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HITCHIN'S EQUATIONS ON A NONORIENTABLE MANIFOLD

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Abstract. We define Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ for a principal bundle P, whose structure group is a compact semisimple Lie group K, over a compact non-orientable Riemannian manifold M. We use the Donaldson-Corlette correspondence, which identifies Hitchin's moduli space with the moduli space of flat $K^{\mathbb{C}}$-connections, which remains valid when M is non-orientable. This enables us to study Hitchin's moduli space both by gauge theoretical methods and algebraically by using representation varieties. If the orientable double cover $\tilde{M}$ of M is a Kähler manifold with odd complex dimension and if the Kähler form is odd under the non-trivial deck transformation τ on $\tilde{M}$, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ of the pull-back bundle $\tilde{P} \to \tilde{M}$ has a hyper-Kähler structure and admits an involution induced by τ. The fixed-point set $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\tau$ is symplectic or Lagrangian with respect to various symplectic structures on $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$. We show that there is a local diffeomorphism from $\mathcal{M}^{\mathrm{Hitchin}}(P)$ to $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\tau$. We compare the gauge theoretical constructions with the algebraic approach using representation varieties.

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1. Introduction

Let M be a compact orientable Riemannian manifold and let K be a connected compact Lie group. Given a principal K-bundle $P \to M$, let $\mathcal{A}(P)$ be the space of connections and let $\mathcal{G}(P)$ be the group of gauge transformations on P. Consider Hitchin's equations

$$F_A - \tfrac{1}{2}[\psi, \psi] = 0, \quad d_A\psi = 0, \quad d_A^*\psi = 0 \tag{1.1}$$

on the pairs $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \mathrm{ad}\, P)$. Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is the set of space of solutions (A, ψ) to (1.1) modulo $\mathcal{G}(P)$ [15, 29]. On the other hand, let $G = K^{\mathbb{C}}$ be the complexification of K and let $P^{\mathbb{C}} = P \times_K G$, which is a principal bundle with structure group G. The moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ of flat G-connections on $P^{\mathbb{C}}$, also known as the de Rham moduli space, is the space of flat reductive connections of $P^{\mathbb{C}}$ modulo $\mathcal{G}(P)^{\mathbb{C}} \cong \mathcal{G}(P^{\mathbb{C}})$. A theorem of Donaldson [8] and Corlette [7] states that the moduli spaces $\mathcal{M}^{\mathrm{Hitchin}}(P)$ and $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ are homeomorphic. The smooth part of $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a Kähler manifold with a complex structure $\bar{J}$ induced by that on G.

Suppose in addition that M is a Kähler manifold. Then there is another complex structure $\bar{I}$ on $\mathcal{M}^{\mathrm{Hitchin}}(P)$ induced by that on M, and a third one given by $\bar{K} = \bar{I}\bar{J}$. The three complex structures $\bar{I}, \bar{J}, \bar{K}$ and their corresponding Kähler forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$ form a hyper-Kähler structure on (the smooth part of) $\mathcal{M}^{\mathrm{Hitchin}}(P)$ [15, 29]. This hyper-Kähler structure comes from an infinite dimensional version of a hyper-Kähler quotient [16] of the tangent bundle $T\mathcal{A}(P)$, which is hyper-Kähler, by the action of $\mathcal{G}(P)$, which is Hamiltonian with respect to each of the Kähler forms $\omega_I, \omega_J, \omega_K$ on $T\mathcal{A}(P)$. When M is a compact orientable surface, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is equal to the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(P) := T\mathcal{A}(P) /\!/\!/_0 \mathcal{G}(P)$ [15]. It plays an important role in mirror symmetry and geometric Langlands program [14, 21]. When M is higher dimensional, $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\mathrm{HK}}(P)$ [29].

For a compact Lie group K, the moduli space of flat K-connections on a compact orientable surface was already studied in a celebrated work of Atiyah and Bott [1]. When M is a compact, nonorientable surface, the moduli space of flat K-connections was studied in [17, 19] through an involution on the space of connections over its orientable double cover $\tilde{M}$, induced by lifting the deck transformation on $\tilde{M}$ to the pull-back $\tilde{P} \to \tilde{M}$ of the given K-bundle $P \to M$ so that the quotient of $\tilde{P}$ by the involution is the original bundle P itself. This involution acts trivially on the structure group K. If instead one considers an involution on the bundle over $\tilde{M}$ that acts nontrivially on the fibers (such as the complex conjugation), then the fixed points give rise to the moduli space of real or quaternionic vector bundles over a real algebraic curve. This was studied thoroughly in [4, 27], for example when $K = U(n)$.

Key words and phrases. moduli spaces, non-orientable manifolds, symplectic and hyper-Kähler geometry, representation varieties.

In this paper, we study Hitchin's equations on a non-orientable manifold. Let M be a compact connected non-orientable Riemannian manifold and let $P \to M$ be a principal K-bundle over M, where K is a compact connected Lie group. The de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$, i.e., the moduli space of flat connections on $P^{\mathbb{C}}$, does not depend on the orientability of M. On the other hand, Hitchin's equations (1.1) on the pairs $(A,\psi) \in \mathcal{A}(P) \times \Omega^1(M, \mathrm{ad}\,P)$ still make sense (see subsection 2.2). We define Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ as the quotient of the space of pairs (A,ψ) satisfying (1.1) by the group $\mathcal{G}(P)$ of gauge transformations on P. We explain that the homeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(P) \cong \mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ of Donaldon-Corlette remains valid when M is non-orientable (Theorem 2.2).

If the oriented cover $\tilde{M}$ of M is a Kähler manifold, then for the pull-back bundle $\tilde{P} := \pi^* P$ over $\tilde{M}$, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ is hyper-Kähler with complex structures $\bar{I}, \bar{J}, \bar{K}$ and Kähler forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. If the Kähler form ω on $\tilde{M}$ satisfies $\tau^*\omega = -\omega$ (the complex dimension of $\tilde{M}$ must be odd for τ to be orientation reversing), then τ induces an involution (still denoted by τ) on $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ that satisfies $\tau^*\bar{\omega}_I = -\bar{\omega}_I$, $\tau^*\bar{\omega}_J = \bar{\omega}_J$ and $\tau^*\bar{\omega}_K = -\bar{\omega}_K$. Consequently, the fixed-point set $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$ is Lagrangian in $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ with respect to $\bar{\omega}_I, \bar{\omega}_K$ and symplectic with respect to $\bar{\omega}_J$. This is known as an (A,B,A)-brane in [21]. We discover that Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ (where M is non-orientable) is related to $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$ by a local diffeomorphism. Our main results are summarized in the following main theorem. For simplicity, we restrict to certain smooth parts $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$, $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ$ and $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ$ of the respective spaces (see subsection 2.3 for details).

Theorem 1.1. *Let M be a compact non-orientable manifold and let $\pi\colon \tilde{M} \to M$ be its oriented cover on which there is a non-trivial deck transformation τ. Let K be a compact connected Lie group. Given a principal K-bundle $P \to M$, let $\tilde{P} = \pi^* P$ be its pull-back to $\tilde{M}$. Suppose that $\tilde{M}$ is a Kähler manifold of odd complex dimension and the Kähler form ω on $\tilde{M}$ satisfies $\tau^*\omega = -\omega$. Then*
(1) $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ /\!\!/_0\, \mathcal{G}(P)$, which is a symplectic quotient.
(2) $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau$ is Kähler and totally geodesic in $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ$ with respect to $\bar{J}, \bar{\omega}_J$ and totally real and Lagrangian with respect to $\bar{I}, \bar{K}$ and $\bar{\omega}_I, \bar{\omega}_K$.
(3) there is a local Kähler diffeomorphism from $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$ to $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau$.

The theorem of Donaldson and Corlette in the non-orientable setup (Theorem 2.2) enable us to identify Hitchin's moduli space associated to an orientable or non-orientable manifold with the moduli space of flat connections and therefore the representation varieties. Let Γ be a finitely generated group and let G be a connected complex semi-simple Lie group. The representation variety, $\mathrm{Hom}(\Gamma, G)/\!\!/G := \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/G$, is the quotient of the space of reductive homomorphisms from Γ to G by the conjugation action of G. When Γ is the fundamental group of a compact manifold M, the representation variety is also called the Betti moduli space of M; it is homeomorphic to the union of the de Rham moduli spaces $\mathcal{M}^{\mathrm{dR}}(P)$ associated to principal G-bundles $P \to M$ of various topology. When M is non-orientable, let $\tilde{\Gamma}$ be the fundamental group of the oriented cover $\tilde{M}$. Then there is a short exact sequence $1 \to \tilde{\Gamma} \to \Gamma \to \mathbb{Z}_2 \to 1$ and τ acts as an involution on the representation variety $\mathrm{Hom}(\tilde{\Gamma}, G)/\!\!/G$ (Lemma 3.3). We study the relation of representation varieties associated to Γ and $\tilde{\Gamma}$ from an algebraic point of view. Let $PG = G/Z(G)$, where $Z(G)$ is the center of G. Our main results are summarized in the following theorem.

Theorem 1.2. *Let G be a connected complex semi-simple Lie group. Let M be a compact non-orientable manifold and let $\tilde{M}$ be its oriented cover on which there is a non-trivial deck transformation τ. Denote $\Gamma = \pi_1(M)$ and $\tilde{\Gamma} = \pi_1(\tilde{M})$ with some chosen base points. Then*
(1) there exists a continuous map L from $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau$ to $Z(G)/2Z(G)$. Consequently, $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\mathrm{good}}$, where $\mathcal{N}_r^{\mathrm{good}}$ is the preimage of $r \in Z(G)/2Z(G)$.
(2) there exists a $|Z(G)/2Z(G)|$-sheeted Galois covering map from
$\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma, G)/G$ to $\mathcal{N}_0^{\mathrm{good}}$.
In particular, if $|Z(G)|$ is odd, then there exists a bijection from $\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma, G)/G$ to $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau$. The above statements are true if $\mathrm{Hom}^{\mathrm{good}}$ is replaced by $\mathrm{Hom}^{\mathrm{irr}}$.
If in addition $M = \Sigma$ is a compact non-orientable surface and G is simple and simply connected, then
(3) there exists a surjective map from $(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)/G)^\tau$ to $\mathrm{Hom}_\tau^{\mathrm{irr}}(\Gamma, PG)/PG$ that maps $\mathcal{N}_r^{\mathrm{irr}}$ to flat PG-bundles on Σ whose topological type is given by $r \in Z(G)/2Z(G) \cong H^2(\Sigma, Z(G))$. In particular, $\mathcal{N}_0^{\mathrm{irr}}$ maps to the topologically trivial flat PG-bundles on Σ.

Here $\mathrm{Hom}^{\mathrm{good}}$, following the terminology of [20], denotes the "good" part of the space of homomorphisms that are reductive and whose stabilizer is $Z(G)$, whereas $\mathrm{Hom}^{\mathrm{irr}}$ is the space of homomorphisms whose composition with the adjoint representation of G is an irreducible representation (see subsection 3.1 for details). $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$ is the set of homomorphisms from Γ to G whose restriction to $\tilde{\Gamma}$ is "good". $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)/\!\!/G$ is not smooth in general, but contains a smooth part $(\mathcal{M}^{\mathrm{flat}}(P^{\mathbb{C}}))^{\circ}$ (upon identification of moduli spaces). By parts (1) and (2) of the theorem, there is a local homeomorphism $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)/G \to (\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^{\tau}$ (see also Corollary 3.7), which in fact restricts to the local diffeomorphism $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^{\circ} \to (\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^{\circ})^{\tau}$ in part (3) of Theorem 1.1 but is now more accurately described using representation varieties. Also, for $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$ such that $[\phi] \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G$ is fixed by τ, $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ. In the gauge-theoretic language, ϕ corresponds to a flat connection on $\tilde{M}$ and represents a point fixed by τ in the de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})$, while extension of ϕ to Γ means that the flat connection on $\tilde{M}$ is the pull-back of a flat connection on M. Flat connections on $\tilde{M}$ that are not pull-backs from M correspond to flat PG-bundles over M (where $PG = G/Z(G)$). This is shown in part (3) of Theorem 1.2 and then discussed in greater generality in the last section.

