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**Article:**

Ho, Nankuo, Wilkin, Graeme Peter Desmond and Wu, Siye (2018) Hitchin's equations on a nonorientable manifold. *Communications in Analysis and Geometry*. pp. 857-886. ISSN: 1019-8385

<https://doi.org/10.4310/CAG.2018.v26.n4.a6>

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# HITCHIN'S EQUATIONS ON A NONORIENTABLE MANIFOLD

NAN-KUO HO, GRAEME WILKIN, AND SIYE WU

ABSTRACT. We define Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  for a principal bundle  $P$ , whose structure group is a compact semisimple Lie group  $K$ , over a compact non-orientable Riemannian manifold  $M$ . We use the Donaldson-Corlette correspondence, which identifies Hitchin's moduli space with the moduli space of flat  $K^{\mathbb{C}}$ -connections, which remains valid when  $M$  is non-orientable. This enables us to study Hitchin's moduli space both by gauge theoretical methods and algebraically by using representation varieties. If the orientable double cover  $\tilde{M}$  of  $M$  is a Kähler manifold with odd complex dimension and if the Kähler form is odd under the non-trivial deck transformation  $\tau$  on  $\tilde{M}$ , Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$  of the pull-back bundle  $\tilde{P} \rightarrow \tilde{M}$  has a hyper-Kähler structure and admits an involution induced by  $\tau$ . The fixed-point set  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^{\tau}$  is symplectic or Lagrangian with respect to various symplectic structures on  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ . We show that there is a local diffeomorphism from  $\mathcal{M}^{\text{Hitchin}}(P)$  to  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^{\tau}$ . We compare the gauge theoretical constructions with the algebraic approach using representation varieties.

2010 Mathematics Subject Classification: 53D30, 58D27

## 1. INTRODUCTION

Let  $M$  be a compact orientable Riemannian manifold and let  $K$  be a connected compact Lie group. Given a principal  $K$ -bundle  $P \rightarrow M$ , let  $\mathcal{A}(P)$  be the space of connections and let  $\mathcal{G}(P)$  be the group of gauge transformations on  $P$ . Consider Hitchin's equations

$$(1.1) \quad F_A - \frac{1}{2}[\psi, \psi] = 0, \quad d_A \psi = 0, \quad d_A^* \psi = 0$$

on the pairs  $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$ . Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  is the set of space of solutions  $(A, \psi)$  to (1.1) modulo  $\mathcal{G}(P)$  [15, 29]. On the other hand, let  $G = K^{\mathbb{C}}$  be the complexification of  $K$  and let  $P^{\mathbb{C}} = P \times_K G$ , which is a principal bundle with structure group  $G$ . The moduli space  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$  of flat  $G$ -connections on  $P^{\mathbb{C}}$ , also known as the de Rham moduli space, is the space of flat reductive connections of  $P^{\mathbb{C}}$  modulo  $\mathcal{G}(P)^{\mathbb{C}} \cong \mathcal{G}(P^{\mathbb{C}})$ . A theorem of Donaldson [8] and Corlette [7] states that the moduli spaces  $\mathcal{M}^{\text{Hitchin}}(P)$  and  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$  are homeomorphic. The smooth part of  $\mathcal{M}^{\text{Hitchin}}(P)$  is a Kähler manifold with a complex structure  $\bar{J}$  induced by that on  $G$ .

Suppose in addition that  $M$  is a Kähler manifold. Then there is another complex structure  $\bar{I}$  on  $\mathcal{M}^{\text{Hitchin}}(P)$  induced by that on  $M$ , and a third one given by  $\bar{K} = \bar{I}\bar{J}$ . The three complex structures  $\bar{I}, \bar{J}, \bar{K}$  and their corresponding Kähler forms  $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$  form a hyper-Kähler structure on (the smooth part of)  $\mathcal{M}^{\text{Hitchin}}(P)$  [15, 29]. This hyper-Kähler structure comes from an infinite dimensional version of a hyper-Kähler quotient [16] of the tangent bundle  $T\mathcal{A}(P)$ , which is hyper-Kähler, by the action of  $\mathcal{G}(P)$ , which is Hamiltonian with respect to each of the Kähler forms  $\omega_I, \omega_J, \omega_K$  on  $T\mathcal{A}(P)$ . When  $M$  is a compact orientable surface, Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  is equal to the hyper-Kähler quotient  $\mathcal{M}^{\text{HK}}(P) := T\mathcal{A}(P) \mathbin{\!/\!/\!/_0} \mathcal{G}(P)$  [15]. It plays an important role in mirror symmetry and geometric Langlands program [14, 21]. When  $M$  is higher dimensional,  $\mathcal{M}^{\text{Hitchin}}(P)$  is a hyper-Kähler subspace in  $\mathcal{M}^{\text{HK}}(P)$  [29].

For a compact Lie group  $K$ , the moduli space of flat  $K$ -connections on a compact orientable surface was already studied in a celebrated work of Atiyah and Bott [1]. When  $M$  is a compact, nonorientable surface, the moduli space of flat  $K$ -connections was studied in [17, 19] through an involution on the space of connections over its orientable double cover  $\tilde{M}$ , induced by lifting the deck transformation on  $\tilde{M}$  to the pull-back  $\tilde{P} \rightarrow \tilde{M}$  of the given  $K$ -bundle  $P \rightarrow M$  so that the quotient of  $\tilde{P}$  by the involution is the original bundle  $P$  itself. This involution acts trivially on the structure group  $K$ . If instead one considers an involution on the bundle over  $\tilde{M}$  that acts nontrivially on the fibers (such as the complex conjugation), then the fixed points give rise to the moduli space of real or quaternionic vector bundles over a real algebraic curve. This was studied thoroughly in [4, 27], for example when  $K = U(n)$ .

*Key words and phrases.* moduli spaces, non-orientable manifolds, symplectic and hyper-Kähler geometry, representation varieties.

In this paper, we study Hitchin's equations on a non-orientable manifold. Let  $M$  be a compact connected non-orientable Riemannian manifold and let  $P \rightarrow M$  be a principal  $K$ -bundle over  $M$ , where  $K$  is a compact connected Lie group. The de Rham moduli space  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ , i.e., the moduli space of flat connections on  $P^{\mathbb{C}}$ , does not depend on the orientability of  $M$ . On the other hand, Hitchin's equations (1.1) on the pairs  $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$  still make sense (see subsection 2.2). We define Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  as the quotient of the space of pairs  $(A, \psi)$  satisfying (1.1) by the group  $\mathcal{G}(P)$  of gauge transformations on  $P$ . We explain that the homeomorphism  $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$  of Donaldson-Corlette remains valid when  $M$  is non-orientable (Theorem 2.2).

If the oriented cover  $\tilde{M}$  of  $M$  is a Kähler manifold, then for the pull-back bundle  $\tilde{P} := \pi^*P$  over  $\tilde{M}$ , Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$  is hyper-Kähler with complex structures  $\bar{I}, \bar{J}, \bar{K}$  and Kähler forms  $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$ . If the Kähler form  $\omega$  on  $\tilde{M}$  satisfies  $\tau^*\omega = -\omega$  (the complex dimension of  $\tilde{M}$  must be odd for  $\tau$  to be orientation reversing), then  $\tau$  induces an involution (still denoted by  $\tau$ ) on  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$  that satisfies  $\tau^*\bar{\omega}_I = -\bar{\omega}_I$ ,  $\tau^*\bar{\omega}_J = \bar{\omega}_J$  and  $\tau^*\bar{\omega}_K = -\bar{\omega}_K$ . Consequently, the fixed-point set  $(\mathcal{M}^{\text{Hitchin}}(\tilde{P}))^\tau$  is Lagrangian in  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$  with respect to  $\bar{\omega}_I, \bar{\omega}_K$  and symplectic with respect to  $\bar{\omega}_J$ . This is known as an (A,B,A)-brane in [21]. We discover that Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  (where  $M$  is non-orientable) is related to  $(\mathcal{M}^{\text{Hitchin}}(\tilde{P}))^\tau$  by a local diffeomorphism. Our main results are summarized in the following main theorem. For simplicity, we restrict to certain smooth parts  $\mathcal{M}^{\text{Hitchin}}(P)^\circ$ ,  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$  and  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ$  of the respective spaces (see subsection 2.3 for details).

**Theorem 1.1.** *Let  $M$  be a compact non-orientable manifold and let  $\pi: \tilde{M} \rightarrow M$  be its oriented cover on which there is a non-trivial deck transformation  $\tau$ . Let  $K$  be a compact connected Lie group. Given a principal  $K$ -bundle  $P \rightarrow M$ , let  $\tilde{P} = \pi^*P$  be its pull-back to  $\tilde{M}$ . Suppose that  $\tilde{M}$  is a Kähler manifold of odd complex dimension and the Kähler form  $\omega$  on  $\tilde{M}$  satisfies  $\tau^*\omega = -\omega$ . Then*

- (1)  $\mathcal{M}^{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ //_0 \mathcal{G}(P)$ , which is a symplectic quotient.
- (2)  $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$  is Kähler and totally geodesic in  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$  with respect to  $\bar{J}, \bar{\omega}_J$  and totally real and Lagrangian with respect to  $\bar{I}, \bar{K}$  and  $\bar{\omega}_I, \bar{\omega}_K$ .
- (3) there is a local Kähler diffeomorphism from  $\mathcal{M}^{\text{Hitchin}}(P)^\circ$  to  $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$ .

The theorem of Donaldson and Corlette in the non-orientable setup (Theorem 2.2) enable us to identify Hitchin's moduli space associated to an orientable or non-orientable manifold with the moduli space of flat connections and therefore the representation varieties. Let  $\Gamma$  be a finitely generated group and let  $G$  be a connected complex semi-simple Lie group. The representation variety,  $\text{Hom}(\Gamma, G) // G := \text{Hom}^{\text{red}}(\Gamma, G) / G$ , is the quotient of the space of reductive homomorphisms from  $\Gamma$  to  $G$  by the conjugation action of  $G$ . When  $\Gamma$  is the fundamental group of a compact manifold  $M$ , the representation variety is also called the Betti moduli space of  $M$ ; it is homeomorphic to the union of the de Rham moduli spaces  $\mathcal{M}^{\text{dR}}(P)$  associated to principal  $G$ -bundles  $P \rightarrow M$  of various topology. When  $M$  is non-orientable, let  $\tilde{\Gamma}$  be the fundamental group of the oriented cover  $\tilde{M}$ . Then there is a short exact sequence  $1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 1$  and  $\tau$  acts as an involution on the representation variety  $\text{Hom}(\tilde{\Gamma}, G) // G$  (Lemma 3.3). We study the relation of representation varieties associated to  $\Gamma$  and  $\tilde{\Gamma}$  from an algebraic point of view. Let  $PG = G/Z(G)$ , where  $Z(G)$  is the center of  $G$ . Our main results are summarized in the following theorem.

**Theorem 1.2.** *Let  $G$  be a connected complex semi-simple Lie group. Let  $M$  be a compact non-orientable manifold and let  $\tilde{M}$  be its oriented cover on which there is a non-trivial deck transformation  $\tau$ . Denote  $\Gamma = \pi_1(M)$  and  $\tilde{\Gamma} = \pi_1(\tilde{M})$  with some chosen base points. Then*

- (1) there exists a continuous map  $L$  from  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) // G)^\tau$  to  $Z(G) / 2Z(G)$ . Consequently,  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) // G)^\tau = \bigcup_{r \in Z(G) / 2Z(G)} \mathcal{N}_r^{\text{good}}$ , where  $\mathcal{N}_r^{\text{good}}$  is the preimage of  $r \in Z(G) / 2Z(G)$ .
- (2) there exists a  $|Z(G) / 2Z(G)|$ -sheeted Galois covering map from  $\text{Hom}_\tau^{\text{good}}(\Gamma, G) // G$  to  $\mathcal{N}_0^{\text{good}}$ .

*In particular, if  $|Z(G)|$  is odd, then there exists a bijection from  $\text{Hom}_\tau^{\text{good}}(\Gamma, G) // G$  to  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) // G)^\tau$ . The above statements are true if  $\text{Hom}^{\text{good}}$  is replaced by  $\text{Hom}^{\text{irr}}$ .*

*If in addition  $M = \Sigma$  is a compact non-orientable surface and  $G$  is simple and simply connected, then*

- (3) there exists a surjective map from  $(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G) // G)^\tau$  to  $\text{Hom}_\tau^{\text{irr}}(\Gamma, PG) // PG$  that maps  $\mathcal{N}_r^{\text{irr}}$  to flat  $PG$ -bundles on  $\Sigma$  whose topological type is given by  $r \in Z(G) / 2Z(G) \cong H^2(\Sigma, Z(G))$ . In particular,  $\mathcal{N}_0^{\text{irr}}$  maps to the topologically trivial flat  $PG$ -bundles on  $\Sigma$ .

