{C}$ and fails to have some property, then clearly quasi-$\mathcal{C}$ cannot have the property. Thus Quasimonotone is not closed under edge deletions, since Monotone is not.

However, if $\mathcal{C}$ is closed under edge deletions, so is quasi-$\mathcal{C}$. To see this note that, for $G' = G \setminus e$, either $G'[L{:}R] = G[L{:}R]$ or $G'[L{:}R] = G[L{:}R] \setminus e$ holds, so $G'[L{:}R] \in \mathcal{C}$, and hence $G' \in$ quasi-$\mathcal{C}$.

On the other hand, if $\mathcal{C}$ is closed under edge contraction, quasi-$\mathcal{C}$ is unlikely to have this property. Any class which includes the cycle of length ℓ, but excludes the cycle of length $\ell' < \ell$, is clearly not closed under edge contraction. Thus quasi-LinearForests is not closed under edge contraction, even though LinearForests is, since it includes all odd cycles, but no even cycle. In particular, quasi-$\mathcal{C}$ is unlikely to be *minor-closed*, even when $\mathcal{C}$ has this property, since this requires closure under edge contractions.

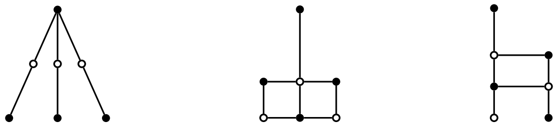
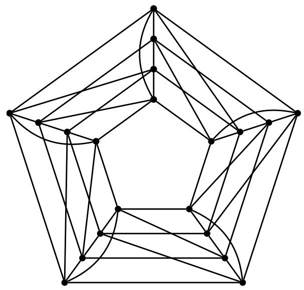
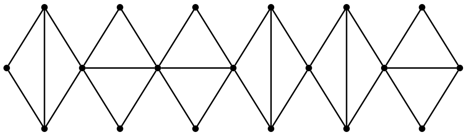


Fig. 2. The tripod, the stirrer and the armchair.



(a)



(b)

Fig. 3. Two quasimonotone graphs.

3. The structure of quasimonotone graphs

3.1. Flaws and pre-holes

A bipartite graph is *monotone* if and only if the rows and columns of its biadjacency matrix can be permuted such that the ones appear consecutively and the boundaries of these intervals are monotonic functions of the row or column index. That is, all the ones are in a staircase-shaped region in the biadjacency matrix. Equivalent characterisations exist, see [12] (Lemma 1.46 on page 52) or [2] (Proposition 6.2.1 on page 93):

- A graph is monotone if and only if it is AT-free bipartite.
- A bipartite graph is monotone if and only if it does not contain a hole, tripod, stirrer or armchair as induced subgraph. The tripod, stirrer and armchair are depicted in Fig. 2.

For references to further characterisations see [4]. Monotone graphs are also called *bipartite permutation graphs* [15] and *proper interval bigraphs* [10].

We let Monotone denote the class of monotone graphs, and then we let Quasimonotone denote the class quasi-Monotone. Two example graphs are shown in Fig. 3.

Let Flaw be the class containing all pre-tripods, pre-stirrers and pre-armchairs. We will say that any graph in Flaw is a *flaw*. A *flawless* graph G will be one which contains no flaw as an induced subgraph. Since all flaws have seven vertices, we can test in $O(n^7)$ time whether an input graph G on n vertices is flawless. Let Flawless denote the class of flawless graphs.

Therefore, quasimonotone graphs are characterised by the absence of pre-holes, pre-tripods, pre-stirrers and pre-armchairs. Let Quasimonotone be the class of quasimonotone graphs.

Clearly Quasimonotone $\subseteq$ EvenHoleFree $\cap$ Flawless, but equality does not hold, as we now discuss.

Let $P = (p_1, p_2, \ldots, p_\ell)$ be a path in G. The *alternating bipartition* L, R of P assigns $L = \{p_1, p_3, \ldots\}$ and $R = \{p_2, p_4, \ldots\}$. We will say that P is *pre-chordless* if it is an induced path in $G[L{:}R]$. In particular, any induced path in G is pre-chordless. Similarly, let $C = (p_1, p_2, \ldots, p_\ell)$ be an even cycle in G. Then C is a pre-hole if it is a hole in $G[L{:}R]$. Thus C must be an even cycle, and all chords must run between L and L or R and R in an alternating bipartition L, R of C. This is equivalent to requiring that C has no odd chord. The alternating partition is inconsistent for an odd cycle, so an odd cycle C cannot be a pre-hole.

From this discussion, it is clear that G contains no pre-hole if and only if it is odd-chordal, as defined in Section 2.1. Hence Quasimonotone = Flawless $\cap$ OddChordal. Given an input graph G, we wish to test whether or not $G \in$ Quasimonotone. We can test whether $G \in$ Flawless in polynomial time, but we do not know how to determine whether $G \in$ OddChordal. Thus it is not clear that this Quasimonotone can be recognised in polynomial time, since pre-holes can be of arbitrary size in Flawless. See Fig. 4 for a family of such pre-holes.

The main contribution of this paper will be to show that the recognition problem for Quasimonotone is indeed in polynomial time. We will not be too concerned with the efficiency of our algorithm beyond polynomiality, so the bounds we prove will often be far from optimal.

Fig. 4. An infinite family of pre-holes.

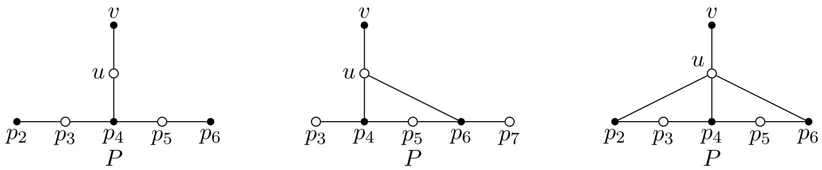


Fig. 5. Cases in the proof of Lemma 3.

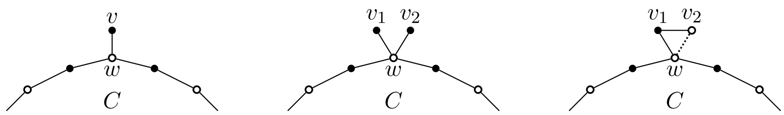


Fig. 6. Cases in the proof of Lemma 5.

3.2. Properties of flawless graphs

First we prove a useful lemma about graphs in FLAWLESS.

Lemma 3. *Let $G \in$ FLAWLESS. Let $P = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ be a pre-chordless path in G, $(p_2, p_3, p_4, p_5, p_6)$ be a hole in G, or $(p_1, p_2, p_3, p_4, p_5, p_6)$ be a pre-hole in G. If $v \notin P$ is such that* $\operatorname{dist}(v, P) = \operatorname{dist}(v, p_4)$*, then* $\operatorname{dist}(v, p_4) = 1$.

Proof. Since $\operatorname{dist}(v, P) = \operatorname{dist}(v, p_4)$ the shortest path from v to p_4 cannot use any edge of P. Therefore, suppose, without loss of generality, that $\operatorname{dist}(v, P) = 2$, and (v, u, p_4) is the shortest path from v to P. Consider the alternating bipartition of P extended to $u \in R$ and $v \in L$, as shown by the black (L) and white (R) vertices in Fig. 5. There are no edges from v to P, since $\operatorname{dist}(v, P) > 1$, and there are no edges between vertices of P in $G[L{:}R]$, since P is pre-chordless.

Thus the only possible edges in $G[L{:}R]$, other than in P and the 2-path (v, u, p_3), are those joining u to a vertex in L. There are three cases, where none, one or both of these edges are present, as shown in Fig. 5. Note that the "one" case has a symmetric version, where the vertices in P are p_1, p_2, p_3, p_4, p_5, and the edge up_2 is present. But these graphs are the tripod, armchair and stirrer, respectively, contradicting G being flawless.

If $(p_2, p_3, p_4, p_5, p_6)$ is a hole, or $(p_1, p_2, p_3, p_4, p_5, p_6)$ is a pre-hole, we need only observe that the configurations in Fig. 5 exist, if P is allowed to "wrap around" the (pre-)hole. That is, if p_1, p_7 are interpreted as p_6, p_2 respectively. □

Note that any subpath of a pre-hole or odd hole C is pre-chordless.

Lemma 4. *Every odd hole or pre-hole in a connected flawless graph is dominating.*

Proof. Let C be an odd hole or pre-hole in the connected flawless graph G. We show $\operatorname{dist}(v, C) \le 1$ for every vertex v of G.

If $v \in C$, this is obvious. Otherwise, let w be a vertex such that $\operatorname{dist}(v, C) = \operatorname{dist}(v, w)$. Consider the subpath $P = (p_1, p_2, \ldots, p_7)$ of C such that $w = p_4$, where this path wraps around C if $|C| < 7$. Since C is a hole or a pre-hole, P is pre-chordless. The result then follows from Lemma 3. □

If C is an odd hole we will call $n(C) = \{v \in V : \operatorname{dist}(v, C) \le 1\}$, the neighbourhood of C. Thus, if G is connected, then $G = N(C)$ for any odd hole $C \subseteq G$.

Lemma 5. *Suppose $G \in$ FLAWLESS $\cap$ EVENHOLEFREE, and that C is an odd hole in G, of length at least seven. Then every vertex $v \in V$ has at most three neighbours in C. If there are two neighbours, w, x, then* $\operatorname{dist}_C(w, x) = 2$. *If there are three neighbours, w, x, y, then* $\operatorname{dist}_C(w, x) = \operatorname{dist}_C(x, y) = 2$. *If C is a short odd hole in G, then v has at most two neighbours on C.*

Proof. If $v \in C$, v has exactly two neighbours in C, so the lemma is true. Thus suppose $v \notin C$. If w is the only neighbour of $v \in C$, then $G[\{v\} \cup C]$ is the graph shown in Fig. 6. Note that w may have several such neighbours $v_1, v_2, \ldots$, but no two can be connected by an edge, since otherwise there is a bipartition containing a tripod, see Fig. 6. If v_1, v_2 are leaves in G, then they are false twin.

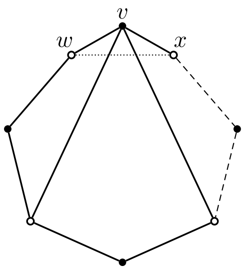


Fig. 7. Cases in the proof of Lemma 5.

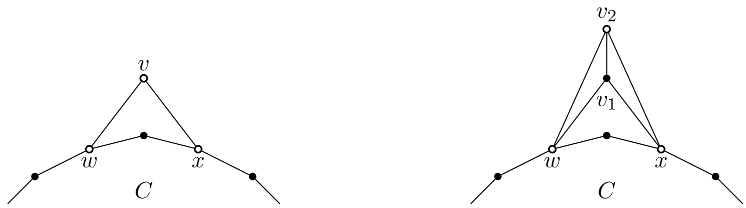


Fig. 8. Cases in the proof of Lemma 5.

