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HITCHIN'S EQUATIONS ON A NONORIENTABLE MANIFOLD

NAN-KUO HO, GRAEME WILKIN, AND SIYE WU

ABSTRACT. We define Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ for a principal bundle P , whose structure group is a compact semisimple Lie group K , over a compact non-orientable Riemannian manifold M . We use the Donaldson-Corlette correspondence, which identifies Hitchin's moduli space with the moduli space of flat $K^{\mathbb{C}}$ -connections, which remains valid when M is non-orientable. This enables us to study Hitchin's moduli space both by gauge theoretical methods and algebraically by using representation varieties. If the orientable double cover \tilde{M} of M is a Kähler manifold with odd complex dimension and if the Kähler form is odd under the non-trivial deck transformation τ on \tilde{M} , Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ of the pull-back bundle $\tilde{P} \rightarrow \tilde{M}$ has a hyper-Kähler structure and admits an involution induced by τ . The fixed-point set $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^{\tau}$ is symplectic or Lagrangian with respect to various symplectic structures on $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$. We show that there is a local diffeomorphism from $\mathcal{M}^{\text{Hitchin}}(P)$ to $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^{\tau}$. We compare the gauge theoretical constructions with the algebraic approach using representation varieties.

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1. INTRODUCTION

Let M be a compact orientable Riemannian manifold and let K be a connected compact Lie group. Given a principal K -bundle $P \rightarrow M$, let $\mathcal{A}(P)$ be the space of connections and let $\mathcal{G}(P)$ be the group of gauge transformations on P . Consider Hitchin's equations

$$(1.1) \quad F_A - \frac{1}{2}[\psi, \psi] = 0, \quad d_A \psi = 0, \quad d_A^* \psi = 0$$

on the pairs $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$. Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is the set of space of solutions (A, ψ) to (1.1) modulo $\mathcal{G}(P)$ [15, 29]. On the other hand, let $G = K^{\mathbb{C}}$ be the complexification of K and let $P^{\mathbb{C}} = P \times_K G$, which is a principal bundle with structure group G . The moduli space $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ of flat G -connections on $P^{\mathbb{C}}$, also known as the de Rham moduli space, is the space of flat reductive connections of $P^{\mathbb{C}}$ modulo $\mathcal{G}(P)^{\mathbb{C}} \cong \mathcal{G}(P^{\mathbb{C}})$. A theorem of Donaldson [8] and Corlette [7] states that the moduli spaces $\mathcal{M}^{\text{Hitchin}}(P)$ and $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ are homeomorphic. The smooth part of $\mathcal{M}^{\text{Hitchin}}(P)$ is a Kähler manifold with a complex structure \bar{J} induced by that on G .

Suppose in addition that M is a Kähler manifold. Then there is another complex structure \bar{I} on $\mathcal{M}^{\text{Hitchin}}(P)$ induced by that on M , and a third one given by $\bar{K} = \bar{I}\bar{J}$. The three complex structures $\bar{I}, \bar{J}, \bar{K}$ and their corresponding Kähler forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$ form a hyper-Kähler structure on (the smooth part of) $\mathcal{M}^{\text{Hitchin}}(P)$ [15, 29]. This hyper-Kähler structure comes from an infinite dimensional version of a hyper-Kähler quotient [16] of the tangent bundle $T\mathcal{A}(P)$, which is hyper-Kähler, by the action of $\mathcal{G}(P)$, which is Hamiltonian with respect to each of the Kähler forms $\omega_I, \omega_J, \omega_K$ on $T\mathcal{A}(P)$. When M is a compact orientable surface, Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is equal to the hyper-Kähler quotient $\mathcal{M}^{\text{HK}}(P) := T\mathcal{A}(P) \mathbin{\!/\!/\!/_0} \mathcal{G}(P)$ [15]. It plays an important role in mirror symmetry and geometric Langlands program [14, 21]. When M is higher dimensional, $\mathcal{M}^{\text{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\text{HK}}(P)$ [29].

For a compact Lie group K , the moduli space of flat K -connections on a compact orientable surface was already studied in a celebrated work of Atiyah and Bott [1]. When M is a compact, nonorientable surface, the moduli space of flat K -connections was studied in [17, 19] through an involution on the space of connections over its orientable double cover \tilde{M} , induced by lifting the deck transformation on \tilde{M} to the pull-back $\tilde{P} \rightarrow \tilde{M}$ of the given K -bundle $P \rightarrow M$ so that the quotient of \tilde{P} by the involution is the original bundle P itself. This involution acts trivially on the structure group K . If instead one considers an involution on the bundle over \tilde{M} that acts nontrivially on the fibers (such as the complex conjugation), then the fixed points give rise to the moduli space of real or quaternionic vector bundles over a real algebraic curve. This was studied thoroughly in [4, 27], for example when $K = U(n)$.

Key words and phrases. moduli spaces, non-orientable manifolds, symplectic and hyper-Kähler geometry, representation varieties.

In this paper, we study Hitchin's equations on a non-orientable manifold. Let M be a compact connected non-orientable Riemannian manifold and let $P \rightarrow M$ be a principal K -bundle over M , where K is a compact connected Lie group. The de Rham moduli space $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$, i.e., the moduli space of flat connections on $P^{\mathbb{C}}$, does not depend on the orientability of M . On the other hand, Hitchin's equations (1.1) on the pairs $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$ still make sense (see subsection 2.2). We define Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ as the quotient of the space of pairs (A, ψ) satisfying (1.1) by the group $\mathcal{G}(P)$ of gauge transformations on P . We explain that the homeomorphism $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ of Donaldson-Corlette remains valid when M is non-orientable (Theorem 2.2).

If the oriented cover \tilde{M} of M is a Kähler manifold, then for the pull-back bundle $\tilde{P} := \pi^*P$ over \tilde{M} , Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ is hyper-Kähler with complex structures $\bar{I}, \bar{J}, \bar{K}$ and Kähler forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. If the Kähler form ω on \tilde{M} satisfies $\tau^*\omega = -\omega$ (the complex dimension of \tilde{M} must be odd for τ to be orientation reversing), then τ induces an involution (still denoted by τ) on $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ that satisfies $\tau^*\bar{\omega}_I = -\bar{\omega}_I$, $\tau^*\bar{\omega}_J = \bar{\omega}_J$ and $\tau^*\bar{\omega}_K = -\bar{\omega}_K$. Consequently, the fixed-point set $(\mathcal{M}^{\text{Hitchin}}(\tilde{P}))^\tau$ is Lagrangian in $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ with respect to $\bar{\omega}_I, \bar{\omega}_K$ and symplectic with respect to $\bar{\omega}_J$. This is known as an (A,B,A)-brane in [21]. We discover that Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ (where M is non-orientable) is related to $(\mathcal{M}^{\text{Hitchin}}(\tilde{P}))^\tau$ by a local diffeomorphism. Our main results are summarized in the following main theorem. For simplicity, we restrict to certain smooth parts $\mathcal{M}^{\text{Hitchin}}(P)^\circ$, $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$ and $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ$ of the respective spaces (see subsection 2.3 for details).

Theorem 1.1. *Let M be a compact non-orientable manifold and let $\pi: \tilde{M} \rightarrow M$ be its oriented cover on which there is a non-trivial deck transformation τ . Let K be a compact connected Lie group. Given a principal K -bundle $P \rightarrow M$, let $\tilde{P} = \pi^*P$ be its pull-back to \tilde{M} . Suppose that \tilde{M} is a Kähler manifold of odd complex dimension and the Kähler form ω on \tilde{M} satisfies $\tau^*\omega = -\omega$. Then*

- (1) $\mathcal{M}^{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ //_0 \mathcal{G}(P)$, which is a symplectic quotient.
- (2) $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$ is Kähler and totally geodesic in $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$ with respect to $\bar{J}, \bar{\omega}_J$ and totally real and Lagrangian with respect to \bar{I}, \bar{K} and $\bar{\omega}_I, \bar{\omega}_K$.
- (3) there is a local Kähler diffeomorphism from $\mathcal{M}^{\text{Hitchin}}(P)^\circ$ to $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$.