For example, let $G = SL(2,\mathbb{C})$, M a compact nonorientable surface and $\tilde{M}$ its orientable double cover. Then $(\mathrm{Hom}^{\mathrm{good}}(\pi_1(\tilde{M}), G)/G)^{\tau}$ is labeled by $Z(G)/2Z(G) = \mathbb{Z}_2$, i.e., $(\mathrm{Hom}^{\mathrm{good}}(\pi_1(\tilde{M}), G)/G)^{\tau} = \bigcup_{r\in\mathbb{Z}_2} \mathcal{N}^{\mathrm{good}}_r$. An element of $(\mathrm{Hom}^{\mathrm{good}}(\pi_1(\tilde{M}), G)/G)^{\tau}$ is mapped by map L in Theorem 1.2(1) (defined in Proposition 3.4) to the null element of $\mathbb{Z}_2$ if and only if it represents a flat connection on $\tilde{M}$ that is the pull-back of a flat connection on M. The natural map from $\mathrm{Hom}^{\mathrm{good}}(\pi_1(M), G)/G$ to $(\mathrm{Hom}^{\mathrm{good}}(\pi_1(\tilde{M}), G)/G)^{\tau}$ is not surjective; it is a $\mathbb{Z}_2$-sheeted Galois covering map onto $\mathcal{N}^{\mathrm{good}}_0$, and $\mathcal{N}^{\mathrm{good}}_1$ is not in the image. $\mathcal{N}^{\mathrm{irr}}_0$ corresponds to the space of topologically trivial flat $PSL(2,\mathbb{C})$-bundles over M while $\mathcal{N}^{\mathrm{irr}}_1$ corresponds to that of topologically nontrivial flat $PSL(2,\mathbb{C})$-bundles over M.

The rest of this paper is organized as follows. In Section 2, we review the basic setup in the orientable case and explain the Donaldson-Corlette theorem for bundles over non-orientable manifolds. We then study finite dimensional symplectic and hyper-Kähler manifolds with an involution and apply the results to the gauge theoretical setting to prove Theorem 1.1. In Section 3, we study flat G-connections by representation varieties. We show that a flat connection on M is reductive if and only if its pull-back to $\tilde{M}$ is reductive. We then define the continuous map in part (1) of Theorem 1.2 and prove the rest of the theorem. In Section 4, we relate the components $\mathcal{N}^{\mathrm{good}}_r$ $(r \neq 0)$ in Theorem 1.2 to G-bundles over $\tilde{M}$ admitting an involution up to $Z(G)$.

We note that in order to study the moduli space of G-bundles over the nonorientable manifold M itself, our involution is fixed-point free on $\tilde{M}$ and is the identity map on G. During the revision of this paper, we came across a few related works. We thank O. García-Prada for pointing out to us the paper [3], where their anti-holomorphic involution acts both on the manifold $\tilde{M}$ and on the structure group G, thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with a special case of part (2) of our Theorem 1.1 when $\tilde{M}$ is a surface, the anti-holomorphic involution on the surface is allowed to have fixed points.

2. The gauge-theoretic perspective

2.1. **Basic setup in the orientable case.** Let K be a connected compact Lie group and let $G = K^{\mathbb{C}}$ be its complexification. Given a principal K-bundle P over a compact orientable manifold M, $P^{\mathbb{C}} = P \times_K G$ is a principal bundle whose structure group is G. The set $\mathcal{A}(P)$ of connections on P is an affine space modeled on $\Omega^1(M, \mathrm{ad}\, P)$. At each $A \in \mathcal{A}(P)$, the tangent space is $T_A\mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}\, P)$. The total space of the tangent bundle over $\mathcal{A}(P)$ is $T\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1(M, \mathrm{ad}\, P)$. At $(A, \psi) \in T\mathcal{A}(P)$, the tangent space is $T_{(A,\psi)}T\mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}\, P)^{\oplus 2}$. There is a translation invariant complex structure J on $T\mathcal{A}(P)$ given by $J(\alpha, \varphi) = (\varphi, -\alpha)$. The space $T\mathcal{A}(P)$ can be naturally identified with $\mathcal{A}(P^{\mathbb{C}})$, the set of connections on $P^{\mathbb{C}} \to M$, via $(A, \psi) \mapsto A - \sqrt{-1}\psi$, under which J corresponds to the complex structure on $\mathcal{A}(P^{\mathbb{C}})$ induced by $G = K^{\mathbb{C}}$. The covariant derivative on $\Omega^{\bullet}(M, \mathrm{ad}\, P^{\mathbb{C}})$ is $D := d_A - \sqrt{-1}\psi$, where d_A denotes the covariant derivative of $A \in \mathcal{A}(P)$ and ψ acts by bracket.

The group of gauge transformations on P is $\mathcal{G}(P) \cong \Gamma(M, \mathrm{Ad}\, P)$. It acts on $\mathcal{A}(P)$ via $A \mapsto g \cdot A$, where $d_{g\cdot A} = g \circ d_A \circ g^{-1}$ and on $T\mathcal{A}(P)$ via $g\colon (A, \psi) \mapsto (g \cdot A, \mathrm{Ad}_g\, \psi)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves J, there is a holomorphic $\mathcal{G}(P)^{\mathbb{C}}$ action on $(T\mathcal{A}(P), J)$. In fact, the complexification $\mathcal{G}(P)^{\mathbb{C}}$ can be naturally identified with $\mathcal{G}(P^{\mathbb{C}}) \cong \Gamma(M, \mathrm{Ad}\, P^{\mathbb{C}})$, and the action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P)$ corresponds to the complex gauge transformations on $\mathcal{A}(P^{\mathbb{C}})$, i.e., $g \in \mathcal{G}(P^{\mathbb{C}})\colon D \mapsto g \circ D \circ g^{-1}$. Let

$$\begin{aligned}\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) &= \{A - \sqrt{-1}\psi \in \mathcal{A}(P^{\mathbb{C}}) : F_{A-\sqrt{-1}\psi} = 0\}\\ &= \big\{(A,\psi) \in T\mathcal{A} : F_A - \tfrac{1}{2}[\psi,\psi] = 0, d_A\psi = 0\big\}\end{aligned}$$

be the set of flat connections on $P^{\mathbb{C}}$. Since the vanishing of $F_{A-\sqrt{-1}\psi}$ is a holomorphic condition, $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ is a complex subset of $\mathcal{A}(P^{\mathbb{C}})$; it is also invariant under $\mathcal{G}(P^{\mathbb{C}})$. The holonomy group $\mathrm{Hol}(A)$ of $A \in \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ can be identified as a subgroup of G, up to a conjugation in G. A flat connection A on $P^{\mathbb{C}}$ is *reductive* if the closure of $\mathrm{Hol}(A)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\mathrm{Hol}(A)$; let $\mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})$ be the set of such. It can be shown that a flat connection is reductive if and only if its orbit under $\mathcal{G}(P^{\mathbb{C}})$ is closed [7]. The *de Rham moduli space*, or the moduli space of reductive flat connections on $P^{\mathbb{C}}$, is

$$\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})/\!\!/\mathcal{G}(P^{\mathbb{C}}) = \mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}}).$$

It has an induced complex structure $\bar{J}$ on its smooth part.

Assume that M has a Riemannian structure and choose an invariant inner product $(\cdot,\cdot)$ on the Lie algebra $\mathfrak{k}$ of K. Then there is a symplectic structure on $T\mathcal{A}(P)$, with which J is compatible, given by

$$\omega_J((\alpha_1,\varphi_1),(\alpha_2,\varphi_2)) = \int_M (\varphi_2, \wedge * \alpha_1) - (\varphi_1, \wedge * \alpha_2), \tag{2.1}$$

where $\alpha_1,\alpha_2,\varphi_1,\varphi_2 \in \Omega^{1,0}(M,\mathrm{ad}\,P)$, such that $(T\mathcal{A}(P),\omega_J)$ is Kähler. The subset $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ is Kähler in $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$. We identify the Lie algebra $\mathrm{Lie}(\mathcal{G}(P)) \cong \Omega^0(M,\mathrm{ad}\,P)$ with its dual by the inner product on $\Omega^0(M,\mathrm{ad}\,P)$. The action of $\mathcal{G}(P)$ on $(T\mathcal{A}(P),\omega_J)$ is Hamiltonian, with moment map

$$\mu_J(A,\psi) = d_A^*\psi \in \Omega^0(M,\mathrm{ad}\,P). \tag{2.2}$$

Let

$$\begin{aligned}\mathcal{A}^{\mathrm{Hitchin}}(P) &= \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0)\\ &=\big\{(A,\psi) \in T\mathcal{A} : F_A - \tfrac{1}{2}[\psi,\psi] = 0, d_A\psi = 0, d_A^*\psi = 0\big\},\end{aligned}$$

the set of pairs (A,ψ) satisfying Hitchin's equations (1.1), and let the quotient space $\mathcal{M}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{Hitchin}}(P)/\mathcal{G}(P)$ be *Hitchin's moduli space*. A theorem of Donaldson [8] and Corlette [7] states that if M is compact and if the structure group G is semisimple, then $\mathcal{M}^{\mathrm{Hitchin}}(P) \cong \mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$.

Suppose that M is a compact Kähler manifold of complex dimension n and let ω be the Kähler form on M. Then there is a complex structure on $T\mathcal{A}(P)$ given by

$$I\colon (\alpha,\varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\alpha, -\varphi)) = \frac{1}{(n-1)!}\,\Lambda^{n-1}(*\alpha, -*\varphi),$$

where $(\alpha,\varphi) \in \Omega^1(M,\mathrm{ad}\,P)^{\oplus 2} \cong T_{(A,\psi)}T\mathcal{A}(P)$ and the map

$$\Lambda\colon \Omega^\bullet(M,\mathrm{ad}\,P) \to \Omega^{\bullet-2}(M,\mathrm{ad}\,P)$$

is the contraction by ω. With respect to I, we have

$$T^{1,0}_{(A,\psi)}T\mathcal{A}(P) \cong \Omega^{0,1}(M,\mathrm{ad}\,P^{\mathbb{C}}) \oplus \Omega^{1,0}(M,\mathrm{ad}\,P^{\mathbb{C}})$$

for any $(A,\psi) \in T\mathcal{A}(P)$. This complex structure I is compatible with a symplectic form ω_I on $T\mathcal{A}(P)$ given by

$$\omega_I((\alpha_1,\varphi_1),(\alpha_2,\varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge \big((\alpha_1, \wedge\alpha_2) - (\varphi_1, \wedge\varphi_2)\big),$$

where $\alpha_1,\alpha_2,\varphi_1,\varphi_2 \in \Omega^1(M,\mathrm{ad}\,P)$. The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is also Hamiltonian with respect to ω_I and the moment map is

$$\mu_I(A,\psi) = \Lambda\,(F_A - \tfrac{1}{2}[\psi,\psi]) \in \Omega^0(M,\mathrm{ad}\,P),$$

where $F_A \in \Omega^2(M,\mathrm{ad}\,P)$ is the curvature of A. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves I, there is a holomorphic $\mathcal{G}(P^{\mathbb{C}})$ action on $(T\mathcal{A}(P), I)$. For any $(A,\psi) \in T\mathcal{A}(P)$, write $\psi = \sqrt{-1}(\phi - \phi^*)$, where $\phi \in \Omega^{1,0}(M,\mathrm{ad}\,P^{\mathbb{C}})$, $\phi^* \in \Omega^{0,1}(M,\mathrm{ad}\,P^{\mathbb{C}})$. Here $\phi \mapsto \phi^*$ is induced by the conjugation on $G = K^{\mathbb{C}}$ preserving the compact form K. Then $D = d_A - \sqrt{-1}\psi = D' + D''$, where $D' = \partial_A - \phi^*$, $D'' = \bar\partial_A + \phi$. The action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P) \cong \mathcal{A}(P^{\mathbb{C}})$ can be described by $g \in \mathcal{G}(P^{\mathbb{C}})\colon D'' \mapsto g \circ D'' \circ g^{-1}$.