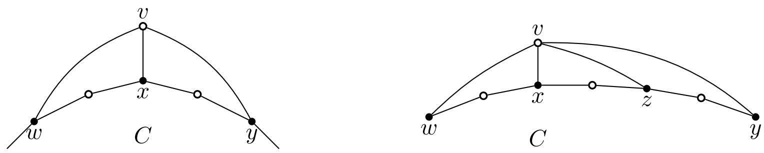


Fig. 9. Cases in the proof of Lemma 5.

Now suppose v has two neighbours w, x on C. Since $\ell = |C| \geq 7$ is odd, we may assume $\nu = \mathrm{dist}_C(w, x) \leq (\ell - 1)/2$. Suppose first that ν is odd. Then we obtain an even cycle C' by omitting $\nu - 1$ vertices of C and adding v. Consider the alternating bipartition L:R of C'. See Fig. 7, with black nodes L, and white nodes R. Now C' has even length $\ell' = \ell - \nu + 2 \geq (\ell + 5)/2 \geq 6$. Then, since C is chordless, there is an even hole in $G[L{:}R]$, unless v is adjacent to every vertex of L. However, since $\ell \geq 7$, $G[L{:}R]$ contains a stirrer, as shown in Fig. 7, contradicting $G \in$ Flawless. Thus $\mathrm{dist}_C(w, x)$ cannot be odd.

Thus suppose $\mathrm{dist}_C(w, x) = \nu \leq (\ell - 1)/2$ is even and $\nu > 2$, so $\nu \geq 4$. Then v, w, x lie on a chordless cycle in G of even length $\nu + 2 \geq 6$. This is an even hole, contradicting $G \in$ EvenHoleFree, so we must have $\mathrm{dist}_C(w, x) = 2$. Then $G[\{v\} \cup C]$ is the graph shown in Fig. 8, and G has another odd hole of length ℓ, passing through v.

There can be several vertices $v_1, v_2, \ldots$ with neighbours w and x, but there can be no edge between any pair of these vertices. Otherwise there is a bipartition containing an armchair, see Fig. 8. If these vertices have neighbours only in C, then they are all false twins.

Now suppose v has at least three neighbours w, x, y on C, where w, y are such that $\mathrm{dist}_C(w, y)$ is maximised. Consider the alternating bipartition L:R of the $(w, \ldots, y)$ path in C, extended to $v \in R$. Then v must be adjacent to every vertex in R between w and y, since otherwise there is an even hole in $G[L{:}R]$. If v has exactly three neighbours, $G[L{:}R]$ contains the subgraph shown in Fig. 9, with L, R the white and black vertices, respectively. Now v cannot have a fourth neighbour z on C. Otherwise, G has a stirrer, involving v, the $(w, \ldots, z)$ path, and y, as shown in Fig. 9.

Finally, if v has three neighbours in C, as shown in Fig. 9, then $\mathrm{dist}(w, y) = 2$, and $\mathrm{dist}_C(w, y) = 4$. Thus C is not a short odd hole, a contradiction. $\square$

The following is similar to Lemma 5, but the details of the proof are slightly different.

Lemma 6. *Let C be a pre-hole in $G \in$ Flawless. Then every vertex $v \in C$ has at most five neighbours in C. Two of these are via edges of C, so v is incident to at most three chords. If there are two chords, vw, vx, then* $\mathrm{dist}_C(w, x) = 2$. *If there are three chords, vw, vx, vy, then* $\mathrm{dist}_C(w, x) = \mathrm{dist}_C(x, y) = 2$.

Proof. Otherwise, v must have at least four chords. These must be even chords to c_0, c_2, c_4, c_6, where $P = (c_0, c_1, \ldots, c_6, c_7)$ is a subpath of C, since C is a pre-hole and G has no even holes. We now move v from L to R. The only new edges which appear in $G[L{:}R]$ are those adjacent to v. But now $c_0, v, c_3, c_4, c_5, c_6, c_7$ induce an armchair in $G[L{:}R]$, contradicting $G \in$ Flawless. See Fig. 10. $\square$

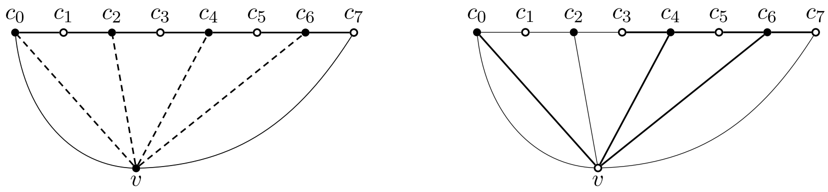


Fig. 10. An armchair.

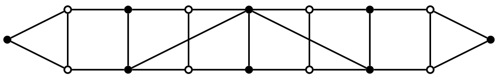


Fig. 11. A pre-hole with a vertex of degree 5.

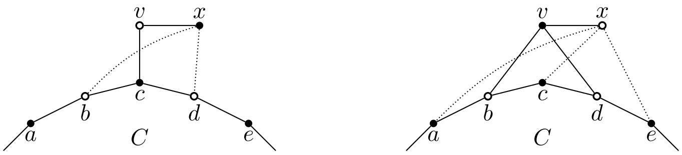


Fig. 12. Cases in the proof of Lemma 6.

The degree bound of Lemma 6 is tight. See Fig. 11.

Lemma 7. *Let C be an odd hole in $G \in$ Flawless. Suppose that $v \notin C$ has a neighbour $x \notin C$. Then there are vertices $w, y \in C$ such that $(vxyw)$ is a quadrangle.*

Proof. From Lemmas 4 and 5, v and x have at least one, and at most three, neighbours on C. If either has two neighbours then these are at distance 2 on C. Observe that it suffices to prove that w, y exist so that $vxyw$ is simply a 4-cycle. If it is not a quadrangle, either v or x has two adjacent neighbours on C, contradicting Lemma 5.

Suppose v has exactly one neighbour c on C. Then we have the first configuration shown in Fig. 12, with the bipartition indicated by the black and white vertices on $G[S]$, where $S = \{a, b, c, d, e, v, x\}$. This contains a tripod, unless x is adjacent to b or d or both. If is adjacent to b, then the Lemma follows with $w = c$ and $y = b$. If is adjacent to d, then the Lemma follows with $w = c$ and $y = d$.

Now suppose v has two neighbours b, d on C. Then, by Lemma 5, we have the second configuration in Fig. 12, with the bipartition on $G[S]$ shown in black and white. If v has a third neighbour z on C, then $z \notin S$ so we may ignore it. Then $G[S]$ contains an armchair, unless x is adjacent to at least one of the black vertices a, c, e. If $xa \in E$, we take $w = b, y = a$, if $xc \in E$, we take $w = b, y = c$, and if $xe \in E$, we take $w = d, y = e$. □

3.3. Determining a short odd hole

We can test whether G contains a hole in time $O(|E|^2)$, using the algorithm of [14]. Moreover, the algorithm returns a hole if one exists. If the hole is even, we can conclude $G \notin$ Quasimonotone. If $G \in$ Flawless, we will show that it has a well-defined structure, so it is possible that there is a faster algorithm than [14] for detecting a hole. However, we will not pursue this here.

We begin with a simple result.

Lemma 8. *If C is an odd cycle in a graph G, there is a triangle or an odd hole C' in G.*

Proof. The claim is clearly true if $|C| \le 3$. Otherwise, assume by induction that it is true for all cycles shorter than C. If C is not already a hole, it has a chord that divides it into a smaller odd cycle C_1, and an even cycle C_1'. The lemma now follows by induction on C_1. □

The proof of Lemma 8 can easily be turned into an efficient algorithm to find C'. Let C be an odd hole in a graph G. Then C will be called a *short* odd hole in G if $\mathrm{dist}(v, w) = \mathrm{dist}_C(v, w)$ for all pairs $v, w \in C$.

Lemma 9. *If G is a triangle-free graph containing an odd hole C, then G contains a short odd hole.*

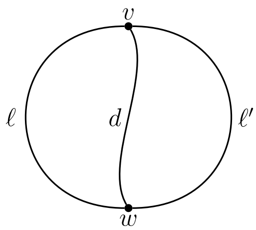


Fig. 13. Odd hole C with a shorter vw path.

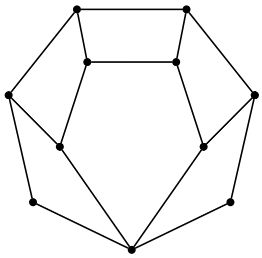


Fig. 14. Short odd holes of unequal size.

Proof. Clearly $\text{dist}(v, w) \le \text{dist}_C(v, w)$ for all pairs $v, w \in C$. Thus, suppose that C is an odd hole in G, but there is a pair v, w such that $\text{dist}(v, w) = d < \ell = \text{dist}_C(v, w)$, and let $\ell' = |C| - \ell \ge \ell > d$. Thus one of ℓ, ℓ' is odd and the other even. See Fig. 13.

We may assume that the shortest vw path P has no internal vertex in common with C. Otherwise, we may choose a different pair v, w for which this is true. Thus we can form two cycles C_1, C_2 of lengths $\ell+d, \ell'+d$. Now one of $\ell+d, \ell'+d$ is odd and the other even. Also $\max\{\ell + d, \ell' + d\} = \ell' + d = |C| - \ell + d < |C|$. Thus we have an odd cycle, C_1 say, with $|C_1| < |C|$. Now, by Lemmas 11 and 8, C_1 contains an odd hole H, and we have $|H| \le |C_1| < |C|$. We can now check whether H is a short hole. This process must clearly terminate with a short hole, since the hole becomes progressively shorter. $\square$

Note that the proof of Lemma 9 gives an efficient algorithm for finding a short odd hole H, given any odd hole C. Clearly the shortest hole in G is a short hole, but the converse need not be true in general, even for quasimonotone graphs. See Fig. 14, which has a short 5-hole and a short 7-hole.

We will also use the following simple corollary.

Corollary 10. *If C is a short odd hole in a graph G,* $\text{diam}(G) \ge \text{diam}(C) = (|C| - 1)/2$. $\square$

We can make similar definitions for pre-holes. Thus, if C is a pre-hole, $G' = G[C]$, and $L{:}R$ is the alternating bipartition of C, then $G'[L{:}R]$ contains no edge other than those of C. A *minimal* pre-hole C is such that $G[C]$ contains no pre-hole with fewer than $|C|$ vertices. Clearly, any graph which contains a pre-hole contains a minimal pre-hole.

4. Flawless graphs containing a long hole

4.1. Triangles

Lemma 11. *Let G be a quasimonotone graph containing an odd hole C of size at least 7. Then G contains no triangle that has a vertex in C.*

Proof. Since C is an odd hole, there are only two cases.