The theorem of Donaldson and Corlette in the non-orientable setup (Theorem 2.2) enable us to identify Hitchin's moduli space associated to an orientable or non-orientable manifold with the moduli space of flat connections and therefore the representation varieties. Let Γ be a finitely generated group and let G be a connected complex semi-simple Lie group. The representation variety, $\text{Hom}(\Gamma, G) // G := \text{Hom}^{\text{red}}(\Gamma, G) / G$, is the quotient of the space of reductive homomorphisms from Γ to G by the conjugation action of G . When Γ is the fundamental group of a compact manifold M , the representation variety is also called the Betti moduli space of M ; it is homeomorphic to the union of the de Rham moduli spaces $\mathcal{M}^{\text{dR}}(P)$ associated to principal G -bundles $P \rightarrow M$ of various topology. When M is non-orientable, let $\tilde{\Gamma}$ be the fundamental group of the oriented cover \tilde{M} . Then there is a short exact sequence $1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 1$ and τ acts as an involution on the representation variety $\text{Hom}(\tilde{\Gamma}, G) // G$ (Lemma 3.3). We study the relation of representation varieties associated to Γ and $\tilde{\Gamma}$ from an algebraic point of view. Let $PG = G/Z(G)$, where $Z(G)$ is the center of G . Our main results are summarized in the following theorem.

Theorem 1.2. *Let G be a connected complex semi-simple Lie group. Let M be a compact non-orientable manifold and let \tilde{M} be its oriented cover on which there is a non-trivial deck transformation τ . Denote $\Gamma = \pi_1(M)$ and $\tilde{\Gamma} = \pi_1(\tilde{M})$ with some chosen base points. Then*

- (1) there exists a continuous map L from $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) / G)^\tau$ to $Z(G) / 2Z(G)$. Consequently, $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) / G)^\tau = \bigcup_{r \in Z(G) / 2Z(G)} \mathcal{N}_r^{\text{good}}$, where $\mathcal{N}_r^{\text{good}}$ is the preimage of $r \in Z(G) / 2Z(G)$.
- (2) there exists a $|Z(G) / 2Z(G)|$ -sheeted Galois covering map from $\text{Hom}_\tau^{\text{good}}(\Gamma, G) / G$ to $\mathcal{N}_0^{\text{good}}$.

In particular, if $|Z(G)|$ is odd, then there exists a bijection from $\text{Hom}_\tau^{\text{good}}(\Gamma, G) / G$ to $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G) / G)^\tau$. The above statements are true if Hom^{good} is replaced by Hom^{irr} .

If in addition $M = \Sigma$ is a compact non-orientable surface and G is simple and simply connected, then

- (3) there exists a surjective map from $(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G) / G)^\tau$ to $\text{Hom}_\tau^{\text{irr}}(\Gamma, PG) / PG$ that maps $\mathcal{N}_r^{\text{irr}}$ to flat PG -bundles on Σ whose topological type is given by $r \in Z(G) / 2Z(G) \cong H^2(\Sigma, Z(G))$. In particular, $\mathcal{N}_0^{\text{irr}}$ maps to the topologically trivial flat PG -bundles on Σ .

Here Hom^{good} , following the terminology of [20], denotes the “good” part of the space of homomorphisms that are reductive and whose stabilizer is $Z(G)$, whereas Hom^{irr} is the space of homomorphisms whose composition with the adjoint representation of G is an irreducible representation (see subsection 3.1 for details). $\text{Hom}_\tau^{\text{good}}(\Gamma, G)$ is the set of homomorphisms from Γ to G whose restriction to $\tilde{\Gamma}$ is “good”. $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$ is not smooth in general, but contains a smooth part $(\mathcal{M}^{\text{flat}}(P^{\mathbb{C}}))^\circ$ (upon identification of moduli spaces). By parts (1) and (2) of the theorem, there is a local homeomorphism $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G \rightarrow (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ (see also Corollary 3.7), which in fact restricts to the local diffeomorphism $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ \rightarrow (\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$ in part (3) of Theorem 1.1 but is now more accurately described using representation varieties. Also, for $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ such that $[\phi] \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$ is fixed by τ , $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ . In the gauge-theoretic language, ϕ corresponds to a flat connection on \tilde{M} and represents a point fixed by τ in the de Rham moduli space $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})$, while extension of ϕ to Γ means that the flat connection on \tilde{M} is the pull-back of a flat connection on M . Flat connections on \tilde{M} that are not pull-backs from M correspond to flat PG -bundles over M (where $PG = G/Z(G)$). This is shown in part (3) of Theorem 1.2 and then discussed in greater generality in the last section.

For example, let $G = SL(2, \mathbb{C})$, M a compact nonorientable surface and \tilde{M} its orientable double cover. Then $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$ is labeled by $Z(G)/2Z(G) = \mathbb{Z}_2$, i.e., $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau = \bigcup_{r \in \mathbb{Z}_2} \mathcal{N}_r^{\text{good}}$. An element of $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$ is mapped by map L in Theorem 1.2(1) (defined in Proposition 3.4) to the null element of \mathbb{Z}_2 if and only if it represents a flat connection on \tilde{M} that is the pull-back of a flat connection on M . The natural map from $\text{Hom}^{\text{good}}(\pi_1(M), G)/G$ to $(\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau$ is not surjective; it is a \mathbb{Z}_2 -sheeted Galois covering map onto $\mathcal{N}_0^{\text{good}}$, and $\mathcal{N}_1^{\text{good}}$ is not in the image. $\mathcal{N}_0^{\text{irr}}$ corresponds to the space of topologically trivial flat $PSL(2, \mathbb{C})$ -bundles over M while $\mathcal{N}_1^{\text{irr}}$ corresponds to that of topologically nontrivial flat $PSL(2, \mathbb{C})$ -bundles over M .

The rest of this paper is organized as follows. In Section 2, we review the basic setup in the orientable case and explain the Donaldson-Corlette theorem for bundles over non-orientable manifolds. We then study finite dimensional symplectic and hyper-Kähler manifolds with an involution and apply the results to the gauge theoretical setting to prove Theorem 1.1. In Section 3, we study flat G -connections by representation varieties. We show that a flat connection on M is reductive if and only if its pull-back to \tilde{M} is reductive. We then define the continuous map in part (1) of Theorem 1.2 and prove the rest of the theorem. In Section 4, we relate the components $\mathcal{N}_r^{\text{good}}$ ($r \neq 0$) in Theorem 1.2 to G -bundles over \tilde{M} admitting an involution up to $Z(G)$.

We note that in order to study the moduli space of G -bundles over the nonorientable manifold M itself, our involution is fixed-point free on \tilde{M} and is the identity map on G . During the revision of this paper, we came across a few related works. We thank O. García-Prada for pointing out to us the paper [3], where their anti-holomorphic involution acts both on the manifold \tilde{M} and on the structure group G , thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with a special case of part (2) of our Theorem 1.1 when \tilde{M} is a surface, the anti-holomorphic involution on the surface is allowed to have fixed points.

2. THE GAUGE-THEORETIC PERSPECTIVE

2.1. Basic setup in the orientable case. Let K be a connected compact Lie group and let $G = K^{\mathbb{C}}$ be its complexification. Given a principal K -bundle P over a compact orientable manifold M , $P^{\mathbb{C}} = P \times_K G$ is a principal bundle whose structure group is G . The set $\mathcal{A}(P)$ of connections on P is an affine space modeled on $\Omega^1(M, \text{ad } P)$. At each $A \in \mathcal{A}(P)$, the tangent space is $T_A \mathcal{A}(P) \cong \Omega^1(M, \text{ad } P)$. The total space of the tangent bundle over $\mathcal{A}(P)$ is $T\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1(M, \text{ad } P)$. At $(A, \psi) \in T\mathcal{A}(P)$, the tangent space is $T_{(A, \psi)} T\mathcal{A}(P) \cong \Omega^1(M, \text{ad } P)^{\oplus 2}$. There is a translation invariant complex structure J on $T\mathcal{A}(P)$ given by $J(\alpha, \varphi) = (\varphi, -\alpha)$. The space $T\mathcal{A}(P)$ can be naturally identified with $\mathcal{A}(P^{\mathbb{C}})$, the set of connections on $P^{\mathbb{C}} \rightarrow M$, via $(A, \psi) \mapsto A - \sqrt{-1}\psi$, under which J corresponds to the complex structure on $\mathcal{A}(P^{\mathbb{C}})$ induced by $G = K^{\mathbb{C}}$. The covariant derivative on $\Omega^\bullet(M, \text{ad } P^{\mathbb{C}})$ is $D := d_A - \sqrt{-1}\psi$, where d_A denotes the covariant derivative of $A \in \mathcal{A}(P)$ and ψ acts by bracket.