Let $\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})$ be the set of Higgs pairs (A,ϕ), i.e., $A \in \mathcal{A}(P)$ and $\phi \in \Omega^{1,0}(M,\mathrm{ad}\,P^{\mathbb{C}})$ satisfying $(D'')^2 = 0$, or

$$\bar\partial_A^2 = 0, \quad \bar\partial_A\phi = 0, \quad [\phi,\phi] = 0.$$

Then $\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})$ is a Kähler subspace of $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$ respect to I. Let $\mathcal{A}^{\mathrm{sst}}(P^{\mathbb{C}})$ be the set of semistable Higgs pairs and let $\mathcal{A}^{\mathrm{pst}}(P^{\mathbb{C}})$ be the set polystable Higgs pairs. (The notions of stable, semistable and polystable Higgs pairs were introduced in [15, 30, 31].) The *moduli space of polystable Higgs pairs* or the *Dolbeault moduli space* is

$$\mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}}) = (\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\mathrm{sst}}(P^{\mathbb{C}}))/\!\!/\mathcal{G}(P^{\mathbb{C}}) = (\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\mathrm{pst}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}}).$$

It has a complex structure induced by I. It can be shown [30, Lemma 1.1] that $\mathcal{A}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$. A theorem of Hitchin [15] and Simpson [29] states that if M is compact and Kähler and the bundle P has vanishing first and second Chern classes, then $\mathcal{M}^{\mathrm{Hitchin}}(P) \cong \mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}})$.

There is a third complex structure on $T\mathcal{A}(P)$ defined by

$$K = IJ = -JI \colon (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\varphi, \alpha)) = \frac{1}{(n-1)!} \Lambda^{n-1}(*\varphi, *\alpha),$$

which is compatible with the symplectic form

$$\omega_K((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge \big((\alpha_1, \wedge\varphi_2) - (\alpha_2, \wedge\varphi_1)\big).$$

The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to ω_K and the moment map is

$$\mu_K(A, \psi) = \Lambda\,(d_A\psi) \in \Omega^0(M, \operatorname{ad} P).$$

Moreover, the action preserves K and therefore extends to another holomorphic action of $\mathcal{G}(P)^{\mathbb{C}}$. The three complex structures I, J, K define a hyper-Kähler structure on $T\mathcal{A}(P)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to all three symplectic forms, we have a hyper-Kähler moment map $\mu = (\mu_I, \mu_J, \mu_K) \colon T\mathcal{A}(P) \to (\Omega^0(M, \operatorname{ad} P))^{\oplus 3}$. The *hyper-Kähler quotient* [16] is $\mathcal{M}^{\mathrm{HK}}(P) = \mu^{-1}(0)/\mathcal{G}(P)$, with complex structures $\bar{I}, \bar{J}, \bar{K}$ and symplectic forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. By the theorems of Donaldson-Corlette and of Hitchin-Simpson, the Hitchin moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a complex space with respect to both $\bar{I}$ and $\bar{J}$. Therefore $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\mathrm{HK}}(P)$ [10, Theorem 8.3.1].

When $M = \Sigma$ is an orientable surface, $\Lambda \colon \Omega^2(\Sigma, \operatorname{ad} P) \to \Omega^0(\Sigma, \operatorname{ad} P)$ is an isomorphism. So $\mathcal{A}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$ coincides with $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$. Thus the moduli spaces $\mathcal{M}^{\mathrm{Hitchin}}(P) \cong \mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}})$ coincide with the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(P)$ [15].

2.2. **Moduli space of Hitchin's equations on a non-orientable manifold.** Now suppose M is a compact non-orientable manifold. Let $\pi \colon \tilde{M} \to M$ be its oriented cover and let $\tau \colon \tilde{M} \to \tilde{M}$ be the non-trivial deck transformation. Given a principal K-bundle $P \to M$, let $\tilde{P} = \pi^* P \to \tilde{M}$ be its pull-back to $\tilde{M}$. Since $\pi \circ \tau = \pi$, the τ action can be lifted to $\tilde{P} = \tilde{M} \times_M P$ as a K-bundle involution (i.e., the lifted involution commutes with the right K-action on $\tilde{P}$), and hence to the associated bundles $\operatorname{Ad} \tilde{P}$ and $\operatorname{ad} \tilde{P}$. Consequently, τ acts on the space of connections $\mathcal{A}(\tilde{P})$ by pull-back $A \mapsto \tau^* A$ and on the group of gauge transformations $\mathcal{G}(\tilde{P})$ by $g \mapsto \tau^* g := \tau^{-1} \circ g \circ \tau$. The τ-invariant subsets are $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$ and $(\mathcal{G}(\tilde{P}))^\tau \cong \mathcal{G}(P)$. In fact, the inclusion map $\mathcal{A}(P) \hookrightarrow \mathcal{A}(\tilde{P})$ onto the τ-invariant part is the pull-back via π of connections on P to those on $\tilde{P}$. Since $\mathcal{A}(\tilde{P})$ is an affine space modeled on $\Omega^1(\tilde{M}, \operatorname{ad} \tilde{P})$, the differential τ_* of $\tau \colon \mathcal{A}(\tilde{P}) \to \mathcal{A}(\tilde{P})$ can be identified with a linear involution on $\Omega^1(\tilde{M}, \operatorname{ad} \tilde{P})$ given by $\alpha \mapsto \tau^* \alpha$.

A Riemannian metric on a non-orientable manifold M pulls back to a Riemannian metric on $\tilde{M}$. Assuming that M is compact, we define an inner product on the space $\Omega^\bullet(M)$ of differential forms on M by

$$\langle \alpha, \beta \rangle = \tfrac{1}{2} \int_{\tilde{M}} \pi^* \alpha \wedge \tilde{*}\, \pi^* \beta$$

for $\alpha, \beta \in \Omega^\bullet(M)$, where $\tilde{*}$ is the Hodge star operator on $\tilde{M}$. Alternatively, the Hodge star $*$ on M maps a form on M to one valued in the orientation line bundle over M, and if α, β are of the same degree, then $\alpha \wedge *\beta$ is a top-degree form on M valued in the orientation line bundle, which can be integrated over M. We still have $\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$. More generally, there is an inner product on the space $\Omega^\bullet(M, \operatorname{ad} P)$ of forms valued in $\operatorname{ad} P$. Therefore $\mathcal{A}(P)$ admits a Riemannian structure, which is half of the restriction of the Riemannian structure on $\mathcal{A}(\tilde{P})$ to the τ-invariant subspace $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$.

Consider the tangent bundle $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(\tilde{M}, \operatorname{ad} \tilde{P})$ of $\mathcal{A}(\tilde{P})$. It has a τ-action given by $\tau \colon (A, \psi) \mapsto (\tau^* A, \tau^* \psi)$, which is holomorphic with respect to the complex structure J. Therefore the fixed point set $(T\mathcal{A}(\tilde{P}))^\tau \cong T\mathcal{A}(P)$ is a complex subspace in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. With respect to the induced Riemannian structure on $T\mathcal{A}(\tilde{P})$,

$\tau\colon T\mathcal{A}(\tilde{P}) \to T\mathcal{A}(\tilde{P})$ is an isometry. Since τ also acts holomorphically on $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong T\mathcal{A}(\tilde{P})$, $(T\mathcal{A}(\tilde{P}))^\tau$ is a Kähler and totally geodesic subspace in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cong (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ is also Kähler and totally geodesic in $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$. We summarize the above discussion in the following lemma.

Lemma 2.1. *Given a compact non-orientable manifold M with oriented double cover $\pi\colon \tilde{M} \to M$ and a principal K-bundle $P \to M$, the non-trivial deck transformation τ on $\tilde{M}$ lifts to an involution (also denoted by τ) on $\tilde{P} = \pi^* P$ and acts as involutions on the space of connections $\mathcal{A}(\tilde{P})$ and on $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, the τ-invariant subspaces $\mathcal{A}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}(P^{\mathbb{C}})$ and $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ are Kähler and totally geodesic subspaces in $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong T\mathcal{A}(\tilde{P})$ and $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$, respectively.*

On a non-orientable manifold M, we still have Hitchin's equations (1.1). Here d_A^* is defined as the (formal) adjoint of d_A with respect to the inner products on $\Omega^\bullet(M, \operatorname{ad} P)$. Alternatively, d_A^* is the first order differential operator on M such that on any orientable open set in M, $d_A^* = *^{-1} d_A\, *$; the latter is actually independent of the choice of local orientation. Yet another but related way to explain the operator d_A^* is to consider the Hodge star operator $*$ on a non-orientable manifold M as a map from differential forms to those valued in the orientation bundle over M. Since the latter is a flat real line bundle, $d_A^* = *^{-1} d_A *$ maps $\Omega^1(M, \operatorname{ad} P)$ to $\Omega^0(M, \operatorname{ad} P)$. Finally, d_A^* can be defined as $(\pi^*)^{-1} \circ d_{\pi^* A}^* \circ \pi^*$. Here $d_{\pi^* A}^* = *^{-1} d_{\pi^* A}\, *$ holds globally on $\tilde{M}$ and $\pi^*\colon \Omega^\bullet(M, \operatorname{ad} P) \to \Omega^\bullet(\tilde{M}, \operatorname{ad} \tilde{P})$ is injective. Let

$$\mathcal{A}^{\mathrm{Hitchin}}(P) := \{(A, \psi) \in T\mathcal{A} : F_A - \tfrac{1}{2}[\psi, \psi] = 0, d_A\psi = 0, d_A^*\psi = 0\}.$$

It is clear that $\mathcal{A}^{\mathrm{Hitchin}}(P) = (\mathcal{A}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$.

The notion of reductive connections on P does not depend on the orientability of M, and we still have the moduli space of flat connections $\mathcal{M}^{\mathrm{flat}}(P^{\mathbb{C}}) = \mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}})$. Let $\mathcal{M}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{Hitchin}}(P)/\mathcal{G}(P)$ be Hitchin's moduli space. The following is the Donaldson-Corlette theorem that also applies to the case when M is non-orientable. Equivalently, there exists a unique reduction of structure group from G to K admitting a solution to Hitchin's equations.

Theorem 2.2. *Let M be a compact non-orientable Riemannian manifold. Then for every reductive flat connection D on $P^{\mathbb{C}}$, there exists a gauge transformation $g \in \mathcal{G}(P^{\mathbb{C}})$ (unique up to $\mathcal{G}(P)$ and the stabilizer of D) such that $g \cdot D = d_A - \sqrt{-1}\psi$ with $(A, \psi) \in \mathcal{A}^{\mathrm{Hitchin}}(P)$. As a consequence, we have a homeomorphism $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Hitchin}}(P)$.*

We now explain that Corlette's proof in [7] applies to the case when M is non-orientable. There is a symplectic form ω_J on $T\mathcal{A}(P)$, still given by(2.1), which is half of the restriction of the symplectic form on $T\mathcal{A}(\tilde{P})$. The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian, and the moment map remains (2.2). Recall Corlette's flow equations on the space of flat connections. Let $D = d_A - \sqrt{-1}\psi$ be a flat connection of the $G = K^{\mathbb{C}}$ bundle $P^{\mathbb{C}} \to M$. Then the flow equations are

$$\frac{\partial D}{\partial t} = -D\mu_J(D). \tag{2.3}$$

Equivalently, one can look for a flow of the form $g(t) \cdot D_0$ and solve for $g(t) \in \mathcal{G}(\tilde{P}^{\mathbb{C}})$ using (cf. [7, p. 369])

$$\frac{\partial g}{\partial t} g^{-1} = -\sqrt{-1}\mu_J(g \cdot D_0). \tag{2.4}$$

Corlette shows in [7] that we have existence and uniqueness of solutions to (2.3) and (2.4) for all time. If the initial condition is a reductive flat connection, then there is a sequence converging to a solution to $\mu_J(D) = 0$. Also, the limit is gauge equivalent to the initial flat reductive connection [7]. These arguments are valid when M is non-orientable.