(i) If the triangle has two vertices on C, then we have the situation of Fig. 7 on page 11, which cannot occur.
(ii) If the triangle $T = (v, w, x)$ has one vertex v on C, then consider the graph G' induced by $C \cup T$, so C is the shortest hole in G'. There are two subcases.
  (a) If neither w nor x has another neighbour on v, we have the situation of Fig. 6, which cannot occur.
  (b) If w has another neighbour z on C, but x does not, we have the situation in Fig. 15. This is a pre-armchair, since the edge xv is not in the bipartition shown. If x has another neighbour on C, this must be either y or z. Neither appears in the bipartition shown in Fig. 15. $\square$

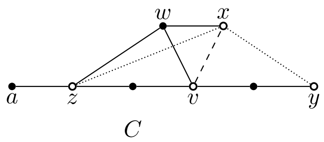


Fig. 15. Case (ii-b) in the proof of Lemma 11.

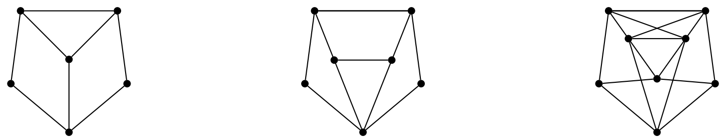


Fig. 16. Cases in the proof of Lemma 12.

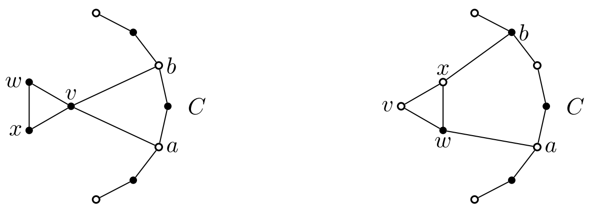


Fig. 17. Cases in the proof of Lemma 12.

However, if C is a 5-hole, there are quasimonotone graphs which contain a triangle with one vertex in C. See Fig. 16.

Lemma 12. *Let G be a quasimonotone graph containing an odd hole C of size at least 7. Then G contains no triangle which is vertex-disjoint from C.*

Proof. Suppose $T = (vwx)$ is such a triangle, and consider the subgraph G' induced by $C \cup T$. By Lemma 7, C is a shortest hole in G', so the vertices of T have degree at least one and at most two in C. Now T is the only triangle in G', since any other triangle would have a vertex in C, contradicting Lemma 11. Thus all vertices of C have degree at most one in T, since a vertex of degree two or more would induce a triangle, using an edge of T. Now suppose some vertex of T, v say, has two neighbours a, b in C, see Fig. 17. Then G' has a hole with $|C'| = |C|$ through a, v and b, with $v \in C'$. Since $|C'| \geq 7$, this contradicts Lemma 11.

Thus T has at most three neighbours in C. Hence there are at least four vertices in C which have no neighbour in T. Let $\nu \geq 2$ be the maximum number of consecutive vertices in C with no neighbour in T. Suppose these are bordered by vertices $a, b \in C$, where $wa, xb \in E$. See Fig. 17. Thus, if ν is odd, there is a pre-hole through a, w, v, x and b, and, if ν is even, there is an even hole through a, w, x and b. See Fig. 17. □

Again, If $|C| = 5$, it is possible to have a triangle which is vertex-disjoint from C. Again Fig. 16 shows an example, but note that this also contains triangles which share an edge with the 5-cycle. It is not difficult to show that, if G contains a 5-cycle and a vertex-disjoint triangle, then G must contain a triangle which shares at least one vertex with the 5-cycle. However, we will not prove this because we make no use of it here.

4.2. Long odd holes

Lemma 13. *Let C, C' be odd holes in a quasimonotone graph G such that $C' \cap C \neq \varnothing$, and $|C|, |C'| \geq 7$. Let $G' = G[(C' \cup C) \setminus (C' \cap C)]$, Then G' has no odd hole or pre-hole.*

Proof. Without loss of generality, we will assume $|C| \leq |C'|$.

If G' has a pre-hole, G is not quasimonotone. So suppose there is an odd hole H in G'. Clearly H must contain edges from both C and C'. Let P be the path $H \cap C$, and P' the path $H \cap C'$. We choose H so that $|P'|$ is minimised. See Fig. 18, where H is bounded by the edges $3'3$ and $5'6$, and $|P'| = 2$.

Suppose any vertex in v the interior of path P' has an edge to a vertex $w \in C$. Clearly $w \notin P$, or H is not a hole. Let v and w be chosen so that $\mathrm{dist}_C(w, P)$ is minimised. Then there is either an odd hole H' with $|C' \cap H'| < |P'|$, contradicting the choice of H, or an even hole, contradicting quasimonotonicity. For example, consider $v = 4'$ in Fig. 18. If $w = 2$, then H' is $(4', 5', 6, 5, 4, 3, 2, 4')$ and $|C' \cap H'| = 1$. If $w = 1$, then $(4', 5', 6, 5, 4, 3, 2, 1, 4')$ is an even hole, and G is not quasimonotone.

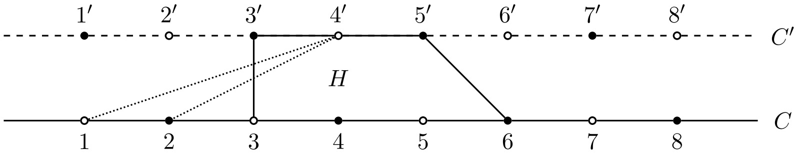


Fig. 18. Odd hole in the proof of Lemma 13.

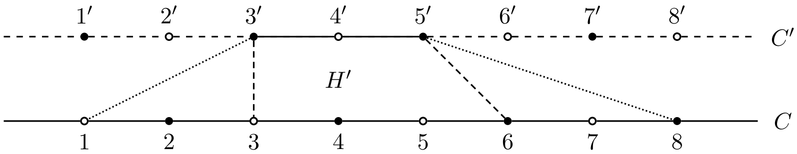


Fig. 19. Cycle in the proof of Lemma 13.

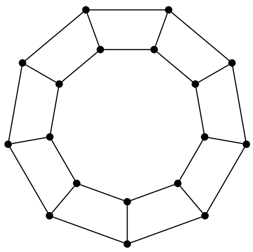


Fig. 20. A 9-prism.

Thus we may assume that no vertex in P' has an edge to C, excepting possible extreme vertices of P'. Now $(C \cup H) \setminus (C \cap H)$ is an even hole, with length $|C| + |H| - 2|C \cap H|$, unless the extreme vertices of P' have edges to $C \setminus P$. If not, G' cannot be quasimonotone. So suppose one of the two extreme vertices v_i has an edge to $w_i \in C \setminus P$ $(i = 1, 2)$. Then, by Lemma 5, w_i is unique and $\mathrm{dist}(w_i, P) = 2$ $(i = 1, 2)$. Now we can construct an even hole in G'. For example, consider $v_1 = 3'$, $v_2 = 5'$ in Fig. 19. The only possibilities for $v_i w_i$ are $3'\,1$ and/or $5'\,8$. Then we can use $3'\,1$ in place of $3'\,3$ and/or $5'\,8$ in place of $5'\,6$ to form an odd cycle H', as shown in Fig. 19. Since H' has no edge to a vertex in $C \setminus H'$, we have an even hole $(C \cup H') \setminus (C \cap H')$. So G is not quasimonotone, a contradiction. □

Corollary 14. *Let C, C' be odd holes in a quasimonotone graph G, such that $C' \cap C \neq \varnothing$. Let $G' = G[(C' \cup C) \setminus (C' \cap C)]$. Then G' is a monotone graph.*

Proof. G' is flawless, and has no holes or pre-holes from Lemma 13, so it is monotone. □

Note that the holes C, C' in Corollary 14 can have different size. See Fig. 14, where G' is a *ladder* (see [6]) with two pendant edges. However, if we have vertex-disjoint odd holes they cannot have different lengths.

A *prism* is the graph given by joining corresponding vertices in two cycles of the same length. It is an *n-prism* if the cycles have length n [11]. See Fig. 20 for an example.

Lemma 15. *Let G be a quasimonotone graph containing an odd hole C. Then G contains no vertex-disjoint hole C' with $|C'| \neq |C|$. Moreover, if $|C| \geq 7$, any two vertex-disjoint holes with $|C'| = |C|$ induce a prism in G.*

Proof. Let $G' = G[C \cup C']$, and suppose $|C'| \neq |C|$. Let C_1 denote the shorter of C, C', and C_2 the longer, so $|C_1| \geq 5$ and $|C_2| \geq 7$. Then every vertex of C_1 has degree in $\{1, 2, 3\}$ in C_2 and every vertex of C_2 has degree in $\{1, 2\}$ in C_1. Since $|C_2| > |C_1|$, there must be a vertex $v \in C_1$ with degree 2 or 3 in C_2, by simple counting. Let a, b be two of these neighbours, such that acb is a subpath of C_2. See Fig. 21. Since G' is flawless, by Lemma 7 every edge of C_1 is in a quadrangle with some edge of C_2, and vice versa. Thus c must be adjacent to a neighbour w of v on C_1, and G' must contain the configuration of Fig. 21. Now G' must contain either ax or dw or both. Otherwise (d, a, v, w, x) is a chordless path of length 4, and must be a subpath of a hole, since d has some edge to C_1, contradicting Lemma 13. Note that $dx \notin E$, since otherwise (d, x, a) or (d, x, w) would be a triangle, contradicting Lemma 11. Whether ax or dv is an edge, this argument can be repeated for the vertices to the left of d and x, or to the right of c and v. Thus there cannot be any more edges than those indicated in Fig. 21, since every vertex of C_2 has degree at most 2 in C_1.

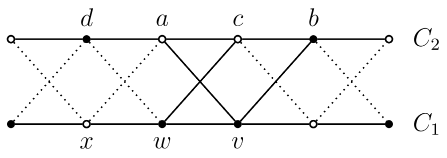


Fig. 21. Crossover.

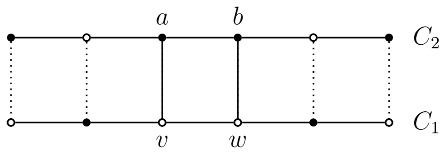


Fig. 22. Matching between C_1 and C_2.

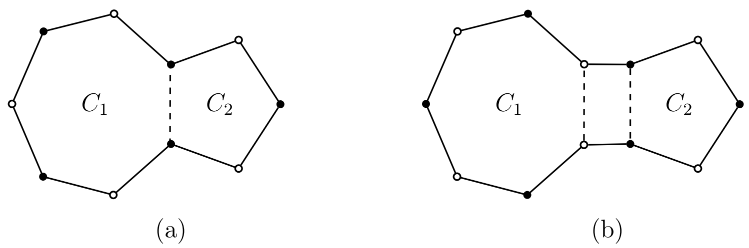


Fig. 23. Preholes with odd holes C_1, C_2.

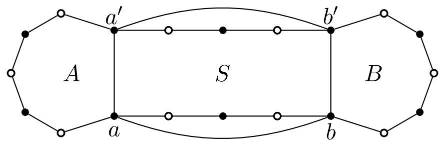


Fig. 24. A possible pre-hole.