Here  $\text{Hom}^{\text{good}}$ , following the terminology of [20], denotes the “good” part of the space of homomorphisms that are reductive and whose stabilizer is  $Z(G)$ , whereas  $\text{Hom}^{\text{irr}}$  is the space of homomorphisms whose composition with the adjoint representation of  $G$  is an irreducible representation (see subsection 3.1 for details).  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)$  is the set of homomorphisms from  $\Gamma$  to  $G$  whose restriction to  $\tilde{\Gamma}$  is “good”.  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$  is not smooth in general, but contains a smooth part  $(\mathcal{M}^{\text{flat}}(P^{\mathbb{C}}))^\circ$  (upon identification of moduli spaces). By parts (1) and (2) of the theorem, there is a local homeomorphism  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G \rightarrow (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$  (see also Corollary 3.7), which in fact restricts to the local diffeomorphism  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ \rightarrow (\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$  in part (3) of Theorem 1.1 but is now more accurately described using representation varieties. Also, for  $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$  such that  $[\phi] \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$  is fixed by  $\tau$ ,  $L([\phi])$  is the obstruction of extending  $\phi$  to a representation of  $\Gamma$ . In the gauge-theoretic language,  $\phi$  corresponds to a flat connection on  $\tilde{M}$  and represents a point fixed by  $\tau$  in the de Rham moduli space  $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})$ , while extension of  $\phi$  to  $\Gamma$  means that the flat connection on  $\tilde{M}$  is the pull-back of a flat connection on  $M$ . Flat connections on  $\tilde{M}$  that are not pull-backs from  $M$  correspond to flat  $PG$ -bundles over  $M$  (where  $PG = G/Z(G)$ ). This is shown in part (3) of Theorem 1.2 and then discussed in greater generality in the last section.

For example, let  $G = SL(2, \mathbb{C})$ ,  $M$  a compact nonorientable surface and  $\tilde{M}$  its orientable double cover. Then  $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$  is labeled by  $Z(G)/2Z(G) = \mathbb{Z}_2$ , i.e.,  $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau = \bigcup_{r \in \mathbb{Z}_2} \mathcal{N}_r^{\text{good}}$ . An element of  $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$  is mapped by map  $L$  in Theorem 1.2(1) (defined in Proposition 3.4) to the null element of  $\mathbb{Z}_2$  if and only if it represents a flat connection on  $\tilde{M}$  that is the pull-back of a flat connection on  $M$ . The natural map from  $\text{Hom}^{\text{good}}(\pi_1(M), G)/G$  to  $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$  is not surjective; it is a  $\mathbb{Z}_2$ -sheeted Galois covering map onto  $\mathcal{N}_0^{\text{good}}$ , and  $\mathcal{N}_1^{\text{good}}$  is not in the image.  $\mathcal{N}_0^{\text{irr}}$  corresponds to the space of topologically trivial flat  $PSL(2, \mathbb{C})$ -bundles over  $M$  while  $\mathcal{N}_1^{\text{irr}}$  corresponds to that of topologically nontrivial flat  $PSL(2, \mathbb{C})$ -bundles over  $M$ .

The rest of this paper is organized as follows. In Section 2, we review the basic setup in the orientable case and explain the Donaldson-Corlette theorem for bundles over non-orientable manifolds. We then study finite dimensional symplectic and hyper-Kähler manifolds with an involution and apply the results to the gauge theoretical setting to prove Theorem 1.1. In Section 3, we study flat  $G$ -connections by representation varieties. We show that a flat connection on  $M$  is reductive if and only if its pull-back to  $\tilde{M}$  is reductive. We then define the continuous map in part (1) of Theorem 1.2 and prove the rest of the theorem. In Section 4, we relate the components  $\mathcal{N}_r^{\text{good}}$  ( $r \neq 0$ ) in Theorem 1.2 to  $G$ -bundles over  $\tilde{M}$  admitting an involution up to  $Z(G)$ .

We note that in order to study the moduli space of  $G$ -bundles over the nonorientable manifold  $M$  itself, our involution is fixed-point free on  $\tilde{M}$  and is the identity map on  $G$ . During the revision of this paper, we came across a few related works. We thank O. García-Prada for pointing out to us the paper [3], where their anti-holomorphic involution acts both on the manifold  $\tilde{M}$  and on the structure group  $G$ , thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with a special case of part (2) of our Theorem 1.1 when  $\tilde{M}$  is a surface, the anti-holomorphic involution on the surface is allowed to have fixed points.

## 2. THE GAUGE-THEORETIC PERSPECTIVE

**2.1. Basic setup in the orientable case.** Let  $K$  be a connected compact Lie group and let  $G = K^{\mathbb{C}}$  be its complexification. Given a principal  $K$ -bundle  $P$  over a compact orientable manifold  $M$ ,  $P^{\mathbb{C}} = P \times_K G$  is a principal bundle whose structure group is  $G$ . The set  $\mathcal{A}(P)$  of connections on  $P$  is an affine space modeled on  $\Omega^1(M, \text{ad } P)$ . At each  $A \in \mathcal{A}(P)$ , the tangent space is  $T_A \mathcal{A}(P) \cong \Omega^1(M, \text{ad } P)$ . The total space of the tangent bundle over  $\mathcal{A}(P)$  is  $T\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$ . At  $(A, \psi) \in T\mathcal{A}(P)$ , the tangent space is  $T_{(A, \psi)} T\mathcal{A}(P) \cong \Omega^1(M, \text{ad } P)^{\oplus 2}$ . There is a translation invariant complex structure  $J$  on  $T\mathcal{A}(P)$  given by  $J(\alpha, \varphi) = (\varphi, -\alpha)$ . The space  $T\mathcal{A}(P)$  can be naturally identified with  $\mathcal{A}(P^{\mathbb{C}})$ , the set of connections on  $P^{\mathbb{C}} \rightarrow M$ , via  $(A, \psi) \mapsto A - \sqrt{-1}\psi$ , under which  $J$  corresponds to the complex structure on  $\mathcal{A}(P^{\mathbb{C}})$  induced by  $G = K^{\mathbb{C}}$ . The covariant derivative on  $\Omega^\bullet(M, \text{ad } P^{\mathbb{C}})$  is  $D := d_A - \sqrt{-1}\psi$ , where  $d_A$  denotes the covariant derivative of  $A \in \mathcal{A}(P)$  and  $\psi$  acts by bracket.

The group of gauge transformations on  $P$  is  $\mathcal{G}(P) \cong \Gamma(M, \text{Ad } P)$ . It acts on  $\mathcal{A}(P)$  via  $A \mapsto g \cdot A$ , where  $d_{g \cdot A} = g \circ d_A \circ g^{-1}$  and on  $T\mathcal{A}(P)$  via  $g : (A, \psi) \mapsto (g \cdot A, \text{Ad}_g \psi)$ . Since the action of  $\mathcal{G}(P)$  on  $T\mathcal{A}(P)$  preserves  $J$ , there is a holomorphic  $\mathcal{G}(P)^{\mathbb{C}}$  action on  $(T\mathcal{A}(P), J)$ . In fact, the complexification  $\mathcal{G}(P)^{\mathbb{C}}$  can be naturally identified with  $\mathcal{G}(P^{\mathbb{C}}) \cong \Gamma(M, \text{Ad } P^{\mathbb{C}})$ , and the action of  $\mathcal{G}(P^{\mathbb{C}})$  on  $T\mathcal{A}(P)$  corresponds to the complex gauge transformations

on  $\mathcal{A}(P^{\mathbb{C}})$ , i.e.,  $g \in \mathcal{G}(P^{\mathbb{C}}): D \mapsto g \circ D \circ g^{-1}$ . Let

$$\begin{aligned} \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) &= \{A - \sqrt{-1}\psi \in \mathcal{A}(P^{\mathbb{C}}) : F_{A - \sqrt{-1}\psi} = 0\} \\ &= \{(A, \psi) \in T\mathcal{A} : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0\} \end{aligned}$$

be the set of flat connections on  $P^{\mathbb{C}}$ . Since the vanishing of  $F_{A - \sqrt{-1}\psi}$  is a holomorphic condition,  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$  is a complex subset of  $\mathcal{A}(P^{\mathbb{C}})$ ; it is also invariant under  $\mathcal{G}(P^{\mathbb{C}})$ . The holonomy group  $\text{Hol}(A)$  of  $A \in \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$  can be identified as a subgroup of  $G$ , up to a conjugation in  $G$ . A flat connection  $A$  on  $P^{\mathbb{C}}$  is *reductive* if the closure of  $\text{Hol}(A)$  in  $G$  is contained in the Levi subgroup of any parabolic subgroup containing  $\text{Hol}(A)$ ; let  $\mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}})$  be the set of such. It can be shown that a flat connection is reductive if and only if its orbit under  $\mathcal{G}(P^{\mathbb{C}})$  is closed [7]. The *de Rham moduli space*, or the moduli space of reductive flat connections on  $P^{\mathbb{C}}$ , is

$$\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) // \mathcal{G}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}}) / \mathcal{G}(P^{\mathbb{C}}).$$

It has an induced complex structure  $\bar{J}$  on its smooth part.

Assume that  $M$  has a Riemannian structure and choose an invariant inner product  $(\cdot, \cdot)$  on the Lie algebra  $\mathfrak{k}$  of  $K$ . Then there is a symplectic structure on  $T\mathcal{A}(P)$ , with which  $J$  is compatible, given by

$$(2.1) \quad \omega_J((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M (\varphi_2, \wedge * \alpha_1) - (\varphi_1, \wedge * \alpha_2),$$

where  $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^{1,0}(M, \text{ad } P)$ , such that  $(T\mathcal{A}(P), \omega_J)$  is Kähler. The subset  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$  is Kähler in  $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$ . We identify the Lie algebra  $\text{Lie}(\mathcal{G}(P)) \cong \Omega^0(M, \text{ad } P)$  with its dual by the inner product on  $\Omega^0(M, \text{ad } P)$ . The action of  $\mathcal{G}(P)$  on  $(T\mathcal{A}(P), \omega_J)$  is Hamiltonian, with moment map

$$(2.2) \quad \mu_J(A, \psi) = d_A^* \psi \in \Omega^0(M, \text{ad } P).$$

Let

$$\begin{aligned} \mathcal{A}^{\text{Hitchin}}(P) &= \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) \\ &= \{(A, \psi) \in T\mathcal{A} : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0, d_A^* \psi = 0\}, \end{aligned}$$

the set of pairs  $(A, \psi)$  satisfying Hitchin's equations (1.1), and let the quotient space  $\mathcal{M}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{Hitchin}}(P) / \mathcal{G}(P)$  be *Hitchin's moduli space*. A theorem of Donaldson [8] and Corlette [7] states that if  $M$  is compact and if the structure group  $G$  is semisimple, then  $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ .

Suppose that  $M$  is a compact Kähler manifold of complex dimension  $n$  and let  $\omega$  be the Kähler form on  $M$ . Then there is a complex structure on  $T\mathcal{A}(P)$  given by

$$I: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\alpha, -\varphi)) = \frac{1}{(n-1)!} \Lambda^{n-1}(*\alpha, -*\varphi),$$

where  $(\alpha, \varphi) \in \Omega^1(M, \text{ad } P)^{\oplus 2} \cong T_{(A, \psi)}T\mathcal{A}(P)$  and the map

$$\Lambda: \Omega^{\bullet}(M, \text{ad } P) \rightarrow \Omega^{\bullet-2}(M, \text{ad } P)$$

is the contraction by  $\omega$ . With respect to  $I$ , we have

$$T_{(A, \psi)}^{1,0}T\mathcal{A}(P) \cong \Omega^{0,1}(M, \text{ad } P^{\mathbb{C}}) \oplus \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$$

for any  $(A, \psi) \in T\mathcal{A}(P)$ . This complex structure  $I$  is compatible with a symplectic form  $\omega_I$  on  $T\mathcal{A}(P)$  given by

$$\omega_I((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \alpha_2) - (\varphi_1, \wedge \varphi_2)),$$

where  $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^1(M, \text{ad } P)$ . The action of  $\mathcal{G}(P)$  on  $T\mathcal{A}(P)$  is also Hamiltonian with respect to  $\omega_I$  and the moment map is

$$\mu_I(A, \psi) = \Lambda(F_A - \frac{1}{2}[\psi, \psi]) \in \Omega^0(M, \text{ad } P),$$

where  $F_A \in \Omega^2(M, \text{ad } P)$  is the curvature of  $A$ . Since the action of  $\mathcal{G}(P)$  on  $T\mathcal{A}(P)$  preserves  $I$ , there is a holomorphic  $\mathcal{G}(P^{\mathbb{C}})$  action on  $(T\mathcal{A}(P), I)$ . For any  $(A, \psi) \in T\mathcal{A}(P)$ , write  $\psi = \sqrt{-1}(\phi - \phi^*)$ , where  $\phi \in \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$ ,  $\phi^* \in \Omega^{0,1}(M, \text{ad } P^{\mathbb{C}})$ . Here  $\phi \mapsto \phi^*$  is induced by the conjugation on  $G = K^{\mathbb{C}}$  preserving the compact form  $K$ . Then  $D = d_A - \sqrt{-1}\psi = D' + D''$ , where  $D' = \partial_A - \phi^*$ ,  $D'' = \bar{\partial}_A + \phi$ . The action of  $\mathcal{G}(P^{\mathbb{C}})$  on  $T\mathcal{A}(P) \cong \mathcal{A}(P^{\mathbb{C}})$  can be described by  $g \in \mathcal{G}(P^{\mathbb{C}}): D'' \mapsto g \circ D'' \circ g^{-1}$ .