We remark that Theorem 2.2 for non-orientable manifolds also follows from the result of the orientable double cover. A flat connection on P is reductive if and only if the pull-back $\pi^* A$ is a flat reductive connection on $\tilde{P}$. (We defer the proof of this statement to Corollary 3.2.) For the bundle $\tilde{P} \to \tilde{M}$, it is easy to check that the right-hand sides of (2.3) and (2.4) define τ-invariant vector fields on $\mathcal{A}(\tilde{P}^{\mathbb{C}})$ and $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, respectively. Since the space $(\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ-invariant connections is closed in $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$ and the space $(\mathcal{G}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ-invariant gauge transformations is closed in $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, Corlette's results on the limit of the flow restrict to the τ-invariant subset as well. That is, the flow on the space of connections is contained in the τ-invariant subset and the limit is a τ-invariant solution to Hitchin's equation. Similarly, the gauge transformation relating to the initial condition is contained in the τ-invariant part of the group of gauge transformations, and the limit is τ-invariant.

2.3. **The Hitchin moduli space and the hyper-Kähler quotient.** Now consider a compact non-orientable manifold M. Suppose its oriented cover $\tilde{M}$ is a Kähler manifold of complex dimension n. Let ω be the Kähler form on $\tilde{M}$. Throughout this subsection, we assume that n is odd and the deck transformation τ on $\tilde{M}$ is an anti-holomorphic involution such that $\tau^*\omega = -\omega$. Then $\tau^*\omega^n = -\omega^n$, which is consistent with the requirement that τ is orientation reversing. The τ-action on $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(M, \operatorname{ad} P)$, $\tau\colon (A,\psi) \mapsto (\tau^*A, \tau^*\psi)$, is an isometry and its differential $\tau_*\colon \Omega^1(M, \operatorname{ad} P)^{\oplus 2} \to \Omega^1(M, \operatorname{ad} P)^{\oplus 2}$ is $\tau_*\colon (\alpha,\varphi) \mapsto (\tau^*\alpha, \tau^*\varphi)$. It is easy to see that $\tau_* \circ I = -I \circ \tau_*$ since τ reverses the orientation of M and that $\tau_* \circ K = -K \circ \tau_*$ since $K = IJ$. So τ acts as an anti-holomorphic involution with respect to both I and K, and $\tau^*\omega_I = -\omega_I$, $\tau^*\omega_K = -\omega_K$. Moreover, since the moment maps μ_I and μ_K on $T\mathcal{A}(\tilde{P})$ involve the contraction Λ by ω, they satisfy $\tau^*(\mu_I(A,\psi)) = -\mu_I(\tau^*A, \tau^*\psi)$, $\tau^*(\mu_K(A,\psi)) = -\mu_K(\tau^*A, \tau^*\psi)$ for all $(A,\psi) \in T\mathcal{A}(\tilde{P})$. The fixed point set $(\mathcal{A}(\tilde{P}))^\tau$ is totally real with respect to the complex structures I and K, and Lagrangian with respect to the symplectic forms ω_I and ω_K [24, 9, 25].

A flat connection $D = d_A - \sqrt{-1}\psi$ on $\tilde{P}^{\mathbb{C}}$ defines an elliptic complex with $D_i\colon \Omega^i(\tilde{M}, \operatorname{ad}\tilde{P}^{\mathbb{C}}) \to \Omega^{i+1}(\tilde{M}, \operatorname{ad}\tilde{P}^{\mathbb{C}})$. Let $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ be the set of flat connections on $\tilde{P}^{\mathbb{C}}$ such that (i) the stabilizer under the $\mathcal{G}(\tilde{P}^{\mathbb{C}})$ action is $Z(G)$, and (ii) the linearization D_1 of the curvature map surjects onto $\ker D_2 \cap \Omega^2(\tilde{M}, [\operatorname{ad}\tilde{P}^{\mathbb{C}}, \operatorname{ad}\tilde{P}^{\mathbb{C}}])$. Notice that when M is a surface, condition (i) implies (ii). The method in [22] and [23, Chapter VII] shows that $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ is a smooth submanifold in $\mathcal{A}(\tilde{P}^{\mathbb{C}})$, and as the action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ on it is free, the subset $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^\circ := (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mathcal{A}^{\mathrm{flat,red}}(\tilde{P}^{\mathbb{C}}))/\mathcal{G}(\tilde{P}^{\mathbb{C}})$ is in the smooth part of the moduli space $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})$ (see also [11] from the point of view of representation varieties). The free action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ or $\mathcal{G}(\tilde{P})/Z(K)$ from condition (i) implies that 0 is a regular value of μ_J on $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$, and the subset $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ := \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0)/\mathcal{G}(\tilde{P})$ is in the smooth part of Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ [15]. By the Donaldson-Corlette theorem, we have the homeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ \cong \mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^\circ$.

On the other hand, for the non-orientable manifold M, let $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ = \{A \in \mathcal{A}(P^{\mathbb{C}}) : \pi^*A \in \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ\}$, $\mathcal{A}^{\mathrm{Hitchin}}(P)^\circ = \mathcal{A}^{\mathrm{Hitchin}}(P) \cap \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ$.
Then $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ := \mathcal{A}^{\mathrm{Hitchin}}(P)^\circ/\mathcal{G}(P)$ is in the smooth part of $\mathcal{M}^{\mathrm{Hitchin}}(P)$, but we will not consider here the smooth points of $\mathcal{M}^{\mathrm{Hitchin}}(P)$ that are outside $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$. By Theorem 2.2 (the analog of the Donaldson-Corlette theorem for non-orientable manifolds), we have a homeomorphism between $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$ and $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^\circ := (\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ \cap \mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}})$.

We now study a general setting. Let (X,ω) be a finite dimensional symplectic manifold with a Hamiltonian action of a compact Lie group K and let $\mu\colon X \to \mathfrak{k}^*$ be the moment map. Suppose as in [25], that there are involutions σ on X and τ on K such that $\sigma(k\cdot x) = \tau(k)\cdot\sigma(x)$ for all $k \in K$ and $x \in X$. Assume that X^σ is not empty. Then K^τ acts on X^σ. We note that τ acts on $\mathfrak{k}$, $\mathfrak{k}^*$, and K^τ is a closed Lie subgroup of K with Lie algebra $\mathfrak{k}^\tau$. Contrary to [25], we assume that the action of (K, K^τ) on (X, X^σ) is symplectic, i.e, we have $\sigma^*\omega = \omega$ and $\sigma^*\mu = \tau\mu$. Then X^σ is a symplectic submanifold in X. Assume that 0 is a regular value of μ and that K acts on $\mu^{-1}(0)$ freely. Since σ preserves $\mu^{-1}(0)$, it descends to a symplectic involution $\bar{\sigma}$ on the (smooth) symplectic quotient $X/\!\!/_0 K = \mu^{-1}(0)/K$ at level 0, and $(X/\!\!/_0 K)^{\bar{\sigma}}$ is a symplectic submanifold.

Lemma 2.3. *In the above setting, the action of K^τ on X^σ is Hamiltonian and the symplectic quotient is $X^\sigma/\!\!/_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$. If $\mu^{-1}(0) \cap X^\sigma \neq \emptyset$, then there exists a symplectic local diffeomorphism from $X^\sigma/\!\!/_0 K^\tau$ to $(X/\!\!/_0 K)^{\bar{\sigma}}$.*

Proof. Let $\mathfrak{k} = \mathfrak{k}^\tau \oplus \mathfrak{q}$ such that $\tau = \pm 1$ on $\mathfrak{k}^\tau$, $\mathfrak{q}$, respectively. It is clear that the action of K^τ on X^σ is Hamiltonian and the moment map μ_τ is the composition $X^\sigma \hookrightarrow X \to \mathfrak{k}^* \to (\mathfrak{k}^\tau)^*$. Since for any $x \in X^\sigma$, $\langle \mu(x), \mathfrak{q}\rangle = 0$, we get $\mu_\tau^{-1}(0) = \mu^{-1}(0) \cap X^\sigma = (\mu^{-1}(0))^\sigma$. By the assumptions, 0 is a regular value of μ_τ, the action of K^τ on $\mu_\tau^{-1}(0)$ is free, and the symplectic quotient is $X^\sigma/\!\!/_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$.

For any $x \in X^\sigma$, the map $\mathfrak{k} \to T_xX$ intertwines τ on $\mathfrak{k}$ and σ on T_xX, and $T_x(K^\tau \cdot x) = (T_x(K\cdot x))^\sigma$. The inclusion $\mu_\tau^{-1}(0) \hookrightarrow \mu^{-1}(0)$ induces a natural map $X^\sigma/\!\!/_0 K^\tau \to (X/\!\!/_0 K)^{\bar{\sigma}}$, whose differentiation at $[x]$ is, after natural symplectic isomorphisms, the linear map $(T_x\mu^{-1}(0))^\sigma/(T_x(K\cdot x))^\sigma \to (T_x\mu^{-1}(0)/T_x(K\cdot x))^{\bar{\sigma}}$. The latter is clearly injective; to show surjectivity, we note that for any $V \in T_x\mu^{-1}(0)$, if $V + T_x(K\cdot x) \in (T_x\mu^{-1}(0)/T_x(K\cdot x))^{\bar{\sigma}}$, then it is the image of $\frac{1}{2}(V + \sigma V) + (T_x(K\cdot x))^\sigma$. The map $X^\sigma/\!\!/_0 K^\tau \to (X/\!\!/_0 K)^{\bar{\sigma}}$ is a local diffeomorphism; it is symplectic because the above linear map is so for each $x \in \mu_\tau^{-1}(0)$. $\square$

Now let X be a hyper-Kähler manifold with complex structures J_i and symplectic structures ω_i $(i = 1, 2, 3)$. Suppose K acts on X and the action is Hamiltonian with respect to all ω_i. Let $\mu = (\mu_1, \mu_2, \mu_3) \colon X \to (\mathfrak{k}^*)^{\oplus 3}$ be the hyper-Kähler moment map. Assume that there are involutions σ on X and τ on K such that $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$ for all $k \in K$ and $x \in X$ and $\sigma^* J_i = (-1)^i J_i$, $\sigma^* \omega_i = (-1)^i \omega_i$, $\sigma^* \mu_i = (-1)^i \tau \mu_i$ for $i = 1, 2, 3$. So the action of (K, K^τ) on (X, X^σ) is symplectic with respect to ω_2 (as above) and anti-symplectic with respect to ω_1, ω_3 (as in [25]). Then X^σ, if non-empty, is Kähler and totally geodesic in X with respect to J_2, ω_2 and is totally real and Lagrangian with respect to J_1, ω_1 and J_3, ω_3. If 0 is a regular value of μ (i.e., 0 is a regular value of each μ_i) and that K acts on $\mu^{-1}(0)$ freely, then $X /\!\!/\!\!/_0 K = \mu^{-1}(0)/K$ is the (smooth) hyper-Kähler quotient at level 0, which has complex structures $\bar{J}_i$ and symplectic structures $\bar{\omega}_i$ $(i = 1, 2, 3)$ [16].