We give C_1 the alternating bipartition except, since $|C_1|$ is odd, both $w, v \in L$. Similarly we give C_1 the alternating bipartition with both $a, c \in R$. This is the alternating partition of an even cycle $C_1 \setminus wv$, wc, $C_2 \setminus ac$, av. We call av, wc a *crossover*, and a pre-hole formed in this way a *crossover pre-hole*. Now we observe that all possible edges other than av, cw have both endpoints in L or both endpoints in R. Thus G' is a pre-hole.

Thus all vertices in C_1 must have only one edge to C_2. (See Fig. 22.) Since these edges form a matching between C_1 and C_2, we must have $|C_1| = |C_2|$, and $G[C \cup C']$ must be a prism. □

5. Preholes in flawless graphs

Lemma 16. *If $G \in$ Flawless and has an odd hole of size $\ell \geq 7$, any minimal pre-hole C in G is either an even hole or*

(a) two odd holes intersecting in an edge or
(b) two disjoint odd holes connected by a quadrangle.

See Fig. 23.

Proof. We may assume that G has no triangles, from Lemmas 11 and 12. Clearly C has at least one even chord e which divides it into two smaller odd cycles C_1 and C_2. By Lemma 8, C_1 and C_2 contain odd holes C_1', C_2'. If $C_1' = C_1$, $C_2' = C_2$, then we are in case (a). Otherwise, we can use Lemma 8 to arrive at two odd holes A in C_1 and B in C_2. Now A and B can have only one chord of C in their boundaries, since C has only even chords. Thus the structure of C is either as shown in Fig. 23, or as shown in Fig. 24. We must show that C cannot be a minimal pre-hole in the latter case.

In Fig. 24, A and B are odd holes and S joins them, and is not a single edge or quadrangle. Thus A and B both have size ℓ, and A, B must induce a prism. Otherwise $G[A \cup B]$ contains a smaller crossover pre-hole, by Lemma 15. Thus, in

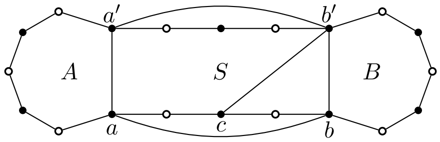


Fig. 25. A possible pre-hole.

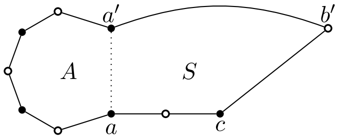


Fig. 26. Smaller pre-hole.

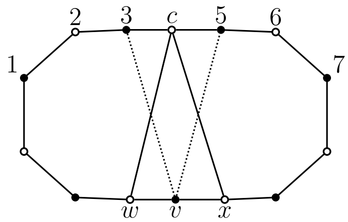


Fig. 27. Preholes with odd holes C_1, C_2.

particular, the edge $a'b'$ must be present, and b' has no other edge to A. Also, we must have $a, b, a', b' \in L$, or we would have a smaller case (b) pre-hole using ab, $a'b'$.

Let c be the vertex nearest a on the path from a to b such that $b'c$ is an even chord of C, as shown in Fig. 25. Note that c is well determined, since $c = b$ is possible.

Consider the cycle C' in G shown in Fig. 26. It is easy to see that C' is a pre-hole, where the certifying bipartition simply moves b' from L to R from that of Fig. 25, as shown.

Thus C' is a pre-hole with size $|C'| < |C|$, so C was not a minimal pre-hole. Thus any minimal pre-hole involving two vertex-disjoint holes of size at least 7 must be as in case (b).

Finally suppose C_1', C_2' are edge- but not vertex-disjoint, so they intersect in a single vertex c. Since $|C_1' \cup C_2'|$ is odd, there must be a vertex $v \in C \setminus (C_1' \cup C_2')$. Since C is minimal, and v is adjacent to both C_1', C_2', by Lemma 7, v must be unique. Since G has no even hole, c, v are the opposite vertices of a quadrangle (c, x, v, w), with $v \in L$, $c, x, w \in R$. Thus we have the configuration shown in Fig. 27.

The vertices such that $\mathrm{dist}_C(c) \leq 3$ form a pre-chordless path $P = (1, 2, 3, c, 5, 6, 7)$, as shown in Fig. 27. Since $|C_1'|, |C_2'| \geq 5$, $\mathrm{dist}_C(v, P) \geq 2$. Suppose v is not adjacent to any vertex of P. Then $\mathrm{dist}(v, P) = \mathrm{dist}(v, c) = 2$, so $vc \in E$ by Lemma 3. This is a contradiction, since C is a pre-hole. Thus v has an edge to P, thus to 1, 3, 5 or 7. By symmetry, suppose either $v1 \in E$ or $v3 \in E$. If $v1 \in E$, $v3 \notin E$, the cycle $(v, 1, 2, 3, c, w)$ is a 6-hole in G, contradicting the minimality of C. Thus $v3 \in E$. But now the quadrangle $(v, 3, c, x)$ separates C into two vertex-disjoint odd cycles $H_1 = (v, 3, 2, 1, \ldots, w)$ and $H_2 = (c, 5, 6, 7, \ldots, x)$. Thus C contains two vertex-disjoint odd holes $H_1' \subseteq H_1$, $H_2' \subseteq H_2$, by Lemma 8. By the minimality of C, we must have $H_1' = H_1$, $H_2' = H_2$, and we are in case (b) again. □

Thus. if G contains an odd hole of size at least 7, minimal pre-holes have only two types, case (a) and case (b). From Lemma 15, case (b) are crossover pre-holes. Examples are shown in Fig. 28.

So let us consider the case (a) pre-holes. We will call these *Möbius pre-holes*, since we will show that such a pre-hole must be a *Möbius ladder* [9,11]. See Fig. 29 for two different drawings of a Möbius ladder. As the name suggests, this is a ladder with a crossover.

Lemma 17. *If C is a Möbius pre-hole in a flawless graph G, then C is a* Möbius ladder.

Proof. Let C have the alternating bipartition such that $a, b \in R$ and ab divides C into odd holes C_1, C_2, with $|C_1| \leq |C_2|$. Note, since $|C_1|, |C_2| \geq 5$, that $|C| \geq 8$. See Fig. 30. If a is incident to more than one chord, then one of C_1, C_2 is not a hole, so C is not a case (a) pre-hole. So ab is the only chord incident to a and, similarly, b.

Let $v \in R \cap C_1$ have distance 2 from a. Then v must have an edge to some $w \in C_2$. Since C is a pre-hole, we must have $w \in R$. Then (a, x, v, w, b, a) is an even cycle, so must have a chord. The only possible chords are from x to the vertices on

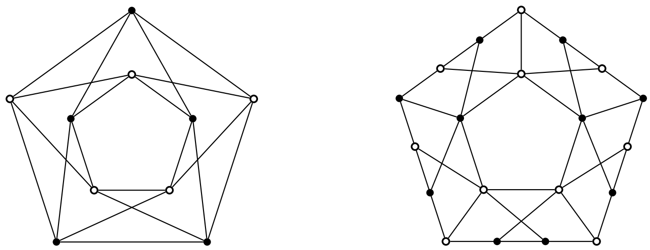


Fig. 28. Crossover pre-holes.

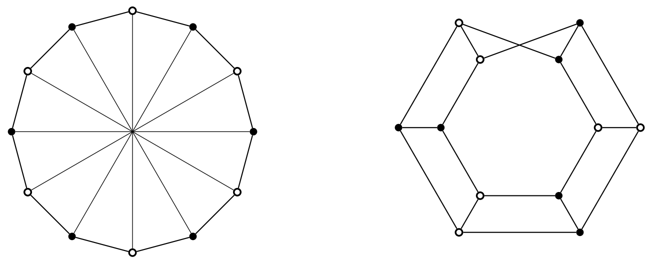


Fig. 29. A Möbius ladder.

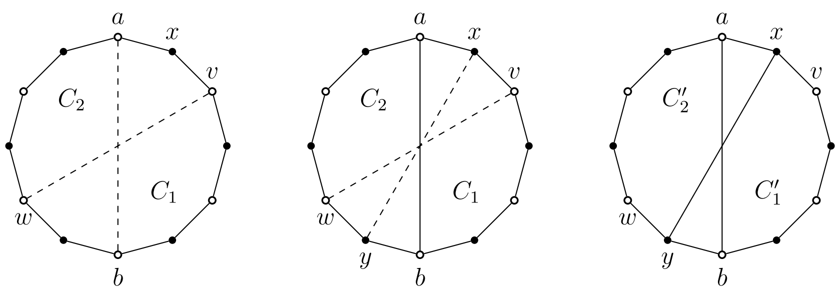


Fig. 30. Implied diameter.

$C_2 \cap L$ between w and b. Thus, in particular, xy must be an edge, where y is adjacent to b on C_2. Now xy divides C into odd holes C'_1, C'_2, so we can repeat the argument to show that vw is an edge, where w is adjacent to y on C_2. We can iterate the argument for all vertices between a and b on C. If $|C_1| < |C_2|$, we will be left with an even hole on $|C_2| - |C_1| + 4$ vertices. So we must have $|C_1| = |C_2|$, and all edges between diametral pairs on C, as in Fig. 29. □

5.1. Splitting

Let G be a flawless graph with a hole C of length $|C| \geq 6$. If $|C|$ is even, we conclude $G \notin$ Quasimonotone, so $|C| \geq 7$ is odd. Thus G does not contain a triangle, from Lemmas 11 and 12. We will assume that this has been tested. We will now show that G must have the annular structure referred to in Section 1.2, rather like a monotone graph with its ends identified.

Now suppose G has a short odd hole C with $C \geq 7$, determined by the procedure of Lemma 9. Thus, by Corollary 10, $\text{diam}(G) \geq \frac{1}{2}(|C| - 1) \geq 3$. Choose any $v \in C$, and consider the graph $\mathcal{G}_v = G[V \setminus N[v]]$. Then $\mathcal{G}_v$ contains no holes, since any hole H in $\mathcal{G}_v$ must be a hole in G. But any hole H in G either contains v, or has a vertex w adjacent to v, by Lemma 4. Since $v, w \notin \mathcal{G}_v$, $H \nsubseteq \mathcal{G}_v$. Neither can $\mathcal{G}_v$ contain a pre-hole, since any pre-hole must contain two holes. Thus $\mathcal{G}_v$ is flawless and contains no holes or pre-holes, so is a monotone graph. Now $\text{diam}(G)$ is at least $\text{diam}(C) = (|C| - 1)/2 \geq 3$. Thus there exists a $w \in C$ such that $N(v) \cap N(w) = \varnothing$.

By definition [6], a graph is monotone if and only if it is bipartite, and its biadjacency matrix has an ordering of rows (L) and columns (R) so that it has the "staircase" structure indicated in Fig. 32. This is symmetrical with respect to rows and columns [6]. That is, the transpose of the biadjacency matrix represents the same monotone graph with L and R interchanged.

A chain graph is a monotone graph in which each vertex in L (resp. R) has an edge to the first vertex in R (resp. L), in this ordering. Thus the biadjacency matrix has the form indicated in Fig. 31. (See [6] for details.)