Let  $\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}})$  be the set of Higgs pairs  $(A, \phi)$ , i.e.,  $A \in \mathcal{A}(P)$  and  $\phi \in \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$  satisfying  $(D'')^2 = 0$ , or

$$\bar{\partial}_A^2 = 0, \quad \bar{\partial}_A \phi = 0, \quad [\phi, \phi] = 0.$$

Then  $\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}})$  is a Kähler subspace of  $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$  respect to  $I$ . Let  $\mathcal{A}^{\text{sst}}(P^{\mathbb{C}})$  be the set of semistable Higgs pairs and let  $\mathcal{A}^{\text{pst}}(P^{\mathbb{C}})$  be the set polystable Higgs pairs. (The notions of stable, semistable and polystable Higgs pairs were introduced in [15, 30, 31].) The *moduli space of polystable Higgs pairs* or the *Dolbeault moduli space* is

$$\mathcal{M}^{\text{Dol}}(P^{\mathbb{C}}) = (\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\text{sst}}(P^{\mathbb{C}})) // \mathcal{G}(P^{\mathbb{C}}) = (\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\text{pst}}(P^{\mathbb{C}})) // \mathcal{G}(P^{\mathbb{C}}).$$

It has a complex structure induced by  $I$ . It can be shown [30, Lemma 1.1] that  $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$ . A theorem of Hitchin [15] and Simpson [29] states that if  $M$  is compact and Kähler and the bundle  $P$  has vanishing first and second Chern classes, then  $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{Dol}}(P^{\mathbb{C}})$ .

There is a third complex structure on  $T\mathcal{A}(P)$  defined by

$$K = IJ = -JI: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\varphi, \alpha)) = \frac{1}{(n-1)!} \Lambda^{n-1}(*\varphi, *\alpha),$$

which is compatible with the symplectic form

$$\omega_K((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \varphi_2) - (\alpha_2, \wedge \varphi_1)).$$

The action of  $\mathcal{G}(P)$  on  $T\mathcal{A}(P)$  is Hamiltonian with respect to  $\omega_K$  and the moment map is

$$\mu_K(A, \psi) = \Lambda(d_A \psi) \in \Omega^0(M, \text{ad } P).$$

Moreover, the action preserves  $K$  and therefore extends to another holomorphic action of  $\mathcal{G}(P)^{\mathbb{C}}$ . The three complex structures  $I, J, K$  define a hyper-Kähler structure on  $T\mathcal{A}(P)$ . Since the action of  $\mathcal{G}(P)$  on  $T\mathcal{A}(P)$  is Hamiltonian with respect to all three symplectic forms, we have a hyper-Kähler moment map  $\mu = (\mu_I, \mu_J, \mu_K): T\mathcal{A}(P) \rightarrow (\Omega^0(M, \text{ad } P))^{\oplus 3}$ . The *hyper-Kähler quotient* [16] is  $\mathcal{M}^{\text{HK}}(P) = \mu^{-1}(0) / \mathcal{G}(P)$ , with complex structures  $\bar{I}, \bar{J}, \bar{K}$  and symplectic forms  $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$ . By the theorems of Donaldson-Corlette and of Hitchin-Simpson, the Hitchin moduli space  $\mathcal{M}^{\text{Hitchin}}(P)$  is a complex space with respect to both  $\bar{I}$  and  $\bar{J}$ . Therefore  $\mathcal{M}^{\text{Hitchin}}(P)$  is a hyper-Kähler subspace in  $\mathcal{M}^{\text{HK}}(P)$  [10, Theorem 8.3.1].

When  $M = \Sigma$  is an orientable surface,  $\Lambda: \Omega^2(\Sigma, \text{ad } P) \rightarrow \Omega^0(\Sigma, \text{ad } P)$  is an isomorphism. So  $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$  coincides with  $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ . Thus the moduli spaces  $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Dol}}(P^{\mathbb{C}})$  coincide with the hyper-Kähler quotient  $\mathcal{M}^{\text{HK}}(P)$  [15].

**2.2. Moduli space of Hitchin's equations on a non-orientable manifold.** Now suppose  $M$  is a compact non-orientable manifold. Let  $\pi: \tilde{M} \rightarrow M$  be its oriented cover and let  $\tau: \tilde{M} \rightarrow \tilde{M}$  be the non-trivial deck transformation. Given a principal  $K$ -bundle  $P \rightarrow M$ , let  $\tilde{P} = \pi^* P \rightarrow \tilde{M}$  be its pull-back to  $\tilde{M}$ . Since  $\pi \circ \tau = \pi$ , the  $\tau$  action can be lifted to  $\tilde{P} = \tilde{M} \times_M P$  as a  $K$ -bundle involution (i.e., the lifted involution commutes with the right  $K$ -action on  $\tilde{P}$ ), and hence to the associated bundles  $\text{Ad } \tilde{P}$  and  $\text{ad } \tilde{P}$ . Consequently,  $\tau$  acts on the space of connections  $\mathcal{A}(\tilde{P})$  by pull-back  $A \mapsto \tau^* A$  and on the group of gauge transformations  $\mathcal{G}(\tilde{P})$  by  $g \mapsto \tau^* g := \tau^{-1} \circ g \circ \tau$ . The  $\tau$ -invariant subsets are  $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$  and  $(\mathcal{G}(\tilde{P}))^\tau \cong \mathcal{G}(P)$ . In fact, the inclusion map  $\mathcal{A}(P) \hookrightarrow \mathcal{A}(\tilde{P})$  onto the  $\tau$ -invariant part is the pull-back via  $\pi$  of connections on  $P$  to those on  $\tilde{P}$ . Since  $\mathcal{A}(\tilde{P})$  is an affine space modeled on  $\Omega^1(\tilde{M}, \text{ad } \tilde{P})$ , the differential  $\tau_*$  of  $\tau: \mathcal{A}(\tilde{P}) \rightarrow \mathcal{A}(\tilde{P})$  can be identified with a linear involution on  $\Omega^1(\tilde{M}, \text{ad } \tilde{P})$  given by  $\alpha \mapsto \tau^* \alpha$ .

A Riemannian metric on a non-orientable manifold  $M$  pulls back to a Riemannian metric on  $\tilde{M}$ . Assuming that  $M$  is compact, we define an inner product on the space  $\Omega^\bullet(M)$  of differential forms on  $M$  by

$$\langle \alpha, \beta \rangle = \frac{1}{2} \int_{\tilde{M}} \pi^* \alpha \wedge \tilde{*} \pi^* \beta$$

for  $\alpha, \beta \in \Omega^\bullet(M)$ , where  $\tilde{*}$  is the Hodge star operator on  $\tilde{M}$ . Alternatively, the Hodge star  $*$  on  $M$  maps a form on  $M$  to one valued in the orientation line bundle over  $M$ , and if  $\alpha, \beta$  are of the same degree, then  $\alpha \wedge * \beta$  is a top-degree form on  $M$  valued in the orientation line bundle, which can be integrated over  $M$ . We still have  $\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$ . More generally, there is an inner product on the space  $\Omega^\bullet(M, \text{ad } P)$  of forms valued in  $\text{ad } P$ . Therefore  $\mathcal{A}(P)$  admits a Riemannian structure, which is half of the restriction of the Riemannian structure on  $\mathcal{A}(\tilde{P})$  to the  $\tau$ -invariant subspace  $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$ .

Consider the tangent bundle  $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(\tilde{M}, \text{ad } \tilde{P})$  of  $\mathcal{A}(\tilde{P})$ . It has a  $\tau$ -action given by  $\tau: (A, \psi) \mapsto (\tau^* A, \tau^* \psi)$ , which is holomorphic with respect to the complex structure  $J$ . Therefore the fixed point set  $(T\mathcal{A}(\tilde{P}))^\tau \cong T\mathcal{A}(P)$  is a complex subspace in  $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ . With respect to the induced Riemannian structure on  $T\mathcal{A}(\tilde{P})$ ,

$\tau: TA(\tilde{P}) \rightarrow TA(\tilde{P})$  is an isometry. Since  $\tau$  also acts holomorphically on  $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong TA(\tilde{P})$ ,  $(TA(\tilde{P}))^\tau$  is a Kähler and totally geodesic subspace in  $TA(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ . Moreover,  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cong (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$  is also Kähler and totally geodesic in  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$ . We summarize the above discussion in the following lemma.

**Lemma 2.1.** *Given a compact non-orientable manifold  $M$  with oriented double cover  $\pi: \tilde{M} \rightarrow M$  and a principal  $K$ -bundle  $P \rightarrow M$ , the non-trivial deck transformation  $\tau$  on  $\tilde{M}$  lifts to an involution (also denoted by  $\tau$ ) on  $\tilde{P} = \pi^*P$  and acts as involutions on the space of connections  $\mathcal{A}(\tilde{P})$  and on  $TA(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ . Moreover, the  $\tau$ -invariant subspaces  $\mathcal{A}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}(P^{\mathbb{C}})$  and  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$  are Kähler and totally geodesic subspaces in  $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong TA(\tilde{P})$  and  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$ , respectively.*

On a non-orientable manifold  $M$ , we still have Hitchin's equations (1.1). Here  $d_A^*$  is defined as the (formal) adjoint of  $d_A$  with respect to the inner products on  $\Omega^\bullet(M, \text{ad } P)$ . Alternatively,  $d_A^*$  is the first order differential operator on  $M$  such that on any orientable open set in  $M$ ,  $d_A^* = *^{-1}d_A*$ ; the latter is actually independent of the choice of local orientation. Yet another but related way to explain the operator  $d_A^*$  is to consider the Hodge star operator  $*$  on a non-orientable manifold  $M$  as a map from differential forms to those valued in the orientation bundle over  $M$ . Since the latter is a flat real line bundle,  $d_A^* = *^{-1}d_A*$  maps  $\Omega^1(M, \text{ad } P)$  to  $\Omega^0(M, \text{ad } P)$ . Finally,  $d_A^*$  can be defined as  $(\pi^*)^{-1} \circ d_{\pi^*A}^* \circ \pi^*$ . Here  $d_{\pi^*A}^* = *^{-1}d_{\pi^*A}$  holds globally on  $\tilde{M}$  and  $\pi^*: \Omega^\bullet(M, \text{ad } P) \rightarrow \Omega^\bullet(\tilde{M}, \text{ad } \tilde{P})$  is injective. Let

$$\mathcal{A}^{\text{Hitchin}}(P) := \{(A, \psi) \in TA : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0, d_A^*\psi = 0\}.$$

It is clear that  $\mathcal{A}^{\text{Hitchin}}(P) = (\mathcal{A}^{\text{Hitchin}}(\tilde{P}))^\tau$ .

The notion of reductive connections on  $P$  does not depend on the orientability of  $M$ , and we still have the moduli space of flat connections  $\mathcal{M}^{\text{flat}}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat, red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}})$ . Let  $\mathcal{M}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{Hitchin}}(P)/\mathcal{G}(P)$  be Hitchin's moduli space. The following is the Donaldson-Corlette theorem that also applies to the case when  $M$  is non-orientable. Equivalently, there exists a unique reduction of structure group from  $G$  to  $K$  admitting a solution to Hitchin's equations.