Proposition 2.4. *In the above setting, let $Y = \mu_1^{-1}(0) \cap \mu_3^{-1}(0)$. Then*
1. Y is a σ-invariant Kähler submanifold in X with respect to J_2, ω_2 and the symplectic quotient $Y^\sigma /\!\!/_0 K^\tau = (\mu^{-1}(0))^\sigma / K^\tau$ is Kähler;
2. $(X /\!\!/\!\!/_0 K)^{\bar{\sigma}}$ is Kähler and totally geodesic in $X /\!\!/\!\!/_0 K$ with respect to $\bar{J}_2, \bar{\omega}_2$ and is totally real and Lagrangian with respect to $\bar{J}_1, \bar{J}_3$ and $\bar{\omega}_1, \bar{\omega}_3$;
3. if $(\mu^{-1}(0))^\sigma \neq \emptyset$, there is a Kähler (with respect to $\bar{J}_2, \bar{\omega}_2$) local diffeomorphism $Y^\sigma /\!\!/_0 K^\tau \to (X /\!\!/\!\!/_0 K)^{\bar{\sigma}}$.

Proof. 1&3. Let $\mu_c = \mu_3 + \sqrt{-1}\mu_1 \colon X \to \mathfrak{k}^{*\mathbb{C}}$. Then μ_c is holomorphic with respect to J_2 and is equivariant under the action of K. Since 0 is a regular value of μ_c, $Y = \mu_c^{-1}(0)$ is a smooth Kähler submanifold in X on which the action of K is Hamiltonian. Applying Lemma 2.3 to Y, we conclude that the action of K^τ on Y^σ is Hamiltonian and that $(\mu^{-1}(0))^\sigma / K^\tau = (\mu_2^{-1}(0) \cap Y^\sigma)/K^\tau = Y^\sigma /\!\!/_0 K^\tau$. Moreover, there is a local diffeomorphism from $Y^\sigma /\!\!/_0 K^\tau$ to $(Y /\!\!/_0 K)^{\bar{\sigma}} = (X /\!\!/\!\!/_0 K)^{\bar{\sigma}}$ which is symplectic. Since K^τ acts holomorphically on (Y^σ, J_2), the symplectic quotient $Y^\sigma /\!\!/_0 K^\tau$ is Kähler, and the above local diffeomorphism is also Kähler.

2. Since σ preserves $\mu^{-1}(0)$, it descends to an involution $\bar{\sigma}$ on $X /\!\!/\!\!/_0 K$ such that $\bar{\sigma}^* \bar{J}_i = (-1)^i \bar{J}_i$, $\bar{\sigma}^* \bar{\omega}_i = (-1)^i \bar{\omega}_i$ for $i = 1, 2, 3$. The result then follows. □

We now prove Theorem 1.1.

Proof. 1&3. Note that $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ is a τ-invariant Kähler submanifold in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Following [1, 15], we can apply the method in Lemma 2.3 to $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ on which τ acts preserving ω_J and J. Since τ also acts on $\mathcal{G}(\tilde{P})$ and $\mathcal{G}(P) \cong (\mathcal{G}(\tilde{P}))^\tau$, $\mathcal{G}(P)/Z(K)$ acts Hamiltonianly and freely on $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ \cong (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$, which is Kähler with respect to J, ω_J. Thus $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ = (\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0))/\mathcal{G}(P) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ /\!\!/_0 \mathcal{G}(P)$ is a symplectic quotient. Since the latter is non-empty, there is a local Kähler diffeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ \to (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ /\!\!/_0 \mathcal{G}(\tilde{P}))^\tau = (\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau$.

2. The space $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ with I, J, K is hyper-Kähler and the action of $\mathcal{G}(\tilde{P})$ is Hamiltonian with respect to $\omega_I, \omega_J, \omega_K$. Let $(\mu^{-1}(0))^\circ$ be the subset of $\mu^{-1}(0)$ on which $\mathcal{G}(\tilde{P})/Z(K)$ acts freely. Then $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ := (\mu^{-1}(0))^\circ/\mathcal{G}(\tilde{P})$ is the smooth part of the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(\tilde{P})$. The involutions τ on $\mathcal{A}(\tilde{P})$ and $\mathcal{G}(\tilde{P})$ satisfy the conditions of Proposition 2.4. So $(\mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ)^\tau$ is Kähler and totally geodesic with respect to $\bar{J}$ and $\bar{\omega}_J$, and totally real and Lagrangian with respect to $\bar{I}, \bar{K}$ and $\bar{\omega}_I, \bar{\omega}_K$ in $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ$. If M is a non-orientable surface, then $\mu_I^{-1}(0) \cap \mu_K^{-1}(0) = \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$ which implies that $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ = \mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ$. In general, $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ$ is a τ-invariant hyper-Kähler submanifold in $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ$. The results follow from $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau = \mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}) \cap (\mathcal{M}^{\mathrm{HK}}(\tilde{P})^\circ)^\tau$. □

3. The representation variety perspective

3.1. **Representation variety and Betti moduli space.** Let Γ be a finitely generated group and let G be a connected complex Lie group. Then G acts on $\mathrm{Hom}(\Gamma, G)$ by the conjugate action on G. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is *reductive* if the closure of $\phi(\Gamma)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\phi(\Gamma)$; let $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ be the set of such. The condition $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ is equivalent to the statement that the G-orbit $G \cdot \phi$ is closed [13]. It is also equivalent to the condition that the composition of ϕ with the adjoint representation of G is semi-simple (see [26, Section 3] and [28, Theorem 30]). The quotient

$$\mathrm{Hom}(\Gamma, G) /\!\!/ G = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/G$$

is known as the *representation variety* or *character variety.* A reductive representation $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ is *good* [20] if its stabilizer $G_\phi = Z(G)$; let $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ be the set of such. On the other hand, $\phi \in \mathrm{Hom}(\Gamma, G)$ is *Ad-irreducible* if its composition with the adjoint representation of G is an irreducible representation of Γ. Let $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ be the set of such. Notice that this set is empty unless G is simple. Clearly, $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$. In general, $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/G$ may not be smooth, but it is so when Γ is the fundamental group of a compact orientable surface [28, Corollary 50].

Suppose M is a compact manifold and $P^{\mathbb{C}} \to M$ is a principal G-bundle over M. Choose a base point $x_0 \in M$ and let $\Gamma = \pi_1(M, x_0)$ be the fundamental group. Then $\mathrm{Hom}(\Gamma, G)/\!\!/G$ is known as the *Betti moduli space* [30], denoted by $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$. The identification $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$, which we recall briefly now, is well known. Given a flat connection, let $T_\alpha \colon P_{\alpha(0)} \to P_{\alpha(1)}$ be the parallel transport along a path α in M. Fix a point $p_0 \in P_{x_0}$ in the fibre over x_0. For $a \in \pi_1(M, x_0)$, choose a loop α based at x_0 representing a, then $\phi(a)$ is the unique element in G defined by $T_\alpha(p_0) = p_0\phi(a)^{-1}$. If we choose another point in the fibre over x_0, then ϕ differs by a conjugation. Finally, the flat connection is reductive if and only if the corresponding element in $\mathrm{Hom}(\Gamma, G)$ is reductive. Upon identification of the de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ and the Betti moduli spaces $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}}) = \mathrm{Hom}(\Gamma, G)/\!\!/G$, the subset $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/G$ contains the smooth part $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^\circ$ introduced in subsection 2.3; they are equal when M is a compact orientable surface.

If M is non-orientable and $\pi \colon \tilde{M} \to M$ is the oriented cover, we choose a base point $\tilde{x}_0 \in \pi^{-1}(x_0)$ and let $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$. Then there is a short exact sequence

$$1 \to \tilde{\Gamma} \to \Gamma \to \mathbb{Z}_2 \to 1$$

and $\tilde{\Gamma}$ can be identified with an index 2 subgroup in Γ. In the rest of this section, we will study the relation of the representation varieties $\mathrm{Hom}(\Gamma, G)/\!\!/G$ and $\mathrm{Hom}(\tilde{\Gamma}, G)/\!\!/G$ or the Betti moduli spaces $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$ and $\mathcal{M}^{\mathrm{Betti}}(\tilde{P}^{\mathbb{C}})$. Some of the results, when M is a compact non-orientable surface, appeared in [17], which used different methods.

We first establish a useful fact that was used in subsection 2.2.

Lemma 3.1. *Suppose Γ is a finitely generated group and $\tilde{\Gamma}$ is an index 2 subgroup in Γ. Let G be a connected, complex reductive Lie group. Then $\phi \in \mathrm{Hom}(\Gamma, G)$ is reductive if and only if the restriction $\phi|_{\tilde{\Gamma}} \in \mathrm{Hom}(\tilde{\Gamma}, G)$ is reductive.*

Proof. Recall that $\phi \in \mathrm{Hom}(\Gamma, G)$ is reductive if and only if the composition $\mathrm{Ad} \circ \phi$ is a semisimple representation on $\mathfrak{g}$. Similarly, $\phi|_{\tilde{\Gamma}}$ is reductive if and only if $\mathrm{Ad} \circ \phi|_{\tilde{\Gamma}}$ is semisimple. By $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}_2$ and [6], [5, Chap. 3, §9.8, Lemme 2], $\mathrm{Ad} \circ \phi$ is semisimple if only if $\mathrm{Ad} \circ \phi|_{\tilde{\Gamma}}$ is so. The result then follows. □

Corollary 3.2. *Let G be a connected, complex reductive Lie group. Suppose P is a principal G-bundle over a compact non-orientable manifold M whose oriented cover is $\pi \colon \tilde{M} \to M$. Then a flat connection A on P is reductive if and only if the pull-back π^*A is a flat reductive connection on $\tilde{P} := \pi^*P$.*

3.2. **Representation varieties associated to an index 2 subgroup.** Let Γ be a finitely generated group and let $\tilde{\Gamma}$ be an index 2 subgroup in Γ. Let G be a connected complex Lie group and let $Z(G)$ be its center. For any $c \in \Gamma \setminus \tilde{\Gamma}$, we have $\mathrm{Ad}_c|_{\tilde{\Gamma}} \in \mathrm{Aut}(\tilde{\Gamma})$, and the class $[\mathrm{Ad}_c|_{\tilde{\Gamma}}] \in \mathrm{Aut}(\tilde{\Gamma})/\mathrm{Inn}(\tilde{\Gamma})$ is independent of the choice of c. So we have a homomorphism $\mathbb{Z}_2 \cong \{1, \tau\} \to \mathrm{Aut}(\tilde{\Gamma})/\mathrm{Inn}(\tilde{\Gamma})$ given by $\tau \mapsto [\mathrm{Ad}_c|_{\tilde{\Gamma}}]$.