The group of gauge transformations on P is $\mathcal{G}(P) \cong \Gamma(M, \text{Ad } P)$. It acts on $\mathcal{A}(P)$ via $A \mapsto g \cdot A$, where $d_{g \cdot A} = g \circ d_A \circ g^{-1}$ and on $T\mathcal{A}(P)$ via $g: (A, \psi) \mapsto (g \cdot A, \text{Ad}_g \psi)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves J , there is a holomorphic $\mathcal{G}(P)^{\mathbb{C}}$ action on $(T\mathcal{A}(P), J)$. In fact, the complexification $\mathcal{G}(P)^{\mathbb{C}}$ can be naturally identified with $\mathcal{G}(P^{\mathbb{C}}) \cong \Gamma(M, \text{Ad } P^{\mathbb{C}})$, and the action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P)$ corresponds to the complex gauge transformations

on $\mathcal{A}(P^{\mathbb{C}})$, i.e., $g \in \mathcal{G}(P^{\mathbb{C}}): D \mapsto g \circ D \circ g^{-1}$. Let

$$\begin{aligned} \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) &= \{A - \sqrt{-1}\psi \in \mathcal{A}(P^{\mathbb{C}}) : F_{A - \sqrt{-1}\psi} = 0\} \\ &= \{(A, \psi) \in T\mathcal{A} : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0\} \end{aligned}$$

be the set of flat connections on $P^{\mathbb{C}}$. Since the vanishing of $F_{A - \sqrt{-1}\psi}$ is a holomorphic condition, $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$ is a complex subset of $\mathcal{A}(P^{\mathbb{C}})$; it is also invariant under $\mathcal{G}(P^{\mathbb{C}})$. The holonomy group $\text{Hol}(A)$ of $A \in \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$ can be identified as a subgroup of G , up to a conjugation in G . A flat connection A on $P^{\mathbb{C}}$ is *reductive* if the closure of $\text{Hol}(A)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\text{Hol}(A)$; let $\mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}})$ be the set of such. It can be shown that a flat connection is reductive if and only if its orbit under $\mathcal{G}(P^{\mathbb{C}})$ is closed [7]. The *de Rham moduli space*, or the moduli space of reductive flat connections on $P^{\mathbb{C}}$, is

$$\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) // \mathcal{G}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}}) / \mathcal{G}(P^{\mathbb{C}}).$$

It has an induced complex structure \bar{J} on its smooth part.

Assume that M has a Riemannian structure and choose an invariant inner product (\cdot, \cdot) on the Lie algebra \mathfrak{k} of K . Then there is a symplectic structure on $T\mathcal{A}(P)$, with which J is compatible, given by

$$(2.1) \quad \omega_J((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M (\varphi_2, \wedge * \alpha_1) - (\varphi_1, \wedge * \alpha_2),$$

where $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^{1,0}(M, \text{ad } P)$, such that $(T\mathcal{A}(P), \omega_J)$ is Kähler. The subset $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$ is Kähler in $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$. We identify the Lie algebra $\text{Lie}(\mathcal{G}(P)) \cong \Omega^0(M, \text{ad } P)$ with its dual by the inner product on $\Omega^0(M, \text{ad } P)$. The action of $\mathcal{G}(P)$ on $(T\mathcal{A}(P), \omega_J)$ is Hamiltonian, with moment map

$$(2.2) \quad \mu_J(A, \psi) = d_A^* \psi \in \Omega^0(M, \text{ad } P).$$

Let

$$\begin{aligned} \mathcal{A}^{\text{Hitchin}}(P) &= \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) \\ &= \{(A, \psi) \in T\mathcal{A} : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0, d_A^* \psi = 0\}, \end{aligned}$$

the set of pairs (A, ψ) satisfying Hitchin's equations (1.1), and let the quotient space $\mathcal{M}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{Hitchin}}(P) / \mathcal{G}(P)$ be *Hitchin's moduli space*. A theorem of Donaldson [8] and Corlette [7] states that if M is compact and if the structure group G is semisimple, then $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$.

Suppose that M is a compact Kähler manifold of complex dimension n and let ω be the Kähler form on M . Then there is a complex structure on $T\mathcal{A}(P)$ given by

$$I: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\alpha, -\varphi)) = \frac{1}{(n-1)!} \Lambda^{n-1}(*\alpha, -*\varphi),$$

where $(\alpha, \varphi) \in \Omega^1(M, \text{ad } P)^{\oplus 2} \cong T_{(A, \psi)}T\mathcal{A}(P)$ and the map

$$\Lambda: \Omega^{\bullet}(M, \text{ad } P) \rightarrow \Omega^{\bullet-2}(M, \text{ad } P)$$

is the contraction by ω . With respect to I , we have

$$T_{(A, \psi)}^{1,0}T\mathcal{A}(P) \cong \Omega^{0,1}(M, \text{ad } P^{\mathbb{C}}) \oplus \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$$

for any $(A, \psi) \in T\mathcal{A}(P)$. This complex structure I is compatible with a symplectic form ω_I on $T\mathcal{A}(P)$ given by

$$\omega_I((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \alpha_2) - (\varphi_1, \wedge \varphi_2)),$$

where $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^1(M, \text{ad } P)$. The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is also Hamiltonian with respect to ω_I and the moment map is

$$\mu_I(A, \psi) = \Lambda(F_A - \frac{1}{2}[\psi, \psi]) \in \Omega^0(M, \text{ad } P),$$

where $F_A \in \Omega^2(M, \text{ad } P)$ is the curvature of A . Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves I , there is a holomorphic $\mathcal{G}(P^{\mathbb{C}})$ action on $(T\mathcal{A}(P), I)$. For any $(A, \psi) \in T\mathcal{A}(P)$, write $\psi = \sqrt{-1}(\phi - \phi^*)$, where $\phi \in \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$, $\phi^* \in \Omega^{0,1}(M, \text{ad } P^{\mathbb{C}})$. Here $\phi \mapsto \phi^*$ is induced by the conjugation on $G = K^{\mathbb{C}}$ preserving the compact form K . Then $D = d_A - \sqrt{-1}\psi = D' + D''$, where $D' = \partial_A - \phi^*$, $D'' = \bar{\partial}_A + \phi$. The action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P) \cong \mathcal{A}(P^{\mathbb{C}})$ can be described by $g \in \mathcal{G}(P^{\mathbb{C}}): D'' \mapsto g \circ D'' \circ g^{-1}$.

Let $\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}})$ be the set of Higgs pairs (A, ϕ) , i.e., $A \in \mathcal{A}(P)$ and $\phi \in \Omega^{1,0}(M, \text{ad } P^{\mathbb{C}})$ satisfying $(D'')^2 = 0$, or

$$\bar{\partial}_A^2 = 0, \quad \bar{\partial}_A \phi = 0, \quad [\phi, \phi] = 0.$$

Then $\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}})$ is a Kähler subspace of $\mathcal{A}(P^{\mathbb{C}}) \cong T\mathcal{A}(P)$ respect to I . Let $\mathcal{A}^{\text{sst}}(P^{\mathbb{C}})$ be the set of semistable Higgs pairs and let $\mathcal{A}^{\text{pst}}(P^{\mathbb{C}})$ be the set polystable Higgs pairs. (The notions of stable, semistable and polystable Higgs pairs were introduced in [15, 30, 31].) The *moduli space of polystable Higgs pairs* or the *Dolbeault moduli space* is

$$\mathcal{M}^{\text{Dol}}(P^{\mathbb{C}}) = (\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\text{sst}}(P^{\mathbb{C}})) // \mathcal{G}(P^{\mathbb{C}}) = (\mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mathcal{A}^{\text{pst}}(P^{\mathbb{C}})) // \mathcal{G}(P^{\mathbb{C}}).$$

It has a complex structure induced by I . It can be shown [30, Lemma 1.1] that $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$. A theorem of Hitchin [15] and Simpson [29] states that if M is compact and Kähler and the bundle P has vanishing first and second Chern classes, then $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{Dol}}(P^{\mathbb{C}})$.