**Theorem 2.2.** *Let  $M$  be a compact non-orientable Riemannian manifold. Then for every reductive flat connection  $D$  on  $P^{\mathbb{C}}$ , there exists a gauge transformation  $g \in \mathcal{G}(P^{\mathbb{C}})$  (unique up to  $\mathcal{G}(P)$  and the stabilizer of  $D$ ) such that  $g \cdot D = d_A - \sqrt{-1}\psi$  with  $(A, \psi) \in \mathcal{A}^{\text{Hitchin}}(P)$ . As a consequence, we have a homeomorphism  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Hitchin}}(P)$ .*

We now explain that Corlette's proof in [7] applies to the case when  $M$  is non-orientable. There is a symplectic form  $\omega_J$  on  $TA(P)$ , still given by (2.1), which is half of the restriction of the symplectic form on  $TA(\tilde{P})$ . The action of  $\mathcal{G}(P)$  on  $TA(P)$  is Hamiltonian, and the moment map remains (2.2). Recall Corlette's flow equations on the space of flat connections. Let  $D = d_A - \sqrt{-1}\psi$  be a flat connection of the  $G = K^{\mathbb{C}}$  bundle  $P^{\mathbb{C}} \rightarrow M$ . Then the flow equations are

$$(2.3) \quad \frac{\partial D}{\partial t} = -D\mu_J(D).$$

Equivalently, one can look for a flow of the form  $g(t) \cdot D_0$  and solve for  $g(t) \in \mathcal{G}(\tilde{P}^{\mathbb{C}})$  using (cf. [7, p. 369])

$$(2.4) \quad \frac{\partial g}{\partial t}g^{-1} = -\sqrt{-1}\mu_J(g \cdot D_0).$$

Corlette shows in [7] that we have existence and uniqueness of solutions to (2.3) and (2.4) for all time. If the initial condition is a reductive flat connection, then there is a sequence converging to a solution to  $\mu_J(D) = 0$ . Also, the limit is gauge equivalent to the initial flat reductive connection [7]. These arguments are valid when  $M$  is non-orientable.

We remark that Theorem 2.2 for non-orientable manifolds also follows from the result of the orientable double cover. A flat connection on  $P$  is reductive if and only if the pull-back  $\pi^*A$  is a flat reductive connection on  $\tilde{P}$ . (We defer the proof of this statement to Corollary 3.2.) For the bundle  $\tilde{P} \rightarrow \tilde{M}$ , it is easy to check that the right-hand sides of (2.3) and (2.4) define  $\tau$ -invariant vector fields on  $\mathcal{A}(\tilde{P}^{\mathbb{C}})$  and  $\mathcal{G}(\tilde{P}^{\mathbb{C}})$ , respectively. Since the space  $(\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$  of  $\tau$ -invariant connections is closed in  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$  and the space  $(\mathcal{G}(\tilde{P}^{\mathbb{C}}))^\tau$  of  $\tau$ -invariant gauge transformations is closed in  $\mathcal{G}(\tilde{P}^{\mathbb{C}})$ , Corlette's results on the limit of the flow restrict to the  $\tau$ -invariant subset as well. That is, the flow on the space of connections is contained in the  $\tau$ -invariant subset and the limit is a  $\tau$ -invariant solution to Hitchin's equation. Similarly, the gauge transformation relating to the initial condition is contained in the  $\tau$ -invariant part of the group of gauge transformations, and the limit is  $\tau$ -invariant.

**2.3. The Hitchin moduli space and the hyper-Kähler quotient.** Now consider a compact non-orientable manifold  $M$ . Suppose its oriented cover  $\tilde{M}$  is a Kähler manifold of complex dimension  $n$ . Let  $\omega$  be the Kähler form on  $\tilde{M}$ . Throughout this subsection, we assume that  $n$  is odd and the deck transformation  $\tau$  on  $\tilde{M}$  is an anti-holomorphic involution such that  $\tau^*\omega = -\omega$ . Then  $\tau^*\omega^n = -\omega^n$ , which is consistent with the requirement that  $\tau$  is orientation reversing. The  $\tau$ -action on  $\mathcal{TA}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(M, \text{ad } P)$ ,  $\tau: (A, \psi) \mapsto (\tau^*A, \tau^*\psi)$ , is an isometry and its differential  $\tau_*: \Omega^1(M, \text{ad } P)^{\oplus 2} \rightarrow \Omega^1(M, \text{ad } P)^{\oplus 2}$  is  $\tau_*: (\alpha, \varphi) \mapsto (\tau^*\alpha, \tau^*\varphi)$ . It is easy to see that  $\tau_* \circ I = -I \circ \tau_*$  since  $\tau$  reverses the orientation of  $M$  and that  $\tau_* \circ K = -K \circ \tau_*$  since  $K = IJ$ . So  $\tau$  acts as an anti-holomorphic involution with respect to both  $I$  and  $K$ , and  $\tau^*\omega_I = -\omega_I$ ,  $\tau^*\omega_K = -\omega_K$ . Moreover, since the moment maps  $\mu_I$  and  $\mu_K$  on  $\mathcal{TA}(\tilde{P})$  involve the contraction  $\lrcorner$  by  $\omega$ , they satisfy  $\tau^*(\mu_I(A, \psi)) = -\mu_I(\tau^*A, \tau^*\psi)$ ,  $\tau^*(\mu_K(A, \psi)) = -\mu_K(\tau^*A, \tau^*\psi)$  for all  $(A, \psi) \in \mathcal{TA}(\tilde{P})$ . The fixed point set  $(\mathcal{A}(\tilde{P}))^\tau$  is totally real with respect to the complex structures  $I$  and  $K$ , and Lagrangian with respect to the symplectic forms  $\omega_I$  and  $\omega_K$  [24, 9, 25].

A flat connection  $D = d_A - \sqrt{-1}\psi$  on  $\tilde{P}^{\mathbb{C}}$  defines an elliptic complex with  $D_i: \Omega^i(\tilde{M}, \text{ad } \tilde{P}^{\mathbb{C}}) \rightarrow \Omega^{i+1}(\tilde{M}, \text{ad } \tilde{P}^{\mathbb{C}})$ . Let  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$  be the set of flat connections on  $\tilde{P}^{\mathbb{C}}$  such that (i) the stabilizer under the  $\mathcal{G}(\tilde{P}^{\mathbb{C}})$  action is  $Z(G)$ , and (ii) the linearization  $D_1$  of the curvature map surjects onto  $\ker D_2 \cap \Omega^2(\tilde{M}, [\text{ad } \tilde{P}^{\mathbb{C}}, \text{ad } \tilde{P}^{\mathbb{C}}])$ . Notice that when  $M$  is a surface, condition (i) implies (ii). The method in [22] and [23, Chapter VII] shows that  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$  is a smooth submanifold in  $\mathcal{A}(\tilde{P}^{\mathbb{C}})$ , and as the action of  $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$  on it is free, the subset  $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ := (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mathcal{A}^{\text{flat,red}}(\tilde{P}^{\mathbb{C}}))/\mathcal{G}(\tilde{P}^{\mathbb{C}})$  is in the smooth part of the moduli space  $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})$  (see also [11] from the point of view of representation varieties). The free action of  $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$  or  $\mathcal{G}(\tilde{P})/Z(K)$  from condition (i) implies that 0 is a regular value of  $\mu_J$  on  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ , and the subset  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ := \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0)/\mathcal{G}(\tilde{P})$  is in the smooth part of Hitchin's moduli space  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$  [15]. By the Donaldson-Corlette theorem, we have the homeomorphism  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ \cong \mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ$ .

On the other hand, for the non-orientable manifold  $M$ , let  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ = \{A \in \mathcal{A}(P^{\mathbb{C}}) : \pi^*A \in \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ\}$ ,  $\mathcal{A}^{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{Hitchin}}(P) \cap \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ$ .

Then  $\mathcal{M}^{\text{Hitchin}}(P)^\circ := \mathcal{A}^{\text{Hitchin}}(P)^\circ/\mathcal{G}(P)$  is in the smooth part of  $\mathcal{M}^{\text{Hitchin}}(P)$ , but we will not consider here the smooth points of  $\mathcal{M}^{\text{Hitchin}}(P)$  that are outside  $\mathcal{M}^{\text{Hitchin}}(P)^\circ$ . By Theorem 2.2 (the analog of the Donaldson-Corlette theorem for non-orientable manifolds), we have a homeomorphism between  $\mathcal{M}^{\text{Hitchin}}(P)^\circ$  and  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ := (\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cap \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}})$ .

We now study a general setting. Let  $(X, \omega)$  be a finite dimensional symplectic manifold with a Hamiltonian action of a compact Lie group  $K$  and let  $\mu: X \rightarrow \mathfrak{k}^*$  be the moment map. Suppose as in [25], that there are involutions  $\sigma$  on  $X$  and  $\tau$  on  $K$  such that  $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$  for all  $k \in K$  and  $x \in X$ . Assume that  $X^\sigma$  is not empty. Then  $K^\tau$  acts on  $X^\sigma$ . We note that  $\tau$  acts on  $\mathfrak{k}, \mathfrak{k}^*$ , and  $K^\tau$  is a closed Lie subgroup of  $K$  with Lie algebra  $\mathfrak{k}^\tau$ . Contrary to [25], we assume that the action of  $(K, K^\tau)$  on  $(X, X^\sigma)$  is symplectic, i.e., we have  $\sigma^*\omega = \omega$  and  $\sigma^*\mu = \tau\mu$ . Then  $X^\sigma$  is a symplectic submanifold in  $X$ . Assume that 0 is a regular value of  $\mu$  and that  $K$  acts on  $\mu^{-1}(0)$  freely. Since  $\sigma$  preserves  $\mu^{-1}(0)$ , it descends to a symplectic involution  $\bar{\sigma}$  on the (smooth) symplectic quotient  $X//_0 K = \mu^{-1}(0)/K$  at level 0, and  $(X//_0 K)^{\bar{\sigma}}$  is a symplectic submanifold.

**Lemma 2.3.** *In the above setting, the action of  $K^\tau$  on  $X^\sigma$  is Hamiltonian and the symplectic quotient is  $X^\sigma//_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$ . If  $\mu^{-1}(0) \cap X^\sigma \neq \emptyset$ , then there exists a symplectic local diffeomorphism from  $X^\sigma//_0 K^\tau$  to  $(X//_0 K)^{\bar{\sigma}}$ .*

*Proof.* Let  $\mathfrak{k} = \mathfrak{k}^\tau \oplus \mathfrak{q}$  such that  $\tau = \pm 1$  on  $\mathfrak{k}^\tau, \mathfrak{q}$ , respectively. It is clear that the action of  $K^\tau$  on  $X^\sigma$  is Hamiltonian and the moment map  $\mu_\tau$  is the composition  $X^\sigma \hookrightarrow X \rightarrow \mathfrak{k}^* \rightarrow (\mathfrak{k}^\tau)^*$ . Since for any  $x \in X^\sigma$ ,  $\langle \mu(x), \mathfrak{q} \rangle = 0$ , we get  $\mu_\tau^{-1}(0) = \mu^{-1}(0) \cap X^\sigma = (\mu^{-1}(0))^\sigma$ . By the assumptions, 0 is a regular value of  $\mu_\tau$ , the action of  $K^\tau$  on  $\mu_\tau^{-1}(0)$  is free, and the symplectic quotient is  $X^\sigma//_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$ .

For any  $x \in X^\sigma$ , the map  $\mathfrak{k} \rightarrow T_x X$  intertwines  $\tau$  on  $\mathfrak{k}$  and  $\sigma$  on  $T_x X$ , and  $T_x(K^\tau \cdot x) = (T_x(K \cdot x))^\sigma$ . The inclusion  $\mu_\tau^{-1}(0) \hookrightarrow \mu^{-1}(0)$  induces a natural map  $X^\sigma//_0 K^\tau \rightarrow (X//_0 K)^{\bar{\sigma}}$ , whose differentiation at  $[x]$  is, after natural symplectic isomorphisms, the linear map  $(T_x \mu^{-1}(0))^\sigma / (T_x(K \cdot x))^\sigma \rightarrow (T_x \mu^{-1}(0) / T_x(K \cdot x))^{\bar{\sigma}}$ . The latter is clearly injective; to show surjectivity, we note that for any  $V \in T_x \mu^{-1}(0)$ , if  $V + T_x(K \cdot x) \in (T_x \mu^{-1}(0) / T_x(K \cdot x))^{\bar{\sigma}}$ , then it is the image of  $\frac{1}{2}(V + \sigma V) + (T_x(K \cdot x))^\sigma$ . The map  $X^\sigma//_0 K^\tau \rightarrow (X//_0 K)^{\bar{\sigma}}$  is a local diffeomorphism; it is symplectic because the above linear map is so for each  $x \in \mu_\tau^{-1}(0)$ .  $\square$

Now let  $X$  be a hyper-Kähler manifold with complex structures  $J_i$  and symplectic structures  $\omega_i$  ( $i = 1, 2, 3$ ). Suppose  $K$  acts on  $X$  and the action is Hamiltonian with respect to all  $\omega_i$ . Let  $\mu = (\mu_1, \mu_2, \mu_3): X \rightarrow (\mathfrak{k}^*)^{\oplus 3}$  be the hyper-Kähler moment map. Assume that there are involutions  $\sigma$  on  $X$  and  $\tau$  on  $K$  such that  $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$  for all  $k \in K$  and  $x \in X$  and  $\sigma^* J_i = (-1)^i J_i$ ,  $\sigma^* \omega_i = (-1)^i \omega_i$ ,  $\sigma^* \mu_i = (-1)^i \tau \mu_i$  for  $i = 1, 2, 3$ . So the action of  $(K, K^\tau)$  on  $(X, X^\sigma)$  is symplectic with respect to  $\omega_2$  (as above) and anti-symplectic with respect to  $\omega_1, \omega_3$  (as in [25]). Then  $X^\sigma$ , if non-empty, is Kähler and totally geodesic in  $X$  with respect to  $J_2, \omega_2$  and is totally real and Lagrangian with respect to  $J_1, \omega_1$  and  $J_3, \omega_3$ . If 0 is a regular value of  $\mu$  (i.e., 0 is a regular value of each  $\mu_i$ ) and that  $K$  acts on  $\mu^{-1}(0)$  freely, then  $X //_0 K = \mu^{-1}(0)/K$  is the (smooth) hyper-Kähler quotient at level 0, which has complex structures  $\bar{J}_i$  and symplectic structures  $\bar{\omega}_i$  ( $i = 1, 2, 3$ ) [16].