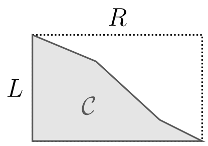


Fig. 31. Chain graph structure.

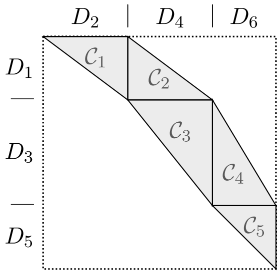


Fig. 32. Decomposition of a monotone graph.

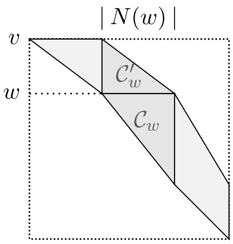


Fig. 33. Neighbourhood of w in $\mathcal{G}_v$.

In the monotone representation, it is an easy observation that the graph has a decomposition into chain graphs, as indicated in Fig. 32, where L is partitioned in $D_1, D_3, \ldots$ and R into $D_2, D_4, \ldots$. This partition was given previously by Brandstädt and Lozin [3], though their proof is based on the representation of a monotone graph as a bipartite permutation graph, and is nontrivial.

Now we have shown that $\mathcal{G}_v$ is monotone, and there is a w such that $N(v) \cap N(w) = \varnothing$. Thus $N(w)$ and its neighbours induce a monotone subgraph $\mathcal{N}_w$ of G, as indicated in Fig. 33. It is easy to see that the vertex set of $\mathcal{N}_w$ is $\{x \in L \cup R : \mathrm{dist}(w, x) \leq 2\}$. Clearly $\mathcal{N}_w$ is the union of two chain graphs C_w, C'_w, with C_w lying in the rows below and including w, and C'_w in the rows above.

We can determine this split using the monotone representation of $\mathcal{G}_v$, with the algorithm of [15]. Then we can construct a representation of the adjacency matrix $A(G)$ of G as indicated in the first diagram in Fig. 34, where $D_2 = N(w)$, $\mathcal{C}_1 = \mathcal{C}_w$ (transposed), and $\mathcal{C}_7 = \mathcal{C}'_w$. The chain graphs $\mathcal{C}_2, \ldots, \mathcal{C}_6$ are a decomposition of the monotone graph $\mathcal{G}_w$. Note that the ordering of the chain graphs in the decomposition is circular, and the second diagram in Fig. 34 gives an equivalent representation to the first, where $\mathcal{C}_1$ (transposed) is moved from the first to the last position.

Suppose there are k chain graphs in the decomposition. In our illustration, Fig. 34, $k = 7$.

Lemma 18. *A flawless graph G which has an odd hole of size at least 7 is quasimonotone if and only if it has such a decomposition and does not contain a pre-hole. If there are k chain graphs in the decomposition, then k is odd, and the shortest hole in G has k vertices.*

Proof. It is clear that k must be odd, since $D_1, D_3, \ldots, D_k$ are the sets of rows.

The only reason that G could fail to be quasimonotone is that it has an even hole or a pre-hole. But any hole H must have at least one edge in each of the chain graphs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$. Otherwise, suppose H has no edge in $\mathcal{C}_i$. If $i = 1$, then H is entirely contained in a monotone graph with decomposition $\mathcal{C}_2, \ldots, \mathcal{C}_k$. If $i = k$, then H is entirely contained in a monotone graph with decomposition $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$. Otherwise, H is entirely contained in a monotone graph with decomposition $\mathcal{C}_{i+1}, \ldots, \mathcal{C}_k, \mathcal{C}_1, \ldots, \mathcal{C}_{i-1}$. This contradicts monotonicity.

Now we observe that H must have an odd number of edges in each chain graph $\mathcal{C}$. This is because the path through $\mathcal{C}$ comprises alternating horizontal and vertices line segments, representing vertices, which meet at nodes representing edges. See Fig. 35. The path must be monotonic within $\mathcal{C}$, or H would have a chord (see Fig. 35), a contradiction. For the

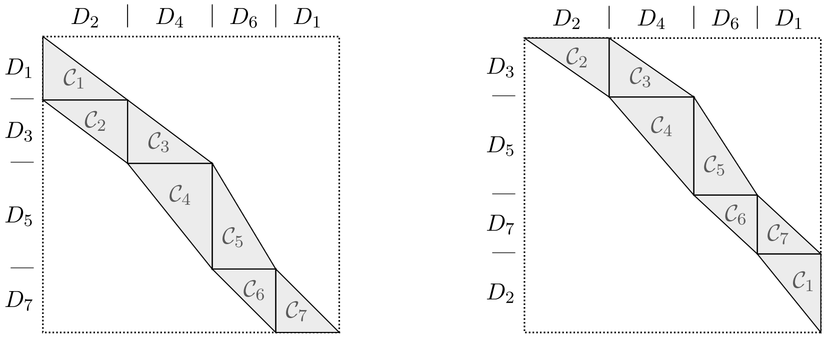


Fig. 34. Decomposition of $A(G)$ for a quasimonotone graph G.

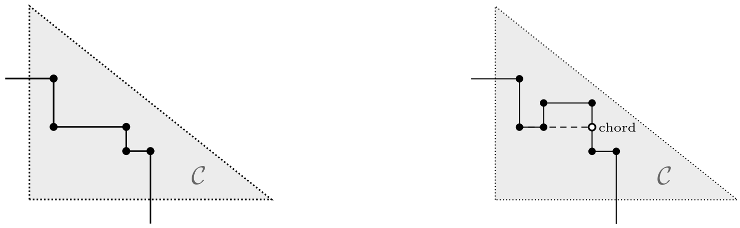


Fig. 35. Path through a chain graph.

same reason, the path cannot pass through $\mathcal{C}$ more than once. So the path must enter $\mathcal{C}$ through a horizontal segment and leave through a vertical segment, or vice versa. Therefore, there must be an odd number of edges of H in $\mathcal{C}$. Hence H has an odd number (k) of odd numbers of edges, and so the total number of edges of H must be odd. Thus any hole H in G must be an odd hole.

Now we observe that the number k of chain graphs in the decomposition of G is the length of a shortest hole. We take exactly one edge in each of the $\mathcal{C}_i$ ($i = 1, 2, \ldots, k$), connected by a zig-zag path of vertices, as indicated in Fig. 34. $\square$

However, it is still possible that G contains a pre-hole. However, the decomposition of Lemma 18 implies that any pre-hole must wind around the annular structure of G. We consider the question of detecting such pre-holes in Section 5.2.

The decomposition of G can clearly be carried out in polynomial time. We check that G is flawless, and has no triangle, or hole of size smaller than 7. We check that G is not a pre-hole. If so, we determine a short odd hole. Then we use the algorithm of [15] to determine the monotone structure of $\mathcal{G}_v$, the monotone structure of $\mathcal{G}_w$, and the split of $\mathcal{N}_w$. If any of these steps fails, G is not quasimonotone. If all succeed, G is quasimonotone, unless it contains a pre-hole. As an example, consider the graph G shown in Fig. 36.

Observe that this procedure really only requires that G be triangle-free, and have diameter at least 3. Thus it can be applied to test quasimonotonicity of some graphs with 5-holes, for example that in Fig. 14.

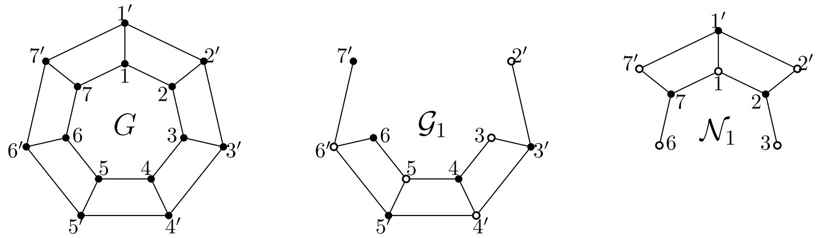
5.2. Recognising pre-holes

Let $G = (V, E)$ be a flawless graph with a hole of size $\ell \geq 7$. Lemma 18 can determine whether or not G is quasimonotone provided it does not contain a pre-hole. We now consider recognition of a pre-hole in such a graph. We use the partition of V from Section 5.1 into independent sets $D_1, \ldots, D_\ell$, where $D_{\ell+1} \equiv D_1$. All edges in E run between D_i and D_{i+1} ($i \in [\ell]$). Let $G_i = G[D_i \cup D_{i+1}]$, with edge set E_i, and let $\overline{G}_i = (V, E \setminus E_i)$. Note that G_i is a chain graph and $\overline{G}_i$ is a monotone graph. Thus $\overline{G}_i$ is bipartite, with bipartition L:R, say, with $D_i, D_{i+1} \in L$.

We search for possible crossovers in G_i. These are pairs $a, b \in D_{i+1}$, $c, d \in D_i$, such that $ac, ad, bc, bd \in E$. We list all such quadruples a, b, c, d, $O(n^4)$ in total, see Fig. 37. Given any quadruple, we attempt to determine vertex disjoint paths P_{ac}, P_{bd} in $\overline{G}_i$ between a, c and b, d or between a, d and b, c. See Fig. 38, cases (a) and (b). We can do this in $O(n|E|) = O(n^3)$ time by network flow. Both paths are even length, since G_i is bipartite and $a, b, c, d \in L$.

If these paths do not exist, we discard this quadruple and consider the next in the list. If these paths do exist, in case (a) we have found a crossover pre-hole P_{ac}, ad, P_{bd}, bc, in case (b) we have found a Möbius pre-hole P_{ad}, bd, P_{bc}, ac. This is clearly a cycle with even length. That it is a pre-hole is certified by reversing the bipartition on P_{ac} in case(a), P_{ad} in case (b), as shown in Fig. 39.

Thus we can detect a pre-hole, or show that none exists, in $O(n^4 \times n^3) = O(n^7)$ time. If a pre-hole exists, we may stop. We have shown that G is not quasimonotone.



$$
\begin{array}{c|cccccccc}
 & 7 & 1' & 2' & 3' & 4' & 5' & 6' & 7' \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Fig. 36. G, $\mathcal{G}_1$, $\mathcal{N}_1$ and the derived $A(G)$.

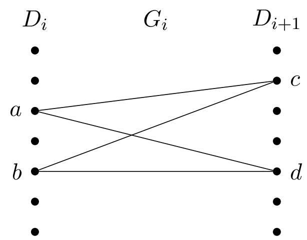


Fig. 37. Possible crossover.

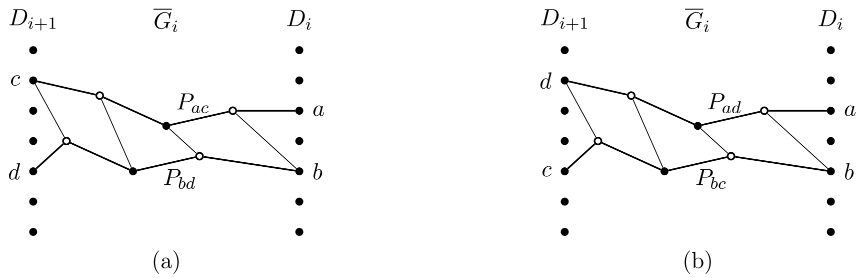


Fig. 38. Vertex-disjoint paths.