Lemma 3.3. *$\mathbb{Z}_2 \cong \{1, \tau\}$ acts on $\mathrm{Hom}(\tilde{\Gamma}, G)/\!\!/G$ and on $\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G$.*

Proof. We define $\tau[\phi] = [\phi \circ \mathrm{Ad}_c]$ for any $\phi \in \mathrm{Hom}(\tilde{\Gamma}, G)$. The action is well-defined since if $[\phi'] = [\phi]$, i.e., $\phi' = \mathrm{Ad}_g \circ \phi$ for some $g \in G$, then $\phi' \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi \circ \mathrm{Ad}_c \sim \phi \circ \mathrm{Ad}_c$. The τ-action is independent of the choice of c because if $c' \in \Gamma \setminus \tilde{\Gamma}$ is another element, then $c'c^{-1} \in \tilde{\Gamma}$ and $\phi \circ \mathrm{Ad}_{c'} = \mathrm{Ad}_{\phi(c'c^{-1})} \circ (\phi \circ \mathrm{Ad}_c) \sim \phi \circ \mathrm{Ad}_c$. We do have a $\mathbb{Z}_2$-action because $\tau^2[\phi] = [\phi \circ \mathrm{Ad}_{c^2}] = [\mathrm{Ad}_{\phi(c^2)} \circ \phi] = [\phi]$. Finally, if ϕ is in $\mathrm{Hom}^{\mathrm{red}}(\tilde{\Gamma}, G)$ or $\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, then so is $\phi \circ \mathrm{Ad}_c$. Thus τ acts on $\mathrm{Hom}(\tilde{\Gamma}, G)/\!\!/G$ and $\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G$. □

Proposition 3.4. *There exists a continuous map*

$$L \colon (\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau \to Z(G)/2Z(G). \tag{3.1}$$

So $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\mathrm{good}}$*, where* $\mathcal{N}_r^{\mathrm{good}} := L^{-1}(r)$*.*

Proof. If $\tau[\phi]=[\phi]$, then there exists $g\in G$ such that $\phi\circ\mathrm{Ad}_c=\mathrm{Ad}_g\circ\phi$. Since $c^2\in\tilde{\Gamma}$, we have $\mathrm{Ad}_{g^2}\circ\phi=\phi\circ\mathrm{Ad}_{c^2}=\mathrm{Ad}_{\phi(c^2)}\circ\phi$. Thus $z:=g^2\phi(c^2)^{-1}\in G_\phi=Z(G)$. If $[\phi']=[\phi]$, i.e., $\phi'=\mathrm{Ad}_h\circ\phi$ for some $h\in G$, then $\phi'\circ\mathrm{Ad}_c=\mathrm{Ad}_{g'}\circ\phi'$ for $g'=\mathrm{Ad}_h g$. Since $g'^2=\mathrm{Ad}_h g^2=z\,\mathrm{Ad}_h\phi(c^2)=z\phi'(c^2)$, we obtain $(g')^2\phi'(c^2)^{-1}=z$.

If $\phi\circ\mathrm{Ad}_{c'}=\mathrm{Ad}_{g'}\circ\phi$ holds for different choices of $c'\in\Gamma\setminus\tilde{\Gamma}$ and $g'\in G$, then $z'=(g')^2\phi(c'^2)^{-1}\in Z(G)$ from the above discussion. On the other hand, we have $\mathrm{Ad}_{g^{-1}g'}\circ\phi=\mathrm{Ad}_{\phi(c^{-1}c')}\circ\phi$ as $c^{-1}c'\in\tilde{\Gamma}$. This gives us $t:=(g')^{-1}g\phi(c^{-1}c')\in G_\phi=Z(G)$. We get

$$\begin{aligned} t^2(g')^2=(tg')^2&=g\phi(c^{-1}c')g\phi(c^{-1}c')=\mathrm{Ad}_g\phi(c^{-1}c')g^2\phi(c^{-1}c')\\ &=\phi(\mathrm{Ad}_c(c^{-1}c'))z\phi(c^2)\phi(c^{-1}c')=\phi((c')^2)z, \end{aligned}$$

i.e., $z'z^{-1}=t^{-2}\in 2Z(G)$. So the map $L\colon[\phi]\mapsto[z]\in Z(G)/2Z(G)$ is well-defined.

Since $\phi\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$, the element $[g]\in G/Z(G)$ is uniquely determined by and depends continuously on ϕ. Therefore $[z]\in Z(G)/2Z(G)$ depends continuously on $[\phi]\in(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau$. □

If $\phi\in\mathrm{Hom}(\Gamma,G)$ satisfies $\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$, then $\phi\in\mathrm{Hom}^{\mathrm{good}}(\Gamma,G)$. However, $\phi\in\mathrm{Hom}^{\mathrm{good}}(\Gamma,G)$ does not imply $\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$. Let

$$\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)=\{\phi\in\mathrm{Hom}(\Gamma,G):\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)\}.$$

We show that if $[\phi]\in(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau$, then $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ.

Lemma 3.5. *The restriction $R\colon[\phi]\mapsto[\phi|_{\tilde{\Gamma}}]$ maps $\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)/G$ surjectively to $\mathcal{N}_0^{\mathrm{good}}$.*

Proof. First, the image $\mathrm{im}(R)\subset\mathcal{N}_0^{\mathrm{good}}$ because for any $\phi\in\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$, $\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$ by definition, so $(\phi|_{\tilde{\Gamma}})\circ\mathrm{Ad}_c=\mathrm{Ad}_{\phi(c)}\circ\phi|_{\tilde{\Gamma}}\sim\phi|_{\tilde{\Gamma}}$ and $L([\phi|_{\tilde{\Gamma}}])=[\phi(c)^2\phi(c^2)^{-1}]=0$. We will show that in fact $\mathrm{im}(R)=\mathcal{N}_0^{\mathrm{good}}$. Let $\phi_0\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$ such that $\tau[\phi_0]=[\phi_0]$ and $L([\phi_0])=0$. Then there exist $g\in G$ and $t\in Z(G)$ such that $\phi_0\circ\mathrm{Ad}_c=\mathrm{Ad}_g\circ\phi_0$ and $g^2\phi(c^2)^{-1}=t^2$. We can extend ϕ_0 to $\phi\in\mathrm{Hom}(\Gamma,G)$ which is uniquely determined by the requirements $\phi|_{\tilde{\Gamma}}=\phi_0$ and $\phi(c)=gt^{-1}$. Since $\phi_0\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$, $\phi\in\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$ and therefore $[\phi_0]\in\mathrm{im}(R)$. □

Proposition 3.6. $R\colon\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)/G\to\mathcal{N}_0^{\mathrm{good}}$ *is a Galois covering map whose structure group is $\{s\in Z(G):s^2=e\}$.*

Proof. We define an action of $\{s\in Z(G):s^2=e\}$ on $\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$. For any such s and $\phi\in\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$, we define $s\cdot\phi$ by $(s\cdot\phi)|_{\tilde{\Gamma}}=\phi|_{\tilde{\Gamma}}$ and $(s\cdot\phi)|_{\Gamma\setminus\tilde{\Gamma}}=s(\phi|_{\Gamma\setminus\tilde{\Gamma}})$ the group multiplication. It is clear that $s\cdot\phi\in\mathrm{Hom}(\Gamma,G)$. Moreover, since $(s\cdot\phi)|_{\tilde{\Gamma}}=\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$, $s\cdot\phi\in\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$.
Clearly, the action descends to a well-defined action on $\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)/G$ by $s\cdot[\phi]=[s\cdot\phi]$ preserving the fibres of R.

We show that this action is free. Suppose $s\cdot[\phi]=[\phi]$, then $s\cdot\phi=\mathrm{Ad}_h\circ\phi$ for some $h\in G$. Since $\phi|_{\tilde{\Gamma}}=(s\cdot\phi)|_{\tilde{\Gamma}}=\mathrm{Ad}_h\circ(\phi|_{\tilde{\Gamma}})\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$, we get $h\in Z(G)$ and hence $s\cdot\phi=\phi$. Then $s\phi(c)=\phi(c)$ implies $s=e$.

It remains to show that the action is transitive on each fibre of R. Let $[\phi],[\phi']\in\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)$ such that $R([\phi])=R([\phi'])$. Then there exists an $h\in G$ such that $\phi'|_{\tilde{\Gamma}}=\mathrm{Ad}_h\circ(\phi|_{\tilde{\Gamma}})$. Thus

$$\begin{aligned} \mathrm{Ad}_{\phi'(c)h}\circ(\phi|_{\tilde{\Gamma}})&=\mathrm{Ad}_{\phi'(c)}\circ(\phi'|_{\tilde{\Gamma}})=(\phi'|_{\tilde{\Gamma}})\circ\mathrm{Ad}_c\\ &=\mathrm{Ad}_h\circ(\phi|_{\tilde{\Gamma}})\circ\mathrm{Ad}_c=\mathrm{Ad}_{h\phi(c)}\circ(\phi|_{\tilde{\Gamma}}). \end{aligned}$$

Hence $s:=\phi(c)^{-1}h^{-1}\phi'(c)h\in Z(G)$ since $\phi|_{\tilde{\Gamma}}\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$. Furthermore

$$s^2=\phi(c)^{-1}sh^{-1}\phi'(c)h=\phi(c^{-2})h^{-1}\phi'(c^2)h=\phi(c^{-2})\phi(c^2)=e.$$

Since we have $(s\cdot\phi)|_{\tilde{\Gamma}}=\phi|_{\tilde{\Gamma}}=\mathrm{Ad}_{h^{-1}}\circ(\phi'|_{\tilde{\Gamma}})$ and $(s\cdot\phi)(c)=s\phi(c)=\phi(c)s=(\mathrm{Ad}_{h^{-1}}\circ\phi')(c)$, we get $s\cdot\phi=\mathrm{Ad}_{h^{-1}}\circ\phi'$, or $[\phi']=[s\cdot\phi]$. □

Corollary 3.7. *Under the above assumptions, there is a local homeomorphism from $\mathrm{Hom}^{\mathrm{good}}_\tau(\Gamma,G)/G$ to $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau$, which restricts to a local diffeomorphism on the smooth part. If $|Z(G)|$ is odd, this local homeomorphism (diffeomorphism, respectively) is a homeomorphism (diffeomorphism, respectively).*

Proof. The first statement follows easily from Propositions 3.4 and 3.6. If $|Z(G)|$ is odd, we get $Z(G)/2Z(G) \cong \{0\}$ and $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau = \mathcal{N}_0^{\mathrm{good}}$ by Proposition 3.4. Furthermore, since $\{s \in Z(G) : s^2 = e\} = \{e\}$, the covering map in Proposition 3.6 is a bijection. □

The involution τ also acts on $\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)/G$. Let

$$\mathrm{Hom}_\tau^{\mathrm{irr}}(\Gamma, G) = \{\phi \in \mathrm{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)\}.$$

By the same idea used in the proof of Propositions 3.4 and 3.6, we get

Corollary 3.8. *If G is simple, there exists a decomposition*

$$(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\mathrm{irr}},$$

where $\mathcal{N}_r^{\mathrm{irr}} = \mathcal{N}_r^{\mathrm{good}} \cap (\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)/G)^\tau$. Furthermore, there exists a Galois covering map $R\colon \mathrm{Hom}_\tau^{\mathrm{irr}}(\Gamma, G)/G \to \mathcal{N}_0^{\mathrm{irr}}$ with structure group $\{s \in Z(G) : s^2 = e\}$. If $|Z(G)|$ is odd, then there is a bijection from $\mathrm{Hom}_\tau^{\mathrm{irr}}(\Gamma, G)/G$ to $(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma}, G)/G)^\tau$.

The results in this subsection show parts (1) and (2) of Theorem 1.2.

3.3. **The Betti moduli space associated to a non-orientable surface.** By subsection 3.2 or parts (1) and (2) of Theorem 1.2, we know that a representation $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$ such that $\tau[\phi] = [\phi]$ can be extended to one on Γ if an only if $L([\phi]) = 0$. When applied to $\Gamma = \pi_1(M)$ and $\tilde{\Gamma} = \pi_1(\tilde{M})$, where M is non-orientable and $\tilde{M}$ is its oriented cover, we conclude that a τ-invariant flat bundle over the $\tilde{M}$ corresponding to $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$ is the pull-back of a flat bundle over M if and only if $L([\phi]) = 0$. We now consider the example when $M = \Sigma$ is a compact non-orientable surface, in which case we can characterize all the components $\mathcal{N}_r^{\mathrm{good}}$ explicitly. The principal G-bundles on Σ are topologically classified by $H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$ whereas those on the oriented cover $\tilde{\Sigma}$ are classified by $H^2(\tilde{\Sigma}, \pi_1(G)) \cong \pi_1(G)$. The classes in these groups are the obstructions of lifting the structure group G of the bundles to its universal cover group.