**Proposition 2.4.** *In the above setting, let  $Y = \mu_1^{-1}(0) \cap \mu_3^{-1}(0)$ . Then*

1.  $Y$  is a  $\sigma$ -invariant Kähler submanifold in  $X$  with respect to  $J_2, \omega_2$  and the symplectic quotient  $Y^\sigma //_0 K^\tau = (\mu^{-1}(0))^\sigma / K^\tau$  is Kähler;
2.  $(X //_0 K)^\sigma$  is Kähler and totally geodesic in  $X //_0 K$  with respect to  $\bar{J}_2, \bar{\omega}_2$  and is totally real and Lagrangian with respect to  $\bar{J}_1, \bar{J}_3$  and  $\bar{\omega}_1, \bar{\omega}_3$ ;
3. if  $(\mu^{-1}(0))^\sigma \neq \emptyset$ , there is a Kähler (with respect to  $\bar{J}_2, \bar{\omega}_2$ ) local diffeomorphism  $Y^\sigma //_0 K^\tau \rightarrow (X //_0 K)^\sigma$ .

*Proof.* 1&3. Let  $\mu_c = \mu_3 + \sqrt{-1}\mu_1: X \rightarrow \mathfrak{k}^{\mathbb{C}}$ . Then  $\mu_c$  is holomorphic with respect to  $J_2$  and is equivariant under the action of  $K$ . Since 0 is a regular value of  $\mu_c$ ,  $Y = \mu_c^{-1}(0)$  is a smooth Kähler submanifold in  $X$  on which the action of  $K$  is Hamiltonian. Applying Lemma 2.3 to  $Y$ , we conclude that the action of  $K^\tau$  on  $Y^\sigma$  is Hamiltonian and that  $(\mu^{-1}(0))^\sigma / K^\tau = (\mu_c^{-1}(0) \cap Y^\sigma) / K^\tau = Y^\sigma //_0 K^\tau$ . Moreover, there is a local diffeomorphism from  $Y^\sigma //_0 K^\tau$  to  $(Y //_0 K)^\sigma = (X //_0 K)^\sigma$  which is symplectic. Since  $K^\tau$  acts holomorphically on  $(Y^\sigma, J_2)$ , the symplectic quotient  $Y^\sigma //_0 K^\tau$  is Kähler, and the above local diffeomorphism is also Kähler.

2. Since  $\sigma$  preserves  $\mu^{-1}(0)$ , it descends to an involution  $\bar{\sigma}$  on  $X //_0 K$  such that  $\bar{\sigma}^* \bar{J}_i = (-1)^i \bar{J}_i$ ,  $\bar{\sigma}^* \bar{\omega}_i = (-1)^i \bar{\omega}_i$  for  $i = 1, 2, 3$ . The result then follows.  $\square$

We now prove Theorem 1.1.

*Proof.* 1&3. Note that  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$  is a  $\tau$ -invariant Kähler submanifold in  $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ . Following [1, 15], we can apply the method in Lemma 2.3 to  $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$  on which  $\tau$  acts preserving  $\omega_J$  and  $J$ . Since  $\tau$  also acts on  $\mathcal{G}(\tilde{P})$  and  $\mathcal{G}(P) \cong (\mathcal{G}(\tilde{P}))^\tau$ ,  $\mathcal{G}(P)/Z(K)$  acts Hamiltonianly and freely on  $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cong (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$ , which is Kähler with respect to  $J, \omega_J$ . Thus  $\mathcal{M}^{\text{Hitchin}}(P)^\circ = (\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0)) / \mathcal{G}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ //_0 \mathcal{G}(P)$  is a symplectic quotient. Since the latter is non-empty, there is a local Kähler diffeomorphism  $\mathcal{M}^{\text{Hitchin}}(P)^\circ \rightarrow (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ //_0 \mathcal{G}(\tilde{P}))^\tau = (\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$ .

2. The space  $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$  with  $I, J, K$  is hyper-Kähler and the action of  $\mathcal{G}(\tilde{P})$  is Hamiltonian with respect to  $\omega_I, \omega_J, \omega_K$ . Let  $(\mu^{-1}(0))^\circ$  be the subset of  $\mu^{-1}(0)$  on which  $\mathcal{G}(\tilde{P})/Z(K)$  acts freely. Then  $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ := (\mu^{-1}(0))^\circ / \mathcal{G}(\tilde{P})$  is the smooth part of the hyper-Kähler quotient  $\mathcal{M}^{\text{HK}}(\tilde{P})$ . The involutions  $\tau$  on  $\mathcal{A}(\tilde{P})$  and  $\mathcal{G}(\tilde{P})$  satisfy the conditions of Proposition 2.4. So  $(\mathcal{M}^{\text{HK}}(\tilde{P})^\circ)^\tau$  is Kähler and totally geodesic with respect to  $\bar{J}$  and  $\bar{\omega}_J$ , and totally real and Lagrangian with respect to  $\bar{I}, \bar{K}$  and  $\bar{\omega}_I, \bar{\omega}_K$  in  $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ$ . If  $M$  is a non-orientable surface, then  $\mu_I^{-1}(0) \cap \mu_K^{-1}(0) = \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$  which implies that  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ = \mathcal{M}^{\text{HK}}(\tilde{P})^\circ$ . In general,  $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$  is a  $\tau$ -invariant hyper-Kähler submanifold in  $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ$ . The results follow from  $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau = \mathcal{M}^{\text{Hitchin}}(\tilde{P}) \cap (\mathcal{M}^{\text{HK}}(\tilde{P})^\circ)^\tau$ .  $\square$

### 3. THE REPRESENTATION VARIETY PERSPECTIVE

**3.1. Representation variety and Betti moduli space.** Let  $\Gamma$  be a finitely generated group and let  $G$  be a connected complex Lie group. Then  $G$  acts on  $\text{Hom}(\Gamma, G)$  by the conjugate action on  $G$ . A representation  $\phi \in \text{Hom}(\Gamma, G)$  is *reductive* if the closure of  $\phi(\Gamma)$  in  $G$  is contained in the Levi subgroup of any parabolic subgroup containing  $\phi(\Gamma)$ ; let  $\text{Hom}^{\text{red}}(\Gamma, G)$  be the set of such. The condition  $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$  is equivalent to the statement that the  $G$ -orbit  $G \cdot \phi$  is closed [13]. It is also equivalent to the condition that the composition of  $\phi$  with the adjoint representation of  $G$  is semi-simple (see [26, Section 3] and [28, Theorem 30]). The quotient

$$\text{Hom}(\Gamma, G) // G = \text{Hom}^{\text{red}}(\Gamma, G) / G$$

is known as the *representation variety* or *character variety*. A reductive representation  $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$  is *good* [20] if its stabilizer  $G_\phi = Z(G)$ ; let  $\text{Hom}^{\text{good}}(\Gamma, G)$  be the set of such. On the other hand,  $\phi \in \text{Hom}(\Gamma, G)$  is *Ad-irreducible* if its composition with the adjoint representation of  $G$  is an irreducible representation of  $\Gamma$ . Let  $\text{Hom}^{\text{irr}}(\Gamma, G)$  be the set of such. Notice that this set is empty unless  $G$  is simple. Clearly,  $\text{Hom}^{\text{irr}}(\Gamma, G) \subset \text{Hom}^{\text{good}}(\Gamma, G)$ . In general,  $\text{Hom}^{\text{good}}(\Gamma, G)/G$  may not be smooth, but it is so when  $\Gamma$  is the fundamental group of a compact orientable surface [28, Corollary 50].

Suppose  $M$  is a compact manifold and  $P^{\mathbb{C}} \rightarrow M$  is a principal  $G$ -bundle over  $M$ . Choose a base point  $x_0 \in M$  and let  $\Gamma = \pi_1(M, x_0)$  be the fundamental group. Then  $\text{Hom}(\Gamma, G)//G$  is known as the *Betti moduli space* [30], denoted by  $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$ . The identification  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$ , which we recall briefly now, is well known. Given a flat connection, let  $T_\alpha: P_{\alpha(0)} \rightarrow P_{\alpha(1)}$  be the parallel transport along a path  $\alpha$  in  $M$ . Fix a point  $p_0 \in P_{x_0}$  in the fibre over  $x_0$ . For  $a \in \pi_1(M, x_0)$ , choose a loop  $\alpha$  based at  $x_0$  representing  $a$ , then  $\phi(a)$  is the unique element in  $G$  defined by  $T_\alpha(p_0) = p_0\phi(a)^{-1}$ . If we choose another point in the fibre over  $x_0$ , then  $\phi$  differs by a conjugation. Finally, the flat connection is reductive if and only if the corresponding element in  $\text{Hom}(\Gamma, G)$  is reductive. Upon identification of the de Rham moduli space  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$  and the Betti moduli spaces  $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}}) = \text{Hom}(\Gamma, G)//G$ , the subset  $\text{Hom}^{\text{good}}(\Gamma, G)/G$  contains the smooth part  $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ$  introduced in subsection 2.3; they are equal when  $M$  is a compact orientable surface.

If  $M$  is non-orientable and  $\pi: \tilde{M} \rightarrow M$  is the oriented cover, we choose a base point  $\tilde{x}_0 \in \pi^{-1}(x_0)$  and let  $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$ . Then there is a short exact sequence

$$1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 1$$

and  $\tilde{\Gamma}$  can be identified with an index 2 subgroup in  $\Gamma$ . In the rest of this section, we will study the relation of the representation varieties  $\text{Hom}(\Gamma, G)//G$  and  $\text{Hom}(\tilde{\Gamma}, G)//G$  or the Betti moduli spaces  $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$  and  $\mathcal{M}^{\text{Betti}}(\tilde{P}^{\mathbb{C}})$ . Some of the results, when  $M$  is a compact non-orientable surface, appeared in [17], which used different methods.

We first establish a useful fact that was used in subsection 2.2.

**Lemma 3.1.** *Suppose  $\Gamma$  is a finitely generated group and  $\tilde{\Gamma}$  is an index 2 subgroup in  $\Gamma$ . Let  $G$  be a connected, complex reductive Lie group. Then  $\phi \in \text{Hom}(\Gamma, G)$  is reductive if and only if the restriction  $\phi|_{\tilde{\Gamma}} \in \text{Hom}(\tilde{\Gamma}, G)$  is reductive.*

*Proof.* Recall that  $\phi \in \text{Hom}(\Gamma, G)$  is reductive if and only if the composition  $\text{Ad} \circ \phi$  is a semisimple representation on  $\mathfrak{g}$ . Similarly,  $\phi|_{\tilde{\Gamma}}$  is reductive if and only if  $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$  is semisimple. By  $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}_2$  and [6], [5, Chap. 3, §9.8, Lemme 2],  $\text{Ad} \circ \phi$  is semisimple if and only if  $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$  is so. The result then follows.  $\square$

**Corollary 3.2.** *Let  $G$  be a connected, complex reductive Lie group. Suppose  $P$  is a principal  $G$ -bundle over a compact non-orientable manifold  $M$  whose oriented cover is  $\pi: \tilde{M} \rightarrow M$ . Then a flat connection  $A$  on  $P$  is reductive if and only if the pull-back  $\pi^*A$  is a flat reductive connection on  $\tilde{P} := \pi^*P$ .*

**3.2. Representation varieties associated to an index 2 subgroup.** Let  $\Gamma$  be a finitely generated group and let  $\tilde{\Gamma}$  be an index 2 subgroup in  $\Gamma$ . Let  $G$  be a connected complex Lie group and let  $Z(G)$  be its center. For any  $c \in \Gamma \setminus \tilde{\Gamma}$ , we have  $\text{Ad}_c|_{\tilde{\Gamma}} \in \text{Aut}(\tilde{\Gamma})$ , and the class  $[\text{Ad}_c|_{\tilde{\Gamma}}] \in \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$  is independent of the choice of  $c$ . So we have a homomorphism  $\mathbb{Z}_2 \cong \{1, \tau\} \rightarrow \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$  given by  $\tau \mapsto [\text{Ad}_c|_{\tilde{\Gamma}}]$ .