There is a third complex structure on $T\mathcal{A}(P)$ defined by

$$K = IJ = -JI: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} * (\omega^{n-1} \wedge (\varphi, \alpha)) = \frac{1}{(n-1)!} \Lambda^{n-1}(*\varphi, *\alpha),$$

which is compatible with the symplectic form

$$\omega_K((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \varphi_2) - (\alpha_2, \wedge \varphi_1)).$$

The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to ω_K and the moment map is

$$\mu_K(A, \psi) = \Lambda(d_A \psi) \in \Omega^0(M, \text{ad } P).$$

Moreover, the action preserves K and therefore extends to another holomorphic action of $\mathcal{G}(P)^{\mathbb{C}}$. The three complex structures I, J, K define a hyper-Kähler structure on $T\mathcal{A}(P)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to all three symplectic forms, we have a hyper-Kähler moment map $\mu = (\mu_I, \mu_J, \mu_K): T\mathcal{A}(P) \rightarrow (\Omega^0(M, \text{ad } P))^{\oplus 3}$. The *hyper-Kähler quotient* [16] is $\mathcal{M}^{\text{HK}}(P) = \mu^{-1}(0) / \mathcal{G}(P)$, with complex structures $\bar{I}, \bar{J}, \bar{K}$ and symplectic forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. By the theorems of Donaldson-Corlette and of Hitchin-Simpson, the Hitchin moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is a complex space with respect to both \bar{I} and \bar{J} . Therefore $\mathcal{M}^{\text{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\text{HK}}(P)$ [10, Theorem 8.3.1].

When $M = \Sigma$ is an orientable surface, $\Lambda: \Omega^2(\Sigma, \text{ad } P) \rightarrow \Omega^0(\Sigma, \text{ad } P)$ is an isomorphism. So $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\text{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$ coincides with $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$. Thus the moduli spaces $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Dol}}(P^{\mathbb{C}})$ coincide with the hyper-Kähler quotient $\mathcal{M}^{\text{HK}}(P)$ [15].

2.2. Moduli space of Hitchin's equations on a non-orientable manifold. Now suppose M is a compact non-orientable manifold. Let $\pi: \tilde{M} \rightarrow M$ be its oriented cover and let $\tau: \tilde{M} \rightarrow \tilde{M}$ be the non-trivial deck transformation. Given a principal K -bundle $P \rightarrow M$, let $\tilde{P} = \pi^*P \rightarrow \tilde{M}$ be its pull-back to \tilde{M} . Since $\pi \circ \tau = \pi$, the τ action can be lifted to $\tilde{P} = \tilde{M} \times_M P$ as a K -bundle involution (i.e., the lifted involution commutes with the right K -action on \tilde{P}), and hence to the associated bundles $\text{Ad } \tilde{P}$ and $\text{ad } \tilde{P}$. Consequently, τ acts on the space of connections $\mathcal{A}(\tilde{P})$ by pull-back $A \mapsto \tau^*A$ and on the group of gauge transformations $\mathcal{G}(\tilde{P})$ by $g \mapsto \tau^*g := \tau^{-1} \circ g \circ \tau$. The τ -invariant subsets are $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$ and $(\mathcal{G}(\tilde{P}))^\tau \cong \mathcal{G}(P)$. In fact, the inclusion map $\mathcal{A}(P) \hookrightarrow \mathcal{A}(\tilde{P})$ onto the τ -invariant part is the pull-back via π of connections on P to those on \tilde{P} . Since $\mathcal{A}(\tilde{P})$ is an affine space modeled on $\Omega^1(\tilde{M}, \text{ad } \tilde{P})$, the differential τ_* of $\tau: \mathcal{A}(\tilde{P}) \rightarrow \mathcal{A}(\tilde{P})$ can be identified with a linear involution on $\Omega^1(\tilde{M}, \text{ad } \tilde{P})$ given by $\alpha \mapsto \tau^*\alpha$.

A Riemannian metric on a non-orientable manifold M pulls back to a Riemannian metric on \tilde{M} . Assuming that M is compact, we define an inner product on the space $\Omega^\bullet(M)$ of differential forms on M by

$$\langle \alpha, \beta \rangle = \frac{1}{2} \int_{\tilde{M}} \pi^* \alpha \wedge \tilde{*} \pi^* \beta$$

for $\alpha, \beta \in \Omega^\bullet(M)$, where $\tilde{*}$ is the Hodge star operator on \tilde{M} . Alternatively, the Hodge star $*$ on M maps a form on M to one valued in the orientation line bundle over M , and if α, β are of the same degree, then $\alpha \wedge *\beta$ is a top-degree form on M valued in the orientation line bundle, which can be integrated over M . We still have $\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$. More generally, there is an inner product on the space $\Omega^\bullet(M, \text{ad } P)$ of forms valued in $\text{ad } P$. Therefore $\mathcal{A}(P)$ admits a Riemannian structure, which is half of the restriction of the Riemannian structure on $\mathcal{A}(\tilde{P})$ to the τ -invariant subspace $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$.

Consider the tangent bundle $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(\tilde{M}, \text{ad } \tilde{P})$ of $\mathcal{A}(\tilde{P})$. It has a τ -action given by $\tau: (A, \psi) \mapsto (\tau^*A, \tau^*\psi)$, which is holomorphic with respect to the complex structure J . Therefore the fixed point set $(T\mathcal{A}(\tilde{P}))^\tau \cong T\mathcal{A}(P)$ is a complex subspace in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. With respect to the induced Riemannian structure on $T\mathcal{A}(\tilde{P})$,

$\tau: TA(\tilde{P}) \rightarrow TA(\tilde{P})$ is an isometry. Since τ also acts holomorphically on $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong TA(\tilde{P})$, $(TA(\tilde{P}))^\tau$ is a Kähler and totally geodesic subspace in $TA(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \cong (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ is also Kähler and totally geodesic in $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$. We summarize the above discussion in the following lemma.

Lemma 2.1. *Given a compact non-orientable manifold M with oriented double cover $\pi: \tilde{M} \rightarrow M$ and a principal K -bundle $P \rightarrow M$, the non-trivial deck transformation τ on \tilde{M} lifts to an involution (also denoted by τ) on $\tilde{P} = \pi^*P$ and acts as involutions on the space of connections $\mathcal{A}(\tilde{P})$ and on $TA(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, the τ -invariant subspaces $\mathcal{A}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}(P^{\mathbb{C}})$ and $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})$ are Kähler and totally geodesic subspaces in $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong TA(\tilde{P})$ and $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$, respectively.*

On a non-orientable manifold M , we still have Hitchin's equations (1.1). Here d_A^* is defined as the (formal) adjoint of d_A with respect to the inner products on $\Omega^\bullet(M, \text{ad } P)$. Alternatively, d_A^* is the first order differential operator on M such that on any orientable open set in M , $d_A^* = *^{-1}d_A*$; the latter is actually independent of the choice of local orientation. Yet another but related way to explain the operator d_A^* is to consider the Hodge star operator $*$ on a non-orientable manifold M as a map from differential forms to those valued in the orientation bundle over M . Since the latter is a flat real line bundle, $d_A^* = *^{-1}d_A*$ maps $\Omega^1(M, \text{ad } P)$ to $\Omega^0(M, \text{ad } P)$. Finally, d_A^* can be defined as $(\pi^*)^{-1} \circ d_{\pi^*A}^* \circ \pi^*$. Here $d_{\pi^*A}^* = *^{-1}d_{\pi^*A}$ holds globally on \tilde{M} and $\pi^*: \Omega^\bullet(M, \text{ad } P) \rightarrow \Omega^\bullet(\tilde{M}, \text{ad } \tilde{P})$ is injective. Let

$$\mathcal{A}^{\text{Hitchin}}(P) := \{(A, \psi) \in TA : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A\psi = 0, d_A^*\psi = 0\}.$$

It is clear that $\mathcal{A}^{\text{Hitchin}}(P) = (\mathcal{A}^{\text{Hitchin}}(\tilde{P}))^\tau$.

The notion of reductive connections on P does not depend on the orientability of M , and we still have the moduli space of flat connections $\mathcal{M}^{\text{flat}}(P^{\mathbb{C}}) = \mathcal{A}^{\text{flat, red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}})$. Let $\mathcal{M}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{Hitchin}}(P)/\mathcal{G}(P)$ be Hitchin's moduli space. The following is the Donaldson-Corlette theorem that also applies to the case when M is non-orientable. Equivalently, there exists a unique reduction of structure group from G to K admitting a solution to Hitchin's equations.