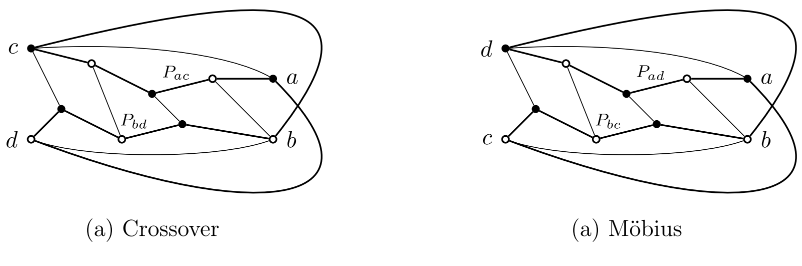


Fig. 39. Preholes.

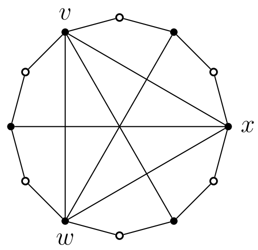


Fig. 40. An internal triangle.

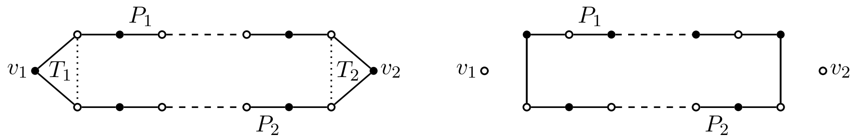


Fig. 41. A pre-hole and its Hamilton subgraph.

6. Flawless graphs without long holes

6.1. Minimal pre-holes in hole-free graphs

The main problem here is to recognise pre-holes. Let C be any minimal pre-hole in a flawless hole-free graph G. A triangle in $G[C]$ will be called an *interior* triangle of C if it has no edge in common with C, a *crossing* triangle if it has one edge in common with C, and a *cap* of C if it has two edges in common with C.

Lemma 19. *If C is a minimal pre-hole in a flawless graph with $|C| > 12$, then $G[C]$ has no interior or crossing triangles, and C is determined by two edge-disjoint caps.*

Proof. Suppose C has an interior triangle, as shown in Fig. 40. The vertices v, w, x partition C into three segments. From Lemma 6, we must have $\text{dist}_C(v, w) \leq 4$, $\text{dist}_C(w, x) \leq 4$ and $\text{dist}_C(x, v) \leq 4$, and hence $|C| \leq 12$.

If $G[C]$ has a crossing triangle, then C has an odd chord, a contradiction. Now, since C is a pre-hole, there must be an even chord u_0v_0 which partitions C into two odd cycles C_1, C_2 with common edge u_0v_0. Suppose that C_1 is not a triangle. Since $G \in$ HoleFree, C_1 must have a chord u_1v_1 partitioning it into an odd cycle C_1' and an even cycle C_2', with $|C_1'| < |C|$. If C_1' is not a triangle we repeat the process until we reach a triangle T_1, which must be a cap, and must be unique. Otherwise we have discovered a crossing or internal triangle of C, a contradiction. This must occur after at most $|C_1|$ repetitions. We then apply the same procedure to C_2, obtaining the second cap T_2. Clearly T_1 and T_2 are edge-disjoint, since they are separated by the chord u_0v_0. Then $|C| \geq 6$ implies that they can share at most one of the vertices u_0, v_0. □

Note that pre-holes with fewer than 12 vertices may contain an interior triangle. See Fig. 46 for an example with 6 vertices. However, the bound 12 is probably far from tight.

Let T_1, T_2 be caps of C, such that $v_i \in T_i$ is adjacent to two edges of C $(i = 1, 2)$. Then there are two edge-disjoint $(v_1, \ldots, v_2)$ paths P_1, P_2 in C. See Fig. 41.

Lemma 20. *Let C, with $|C| > 12$, be a minimal pre-hole in a flawless hole-free graph determined by v_1, v_2, and let $C' = C \setminus \{v_1, v_2\}$. Then $G[C']$ is a Hamilton monotone graph, and all chords of C' connect P_1 to P_2.*

Proof. Clearly $G[C']$ is Hamilton, since $G[C]$ is Hamilton. Now C' cannot be a pre-hole, since it is strictly smaller than C. So $G[C']$ cannot contain a triangle, by Lemma 19. It cannot contain a larger odd cycle, since then it would contain a triangle, by the argument of Lemma 19. Therefore, $G[C']$ is bipartite and, since $G \in$ HoleFree, contains no hole. So, since $G \in$ Flawless, $G[C']$ is a monotone graph. Suppose uv is an edge of $G[C']$ with $u, v \in P_1$. Then, since $G[C]$ has only even chords, the even chord uv and the segment of P_1 between u and v forms an odd cycle, giving a contradiction. □

Thus any minimal pre-hole C comprises a Hamilton monotone graph $G[C']$, to which we add two caps T_1, T_2. We may also add edges from v_1 and v_2 to C', as long as they are even chords in C.

Lemma 21. *Let C be a minimal pre-hole with a cap at $v \in \{v_1, v_2\}$. Then there are at most two chords from v, and both must be connected to either P_1 or P_2.*

Fig. 42. Cases in the proof of Lemma 21.

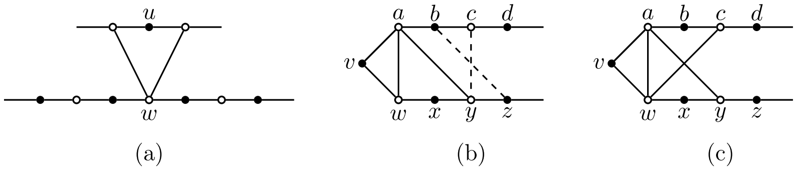


Fig. 43. Cases in the proof of Lemma 22.

Fig. 44. Cases in the proof of Lemma 22.

Proof. The chords must be as shown in Fig. 42(a), since otherwise there is an even hole or a pre-armchair, similarly to Lemma 5. Note that vb must be present if vd is in G. If there is a chord to both P_1 and P_2, we can find a smaller pre-hole by moving v from L to R and using the longer chords from v. See Fig. 42(b). □

Lemma 22. *Let C be a minimal pre-hole with $|C| \geq 8$. Then all vertices in P_1 have a chord to P_2 and vice versa.*

Proof. Suppose first that there are no chords from v_1 or v_2 in $G = (V, E)$.

Let $u \in P_1$ have no edge to P_2. Since $G[C']$ is monotone, u must be in a quadrangle with its neighbours in P_1 and a vertex $w \in P_2$. See Fig. 43(a). Now u is a distance at least 2 from v_1 and v_2, since otherwise it has an edge of T_1 or T_2 to P_2. Thus u is at distance at least 2 from P_2. Now if w is a distance at least 3 from both v_1 and v_2, Lemma 3 implies that u must have an edge to P_1, a contradiction.

Otherwise, we have the situations shown in Fig. 43(b), where the edge bx is absent. Thus ay or cw, or both, must be in G. Suppose that only one is in G and, without loss of generality, that it is ay, as shown in Fig. 43(b). Now either bz or cy, or both, must be present in G. If $bz \in E$, v, a, b, z, y, x, w give a pre-stirrer, a contradiction. So suppose only cy is in E. Then v, w, x, y, z, a, b give a pre-stirrer, again a contradiction. Thus the edge $bx \in E$ unless both $ay, cw \in E$, as shown in Fig. 43(c). In this case we have a shorter pre-hole $(\ldots, d, c, w, v, a, y, z, \ldots)$, after interchanging v, a, w between L and R. Thus we must have $bx \in E$, giving the conclusion. Observe that the configuration of case (a) requires at least 10 vertices, and those of cases (b) and (c) require at least 8. Therefore the conclusion holds only if $|C| \geq 8$.

Now we must consider the effect of chords from v_1 or v_2. These do not affect case (a), since P_2 remains pre-chordless. Also u must remain at distance 2 from P_2, since otherwise we are in case (b) or (c).

In case (b), the only chord from v that can break the flaws in Fig. 43(b) is the edge vx, as shown in Fig. 44(a). In this case, we simply give w the role of v, as shown in Fig. 44(b). Note that the chord vz, if it exists, will now connect P_1 to P_2.

Finally, consider the configuration of Fig. 43(c). By symmetry, we can assume that the chords from v are vb, vd, as shown in Fig. 45(a). The edge vb is absent from the shorter pre-hole in Fig. 45(b), so we have only to consider the edge vd. However, if $vd \in E$, it now connects L to R. Thus there is a shorter pre-hole $(\ldots, z, y, a, v, d, \ldots)$, a contradiction. □

Corollary 23. *Let C be a pre-hole with $|C| \geq 8$. Then, for $v \in C$, $3 \leq \deg_{G[C]}(v) \leq 5$ ($v \notin \{v_1, v_2\}$), $2 \leq \deg_{G[C]}(v) \leq 4$ ($v \in \{v_1, v_2\}$).*

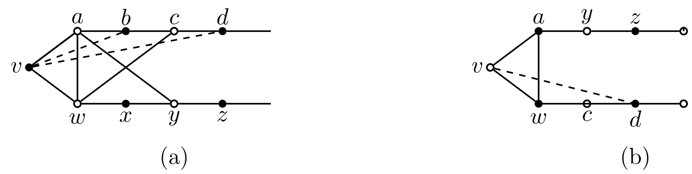


Fig. 45. Cases in the proof of Lemma 22.

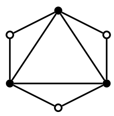


Fig. 46. Prehole with 3 vertices of degree 2.

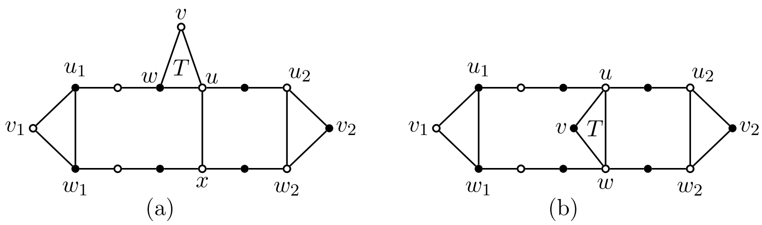


Fig. 47. $|T \cap C'| = 3$.

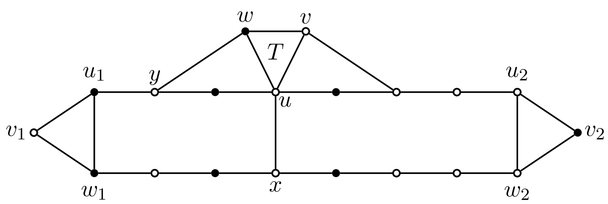


Fig. 48. $|T \cap C'| = 1$.