**Lemma 3.3.**  $\mathbb{Z}_2 \cong \{1, \tau\}$  acts on  $\text{Hom}(\tilde{\Gamma}, G)//G$  and on  $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$ .

*Proof.* We define  $\tau[\phi] = [\phi \circ \text{Ad}_c]$  for any  $\phi \in \text{Hom}(\tilde{\Gamma}, G)$ . The action is well-defined since if  $[\phi'] = [\phi]$ , i.e.,  $\phi' = \text{Ad}_g \circ \phi$  for some  $g \in G$ , then  $\phi' \circ \text{Ad}_c = \text{Ad}_g \circ \phi \circ \text{Ad}_c \sim \phi \circ \text{Ad}_c$ . The  $\tau$ -action is independent of the choice of  $c$  because if  $c' \in \Gamma \setminus \tilde{\Gamma}$  is another element, then  $c'c^{-1} \in \tilde{\Gamma}$  and  $\phi \circ \text{Ad}_{c'} = \text{Ad}_{\phi(c'c^{-1})} \circ (\phi \circ \text{Ad}_c) \sim \phi \circ \text{Ad}_c$ . We do have a  $\mathbb{Z}_2$ -action because  $\tau^2[\phi] = [\phi \circ \text{Ad}_{c^2}] = [\text{Ad}_{\phi(c^2)} \circ \phi] = [\phi]$ . Finally, if  $\phi$  is in  $\text{Hom}^{\text{red}}(\tilde{\Gamma}, G)$  or  $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ , then so is  $\phi \circ \text{Ad}_c$ . Thus  $\tau$  acts on  $\text{Hom}(\tilde{\Gamma}, G)//G$  and  $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$ .  $\square$

**Proposition 3.4.** *There exists a continuous map*

$$(3.1) \quad L: (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau \rightarrow Z(G)/2Z(G).$$

So  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\text{good}}$ , where  $\mathcal{N}_r^{\text{good}} := L^{-1}(r)$ .

*Proof.* If  $\tau[\phi] = [\phi]$ , then there exists  $g \in G$  such that  $\phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi$ . Since  $c^2 \in \tilde{\Gamma}$ , we have  $\text{Ad}_{g^2} \circ \phi = \phi \circ \text{Ad}_{c^2} = \text{Ad}_{\phi(c^2)} \circ \phi$ . Thus  $z := g^2 \phi(c^2)^{-1} \in G_\phi = Z(G)$ . If  $[\phi'] = [\phi]$ , i.e.,  $\phi' = \text{Ad}_h \circ \phi$  for some  $h \in G$ , then  $\phi' \circ \text{Ad}_c = \text{Ad}_{g'} \circ \phi'$  for  $g' = \text{Ad}_h g$ . Since  $g'^2 = \text{Ad}_h g^2 = z \text{Ad}_h \phi(c^2) = z \phi'(c^2)$ , we obtain  $(g')^2 \phi'(c^2)^{-1} = z$ .

If  $\phi \circ \text{Ad}_{c'} = \text{Ad}_{g'} \circ \phi$  holds for different choices of  $c' \in \Gamma \setminus \tilde{\Gamma}$  and  $g' \in G$ , then  $z' = (g')^2 \phi(c'^2)^{-1} \in Z(G)$  from the above discussion. On the other hand, we have  $\text{Ad}_{g^{-1}g'} \circ \phi = \text{Ad}_{\phi(c^{-1}c')} \circ \phi$  as  $c^{-1}c' \in \tilde{\Gamma}$ . This gives us  $t := (g')^{-1} g \phi(c^{-1}c') \in G_\phi = Z(G)$ . We get

$$\begin{aligned} t^2 (g')^2 &= (tg')^2 = g \phi(c^{-1}c') g \phi(c^{-1}c') = \text{Ad}_g \phi(c^{-1}c') g^2 \phi(c^{-1}c') \\ &= \phi(\text{Ad}_c(c^{-1}c')) z \phi(c^2) \phi(c^{-1}c') = \phi((c')^2) z, \end{aligned}$$

i.e.,  $z' z^{-1} = t^{-2} \in 2Z(G)$ . So the map  $L: [\phi] \mapsto [z] \in Z(G)/2Z(G)$  is well-defined.

Since  $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ , the element  $[g] \in G/Z(G)$  is uniquely determined by and depends continuously on  $\phi$ . Therefore  $[z] \in Z(G)/2Z(G)$  depends continuously on  $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ .  $\square$

If  $\phi \in \text{Hom}(\Gamma, G)$  satisfies  $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ , then  $\phi \in \text{Hom}^{\text{good}}(\Gamma, G)$ . However,  $\phi \in \text{Hom}^{\text{good}}(\Gamma, G)$  does not imply  $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ . Let

$$\text{Hom}_\tau^{\text{good}}(\Gamma, G) = \{\phi \in \text{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)\}.$$

We show that if  $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ , then  $L([\phi])$  is the obstruction of extending  $\phi$  to a representation of  $\Gamma$ .

**Lemma 3.5.** *The restriction  $R: [\phi] \mapsto [\phi|_{\tilde{\Gamma}}]$  maps  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$  surjectively to  $\mathcal{N}_0^{\text{good}}$ .*

*Proof.* First, the image  $\text{im}(R) \subset \mathcal{N}_0^{\text{good}}$  because for any  $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$ ,  $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$  by definition, so  $(\phi|_{\tilde{\Gamma}}) \circ \text{Ad}_c = \text{Ad}_{\phi(c)} \circ \phi|_{\tilde{\Gamma}} \sim \phi|_{\tilde{\Gamma}}$  and  $L([\phi|_{\tilde{\Gamma}}]) = [\phi(c)^2 \phi(c^2)^{-1}] = 0$ . We will show that in fact  $\text{im}(R) = \mathcal{N}_0^{\text{good}}$ . Let  $\phi_0 \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$  such that  $\tau[\phi_0] = [\phi_0]$  and  $L([\phi_0]) = 0$ . Then there exist  $g \in G$  and  $t \in Z(G)$  such that  $\phi_0 \circ \text{Ad}_c = \text{Ad}_g \circ \phi_0$  and  $g^2 \phi(c^2)^{-1} = t^2$ . We can extend  $\phi_0$  to  $\phi \in \text{Hom}(\Gamma, G)$  which is uniquely determined by the requirements  $\phi|_{\tilde{\Gamma}} = \phi_0$  and  $\phi(c) = gt^{-1}$ . Since  $\phi_0 \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ ,  $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$  and therefore  $[\phi_0] \in \text{im}(R)$ .  $\square$

**Proposition 3.6.**  *$R: \text{Hom}_\tau^{\text{good}}(\Gamma, G)/G \rightarrow \mathcal{N}_0^{\text{good}}$  is a Galois covering map whose structure group is  $\{s \in Z(G) : s^2 = e\}$ .*

*Proof.* We define an action of  $\{s \in Z(G) : s^2 = e\}$  on  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)$ . For any such  $s$  and  $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$ , we define  $s \cdot \phi$  by  $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}}$  and  $(s \cdot \phi)|_{\Gamma \setminus \tilde{\Gamma}} = s(\phi|_{\Gamma \setminus \tilde{\Gamma}})$  the group multiplication. It is clear that  $s \cdot \phi \in \text{Hom}(\Gamma, G)$ . Moreover, since  $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ ,  $s \cdot \phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$ . Clearly, the action descends to a well-defined action on  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$  by  $s \cdot [\phi] = [s \cdot \phi]$  preserving the fibres of  $R$ .

We show that this action is free. Suppose  $s \cdot [\phi] = [\phi]$ , then  $s \cdot \phi = \text{Ad}_h \circ \phi$  for some  $h \in G$ . Since  $\phi|_{\tilde{\Gamma}} = (s \cdot \phi)|_{\tilde{\Gamma}} = \text{Ad}_h \circ \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ , we get  $h \in Z(G)$  and hence  $s \cdot \phi = \phi$ . Then  $s\phi(c) = \phi(c)$  implies  $s = e$ .

It remains to show that the action is transitive on each fibre of  $R$ . Let  $[\phi], [\phi'] \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$  such that  $R([\phi]) = R([\phi'])$ . Then there exists an  $h \in G$  such that  $\phi'|_{\tilde{\Gamma}} = \text{Ad}_h \circ \phi|_{\tilde{\Gamma}}$ . Thus

$$\begin{aligned} \text{Ad}_{\phi'(c)h} \circ (\phi|_{\tilde{\Gamma}}) &= \text{Ad}_{\phi'(c)} \circ (\phi'|_{\tilde{\Gamma}}) = (\phi'|_{\tilde{\Gamma}}) \circ \text{Ad}_c \\ &= \text{Ad}_h \circ (\phi|_{\tilde{\Gamma}}) \circ \text{Ad}_c = \text{Ad}_{h\phi(c)} \circ (\phi|_{\tilde{\Gamma}}). \end{aligned}$$

Hence  $s := \phi(c)^{-1} h^{-1} \phi'(c) h \in Z(G)$  since  $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ . Furthermore

$$s^2 = \phi(c)^{-1} s h^{-1} \phi'(c) h = \phi(c^{-2}) h^{-1} \phi'(c^2) h = \phi(c^{-2}) \phi(c^2) = e.$$

Since we have  $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} = \text{Ad}_{h^{-1}} \circ (\phi'|_{\tilde{\Gamma}})$  and  $(s \cdot \phi)(c) = s\phi(c) = \phi(c)s = (\text{Ad}_{h^{-1}} \circ \phi')(c)$ , we get  $s \cdot \phi = \text{Ad}_{h^{-1}} \circ \phi'$ , or  $[\phi'] = [s \cdot \phi]$ .  $\square$

**Corollary 3.7.** *Under the above assumptions, there is a local homeomorphism from  $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$  to  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ , which restricts to a local diffeomorphism on the smooth part. If  $|Z(G)|$  is odd, this local homeomorphism (diffeomorphism, respectively) is a homeomorphism (diffeomorphism, respectively).*

*Proof.* The first statement follows easily from Propositions 3.4 and 3.6. If  $|Z(G)|$  is odd, we get  $Z(G)/2Z(G) \cong \{0\}$  and  $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \mathcal{N}_0^{\text{good}}$  by Proposition 3.4. Furthermore, since  $\{s \in Z(G) : s^2 = e\} = \{e\}$ , the covering map in Proposition 3.6 is a bijection.  $\square$

The involution  $\tau$  also acts on  $\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G$ . Let

$$\text{Hom}_\tau^{\text{irr}}(\Gamma, G) = \{\phi \in \text{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)\}.$$

By the same idea used in the proof of Propositions 3.4 and 3.6, we get

**Corollary 3.8.** *If  $G$  is simple, there exists a decomposition*

$$(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\text{irr}},$$

where  $\mathcal{N}_r^{\text{irr}} = \mathcal{N}_r^{\text{good}} \cap (\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau$ . Furthermore, there exists a Galois covering map  $R: \text{Hom}_\tau^{\text{irr}}(\Gamma, G)/G \rightarrow \mathcal{N}_0^{\text{irr}}$  with structure group  $\{s \in Z(G) : s^2 = e\}$ . If  $|Z(G)|$  is odd, then there is a bijection from  $\text{Hom}_\tau^{\text{irr}}(\Gamma, G)/G$  to  $(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau$ .

The results in this subsection show parts (1) and (2) of Theorem 1.2.

**3.3. The Betti moduli space associated to a non-orientable surface.** By subsection 3.2 or parts (1) and (2) of Theorem 1.2, we know that a representation  $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$  such that  $\tau[\phi] = [\phi]$  can be extended to one on  $\Gamma$  if and only if  $L([\phi]) = 0$ . When applied to  $\Gamma = \pi_1(M)$  and  $\tilde{\Gamma} = \pi_1(\tilde{M})$ , where  $M$  is non-orientable and  $\tilde{M}$  is its oriented cover, we conclude that a  $\tau$ -invariant flat bundle over the  $\tilde{M}$  corresponding to  $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$  is the pull-back of a flat bundle over  $M$  if and only if  $L([\phi]) = 0$ . We now consider the example when  $M = \Sigma$  is a compact non-orientable surface, in which case we can characterize all the components  $\mathcal{N}_r^{\text{good}}$  explicitly. The principal  $G$ -bundles on  $\Sigma$  are topologically classified by  $H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$  whereas those on the oriented cover  $\tilde{\Sigma}$  are classified by  $H^2(\tilde{\Sigma}, \pi_1(G)) \cong \pi_1(G)$ . The classes in these groups are the obstructions of lifting the structure group  $G$  of the bundles to its universal cover group.