Theorem 2.2. *Let M be a compact non-orientable Riemannian manifold. Then for every reductive flat connection D on $P^{\mathbb{C}}$, there exists a gauge transformation $g \in \mathcal{G}(P^{\mathbb{C}})$ (unique up to $\mathcal{G}(P)$ and the stabilizer of D) such that $g \cdot D = d_A - \sqrt{-1}\psi$ with $(A, \psi) \in \mathcal{A}^{\text{Hitchin}}(P)$. As a consequence, we have a homeomorphism $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Hitchin}}(P)$.*

We now explain that Corlette's proof in [7] applies to the case when M is non-orientable. There is a symplectic form ω_J on $TA(P)$, still given by (2.1), which is half of the restriction of the symplectic form on $TA(\tilde{P})$. The action of $\mathcal{G}(P)$ on $TA(P)$ is Hamiltonian, and the moment map remains (2.2). Recall Corlette's flow equations on the space of flat connections. Let $D = d_A - \sqrt{-1}\psi$ be a flat connection of the $G = K^{\mathbb{C}}$ bundle $P^{\mathbb{C}} \rightarrow M$. Then the flow equations are

$$(2.3) \quad \frac{\partial D}{\partial t} = -D\mu_J(D).$$

Equivalently, one can look for a flow of the form $g(t) \cdot D_0$ and solve for $g(t) \in \mathcal{G}(\tilde{P}^{\mathbb{C}})$ using (cf. [7, p. 369])

$$(2.4) \quad \frac{\partial g}{\partial t} g^{-1} = -\sqrt{-1}\mu_J(g \cdot D_0).$$

Corlette shows in [7] that we have existence and uniqueness of solutions to (2.3) and (2.4) for all time. If the initial condition is a reductive flat connection, then there is a sequence converging to a solution to $\mu_J(D) = 0$. Also, the limit is gauge equivalent to the initial flat reductive connection [7]. These arguments are valid when M is non-orientable.

We remark that Theorem 2.2 for non-orientable manifolds also follows from the result of the orientable double cover. A flat connection on P is reductive if and only if the pull-back π^*A is a flat reductive connection on \tilde{P} . (We defer the proof of this statement to Corollary 3.2.) For the bundle $\tilde{P} \rightarrow \tilde{M}$, it is easy to check that the right-hand sides of (2.3) and (2.4) define τ -invariant vector fields on $\mathcal{A}(\tilde{P}^{\mathbb{C}})$ and $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, respectively. Since the space $(\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ -invariant connections is closed in $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$ and the space $(\mathcal{G}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ -invariant gauge transformations is closed in $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, Corlette's results on the limit of the flow restrict to the τ -invariant subset as well. That is, the flow on the space of connections is contained in the τ -invariant subset and the limit is a τ -invariant solution to Hitchin's equation. Similarly, the gauge transformation relating to the initial condition is contained in the τ -invariant part of the group of gauge transformations, and the limit is τ -invariant.

2.3. The Hitchin moduli space and the hyper-Kähler quotient. Now consider a compact non-orientable manifold M . Suppose its oriented cover \tilde{M} is a Kähler manifold of complex dimension n . Let ω be the Kähler form on \tilde{M} . Throughout this subsection, we assume that n is odd and the deck transformation τ on \tilde{M} is an anti-holomorphic involution such that $\tau^*\omega = -\omega$. Then $\tau^*\omega^n = -\omega^n$, which is consistent with the requirement that τ is orientation reversing. The τ -action on $\mathcal{TA}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(M, \text{ad } P)$, $\tau: (A, \psi) \mapsto (\tau^*A, \tau^*\psi)$, is an isometry and its differential $\tau_*: \Omega^1(M, \text{ad } P)^{\oplus 2} \rightarrow \Omega^1(M, \text{ad } P)^{\oplus 2}$ is $\tau_*: (\alpha, \varphi) \mapsto (\tau^*\alpha, \tau^*\varphi)$. It is easy to see that $\tau_* \circ I = -I \circ \tau_*$ since τ reverses the orientation of M and that $\tau_* \circ K = -K \circ \tau_*$ since $K = IJ$. So τ acts as an anti-holomorphic involution with respect to both I and K , and $\tau^*\omega_I = -\omega_I$, $\tau^*\omega_K = -\omega_K$. Moreover, since the moment maps μ_I and μ_K on $\mathcal{TA}(\tilde{P})$ involve the contraction \lrcorner by ω , they satisfy $\tau^*(\mu_I(A, \psi)) = -\mu_I(\tau^*A, \tau^*\psi)$, $\tau^*(\mu_K(A, \psi)) = -\mu_K(\tau^*A, \tau^*\psi)$ for all $(A, \psi) \in \mathcal{TA}(\tilde{P})$. The fixed point set $(\mathcal{A}(\tilde{P}))^\tau$ is totally real with respect to the complex structures I and K , and Lagrangian with respect to the symplectic forms ω_I and ω_K [24, 9, 25].

A flat connection $D = d_A - \sqrt{-1}\psi$ on $\tilde{P}^{\mathbb{C}}$ defines an elliptic complex with $D_i: \Omega^i(\tilde{M}, \text{ad } \tilde{P}^{\mathbb{C}}) \rightarrow \Omega^{i+1}(\tilde{M}, \text{ad } \tilde{P}^{\mathbb{C}})$. Let $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ be the set of flat connections on $\tilde{P}^{\mathbb{C}}$ such that (i) the stabilizer under the $\mathcal{G}(\tilde{P}^{\mathbb{C}})$ action is $Z(G)$, and (ii) the linearization D_1 of the curvature map surjects onto $\ker D_2 \cap \Omega^2(\tilde{M}, [\text{ad } \tilde{P}^{\mathbb{C}}, \text{ad } \tilde{P}^{\mathbb{C}}])$. Notice that when M is a surface, condition (i) implies (ii). The method in [22] and [23, Chapter VII] shows that $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ is a smooth submanifold in $\mathcal{A}(\tilde{P}^{\mathbb{C}})$, and as the action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ on it is free, the subset $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ := (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mathcal{A}^{\text{flat,red}}(\tilde{P}^{\mathbb{C}}))/\mathcal{G}(\tilde{P}^{\mathbb{C}})$ is in the smooth part of the moduli space $\mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})$ (see also [11] from the point of view of representation varieties). The free action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ or $\mathcal{G}(\tilde{P})/Z(K)$ from condition (i) implies that 0 is a regular value of μ_J on $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$, and the subset $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ := \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0)/\mathcal{G}(\tilde{P})$ is in the smooth part of Hitchin's moduli space $\mathcal{M}^{\text{Hitchin}}(\tilde{P})$ [15]. By the Donaldson-Corlette theorem, we have the homeomorphism $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ \cong \mathcal{M}^{\text{dR}}(\tilde{P}^{\mathbb{C}})^\circ$.

On the other hand, for the non-orientable manifold M , let $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ = \{A \in \mathcal{A}(P^{\mathbb{C}}) : \pi^*A \in \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ\}$, $\mathcal{A}^{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{Hitchin}}(P) \cap \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ$.

Then $\mathcal{M}^{\text{Hitchin}}(P)^\circ := \mathcal{A}^{\text{Hitchin}}(P)^\circ/\mathcal{G}(P)$ is in the smooth part of $\mathcal{M}^{\text{Hitchin}}(P)$, but we will not consider here the smooth points of $\mathcal{M}^{\text{Hitchin}}(P)$ that are outside $\mathcal{M}^{\text{Hitchin}}(P)^\circ$. By Theorem 2.2 (the analog of the Donaldson-Corlette theorem for non-orientable manifolds), we have a homeomorphism between $\mathcal{M}^{\text{Hitchin}}(P)^\circ$ and $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ := (\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cap \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}})$.

We now study a general setting. Let (X, ω) be a finite dimensional symplectic manifold with a Hamiltonian action of a compact Lie group K and let $\mu: X \rightarrow \mathfrak{k}^*$ be the moment map. Suppose as in [25], that there are involutions σ on X and τ on K such that $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$ for all $k \in K$ and $x \in X$. Assume that X^σ is not empty. Then K^τ acts on X^σ . We note that τ acts on $\mathfrak{k}, \mathfrak{k}^*$, and K^τ is a closed Lie subgroup of K with Lie algebra \mathfrak{k}^τ . Contrary to [25], we assume that the action of (K, K^τ) on (X, X^σ) is symplectic, i.e., we have $\sigma^*\omega = \omega$ and $\sigma^*\mu = \tau\mu$. Then X^σ is a symplectic submanifold in X . Assume that 0 is a regular value of μ and that K acts on $\mu^{-1}(0)$ freely. Since σ preserves $\mu^{-1}(0)$, it descends to a symplectic involution $\bar{\sigma}$ on the (smooth) symplectic quotient $X//_0 K = \mu^{-1}(0)/K$ at level 0, and $(X//_0 K)^\sigma$ is a symplectic submanifold.