A compact non-orientable surface Σ is of the form Σ_k^ℓ ($\ell \geq 0$, $k = 1, 2$), the connected sum of $2\ell + k$ copies of $\mathbb{R}P^2$. Then $\tilde{\Sigma}$ is a compact surface of genus $2\ell + k - 1$. For $k = 1$, we have

$$\pi_1(\Sigma) = \big\langle\, a_i, b_i \,(1 \leq i \leq \ell), c : c^{-2} \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \,\big\rangle,$$

$$\pi_1(\tilde{\Sigma}) = \big\langle\, a_i, b_i, a_i', b_i' \,(1 \leq i \leq \ell) : \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a_i', b_i'] \,\big\rangle.$$

The inclusion $\pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma)$ is given by $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a_i' \mapsto \mathrm{Ad}_c\, b_i$, $b_i' \mapsto \mathrm{Ad}_c\, a_i$ $(1 \leq i \leq \ell)$. For $k = 2$, we have

$$\pi_1(\Sigma) = \big\langle\, a_i, b_i \,(1 \leq i \leq \ell), c, d : d^{-1} c d^{-1} c^{-1} \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \,\big\rangle,$$

$$\pi_1(\tilde{\Sigma}) = \big\langle a_0, b_0, a_i, b_i, a_i', b_i' \,(1 \leq i \leq \ell) : [a_0, b_0] \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a_i', b_i'] \,\big\rangle.$$

The inclusion $\pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma)$ is given by $a_0 \mapsto d^{-1}$, $b_0 \mapsto c^2$, $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a_i' \mapsto \mathrm{Ad}_{d^{-1}c}\, b_i$, $b_i' \mapsto \mathrm{Ad}_{d^{-1}c}\, a_i$ $(1 \leq i \leq \ell)$. In both cases, $c \in \pi_1(\Sigma) \setminus \pi_1(\tilde{\Sigma})$.

While a flat G-bundle over Σ may be non-trivial, its pull-back to $\tilde{\Sigma}$ is always trivial topologically [19]. We assume that G is semi-simple, simply connected and denote $PG = G/Z(G)$. Then $\pi_1(PG) = Z(G)$ and we have $H^2(\Sigma, \pi_1(PG)) \cong Z(G)/2Z(G)$. The map

$$O\colon \mathrm{Hom}(\pi_1(\Sigma), PG)/PG \to Z(G)/2Z(G)$$

that gives the obstruction class can be explicitly described as follows [18]. Let $\phi \in \mathrm{Hom}(\pi_1(\Sigma), PG)$. For $k = 1$, let $\widetilde{\phi(a_i)}$, $\widetilde{\phi(b_i)}$, $\widetilde{\phi(c)} \in G$ be any lifts of $\phi(a_i)$, $\phi(b_i)$, $\phi(c) \in PG$, respectively. Then $O([\phi])$ is the element in $Z(G)/2Z(G)$ represented by $\widetilde{\phi(c)}^2 (\prod_{i=1}^{\ell} [\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}])^{-1} \in Z(G)$. (It is easy to check that the class in $Z(G)/2Z(G)$ is independent of the lifts.) The description of the case $k = 2$ is similar. Consequently, there is a decomposition

$$\mathrm{Hom}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r,$$

where $\mathcal{M}_r = O^{-1}(r)$.

Let $G \to PG$, $g \mapsto \bar{g}$ be the quotient map. Denote the induced map by $\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathrm{Hom}(\pi_1(\Sigma), PG)$, $\phi \mapsto \bar{\phi}$. In this section, we need to be restricted to Ad-irreducible representations. The reason is that ϕ is Ad-irreducible if and only if $\bar{\phi}$ is so, whereas if ϕ is good, $\bar{\phi}$ is not necessarily so and its stabilizer may be larger than $Z(G)$. We have

$$\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG = \bigcup_{r\in Z(G)/2Z(G)} \mathcal{M}_r^{\mathrm{irr}},$$

where $\mathcal{M}_r^{\mathrm{irr}} = \mathcal{M}_r \cap (\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG)$.

Lemma 3.9. *There is a natural map*

$$\Psi\colon (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

satisfying $L = O \circ \Psi$. Consequently, Ψ maps $\mathcal{N}_r^{\mathrm{irr}}$ to $\mathcal{M}_r^{\mathrm{irr}}$ for each $r \in Z(G)/2Z(G)$.

Proof. Given $[\phi] \in (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$, there exists $g \in G$ (which is unique up to $Z(G)$ since $G_\phi = Z(G)$) such that $\mathrm{Ad}_g \circ\phi = \phi \circ \mathrm{Ad}_c$. We define $\check{\phi} \in \mathrm{Hom}(\pi_1(\Sigma), PG)$ by $\check{\phi}|_{\pi_1(\tilde{\Sigma})} = \bar{\phi}$ and $\check{\phi}(c) = \bar{g}$. The representation $\check{\phi}$ is a homomorphism because $\check{\phi}(c)^2 = \bar{g}^2 = \bar{\phi}(c^2)$, which follows from the result $z = g^2\phi(c^2)^{-1} \in Z(G)$ in Proposition 3.4. Since $\bar{\phi} \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)$, we have $\check{\phi} \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)$. We define Ψ by $\Psi([\phi]) = [\check{\phi}]$. To show that $O([\check{\phi}]) = L([\phi]) = [z]$, we work in the case $k = 1$. By using the respective lifts $\phi(a_i)$, $\phi(b_i)$, $g \in G$ of $\check{\phi}(a_i)$, $\check{\phi}(b_i)$, $\check{\phi}(c) \in PG$, we get

$$O([\check{\phi}]) = [g^2(\prod_{i=1}^{\ell}[\phi(a_i), \phi(b_i)])^{-1}] = [g^2\phi(c^2)^{-1}] = [z],$$

where we have used the relation $\prod_{i=1}^{\ell}[\phi(a_i), \phi(b_i)] = c^2$ in $\pi_1(\tilde{\Sigma})$. The case $k = 2$ is similar. $\square$

Proposition 3.10. *The map*

$$\Psi\colon (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

is surjective. Consequently, $\Psi\colon \mathcal{N}_r^{\mathrm{irr}} \to \mathcal{M}_r^{\mathrm{irr}}$ is surjective for each $r \in Z(G)/2Z(G)$.

Proof. Let $[\phi] \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\Sigma, PG)/PG$. Although $\phi(c) \in PG$, $\mathrm{Ad}_{\phi(c)}$ acts on G. We show the case $k = 1$ only. Fix the lifts $\widetilde{\phi(a_i)}$, $\widetilde{\phi(b_i)} \in G$ of $\phi(a_i)$, $\phi(b_i) \in PG$. Define $\tilde{\phi} \in \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)$ by setting $\tilde{\phi}(a_i) = \widetilde{\phi(a_i)}$, $\tilde{\phi}(b_i) = \widetilde{\phi(b_i)}$, $\tilde{\phi}(a_i') = \mathrm{Ad}_{\phi(c)}\tilde{\phi}(b_i)$, $\tilde{\phi}(b_i') = \mathrm{Ad}_{\phi(c)}\tilde{\phi}(a_i)$, for $i = 1, \ldots, \ell$. This indeed defines a representation because

$$\prod_{i=1}^{\ell}[\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \prod_{i=1}^{\ell}[\tilde{\phi}(a_i'), \tilde{\phi}(b_i')] = \prod_{i=1}^{\ell}[\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \ \mathrm{Ad}_{\phi(c)} \prod_{i=1}^{\ell}[\tilde{\phi}(b_i), \tilde{\phi}(a_i)] = e.$$

The last equality is because $\prod_{i=1}^{\ell}[\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \in G$ projects to $\phi(c)^2 \in PG$. Since ϕ is Ad-irreducible, so is $\tilde{\phi}$. $[\tilde{\phi}]$ is τ-invariant because $\tilde{\phi}\circ\mathrm{Ad}_c = \mathrm{Ad}_{\phi(c)}\circ\tilde{\phi}$, which can be checked on the generators: $\tilde{\phi}(\mathrm{Ad}_c a_i) = \tilde{\phi}(b_i') = \mathrm{Ad}_{\phi(c)}\tilde{\phi}(a_i)$, $\tilde{\phi}(\mathrm{Ad}_c a_i') = \mathrm{Ad}_{\phi(c^2)}\tilde{\phi}(b_i) = \mathrm{Ad}_{\phi(c)}\tilde{\phi}(a_i')$, etc. It is then obvious that $\Psi([\tilde{\phi}]) = [\phi]$. $\square$

For the group PG, since $Z(PG)$ is trivial, $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$ does not decompose according to Proposition 3.4 and the map

$$\bar{R}\colon \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG \to (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$$

in Proposition 3.6 is bijective. The map Ψ is in fact the composition of $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$ (induced by $G \to PG$) followed by $\bar{R}^{-1}$. So for each $r \in Z(G)/2Z(G)$, the component $\mathcal{N}_r^{\mathrm{irr}}$ of the fixed point set $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$ corresponds precisely to the component $\mathcal{M}_r^{\mathrm{irr}}$ of $\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$ which consists of flat PG-bundles over Σ of topological type $r \in Z(G)/2Z(G)$. In particular, $\mathcal{N}_0^{\mathrm{irr}}$ corresponds to the component $\mathcal{M}_0^{\mathrm{irr}}$ of topologically trivial flat PG-bundles over Σ.

The results in subsection shows part (3) of Theorem 1.2.

4. Comparison of representation variety and gauge theoretical constructions

Suppose M is a compact non-orientable manifold, $\pi\colon \tilde{M} \to M$ is the oriented cover, and $\tau\colon \tilde{M} \to \tilde{M}$ is the non-trivial deck transformation. In subsection 2.2, we considered the natural lift of τ on $\tilde{P}^{\mathbb{C}} = \pi^*P^{\mathbb{C}}$, where $P^{\mathbb{C}}$ is a principal G-bundle over M. Such a lift, still denoted by τ, is a G-bundle map satisfying $\tau^2 = \mathrm{id}_{\tilde{P}^{\mathbb{C}}}$ and induces involutions on the space $\mathcal{A}(\tilde{P}^{\mathbb{C}})$ of connections on $\tilde{P}^{\mathbb{C}}$ and various moduli spaces. Moduli spaces associated to $P^{\mathbb{C}} \to M$ are then related to the τ-invariant parts of those associated to $\tilde{P}^{\mathbb{C}} \to \tilde{M}$ (cf. Theorem 1.1, especially part 3). This can also be seen in the language of representation varieties (cf. Lemma 3.5, Proposition 3.6 on $\mathcal{N}_0^{\mathrm{good}}$

and Corollary 3.7). To provide a geometric interpretation of the rest of the results in subsections 3.2 and 3.3 on $\mathcal{N}_r^{\mathrm{good}}$ or $\mathcal{N}_r^{\mathrm{irr}}$ when $r \neq 0$, we will need to generalize the setting in gauge theory.

Suppose $Q \to \tilde{M}$ is a principal G-bundle and the non-trivial deck transformation τ on $\tilde{M}$ is lifted to a bundle map τ_Q on Q, which is not necessarily an involution. Let A be an irreducible connection on Q that is invariant under τ_Q up to a gauge transformation, i.e., $\tau_Q^* A = \varphi^* A$ for $\varphi \in \mathcal{G}(Q)$. Since $(\tau_Q \circ \varphi^{-1})^2$ is a gauge transformation on Q which fixes A, it is in the center $Z(G)$. So by modifying τ_Q with a gauge transformation φ, we can assume that τ_Q satisfies $\tau_Q^2 = z \in Z(G)$. In this way, although τ_Q is not strictly an involution, it is so up to a gauge transformation, the right action of z on Q. Since φ and hence τ_Q can be adjusted by an element in $Z(G)$, $z = \tau_Q^2$ is well defined modulo $2Z(G)$. If $z = t^2 \in 2Z(G)$ $(t \in Z(G))$, then z can be absorbed in τ_Q by a redefinition such that τ_Q is an honest involution, and we are back to the situation before. In the general case when $\tau_Q^2 = z \in Z(G)$ is not the identity element, since $Z(G)$ acts trivially on the connections as gauge transformations, the action $\tau_Q^*\colon \mathcal{A}(Q) \to \mathcal{A}(Q)$ of τ_Q on connections is still an honest involution. So we can define the invariant subspace $\mathcal{A}(Q)^{\tau_Q}$ and much of the analysis in subsections 2.2 and 2.3 applies.