A compact non-orientable surface  $\Sigma$  is of the form  $\Sigma_k^\ell$  ( $\ell \geq 0, k = 1, 2$ ), the connected sum of  $2\ell + k$  copies of  $\mathbb{R}P^2$ . Then  $\tilde{\Sigma}$  is a compact surface of genus  $2\ell + k - 1$ . For  $k = 1$ , we have

$$\begin{aligned} \pi_1(\Sigma) &= \langle a_i, b_i (1 \leq i \leq \ell), c : c^{-2} \prod_{i=1}^\ell [a_i, b_i] \rangle, \\ \pi_1(\tilde{\Sigma}) &= \langle a_i, b_i, a'_i, b'_i (1 \leq i \leq \ell) : \prod_{i=1}^\ell [a_i, b_i] \prod_{i=1}^\ell [a'_i, b'_i] \rangle. \end{aligned}$$

The inclusion  $\pi_1(\tilde{\Sigma}) \rightarrow \pi_1(\Sigma)$  is given by  $a_i \mapsto a_i, b_i \mapsto b_i, a'_i \mapsto \text{Ad}_c b_i, b'_i \mapsto \text{Ad}_c a_i$  ( $1 \leq i \leq \ell$ ). For  $k = 2$ , we have

$$\begin{aligned} \pi_1(\Sigma) &= \langle a_i, b_i (1 \leq i \leq \ell), c, d : d^{-1} c d^{-1} c^{-1} \prod_{i=1}^\ell [a_i, b_i] \rangle, \\ \pi_1(\tilde{\Sigma}) &= \langle a_0, b_0, a_i, b_i, a'_i, b'_i (1 \leq i \leq \ell) : [a_0, b_0] \prod_{i=1}^\ell [a_i, b_i] \prod_{i=1}^\ell [a'_i, b'_i] \rangle. \end{aligned}$$

The inclusion  $\pi_1(\tilde{\Sigma}) \rightarrow \pi_1(\Sigma)$  is given by  $a_0 \mapsto d^{-1}, b_0 \mapsto c^2, a_i \mapsto a_i, b_i \mapsto b_i, a'_i \mapsto \text{Ad}_{d^{-1}c} b_i, b'_i \mapsto \text{Ad}_{d^{-1}c} a_i$  ( $1 \leq i \leq \ell$ ). In both cases,  $c \in \pi_1(\Sigma) \setminus \pi_1(\tilde{\Sigma})$ .

While a flat  $G$ -bundle over  $\Sigma$  may be non-trivial, its pull-back to  $\tilde{\Sigma}$  is always trivial topologically [19]. We assume that  $G$  is semi-simple, simply connected and denote  $PG = G/Z(G)$ . Then  $\pi_1(PG) = Z(G)$  and we have  $H^2(\Sigma, \pi_1(PG)) \cong Z(G)/2Z(G)$ . The map

$$O: \text{Hom}(\pi_1(\Sigma), PG)/PG \rightarrow Z(G)/2Z(G)$$

that gives the obstruction class can be explicitly described as follows [18]. Let  $\phi \in \text{Hom}(\pi_1(\Sigma), PG)$ . For  $k = 1$ , let  $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}, \widetilde{\phi(c)} \in G$  be any lifts of  $\phi(a_i), \phi(b_i), \phi(c) \in PG$ , respectively. Then  $O([\phi])$  is the element in  $Z(G)/2Z(G)$  represented by  $\widetilde{\phi(c)}^2 (\prod_{i=1}^\ell [\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}])^{-1} \in Z(G)$ . (It is easy to check that the class in  $Z(G)/2Z(G)$  is independent of the lifts.) The description of the case  $k = 2$  is similar. Consequently, there is a decomposition

$$\text{Hom}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r,$$

where  $\mathcal{M}_r = O^{-1}(r)$ .

Let  $G \rightarrow PG, g \mapsto \bar{g}$  be the quotient map. Denote the induced map by  $\text{Hom}(\pi_1(\Sigma), G) \rightarrow \text{Hom}(\pi_1(\Sigma), PG), \phi \mapsto \bar{\phi}$ . In this section, we need to be restricted to Ad-irreducible representations. The reason is that  $\phi$  is Ad-irreducible

if and only if  $\bar{\phi}$  is so, whereas if  $\phi$  is good,  $\bar{\phi}$  is not necessarily so and its stabilizer may be larger than  $Z(G)$ . We have

$$\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r^{\mathrm{irr}},$$

where  $\mathcal{M}_r^{\mathrm{irr}} = \mathcal{M}_r \cap (\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG)$ .

**Lemma 3.9.** *There is a natural map*

$$\Psi: (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

satisfying  $L = O \circ \Psi$ . Consequently,  $\Psi$  maps  $\mathcal{N}_r^{\mathrm{irr}}$  to  $\mathcal{M}_r^{\mathrm{irr}}$  for each  $r \in Z(G)/2Z(G)$ .

*Proof.* Given  $[\phi] \in (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$ , there exists  $g \in G$  (which is unique up to  $Z(G)$  since  $G_\phi = Z(G)$ ) such that  $\mathrm{Ad}_g \circ \phi = \phi \circ \mathrm{Ad}_c$ . We define  $\check{\phi} \in \mathrm{Hom}(\pi_1(\Sigma), PG)$  by  $\check{\phi}|_{\pi_1(\tilde{\Sigma})} = \bar{\phi}$  and  $\check{\phi}(c) = \bar{g}$ . The representation  $\check{\phi}$  is a homomorphism because  $\check{\phi}(c)^2 = \bar{g}^2 = \bar{\phi}(c^2)$ , which follows from the result  $z = g^2\phi(c^2)^{-1} \in Z(G)$  in Proposition 3.4. Since  $\bar{\phi} \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)$ , we have  $\check{\phi} \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)$ . We define  $\Psi$  by  $\Psi([\phi]) = [\check{\phi}]$ . To show that  $O([\check{\phi}]) = L([\phi]) = [z]$ , we work in the case  $k = 1$ . By using the respective lifts  $\phi(a_i)$ ,  $\phi(b_i)$ ,  $g \in G$  of  $\check{\phi}(a_i)$ ,  $\check{\phi}(b_i)$ ,  $\check{\phi}(c) \in PG$ , we get

$$O([\check{\phi}]) = [g^2(\prod_{i=1}^\ell [\phi(a_i), \phi(b_i)])^{-1}] = [g^2\phi(c^2)^{-1}] = [z],$$

where we have used the relation  $\prod_{i=1}^\ell [\phi(a_i), \phi(b_i)] = c^2$  in  $\pi_1(\tilde{\Sigma})$ . The case  $k = 2$  is similar.  $\square$

**Proposition 3.10.** *The map*

$$\Psi: (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

is surjective. Consequently,  $\Psi: \mathcal{N}_r^{\mathrm{irr}} \rightarrow \mathcal{M}_r^{\mathrm{irr}}$  is surjective for each  $r \in Z(G)/2Z(G)$ .

*Proof.* Let  $[\phi] \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\Sigma, PG)/PG$ . Although  $\phi(c) \in PG$ ,  $\mathrm{Ad}_{\phi(c)}$  acts on  $G$ . We show the case  $k = 1$  only. Fix the lifts  $\widetilde{\phi(a_i)}$ ,  $\widetilde{\phi(b_i)} \in G$  of  $\phi(a_i)$ ,  $\phi(b_i) \in PG$ . Define  $\tilde{\phi} \in \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)$  by setting  $\tilde{\phi}(a_i) = \widetilde{\phi(a_i)}$ ,  $\tilde{\phi}(b_i) = \widetilde{\phi(b_i)}$ ,  $\tilde{\phi}(a'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(b_i)$ ,  $\tilde{\phi}(b'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a_i)$ , for  $i = 1, \dots, \ell$ . This indeed defines a representation because

$$\prod_{i=1}^\ell [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \prod_{i=1}^\ell [\tilde{\phi}(a'_i), \tilde{\phi}(b'_i)] = \prod_{i=1}^\ell [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \mathrm{Ad}_{\phi(c)} \prod_{i=1}^\ell [\tilde{\phi}(b_i), \tilde{\phi}(a_i)] = e.$$

The last equality is because  $\prod_{i=1}^\ell [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \in G$  projects to  $\phi(c)^2 \in PG$ . Since  $\phi$  is Ad-irreducible, so is  $\tilde{\phi}$ .  $[\tilde{\phi}]$  is  $\tau$ -invariant because  $\tilde{\phi} \circ \mathrm{Ad}_c = \mathrm{Ad}_{\phi(c)} \circ \tilde{\phi}$ , which can be checked on the generators:  $\tilde{\phi}(\mathrm{Ad}_c a_i) = \tilde{\phi}(b'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a_i)$ ,  $\tilde{\phi}(\mathrm{Ad}_c a'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(b_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a'_i)$ , etc. It is then obvious that  $\Psi([\tilde{\phi}]) = [\phi]$ .  $\square$

For the group  $PG$ , since  $Z(PG)$  is trivial,  $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$  does not decompose according to Proposition 3.4 and the map

$$\bar{R}: \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG \rightarrow (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$$

in Proposition 3.6 is bijective. The map  $\Psi$  is in fact the composition of  $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$  (induced by  $G \rightarrow PG$ ) followed by  $\bar{R}^{-1}$ . So for each  $r \in Z(G)/2Z(G)$ , the component  $\mathcal{N}_r^{\mathrm{irr}}$  of the fixed point set  $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$  corresponds precisely to the component  $\mathcal{M}_r^{\mathrm{irr}}$  of  $\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$  which consists of flat  $PG$ -bundles over  $\Sigma$  of topological type  $r \in Z(G)/2Z(G)$ . In particular,  $\mathcal{N}_0^{\mathrm{irr}}$  corresponds to the component  $\mathcal{M}_0^{\mathrm{irr}}$  of topologically trivial flat  $PG$ -bundles over  $\Sigma$ .

The results in subsection shows part (3) of Theorem 1.2.

#### 4. COMPARISON OF REPRESENTATION VARIETY AND GAUGE THEORETICAL CONSTRUCTIONS

Suppose  $M$  is a compact non-orientable manifold,  $\pi: \tilde{M} \rightarrow M$  is the oriented cover, and  $\tau: \tilde{M} \rightarrow \tilde{M}$  is the non-trivial deck transformation. In subsection 2.2, we considered the natural lift of  $\tau$  on  $\tilde{P}^C = \pi^* P^C$ , where  $P^C$  is a principal  $G$ -bundle over  $M$ . Such a lift, still denoted by  $\tau$ , is a  $G$ -bundle map satisfying  $\tau^2 = \mathrm{id}_{\tilde{P}^C}$  and induces involutions on the space  $\mathcal{A}(\tilde{P}^C)$  of connections on  $\tilde{P}^C$  and various moduli spaces. Moduli spaces associated to  $P^C \rightarrow M$  are then related to the  $\tau$ -invariant parts of those associated to  $\tilde{P}^C \rightarrow \tilde{M}$  (cf. Theorem 1.1, especially part 3). This can also be seen in the language of representation varieties (cf. Lemma 3.5, Proposition 3.6 on  $\mathcal{N}_0^{\mathrm{good}}$

and Corollary 3.7). To provide a geometric interpretation of the rest of the results in subsections 3.2 and 3.3 on  $\mathcal{N}_r^{\text{good}}$  or  $\mathcal{N}_r^{\text{irr}}$  when  $r \neq 0$ , we will need to generalize the setting in gauge theory.

Suppose  $Q \rightarrow \tilde{M}$  is a principal  $G$ -bundle and the non-trivial deck transformation  $\tau$  on  $\tilde{M}$  is lifted to a bundle map  $\tau_Q$  on  $Q$ , which is not necessarily an involution. Let  $A$  be an irreducible connection on  $Q$  that is invariant under  $\tau_Q$  up to a gauge transformation, i.e.,  $\tau_Q^* A = \varphi^* A$  for  $\varphi \in \mathcal{G}(Q)$ . Since  $(\tau_Q \circ \varphi^{-1})^2$  is a gauge transformation on  $Q$  which fixes  $A$ , it is in the center  $Z(G)$ . So by modifying  $\tau_Q$  with a gauge transformation  $\varphi$ , we can assume that  $\tau_Q$  satisfies  $\tau_Q^2 = z \in Z(G)$ . In this way, although  $\tau_Q$  is not strictly an involution, it is so up to a gauge transformation, the right action of  $z$  on  $Q$ . Since  $\varphi$  and hence  $\tau_Q$  can be adjusted by an element in  $Z(G)$ ,  $z = \tau_Q^2$  is well defined modulo  $2Z(G)$ . If  $z = t^2 \in 2Z(G)$  ( $t \in Z(G)$ ), then  $z$  can be absorbed in  $\tau_Q$  by a redefinition such that  $\tau_Q$  is an honest involution, and we are back to the situation before. In the general case when  $\tau_Q^2 = z \in Z(G)$  is not the identity element, since  $Z(G)$  acts trivially on the connections as gauge transformations, the action  $\tau_Q^*: \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$  of  $\tau_Q$  on connections is still an honest involution. So we can define the invariant subspace  $\mathcal{A}(Q)^{\tau_Q}$  and much of the analysis in subsections 2.2 and 2.3 applies.