Lemma 2.3. *In the above setting, the action of K^τ on X^σ is Hamiltonian and the symplectic quotient is $X^\sigma//_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$. If $\mu^{-1}(0) \cap X^\sigma \neq \emptyset$, then there exists a symplectic local diffeomorphism from $X^\sigma//_0 K^\tau$ to $(X//_0 K)^\sigma$.*

Proof. Let $\mathfrak{k} = \mathfrak{k}^\tau \oplus \mathfrak{q}$ such that $\tau = \pm 1$ on $\mathfrak{k}^\tau, \mathfrak{q}$, respectively. It is clear that the action of K^τ on X^σ is Hamiltonian and the moment map μ_τ is the composition $X^\sigma \hookrightarrow X \rightarrow \mathfrak{k}^* \rightarrow (\mathfrak{k}^\tau)^*$. Since for any $x \in X^\sigma$, $\langle \mu(x), \mathfrak{q} \rangle = 0$, we get $\mu_\tau^{-1}(0) = \mu^{-1}(0) \cap X^\sigma = (\mu^{-1}(0))^\sigma$. By the assumptions, 0 is a regular value of μ_τ , the action of K^τ on $\mu_\tau^{-1}(0)$ is free, and the symplectic quotient is $X^\sigma//_0 K^\tau = (\mu^{-1}(0) \cap X^\sigma)/K^\tau$.

For any $x \in X^\sigma$, the map $\mathfrak{k} \rightarrow T_x X$ intertwines τ on \mathfrak{k} and σ on $T_x X$, and $T_x(K^\tau \cdot x) = (T_x(K \cdot x))^\sigma$. The inclusion $\mu_\tau^{-1}(0) \hookrightarrow \mu^{-1}(0)$ induces a natural map $X^\sigma//_0 K^\tau \rightarrow (X//_0 K)^\sigma$, whose differentiation at $[x]$ is, after natural symplectic isomorphisms, the linear map $(T_x \mu^{-1}(0))^\sigma / (T_x(K \cdot x))^\sigma \rightarrow (T_x \mu^{-1}(0)/T_x(K \cdot x))^\sigma$. The latter is clearly injective; to show surjectivity, we note that for any $V \in T_x \mu^{-1}(0)$, if $V + T_x(K \cdot x) \in (T_x \mu^{-1}(0)/T_x(K \cdot x))^\sigma$, then it is the image of $\frac{1}{2}(V + \sigma V) + (T_x(K \cdot x))^\sigma$. The map $X^\sigma//_0 K^\tau \rightarrow (X//_0 K)^\sigma$ is a local diffeomorphism; it is symplectic because the above linear map is so for each $x \in \mu_\tau^{-1}(0)$. \square

Now let X be a hyper-Kähler manifold with complex structures J_i and symplectic structures ω_i ($i = 1, 2, 3$). Suppose K acts on X and the action is Hamiltonian with respect to all ω_i . Let $\mu = (\mu_1, \mu_2, \mu_3): X \rightarrow (\mathfrak{k}^*)^{\oplus 3}$ be the hyper-Kähler moment map. Assume that there are involutions σ on X and τ on K such that $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$ for all $k \in K$ and $x \in X$ and $\sigma^* J_i = (-1)^i J_i$, $\sigma^* \omega_i = (-1)^i \omega_i$, $\sigma^* \mu_i = (-1)^i \tau \mu_i$ for $i = 1, 2, 3$. So the action of (K, K^τ) on (X, X^σ) is symplectic with respect to ω_2 (as above) and anti-symplectic with respect to ω_1, ω_3 (as in [25]). Then X^σ , if non-empty, is Kähler and totally geodesic in X with respect to J_2, ω_2 and is totally real and Lagrangian with respect to J_1, ω_1 and J_3, ω_3 . If 0 is a regular value of μ (i.e., 0 is a regular value of each μ_i) and that K acts on $\mu^{-1}(0)$ freely, then $X //_0 K = \mu^{-1}(0)/K$ is the (smooth) hyper-Kähler quotient at level 0, which has complex structures \bar{J}_i and symplectic structures $\bar{\omega}_i$ ($i = 1, 2, 3$) [16].

Proposition 2.4. *In the above setting, let $Y = \mu_1^{-1}(0) \cap \mu_3^{-1}(0)$. Then*

1. Y is a σ -invariant Kähler submanifold in X with respect to J_2, ω_2 and the symplectic quotient $Y^\sigma //_0 K^\tau = (\mu^{-1}(0))^\sigma / K^\tau$ is Kähler;
2. $(X //_0 K)^\sigma$ is Kähler and totally geodesic in $X //_0 K$ with respect to $\bar{J}_2, \bar{\omega}_2$ and is totally real and Lagrangian with respect to \bar{J}_1, \bar{J}_3 and $\bar{\omega}_1, \bar{\omega}_3$;
3. if $(\mu^{-1}(0))^\sigma \neq \emptyset$, there is a Kähler (with respect to $\bar{J}_2, \bar{\omega}_2$) local diffeomorphism $Y^\sigma //_0 K^\tau \rightarrow (X //_0 K)^\sigma$.

Proof. 1&3. Let $\mu_c = \mu_3 + \sqrt{-1}\mu_1: X \rightarrow \mathfrak{k}^{\mathbb{C}}$. Then μ_c is holomorphic with respect to J_2 and is equivariant under the action of K . Since 0 is a regular value of μ_c , $Y = \mu_c^{-1}(0)$ is a smooth Kähler submanifold in X on which the action of K is Hamiltonian. Applying Lemma 2.3 to Y , we conclude that the action of K^τ on Y^σ is Hamiltonian and that $(\mu^{-1}(0))^\sigma / K^\tau = (\mu_c^{-1}(0) \cap Y^\sigma) / K^\tau = Y^\sigma //_0 K^\tau$. Moreover, there is a local diffeomorphism from $Y^\sigma //_0 K^\tau$ to $(Y //_0 K)^\sigma = (X //_0 K)^\sigma$ which is symplectic. Since K^τ acts holomorphically on (Y^σ, J_2) , the symplectic quotient $Y^\sigma //_0 K^\tau$ is Kähler, and the above local diffeomorphism is also Kähler.

2. Since σ preserves $\mu^{-1}(0)$, it descends to an involution $\bar{\sigma}$ on $X //_0 K$ such that $\bar{\sigma}^* \bar{J}_i = (-1)^i \bar{J}_i$, $\bar{\sigma}^* \bar{\omega}_i = (-1)^i \bar{\omega}_i$ for $i = 1, 2, 3$. The result then follows. \square

We now prove Theorem 1.1.

Proof. 1&3. Note that $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ is a τ -invariant Kähler submanifold in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Following [1, 15], we can apply the method in Lemma 2.3 to $\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ on which τ acts preserving ω_J and J . Since τ also acts on $\mathcal{G}(\tilde{P})$ and $\mathcal{G}(P) \cong (\mathcal{G}(\tilde{P}))^\tau$, $\mathcal{G}(P)/Z(K)$ acts Hamiltonianly and freely on $\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cong (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$, which is Kähler with respect to J, ω_J . Thus $\mathcal{M}^{\text{Hitchin}}(P)^\circ = (\mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ \cap \mu_J^{-1}(0)) / \mathcal{G}(P) = \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})^\circ //_0 \mathcal{G}(P)$ is a symplectic quotient. Since the latter is non-empty, there is a local Kähler diffeomorphism $\mathcal{M}^{\text{Hitchin}}(P)^\circ \rightarrow (\mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})^\circ //_0 \mathcal{G}(\tilde{P}))^\tau = (\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$.