We now consider flat connections and relate this generalized setting to our results on representation varieties. Choose base points $x_0 \in M$ and $\tilde{x}_0 \in \pi^{-1}(x_0) \subset \tilde{M}$, and let $\mathit{\Gamma} = \pi_1(M, x_0)$, $\tilde{\mathit{\Gamma}} = \pi_1(\tilde{M}, \tilde{x}_0)$. We fix an element $c \in \mathit{\Gamma} \setminus \tilde{\mathit{\Gamma}}$.

Proposition 4.1. *For any $z \in Z(G)$, there is a 1-1 correspondence between the following two sets:*
(1) isomorphism classes of pairs (Q, A), where $Q \to \tilde{M}$ is a principal G-bundle with a G-bundle map τ_Q lifting the deck transformation τ on $\tilde{M}$ satisfying $\tau_Q^2 = z$, A is a τ_Q-invariant flat connection on Q
and
(2) equivalence classes of pairs (ϕ, g) under the diagonal adjoint action of G, where $\phi \in \mathrm{Hom}(\tilde{\mathit{\Gamma}}, G)$ and $g \in G$ satisfy $\phi \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi$ and $g^2 \phi(c^2)^{-1} = z$.

Proof. Given a bundle Q and a τ_Q-invariant flat connection A, let $T_\alpha\colon Q_{\alpha(0)} \to Q_{\alpha(1)}$ be the parallel transport along a path $\alpha\colon [0,1] \to \tilde{M}$. τ_Q-invariance of the connection implies $\tau_Q \circ T_\alpha = T_{\tau\circ\alpha} \circ \tau_Q$ for any path α. Let γ be a path in $\tilde{M}$ from $\tilde{x}_0$ to $\tau(\tilde{x}_0)$ so that $[\pi \circ \gamma] = c$. Choose $q_0 \in Q_{\tilde{x}_0}$ and let $g \in G$ be defined by $T_\gamma q_0 = \tau_Q(q_0) g^{-1}$. On the other hand, define $\phi \in \mathrm{Hom}(\tilde{\mathit{\Gamma}}, G)$ by $T_\alpha q_0 = q_0 \phi(a)^{-1}$ for any $a \in \tilde{\mathit{\Gamma}}$, where α is a loop in $\tilde{M}$ based at $\tilde{x}_0$ such that $[\alpha] = a$. To check the conditions on (ϕ, g), we note that $\tau_Q(T_\alpha q_0) = \tau_Q(q_0)\phi(a)^{-1}$ and

$$T_{\tau\circ\alpha}\tau_Q(q_0) = T_\gamma \circ T_{\gamma\cdot(\tau\circ\alpha)\cdot\gamma^{-1}}(q_0 g) = (T_\gamma q_0)\phi(\mathrm{Ad}_c\, a) g = \tau_Q(q_0)\, \mathrm{Ad}_g^{-1} \phi(\mathrm{Ad}_c\, a).$$

So τ_Q-invariance implies $\phi(\mathrm{Ad}_c\, a) = \mathrm{Ad}_g\, \phi(a)$ for all $a \in \tilde{\mathit{\Gamma}}$. Similar calculations give $\tau_Q(T_\gamma q_0) = \tau_Q(\tau_Q(q_0) g^{-1}) = q_0 z g^{-1}$ and $T_{\tau\circ\gamma}(\tau_Q q_0) = T_{\gamma\cdot(\tau\circ\gamma)}(q_0 g)$
$= q_0 \phi(c^2)^{-1} g$ which imply $g^2 \phi(c^2)^{-1} = z$. If another point $q_0' = q_0 h \in Q_{\tilde{x}_0}$ is chosen (where $h \in G$), then the resulting pair is $(\phi', g') = (\mathrm{Ad}_{h^{-1}} \circ \phi, \mathrm{Ad}_{h^{-1}}\, g)$.

Conversely, given a pair (ϕ, g) satisfying the conditions, we want to construct a bundle Q together with a lifting τ_Q of τ such that $\tau_Q^2 = z$ and a τ_Q-invariant flat connection on Q. Let $\hat{M}$ be the universal covering space of $\tilde{M}$ (and of M). Then $\tilde{\mathit{\Gamma}}$ and $\mathit{\Gamma}$ act on $\hat{M}$, and $\tilde{M} = \hat{M}/\tilde{\mathit{\Gamma}}$, $M = \hat{M}/\mathit{\Gamma}$. Let $Q = \hat{M} \times_{\tilde{\mathit{\Gamma}}} G$, that is, points in Q are equivalence classes $[(x, h)]$, where $x \in \hat{M}$ and $h \in G$, and $(xa, h) \sim (x, \phi(a)h)$ for any $a \in \tilde{\mathit{\Gamma}}$. Let $\tau_Q\colon Q \to Q$ be defined by $\tau_Q\colon [(x, h)] \mapsto [(xc^{-1}, gh)]$. To check that τ_Q is well-defined, we note that for any $a \in \tilde{\mathit{\Gamma}}$, $(xac^{-1}, gh) \sim (xc^{-1}, \phi(\mathrm{Ad}_c\, a) gh) = (xc^{-1}, g\phi(a)h)$. Clearly, τ_Q commutes with the right G-action on Q. Furthermore, $\tau_Q^2 = z$ because $\tau_Q^2\colon [(x, h)] \mapsto [(xc^{-2}, g^2 h)] = [(x, \phi(c^{-2}) g^2 h)] = [(x, h)]z$. It is easy to see that the trivial connection on $\hat{M} \times G$ is $\tilde{\mathit{\Gamma}}$-invariant and descends to a flat connection on Q. The latter is invariant under τ_Q since the trivial connection on $\hat{M} \times G$ is invariant under $(x, h) \mapsto (xc^{-1}, gh)$. Moreover, this connection induces the pair (ϕ, g). ☐

Remark 4.2. We explain the gauge theoretic perspective of the results in subsections 3.2 and 3.3 using the correspondence in Proposition 4.1.

1. As we noted, the τ is lifted to a G-bundle map τ_Q on $Q \to \tilde{M}$ such that $\tau_Q^2 = z \in Z(G)$, then z is determined up to $2Z(G)$. Likewise, $z = g^2 \phi(c^2)^{-1}$ is determined also modulo $2Z(G)$ by $[\phi] \in (\mathrm{Hom}^{\mathrm{good}}(\tilde{\mathit{\Gamma}}, G)/G)^\tau$ (Proposition 3.4). If $\tau_Q^2 = t^2$ for some $t \in Z(G)$, then τ_Q can be redefined as $\tau_Q' = \tau_Q t^{-1}$ so that $(\tau_Q')^2 = \mathrm{id}_Q$. We then have a G-bundle $Q/\tau_Q' \to M$ over the non-orientable manifold M whose pull-back of to $\tilde{M}$ is Q. If a flat connection is invariant under τ_Q, it is also invariant under τ_Q' and hence descends to a flat connection on Q/τ_Q'.

This is the situation in Lemma 3.5 and Proposition 3.6 (where Q/τ'_Q was $P^{\mathbb{C}}$). In fact, from these results, we see that $[z]\in Z(G)/2Z(G)$ is the obstruction to the existence of a flat G-bundle on M whose pull-back to $\tilde{M}$ is Q.
2. In general, $\tau_Q^2\neq\mathrm{id}_Q$ and the quotient of Q by the subgroup generated by τ_Q is a bundle over M with a fibre smaller than G. However, the PG-bundle $\bar{Q}:=Q/Z(G)$ over $\tilde{M}$ does have an honest involution $\tau_{\bar{Q}}$. So $\bar{Q}$ descends to a PG-bundle $\bar{Q}/\tau_{\bar{Q}}$ over M. Moreover, a τ_Q-invariant flat connection on Q descends to a $\tau_{\bar{Q}}$-invariant flat connection on $\bar{Q}$ and hence to a flat PG-connection on $\bar{Q}/\tau_{\bar{Q}}$. The bundle $\bar{Q}/\tau_{\bar{Q}}\to M$ is usually non-trivial as its structure group can not be lifted to G. (Otherwise, Q would be its pull-back to $\tilde{M}$ and would admit a lift τ_Q of τ so that $\tau_Q^2=\mathrm{id}_Q$.) Proposition 3.10 shows that when G is simply connected and when $M=\Sigma$ is a non-orientable surface, the topological type, i.e., the obstruction to lifting the PG-bundle $\bar{Q}/\tau_{\bar{Q}}$ to a G-bundle over M is precisely $[z]\in Z(G)/2Z(G)$.

Remark 4.3. 1. We can use $\tilde{x}_1=\tau(\tilde{x}_0)$ as an another base point of the fundamental group of $\tilde{M}$ so that $\tilde{x}_0$ and $\tilde{x}_1$ play symmetric roles. The image of $\pi_1(\tilde{\Sigma},\tilde{x}_1)$ under π_* can be identified with $\tilde{\Gamma}\subset\Gamma$. The isomorphism $\tau_*\colon\tilde{\Gamma}\to\pi_1(\tilde{\Sigma},\tilde{x}_1)\cong\tilde{\Gamma}$ is then $a\mapsto\mathrm{Ad}_c^{-1}a$. Having chosen $q_0\in Q_{\tilde{x}_0}$, let $q_1=\tau_Q(q_0)\in Q_{\tilde{x}_1}$ and define $\phi_1\colon\pi_1(\tilde{\Sigma},\tilde{x}_1)\to G$ by $T_\alpha q_1=q_1\phi_1([\alpha])^{-1}$, where α is a loop in $\tilde{\Sigma}$ based at $\tilde{x}_1$. Using the identity $\tau_Q\circ T_{\tau\circ\alpha}=T_\alpha\circ\tau_Q$, we obtain $\phi_1([\alpha])=\phi([\tau\circ\alpha])$. Since $\tau_Q^2=z$, we also have the identity $T_\gamma z=\tau_Q\circ T_{\tau\circ\gamma}\circ\tau_Q$. So upon the identification of $Q_{\tilde{x}_0}$ and $Q_{\tilde{x}_1}$ by τ_Q, the parallel transports along γ and $\tau\circ\gamma$ differ by z.
2. When $M=\Sigma$ is a non-orientable surface, the approach of double base points was taken in [17, 19]. Consider for example the case $M=\Sigma_1^\ell$. Let α_i, β_i $(1\le i\le\ell)$ be loops in the oriented cover $\tilde{\Sigma}$ based at $\tilde{x}_0$ and let γ be a path in from $\tilde{x}_0$ to $\tilde{x}_1$ so that $[\pi\circ\alpha_i]=a_i$, $[\pi\circ\beta_i]=b_i$, $[\pi\circ\gamma]=c$. Then an element in $\mathcal{N}_r$ $(r=[z]\in Z(G)/2Z(G))$ can be represented by $(A_i,B_i,C;A'_i,B'_i,C')\in G^{4\ell+2}$ satisfying $A'_i=A_i$, $B'_i=B_i$, $C'=Cz$, where A_i,B_i,C,A'_i,B'_i,C' are the holonomies along the loops or paths $\alpha_i,\beta_i,\gamma,\tau\circ\alpha_i,\tau\circ\beta_i,\tau\circ\gamma$, $(1\le i\le\ell)$, respectively. By the above discussion, we have the pattern $A_i=\phi([\alpha_i])=\phi_1([\tau\circ\alpha_i])=A'_i$, $B_i=\phi([\beta_i])=\phi_1([\tau\circ\beta_i])=B'_i$, $(1\le i\le\ell)$, $C'=Cz$ as in [17, 19].

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