We now consider flat connections and relate this generalized setting to our results on representation varieties. Choose base points  $x_0 \in M$  and  $\tilde{x}_0 \in \pi^{-1}(x_0) \subset \tilde{M}$ , and let  $\Gamma = \pi_1(M, x_0)$ ,  $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$ . We fix an element  $c \in \Gamma \setminus \tilde{\Gamma}$ .

**Proposition 4.1.** *For any  $z \in Z(G)$ , there is a 1-1 correspondence between the following two sets:*

(1) *isomorphism classes of pairs  $(Q, A)$ , where  $Q \rightarrow \tilde{M}$  is a principal  $G$ -bundle with a  $G$ -bundle map  $\tau_Q$  lifting the deck transformation  $\tau$  on  $\tilde{M}$  satisfying  $\tau_Q^2 = z$ ,  $A$  is a  $\tau_Q$ -invariant flat connection on  $Q$*

*and*

(2) *equivalence classes of pairs  $(\phi, g)$  under the diagonal adjoint action of  $G$ , where  $\phi \in \text{Hom}(\tilde{\Gamma}, G)$  and  $g \in G$  satisfy  $\phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi$  and  $g^2 \phi(c^2)^{-1} = z$ .*

*Proof.* Given a bundle  $Q$  and a  $\tau_Q$ -invariant flat connection  $A$ , let  $T_\alpha: Q_{\alpha(0)} \rightarrow Q_{\alpha(1)}$  be the parallel transport along a path  $\alpha: [0, 1] \rightarrow \tilde{M}$ .  $\tau_Q$ -invariance of the connection implies  $\tau_Q \circ T_\alpha = T_{\tau \circ \alpha} \circ \tau_Q$  for any path  $\alpha$ . Let  $\gamma$  be a path in  $\tilde{M}$  from  $\tilde{x}_0$  to  $\tau(\tilde{x}_0)$  so that  $[\pi \circ \gamma] = c$ . Choose  $q_0 \in Q_{\tilde{x}_0}$  and let  $g \in G$  be defined by  $T_\gamma q_0 = \tau_Q(q_0)g^{-1}$ . On the other hand, define  $\phi \in \text{Hom}(\tilde{\Gamma}, G)$  by  $T_\alpha q_0 = q_0 \phi(a)^{-1}$  for any  $a \in \tilde{\Gamma}$ , where  $\alpha$  is a loop in  $\tilde{M}$  based at  $\tilde{x}_0$  such that  $[\alpha] = a$ . To check the conditions on  $(\phi, g)$ , we note that  $\tau_Q(T_\alpha q_0) = \tau_Q(q_0)\phi(a)^{-1}$  and

$$T_{\tau \circ \alpha} \tau_Q(q_0) = T_\gamma \circ T_{\gamma \cdot (\tau \circ \alpha) \cdot \gamma^{-1}}(q_0 g) = (T_\gamma q_0) \phi(\text{Ad}_c a) g = \tau_Q(q_0) \text{Ad}_g^{-1} \phi(\text{Ad}_c a).$$

So  $\tau_Q$ -invariance implies  $\phi(\text{Ad}_c a) = \text{Ad}_g \phi(a)$  for all  $a \in \tilde{\Gamma}$ . Similar calculations give  $\tau_Q(T_\gamma q_0) = \tau_Q(\tau_Q(q_0)g^{-1}) = q_0 z g^{-1}$  and  $T_{\tau \circ \gamma}(\tau_Q q_0) = T_{\gamma \cdot (\tau \circ \gamma)}(q_0 g) = q_0 \phi(c^2)^{-1} g$  which imply  $g^2 \phi(c^2)^{-1} = z$ . If another point  $q'_0 = q_0 h \in Q_{\tilde{x}_0}$  is chosen (where  $h \in G$ ), then the resulting pair is  $(\phi', g') = (\text{Ad}_{h^{-1}} \circ \phi, \text{Ad}_{h^{-1}} g)$ .

Conversely, given a pair  $(\phi, g)$  satisfying the conditions, we want to construct a bundle  $Q$  together with a lifting  $\tau_Q$  of  $\tau$  such that  $\tau_Q^2 = z$  and a  $\tau_Q$ -invariant flat connection on  $Q$ . Let  $\hat{M}$  be the universal covering space of  $\tilde{M}$  (and of  $M$ ). Then  $\tilde{\Gamma}$  and  $\Gamma$  act on  $\hat{M}$ , and  $\tilde{M} = \hat{M}/\tilde{\Gamma}$ ,  $M = \hat{M}/\Gamma$ . Let  $Q = \hat{M} \times_{\tilde{\Gamma}} G$ , that is, points in  $Q$  are equivalence classes  $[(x, h)]$ , where  $x \in \hat{M}$  and  $h \in G$ , and  $(xa, h) \sim (x, \phi(a)h)$  for any  $a \in \tilde{\Gamma}$ . Let  $\tau_Q: Q \rightarrow Q$  be defined by  $\tau_Q: [(x, h)] \mapsto [(xc^{-1}, gh)]$ . To check that  $\tau_Q$  is well-defined, we note that for any  $a \in \tilde{\Gamma}$ ,  $(xac^{-1}, gh) \sim (xc^{-1}, \phi(\text{Ad}_c a)gh) = (xc^{-1}, g\phi(a)h)$ . Clearly,  $\tau_Q$  commutes with the right  $G$ -action on  $Q$ . Furthermore,  $\tau_Q^2 = z$  because  $\tau_Q^2: [(x, h)] \mapsto [(xc^{-2}, g^2h)] = [(x, \phi(c^{-2})g^2h)] = [(x, h)]z$ . It is easy to see that the trivial connection on  $\hat{M} \times G$  is  $\tilde{\Gamma}$ -invariant and descends to a flat connection on  $Q$ . The latter is invariant under  $\tau_Q$  since the trivial connection on  $\hat{M} \times G$  is invariant under  $(x, h) \mapsto (xc^{-1}, gh)$ . Moreover, this connection induces the pair  $(\phi, g)$ .  $\square$

*Remark 4.2.* We explain the gauge theoretic perspective of the results in subsections 3.2 and 3.3 using the correspondence in Proposition 4.1.

1. As we noted, the  $\tau$  is lifted to a  $G$ -bundle map  $\tau_Q$  on  $Q \rightarrow \tilde{M}$  such that  $\tau_Q^2 = z \in Z(G)$ , then  $z$  is determined up to  $2Z(G)$ . Likewise,  $z = g^2 \phi(c^2)^{-1}$  is determined also modulo  $2Z(G)$  by  $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$  (Proposition 3.4). If  $\tau_Q^2 = t^2$  for some  $t \in Z(G)$ , then  $\tau_Q$  can be redefined as  $\tau'_Q = \tau_Q t^{-1}$  so that  $(\tau'_Q)^2 = \text{id}_Q$ . We then have a  $G$ -bundle  $Q/\tau'_Q \rightarrow M$  over the non-orientable manifold  $M$  whose pull-back of to  $\tilde{M}$  is  $Q$ . If a flat connection is invariant under  $\tau_Q$ , it is also invariant under  $\tau'_Q$  and hence descends to a flat connection on  $Q/\tau'_Q$ .

This is the situation in Lemma 3.5 and Proposition 3.6 (where  $Q/\tau'_Q$  was  $P^{\mathbb{C}}$ ). In fact, from these results, we see that  $[z] \in Z(G)/2Z(G)$  is the obstruction to the existence of a flat  $G$ -bundle on  $M$  whose pull-back to  $\tilde{M}$  is  $Q$ .

2. In general,  $\tau_Q^2 \neq \text{id}_Q$  and the quotient of  $Q$  by the subgroup generated by  $\tau_Q$  is a bundle over  $M$  with a fibre smaller than  $G$ . However, the  $PG$ -bundle  $\bar{Q} := Q/Z(G)$  over  $\tilde{M}$  does have an honest involution  $\tau_{\bar{Q}}$ . So  $\bar{Q}$  descends to a  $PG$ -bundle  $\bar{Q}/\tau_{\bar{Q}}$  over  $M$ . Moreover, a  $\tau_Q$ -invariant flat connection on  $Q$  descends to a  $\tau_{\bar{Q}}$ -invariant flat connection on  $\bar{Q}$  and hence to a flat  $PG$ -connection on  $\bar{Q}/\tau_{\bar{Q}}$ . The bundle  $\bar{Q}/\tau_{\bar{Q}} \rightarrow M$  is usually non-trivial as its structure group can not be lifted to  $G$ . (Otherwise,  $Q$  would be its pull-back to  $\tilde{M}$  and would admit a lift  $\tau_Q$  of  $\tau$  so that  $\tau_Q^2 = \text{id}_Q$ .) Proposition 3.10 shows that when  $G$  is simply connected and when  $M = \Sigma$  is a non-orientable surface, the topological type, i.e., the obstruction to lifting the  $PG$ -bundle  $\bar{Q}/\tau_{\bar{Q}}$  to a  $G$ -bundle over  $M$  is precisely  $[z] \in Z(G)/2Z(G)$ .

*Remark 4.3.* 1. We can use  $\tilde{x}_1 = \tau(\tilde{x}_0)$  as another base point of the fundamental group of  $\tilde{M}$  so that  $\tilde{x}_0$  and  $\tilde{x}_1$  play symmetric roles. The image of  $\pi_1(\tilde{\Sigma}, \tilde{x}_1)$  under  $\pi_*$  can be identified with  $\tilde{\Gamma} \subset \Gamma$ . The isomorphism  $\tau_*: \tilde{\Gamma} \rightarrow \pi_1(\tilde{\Sigma}, \tilde{x}_1) \cong \tilde{\Gamma}$  is then  $a \mapsto \text{Ad}_c^{-1} a$ . Having chosen  $q_0 \in Q_{\tilde{x}_0}$ , let  $q_1 = \tau_Q(q_0) \in Q_{\tilde{x}_1}$  and define  $\phi_1: \pi_1(\tilde{\Sigma}, \tilde{x}_1) \rightarrow G$  by  $T_\alpha q_1 = q_1 \phi_1([\alpha])^{-1}$ , where  $\alpha$  is a loop in  $\tilde{\Sigma}$  based at  $\tilde{x}_1$ . Using the identity  $\tau_Q \circ T_{\tau \circ \alpha} = T_\alpha \circ \tau_Q$ , we obtain  $\phi_1([\alpha]) = \phi([\tau \circ \alpha])$ . Since  $\tau_Q^2 = z$ , we also have the identity  $T_\gamma z = \tau_Q \circ T_{\tau \circ \gamma} \circ \tau_Q$ . So upon the identification of  $Q_{\tilde{x}_0}$  and  $Q_{\tilde{x}_1}$  by  $\tau_Q$ , the parallel transports along  $\gamma$  and  $\tau \circ \gamma$  differ by  $z$ .

2. When  $M = \Sigma$  is a non-orientable surface, the approach of double base points was taken in [17, 19]. Consider for example the case  $M = \Sigma_1^f$ . Let  $\alpha_i, \beta_i$  ( $1 \leq i \leq \ell$ ) be loops in the oriented cover  $\tilde{\Sigma}$  based at  $\tilde{x}_0$  and let  $\gamma$  be a path in from  $\tilde{x}_0$  to  $\tilde{x}_1$  so that  $[\pi \circ \alpha_i] = a_i$ ,  $[\pi \circ \beta_i] = b_i$ ,  $[\pi \circ \gamma] = c$ . Then an element in  $\mathcal{N}_r$  ( $r = [z] \in Z(G)/2Z(G)$ ) can be represented by  $(A_i, B_i, C; A'_i, B'_i, C') \in G^{4\ell+2}$  satisfying  $A'_i = A_i$ ,  $B'_i = B_i$ ,  $C' = Cz$ , where  $A_i, B_i, C, A'_i, B'_i, C'$  are the holonomies along the loops or paths  $\alpha_i, \beta_i, \gamma, \tau \circ \alpha_i, \tau \circ \beta_i, \tau \circ \gamma$ , ( $1 \leq i \leq \ell$ ), respectively. By the above discussion, we have the pattern  $A_i = \phi([\alpha_i]) = \phi_1([\tau \circ \alpha_i]) = A'_i$ ,  $B_i = \phi([\beta_i]) = \phi_1([\tau \circ \beta_i]) = B'_i$ , ( $1 \leq i \leq \ell$ ),  $C' = Cz$  as in [17, 19].

#### ACKNOWLEDGMENTS

We would like to thank U. Bruzzo, W. Goldman and E. Xia for useful discussions.

The research of N.H. was supported by grant number 99-2115-M-007-008-MY3 from the National Science Council of Taiwan. The research of G.W. was supported by grant number R-146-000-200-112 from the National University of Singapore. The research of S.W. was partially supported by RGC grant HKU705612P (Hong Kong) and by MOST grant 105-2115-M-007-001-MY2 (Taiwan).

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