2. The space $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ with I, J, K is hyper-Kähler and the action of $\mathcal{G}(\tilde{P})$ is Hamiltonian with respect to $\omega_I, \omega_J, \omega_K$. Let $(\mu^{-1}(0))^\circ$ be the subset of $\mu^{-1}(0)$ on which $\mathcal{G}(\tilde{P})/Z(K)$ acts freely. Then $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ := (\mu^{-1}(0))^\circ / \mathcal{G}(\tilde{P})$ is the smooth part of the hyper-Kähler quotient $\mathcal{M}^{\text{HK}}(\tilde{P})$. The involutions τ on $\mathcal{A}(\tilde{P})$ and $\mathcal{G}(\tilde{P})$ satisfy the conditions of Proposition 2.4. So $(\mathcal{M}^{\text{HK}}(\tilde{P})^\circ)^\tau$ is Kähler and totally geodesic with respect to \bar{J} and $\bar{\omega}_J$, and totally real and Lagrangian with respect to \bar{I}, \bar{K} and $\bar{\omega}_I, \bar{\omega}_K$ in $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ$. If M is a non-orientable surface, then $\mu_I^{-1}(0) \cap \mu_K^{-1}(0) = \mathcal{A}^{\text{flat}}(\tilde{P}^{\mathbb{C}})$ which implies that $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ = \mathcal{M}^{\text{HK}}(\tilde{P})^\circ$. In general, $\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ$ is a τ -invariant hyper-Kähler submanifold in $\mathcal{M}^{\text{HK}}(\tilde{P})^\circ$. The results follow from $(\mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ)^\tau = \mathcal{M}^{\text{Hitchin}}(\tilde{P}) \cap (\mathcal{M}^{\text{HK}}(\tilde{P})^\circ)^\tau$. \square

3. THE REPRESENTATION VARIETY PERSPECTIVE

3.1. Representation variety and Betti moduli space. Let Γ be a finitely generated group and let G be a connected complex Lie group. Then G acts on $\text{Hom}(\Gamma, G)$ by the conjugate action on G . A representation $\phi \in \text{Hom}(\Gamma, G)$ is *reductive* if the closure of $\phi(\Gamma)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\phi(\Gamma)$; let $\text{Hom}^{\text{red}}(\Gamma, G)$ be the set of such. The condition $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$ is equivalent to the statement that the G -orbit $G \cdot \phi$ is closed [13]. It is also equivalent to the condition that the composition of ϕ with the adjoint representation of G is semi-simple (see [26, Section 3] and [28, Theorem 30]). The quotient

$$\text{Hom}(\Gamma, G) // G = \text{Hom}^{\text{red}}(\Gamma, G) / G$$

is known as the *representation variety* or *character variety*. A reductive representation $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$ is *good* [20] if its stabilizer $G_\phi = Z(G)$; let $\text{Hom}^{\text{good}}(\Gamma, G)$ be the set of such. On the other hand, $\phi \in \text{Hom}(\Gamma, G)$ is *Ad-irreducible* if its composition with the adjoint representation of G is an irreducible representation of Γ . Let $\text{Hom}^{\text{irr}}(\Gamma, G)$ be the set of such. Notice that this set is empty unless G is simple. Clearly, $\text{Hom}^{\text{irr}}(\Gamma, G) \subset \text{Hom}^{\text{good}}(\Gamma, G)$. In general, $\text{Hom}^{\text{good}}(\Gamma, G)/G$ may not be smooth, but it is so when Γ is the fundamental group of a compact orientable surface [28, Corollary 50].

Suppose M is a compact manifold and $P^{\mathbb{C}} \rightarrow M$ is a principal G -bundle over M . Choose a base point $x_0 \in M$ and let $\Gamma = \pi_1(M, x_0)$ be the fundamental group. Then $\text{Hom}(\Gamma, G)//G$ is known as the *Betti moduli space* [30], denoted by $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$. The identification $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$, which we recall briefly now, is well known. Given a flat connection, let $T_\alpha: P_{\alpha(0)} \rightarrow P_{\alpha(1)}$ be the parallel transport along a path α in M . Fix a point $p_0 \in P_{x_0}$ in the fibre over x_0 . For $a \in \pi_1(M, x_0)$, choose a loop α based at x_0 representing a , then $\phi(a)$ is the unique element in G defined by $T_\alpha(p_0) = p_0\phi(a)^{-1}$. If we choose another point in the fibre over x_0 , then ϕ differs by a conjugation. Finally, the flat connection is reductive if and only if the corresponding element in $\text{Hom}(\Gamma, G)$ is reductive. Upon identification of the de Rham moduli space $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})$ and the Betti moduli spaces $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}}) = \text{Hom}(\Gamma, G)//G$, the subset $\text{Hom}^{\text{good}}(\Gamma, G)/G$ contains the smooth part $\mathcal{M}^{\text{dR}}(P^{\mathbb{C}})^\circ$ introduced in subsection 2.3; they are equal when M is a compact orientable surface.

If M is non-orientable and $\pi: \tilde{M} \rightarrow M$ is the oriented cover, we choose a base point $\tilde{x}_0 \in \pi^{-1}(x_0)$ and let $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$. Then there is a short exact sequence

$$1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 1$$

and $\tilde{\Gamma}$ can be identified with an index 2 subgroup in Γ . In the rest of this section, we will study the relation of the representation varieties $\text{Hom}(\Gamma, G)//G$ and $\text{Hom}(\tilde{\Gamma}, G)//G$ or the Betti moduli spaces $\mathcal{M}^{\text{Betti}}(P^{\mathbb{C}})$ and $\mathcal{M}^{\text{Betti}}(\tilde{P}^{\mathbb{C}})$. Some of the results, when M is a compact non-orientable surface, appeared in [17], which used different methods.

We first establish a useful fact that was used in subsection 2.2.

Lemma 3.1. *Suppose Γ is a finitely generated group and $\tilde{\Gamma}$ is an index 2 subgroup in Γ . Let G be a connected, complex reductive Lie group. Then $\phi \in \text{Hom}(\Gamma, G)$ is reductive if and only if the restriction $\phi|_{\tilde{\Gamma}} \in \text{Hom}(\tilde{\Gamma}, G)$ is reductive.*

Proof. Recall that $\phi \in \text{Hom}(\Gamma, G)$ is reductive if and only if the composition $\text{Ad} \circ \phi$ is a semisimple representation on \mathfrak{g} . Similarly, $\phi|_{\tilde{\Gamma}}$ is reductive if and only if $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$ is semisimple. By $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}_2$ and [6], [5, Chap. 3, §9.8, Lemme 2], $\text{Ad} \circ \phi$ is semisimple if and only if $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$ is so. The result then follows. \square

Corollary 3.2. *Let G be a connected, complex reductive Lie group. Suppose P is a principal G -bundle over a compact non-orientable manifold M whose oriented cover is $\pi: \tilde{M} \rightarrow M$. Then a flat connection A on P is reductive if and only if the pull-back π^*A is a flat reductive connection on $\tilde{P} := \pi^*P$.*

3.2. Representation varieties associated to an index 2 subgroup. Let Γ be a finitely generated group and let $\tilde{\Gamma}$ be an index 2 subgroup in Γ . Let G be a connected complex Lie group and let $Z(G)$ be its center. For any $c \in \Gamma \setminus \tilde{\Gamma}$, we have $\text{Ad}_c|_{\tilde{\Gamma}} \in \text{Aut}(\tilde{\Gamma})$, and the class $[\text{Ad}_c|_{\tilde{\Gamma}}] \in \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$ is independent of the choice of c . So we have a homomorphism $\mathbb{Z}_2 \cong \{1, \tau\} \rightarrow \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$ given by $\tau \mapsto [\text{Ad}_c|_{\tilde{\Gamma}}]$.

Lemma 3.3. $\mathbb{Z}_2 \cong \{1, \tau\}$ acts on $\text{Hom}(\tilde{\Gamma}, G)//G$ and on $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$.

Proof. We define $\tau[\phi] = [\phi \circ \text{Ad}_c]$ for any $\phi \in \text{Hom}(\tilde{\Gamma}, G)$. The action is well-defined since if $[\phi'] = [\phi]$, i.e., $\phi' = \text{Ad}_g \circ \phi$ for some $g \in G$, then $\phi' \circ \text{Ad}_c = \text{Ad}_g \circ \phi \circ \text{Ad}_c \sim \phi \circ \text{Ad}_c$. The τ -action is independent of the choice of c because if $c' \in \Gamma \setminus \tilde{\Gamma}$ is another element, then $c'c^{-1} \in \tilde{\Gamma}$ and $\phi \circ \text{Ad}_{c'} = \text{Ad}_{\phi(c'c^{-1})} \circ (\phi \circ \text{Ad}_c) \sim \phi \circ \text{Ad}_c$. We do have a \mathbb{Z}_2 -action because $\tau^2[\phi] = [\phi \circ \text{Ad}_{c^2}] = [\text{Ad}_{\phi(c^2)} \circ \phi] = [\phi]$. Finally, if ϕ is in $\text{Hom}^{\text{red}}(\tilde{\Gamma}, G)$ or $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, then so is $\phi \circ \text{Ad}_c$. Thus τ acts on $\text{Hom}(\tilde{\Gamma}, G)//G$ and $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$. \square

Proposition 3.4. *There exists a continuous map*

$$(3.1) \quad L: (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau \rightarrow Z(G)/2Z(G).$$

So $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\text{good}}$, where $\mathcal{N}_r^{\text{good}} := L^{-1}(r)$.