Proof. Follows directly from Lemmas 6 and 22. □

Note that there is a pre-hole with six vertices and three vertices of degree 2, and an interior triangle.

Let $T_1 = \{v_1, u_1, w_1\}, T_2 = \{v_2, u_2, w_2\}$ be any two edge-disjoint triangles in a flawless graph G. Let M be the component of $G \setminus \{v_1, v_2\}$ containing u_1w_1, u_2w_2, if such a component exists. If M does not exist then v_1, v_2 clearly do not determine a pre-hole. However, if it does

Lemma 24. $C = (v_1, u_1, \ldots, u_2, v_2, w_2, \ldots, w_1, v_1)$ *determines a minimal pre-hole if and only if M is a monotone graph containing two vertex-disjoint paths between u_1, u_2 and v_1, v_2.*

Proof. Since $C' \subseteq M$, the condition is certainly sufficient. Now, if there are not vertex-disjoint paths $P_1 = (u_1, \ldots, u_2)$, $P_1 = (v_1, \ldots, v_2)$, then C cannot be a pre-hole. So suppose that M is not monotone, and hence must contain a triangle $T = \{u, v, w\}$. There are four cases depending on the number of vertices in $T \cap C'$.

(i) $|T \cap C'| = 3$. Then C' is not monotone, so this cannot occur, by Lemma 20.

(ii) $|T \cap C'| = 2$. There are two subcases:

  (a) $|T \cap P_1| = 2$. See Fig. 47(a). In this case, with v to the left of u, let x be the neighbour of u in P_2 furthest to the left of w_2. Then, on moving u from R to L, $(v_1, u_1, \ldots, w, v, w, x, \ldots, w_1, v_1)$ is a shorter pre-hole.

  (b) $|T \cap P_1| = |T \cap P_2| = 1$. See Fig. 47(b). In this case, one edge of the triangle is a chord of C'. Then $(v, u, \ldots, u_2, v_2, w_2, \ldots, w, v)$ is a shorter pre-hole.

(iii) $|T \cap C'| = 1$. Then, using the arguments of Lemma 11, the situation is as shown in Fig. 48. Here x is the neighbour of u in P_2 furthest to the left of w_2. Then, on moving v from R to L, there is a shorter pre-hole $(v_1, u_1, \ldots, y, w, v, u, x, \ldots, w_1, v_1)$.

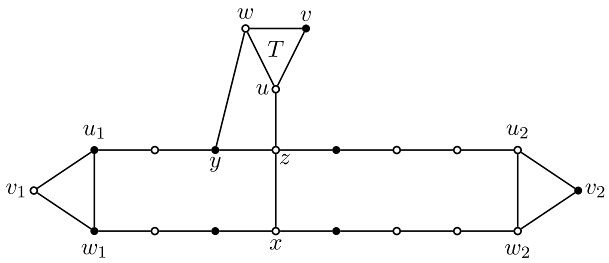


Fig. 49. $|T \cap C'| = 0$.

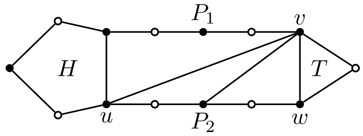


Fig. 50. A pre-hole determined by a 5-hole and a triangle.

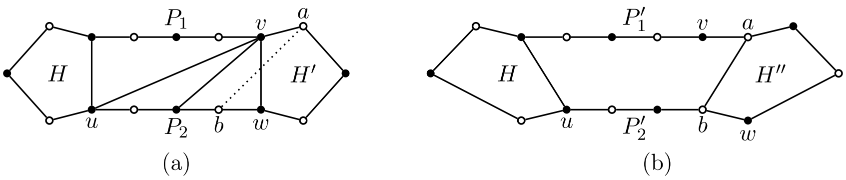


Fig. 51. A pre-hole determined by two 5-holes.

(iv) $|T \cap C'| = 0$. Then, again using the arguments of Lemma 11, the situation is as in Fig. 49. Here x is the neighbour of z in P_2 furthest to the left of w_2. Then, on moving z from R to L, there is a shorter pre-hole $(v_1, u_1, \dots, y, w, v, u, z, x, \dots, w_1, v_1)$.

In all cases, we have a contradiction to the minimality of C, and hence M can have no triangle, and so is monotone. Note also, in all cases, that if there are edges other than those shown, they are either irrelevant, if they are L: L or R: R, or can be used to shorten the pre-hole further, if they are L:R. See Fig. 45, for example. □

Lemma 24 implies a polynomial time algorithm for detecting a minimal pre-hole, in a similar way to the algorithm of Section 5.2.

6.2. Preholes containing 5-holes and triangles

It remains to consider pre-holes in graphs which contain 5-holes, and may also contain triangles. Preholes determined by two triangles will be dealt with as in Section 6.1.

Lemma 25. *Let C be a minimal pre-hole in a flawless graph G which contains no odd hole of size greater than five. If C connects a 5-hole and a triangle, or if C connects two 5-holes, then $|C| \le 12$.*

Proof. The situation is as shown in Fig. 50. The pre-hole C connects a hole H and a triangle T, though the argument applies equally if T is replaced by a 5-hole H', as indicated in Fig. 51.

Now H and T are joined by two paths P_1, P_2, as shown. We will not assume that these paths are of equal length. Let v be the unique vertex in $T \cap P_1$, and w the unique vertex in $T \cap P_2$. From Lemma 6, v is incident to at most three chords in C, which must be at distance 2 on C. However, by Lemma 3, v must be adjacent to $u \in H$. Since $\text{dist}(u, w) \le 4$, $|P_2| \le 4$. The same argument applied to w gives $|P_1| \le 4$. Thus $|C| \le 12$.

If T is replaced by H', let us assume that P_2 is the shorter path, and is as short as possible. Then using the argument above, $|P_1|, |P_2| \le 4$. If $|P_2| \ge 3$, v must have three edges to $P_2 \cup H$. (In Fig. 51(a), $|P_2| = 4$.) Then $\{u, \dots, w, v, a\}$ form a stirrer, unless ab is an edge. If so, C has a representation with $|P_1'| > 4$, as shown in Fig. 51(b), so C cannot be a minimal pre-hole. It follows that $|P_1|, |P_2| \le 2$, and hence $|C| \le 12$. □

It follows that there is an $O(n^{12})$ time algorithm for detecting all minimal pre-holes in a graph with no holes of length greater than 5, using simple enumeration.

7. Recognition algorithm

Algorithm 1 is a summary as pseudocode.

```
input  : a connected graph G = (V, E)
output: accept if G is quasi-monotone, reject otherwise
1  begin
2      if G contains a flaw or a pre-hole of length 12 or less then reject;
3      if G contains a hole then
4          find a hole C in G;
5          if |C| is even or G contains a triangle then reject;          /* Lemma 19 */
6          reduce it to a short hole C′;                                /* Lemma 9 */
7          if |C′| is even then reject;
8      if C′ is defined and |C′| ≥ 7 then
9          choose vertices v, w ∈ V with dist(v, w) ≥ 3;
10         if 𝒢_v and 𝒢_w are monotone then
11             partition V into independent sets D_1, D_2, ..., D_ℓ;     /* Section 5.1 */
12             for i ← 1 to ℓ do
13                 for every 4-cycle (a, c, b, d) in G_i do
14                     if disjoint paths P_ad and P_bc in Ḡ_i exist then reject;   /* Section 5.2 */
15             accept
16         else
17             reject
18     else
19         𝒯 ← the set of all triangles in G;                            /* Lemma 25 */
20         for every pair of vertex-disjoint triangles T_1 ∈ 𝒯 and T_2 ∈ 𝒯 do
21             U_1 ← the vertex set of the component of G \ T_2 containing T_1;
22             U_2 ← the vertex set of the component of G \ T_1 containing T_2;
23             U_3 ← U_1 ∩ U_2; U_4 ← T_1 ∪ T_2 ∪ U_3;
24             if G[U_3] is bipartite and |U_4| > 12 and G[U_4] contains two disjoint T_1–T_2-paths then reject;
25         accept
```

Algorithm 1: Algorithm recognising quasi-monotone graphs

For the run-time analysis we assume a connected graph $G = (V, E)$ as input with $|V| = n$ and $|E| = m$. Line 2 can be executed in time $O(n^{12})$. In line 3 the algorithm can find a hole in G if there is one in time $O(m^2)$, see [14]. In line 5 we shorten a long hole, $O(nm)$. The tests in lines 5, 7 and 8 require time $O(n)$. In the same time we choose v and w in line 8. The graphs $\mathcal{G}_v$ and $\mathcal{G}_w$ can be constructed and recognised as monotone graphs in linear time [15]. This also gives the partition into chain graphs in line 10. In lines 12 and 13 we consider $O(n^4)$ quadrangles. The body of the for-loops in line 14 can be implemented by max flow in time $O(n^3)$. That is, the then-branch of the conditional statement starting at line 8, which deals with holes of length seven or more, requires time $O(n^7)$ in total.

In the else-branch we only consider triangles, since every pre-hole containing a five-hole has length at most twelve by Lemma 25, and was therefore detected in line 2. The set $\mathcal{T}$ in line 18 can be constructed in time $O(n^3)$, and n^3 also a bound on its size. Therefore the for-loop starting in line 20 is executed at most n^6 times. In lines 13–24 we construct U_1, U_2, $G[U_3]$ and $G[U_4]$ in linear time. The disjoint paths in line 24 can be found in time $O(n^3)$, which gives a total time of $O(n^9)$ for the else-branch. Consequently line 2 determines the overall running time of $O(n^{12})$.

8. Recognising a pre-hole

The recognition algorithm finds a pre-hole in a flawless graph, if there is one. However, it relies heavily on the absence of flaws. Therefore it is quite natural to ask whether we can find a pre-hole in any graph in polynomial time. This is exactly the recognition problem for the class OddChordal.

Here we consider a related problem: Given a graph, is it a pre-hole? This is the question of whether the graph is a cycle with only even chords. We will show that this is an $\mathbb{NP}$-complete problem. Of course, this does not mean that the recognition of odd-chordal graphs is $\mathbb{NP}$-complete, since that is the problem of determining whether the graph *contains* a pre-hole. To illustrate the difference, consider the question of whether a graph is a cycle, having either odd or even chords. This is $\mathbb{NP}$-complete, since it is the Hamilton cycle problem. By contrast, the question of whether a graph *contains* a cycle is easy. It simply involves determining whether the graph is a forest.

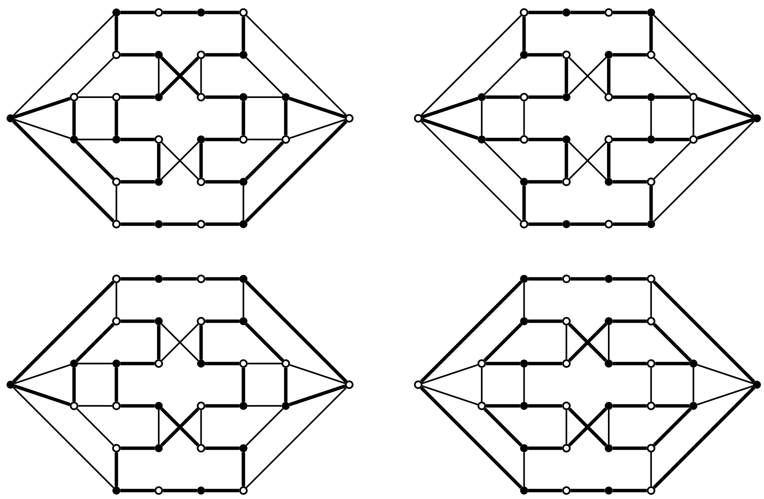


Fig. 52. Three certifying bipartitions of a graph G. For the fourth bipartition L, R the graph $G[L{:}R]$ is 2-regular but disconnected.

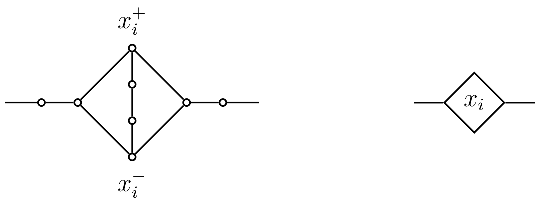


Fig. 53. The truth assignment component for x_i, on the left in full and on the right symbolically.

A graph $G = (V, E)$ is a pre-hole if there is a bipartition L, R of V such that $G[L{:}R]$ is a hole. That is, $G[L{:}R]$ is a Hamilton cycle of G and every other edge in E has both endpoints in L or both in R. If G is a pre-hole then such an L, R is called *certifying bipartition*. For example, the graph G depicted in Fig. 52 is a pre-hole. Three certifying bipartitions are shown. G has no more certifying bipartitions. To see this, start from a bipartition of one of the triangles, and observe that it uniquely extends to a bipartition of the whole graph.

The decision problem PH asks, given a graph G, whether G is a pre-hole. Clearly PH is a problem in $\mathbb{NP}$. We show it is $\mathbb{NP}$-complete by a reduction from NAE3SAT.

An instance of NAE3SAT is a boolean formula $\varphi = \bigwedge_{j=1}^{m} c_j$ in CNF. Each clause $c_j = \ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3}$ consists of exactly three literals. If $X = \{x_1, x_2, \ldots, x_n\}$ is the set of variables occurring in φ then every literal $\ell_{j,k}$ is either a variable x_i or its negation $\neg x_i$. For a truth assignment $a : X \to \{0, 1\}$ let $\overline{a} : X \to \{0, 1\}$ be defined by $\overline{a}(x) = a(\neg x)$ for all $x \in X$. The instance φ of NAE3SAT is accepted if there is a truth assignment a such that $a(\varphi) = 1$ and $\overline{a}(\varphi) = 1$. That is, for each clause c_j not all literals receive equal truth value.

Given an instance φ of NAE3SAT we construct a graph $G = (V, E)$ as follows:

(a) For each variable we create a *truth assignment component* (tac) as shown in Fig. 53, which also defines the vertices x_i^+ and x_i^-.
(b) For each clause we create a *satisfaction test component* (stc) as shown in Fig. 54, which also defines the vertices $b_{j,k}$ and $d_{j,k}$. The stc is obtained from the graph G shown in Fig. 52 by cutting an edge that connects two vertices of degree two into two half-edges.
(c) We link these components in a circular way as shown in Fig. 55.
(d) We add the edges in the set F which is

$$\{\{x_i^+, b_{j,k}\}, \{x_i^-, d_{j,k}\} \mid \ell_{j,k} = x_i\}\} \cup \{\{x_i^+, d_{j,k}\}, \{x_i^-, b_{j,k}\} \mid \ell_{j,k} = \neg x_i\}\}$$

where $1 \le i \le n$, $1 \le j \le m$ and $1 \le k \le 3$.

This completes the construction of G.

Next we assume a is a truth assignment such that $a(\varphi) = 1$ and $\overline{a}(\varphi) = 1$. We construct a bipartition (L, R) of G that certifies that G is a pre-hole. Inside each tac for x_i we put $x_i^+ \in R$ and $x_i^- \in L$ if $a(x_i) = 1$ and the other way around if $a(x_i) = 0$, see Fig. 56. The vertices $b_{j,k}$ and $d_{j,k}$ are put in the L or R such that all edges in F have both endpoints in the

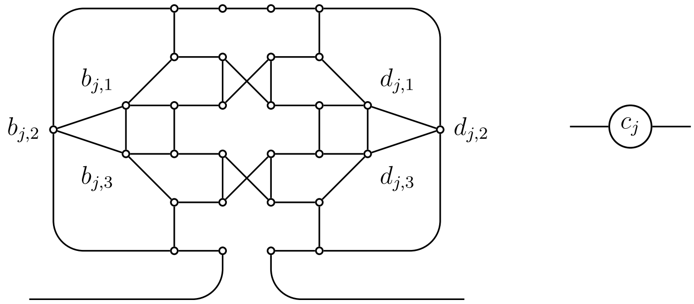


Fig. 54. The satisfaction test component for c_j, on the left in full and on the right symbolically.

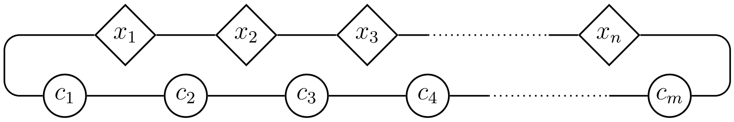


Fig. 55. The components linked.

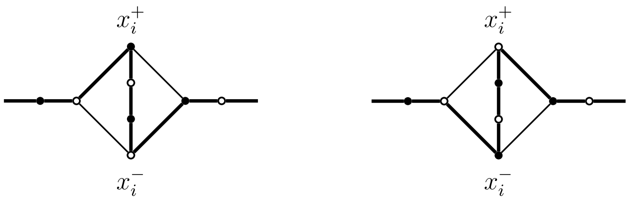


Fig. 56. The bipartition for $a(x_i) = 1$ on the left, and for $a(x_i) = 0$ on the right.

same partite set. Now the triangles in the stc's are partitioned into two nonempty sets because now all literals in one clause have the same truth value. The bipartition of the triangles extends to the whole stc as shown in Fig. 52.

Finally we assume a certifying bipartition (L, R) of the vertices of G. We define a truth assignment a by $a(x_i) = 1$ if and only if $x_i^+ \in R$. To see $a(\varphi) = 1$ and $\overline{a}(\varphi) = 1$ observe:

(a) Every edge of G that is incident to a vertex of degree 2 is contained in every Hamilton cycle of G. That is, its endpoints belong to different sides of the bipartition.
(b) Every edge in F is contained in no Hamilton cycle of G. That is, its endpoints belong to the same side of the bipartition.
(c) Consequently, the bipartition of each component is one of the cases depicted in Fig. 52 (for stc) or 56 (for tac).

Now assume there is a clause c_j such that a assigns the same truth value to all three literals. Then one of the triangles in the corresponding stc has all vertices in L and the other one all vertices in R, see the bottom-right bipartition in Fig. 52. This contradicts the connectedness of $G[L{:}R]$. Hence $a(\varphi) = 1$ and $\overline{a}(\varphi) = 1$.

9. Conclusion and discussion

In [7] we considered the problem of ergodicity and rapid mixing of the switch chain in hereditary graph classes. We gave a complete answer to the ergodicity question, and showed rapid mixing for the new class of quasimonotone graphs. This led us to introduce a new "quasi-" operator on bipartite graph classes, which is of independent interest. Quasimonotone graphs are a particular case of this construction. Another interesting class is the class of odd-chordal graphs, which are the quasi-chordal bipartite graphs. This is close to the largest class for which the switch chain is ergodic.

In this paper, we have investigated recognition of the quasimonotone graphs, and shown that this is in $\mathbb{P}$. This is intended only to be a proof-of-concept. Our algorithms are far from optimal, and can certainly be improved. However, we do not believe that this class can be recognised in linear time, as for monotone graphs.

A more straightforward approach to recognising quasimonotone graphs would be provided by a polynomial time recognition algorithm for odd-chordal graphs. This is equivalent to the detection of pre-holes in a graph. We have considered this question, but we leave it as an open problem. The only evidence we can provide is that it is $\mathbb{NP}$-complete to determine if a graph is a pre-hole, which may be a harder question, Nonetheless, the $\mathbb{NP}$-completeness proof suggests that an efficient algorithm for recognising odd-chordal graphs may be elusive.

References

[1] Kenneth P. Bogart, Douglas B. West, A short proof that 'proper = unit', Discrete Math. 201 (1) (1999) 21–23.
[2] Andreas Brandstädt, Van Bang Le, Jeremy P. Spinrad, Spinrad Graph Classes: A Survey, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
[3] Andreas Brandstädt, Vadim V. Lozin, On the linear structure and clique-width of bipartite permutation graphs, Ars Combin. 67 (2003) 273–281.
[4] David E. Brown, J. Richard Lundgren, Characterizations for unit interval bigraphs, Congr. Numer. 206 (2010) 5–17.
[5] Persi Diaconis, Ronald Graham, Susan P. Holmes, Statistical problems involving permutations with restricted positions, in: State of the Art in Probability and Statistics, in: Lecture Notes–Monograph Series, vol. 36, Institute of Mathematical Statistics, 2001, pp. 195–222.
[6] Martin Dyer, Mark Jerrum, Haiko Müller, On the switch Markov chain for perfect matchings, J. ACM 64 (2017) Article 2.
[7] Martin Dyer, Haiko Müller, Counting perfect matchings and the switch chain, CoRR abs/170505790 (2017).
[8] Martin Dyer, Haiko Müller, Quasimonotone graphs, in: Proceedings of the 44th International Workshop on Graph-Theoretic Concepts in Computer Science, in: Lecture Notes in Computer Science, vol. 11159, Springer-Verlag, 2018, pp. 190–202.
[9] Richard K. Guy, Frank Harary, On the Möbius ladders, Canad. Math. Bull. 10 (1967) 493–496.
[10] Pavol Hell, Jing Huang, Interval bigraphs and circular arc graphs, J. Graph Theory 46 (2004) 313–327.
[11] Milan Hladnik, Dragan Marušič, Tomasž. Pisanski, Cyclic Haar graphs, Discrete Math. 244 (1) (2002) 137–152.
[12] Ekkehard Köhler, Graphs Without Asteroidal Triples (Ph.D. thesis), TU Berlin, 1999.
[13] Anna Lubiw, Doubly lexical orderings of matrices, SIAM J. Comput. 16 (5) (1987) 854–879.
[14] Stavros D. Nikolopoulos, Leonidas Palios, Detecting holes and antiholes in graphs, Algorithmica 47 (2) (2007) 119–138.
[15] Jeremy Spinrad, Andreas Brandstädt, Lorna Stewart, Bipartite permutation graphs, Discrete Appl. Math. 18 (3) (1987) 279–292.