Proof. If $\tau[\phi] = [\phi]$, then there exists $g \in G$ such that $\phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi$. Since $c^2 \in \tilde{\Gamma}$, we have $\text{Ad}_{g^2} \circ \phi = \phi \circ \text{Ad}_{c^2} = \text{Ad}_{\phi(c^2)} \circ \phi$. Thus $z := g^2 \phi(c^2)^{-1} \in G_\phi = Z(G)$. If $[\phi'] = [\phi]$, i.e., $\phi' = \text{Ad}_h \circ \phi$ for some $h \in G$, then $\phi' \circ \text{Ad}_c = \text{Ad}_{g'} \circ \phi'$ for $g' = \text{Ad}_h g$. Since $g'^2 = \text{Ad}_h g^2 = z \text{Ad}_h \phi(c^2) = z \phi'(c^2)$, we obtain $(g')^2 \phi'(c^2)^{-1} = z$.

If $\phi \circ \text{Ad}_{c'} = \text{Ad}_{g'} \circ \phi$ holds for different choices of $c' \in \Gamma \setminus \tilde{\Gamma}$ and $g' \in G$, then $z' = (g')^2 \phi(c'^2)^{-1} \in Z(G)$ from the above discussion. On the other hand, we have $\text{Ad}_{g^{-1}g'} \circ \phi = \text{Ad}_{\phi(c^{-1}c')} \circ \phi$ as $c^{-1}c' \in \tilde{\Gamma}$. This gives us $t := (g')^{-1} g \phi(c^{-1}c') \in G_\phi = Z(G)$. We get

$$\begin{aligned} t^2 (g')^2 &= (tg')^2 = g \phi(c^{-1}c') g \phi(c^{-1}c') = \text{Ad}_g \phi(c^{-1}c') g^2 \phi(c^{-1}c') \\ &= \phi(\text{Ad}_c(c^{-1}c')) z \phi(c^2) \phi(c^{-1}c') = \phi((c')^2) z, \end{aligned}$$

i.e., $z' z^{-1} = t^{-2} \in 2Z(G)$. So the map $L: [\phi] \mapsto [z] \in Z(G)/2Z(G)$ is well-defined.

Since $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, the element $[g] \in G/Z(G)$ is uniquely determined by and depends continuously on ϕ . Therefore $[z] \in Z(G)/2Z(G)$ depends continuously on $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$. \square

If $\phi \in \text{Hom}(\Gamma, G)$ satisfies $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, then $\phi \in \text{Hom}^{\text{good}}(\Gamma, G)$. However, $\phi \in \text{Hom}^{\text{good}}(\Gamma, G)$ does not imply $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$. Let

$$\text{Hom}_\tau^{\text{good}}(\Gamma, G) = \{\phi \in \text{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)\}.$$

We show that if $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$, then $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ .

Lemma 3.5. *The restriction $R: [\phi] \mapsto [\phi|_{\tilde{\Gamma}}]$ maps $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$ surjectively to $\mathcal{N}_0^{\text{good}}$.*

Proof. First, the image $\text{im}(R) \subset \mathcal{N}_0^{\text{good}}$ because for any $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$, $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ by definition, so $(\phi|_{\tilde{\Gamma}}) \circ \text{Ad}_c = \text{Ad}_{\phi(c)} \circ \phi|_{\tilde{\Gamma}} \sim \phi|_{\tilde{\Gamma}}$ and $L([\phi|_{\tilde{\Gamma}}]) = [\phi(c)^2 \phi(c^2)^{-1}] = 0$. We will show that in fact $\text{im}(R) = \mathcal{N}_0^{\text{good}}$. Let $\phi_0 \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ such that $\tau[\phi_0] = [\phi_0]$ and $L([\phi_0]) = 0$. Then there exist $g \in G$ and $t \in Z(G)$ such that $\phi_0 \circ \text{Ad}_c = \text{Ad}_g \circ \phi_0$ and $g^2 \phi(c^2)^{-1} = t^2$. We can extend ϕ_0 to $\phi \in \text{Hom}(\Gamma, G)$ which is uniquely determined by the requirements $\phi|_{\tilde{\Gamma}} = \phi_0$ and $\phi(c) = gt^{-1}$. Since $\phi_0 \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$ and therefore $[\phi_0] \in \text{im}(R)$. \square

Proposition 3.6. *$R: \text{Hom}_\tau^{\text{good}}(\Gamma, G)/G \rightarrow \mathcal{N}_0^{\text{good}}$ is a Galois covering map whose structure group is $\{s \in Z(G) : s^2 = e\}$.*

Proof. We define an action of $\{s \in Z(G) : s^2 = e\}$ on $\text{Hom}_\tau^{\text{good}}(\Gamma, G)$. For any such s and $\phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$, we define $s \cdot \phi$ by $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}}$ and $(s \cdot \phi)|_{\Gamma \setminus \tilde{\Gamma}} = s(\phi|_{\Gamma \setminus \tilde{\Gamma}})$ the group multiplication. It is clear that $s \cdot \phi \in \text{Hom}(\Gamma, G)$. Moreover, since $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, $s \cdot \phi \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$. Clearly, the action descends to a well-defined action on $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$ by $s \cdot [\phi] = [s \cdot \phi]$ preserving the fibres of R .

We show that this action is free. Suppose $s \cdot [\phi] = [\phi]$, then $s \cdot \phi = \text{Ad}_h \circ \phi$ for some $h \in G$. Since $\phi|_{\tilde{\Gamma}} = (s \cdot \phi)|_{\tilde{\Gamma}} = \text{Ad}_h \circ \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, we get $h \in Z(G)$ and hence $s \cdot \phi = \phi$. Then $s\phi(c) = \phi(c)$ implies $s = e$.

It remains to show that the action is transitive on each fibre of R . Let $[\phi], [\phi'] \in \text{Hom}_\tau^{\text{good}}(\Gamma, G)$ such that $R([\phi]) = R([\phi'])$. Then there exists an $h \in G$ such that $\phi'|_{\tilde{\Gamma}} = \text{Ad}_h \circ \phi|_{\tilde{\Gamma}}$. Thus

$$\begin{aligned} \text{Ad}_{\phi'(c)h} \circ (\phi|_{\tilde{\Gamma}}) &= \text{Ad}_{\phi'(c)} \circ (\phi'|_{\tilde{\Gamma}}) = (\phi'|_{\tilde{\Gamma}}) \circ \text{Ad}_c \\ &= \text{Ad}_h \circ (\phi|_{\tilde{\Gamma}}) \circ \text{Ad}_c = \text{Ad}_{h\phi(c)} \circ (\phi|_{\tilde{\Gamma}}). \end{aligned}$$

Hence $s := \phi(c)^{-1} h^{-1} \phi'(c) h \in Z(G)$ since $\phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$. Furthermore

$$s^2 = \phi(c)^{-1} s h^{-1} \phi'(c) h = \phi(c^{-2}) h^{-1} \phi'(c^2) h = \phi(c^{-2}) \phi(c^2) = e.$$

Since we have $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} = \text{Ad}_{h^{-1}} \circ (\phi'|_{\tilde{\Gamma}})$ and $(s \cdot \phi)(c) = s\phi(c) = \phi(c)s = (\text{Ad}_{h^{-1}} \circ \phi')(c)$, we get $s \cdot \phi = \text{Ad}_{h^{-1}} \circ \phi'$, or $[\phi'] = [s \cdot \phi]$. \square

Corollary 3.7. *Under the above assumptions, there is a local homeomorphism from $\text{Hom}_\tau^{\text{good}}(\Gamma, G)/G$ to $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$, which restricts to a local diffeomorphism on the smooth part. If $|Z(G)|$ is odd, this local homeomorphism (diffeomorphism, respectively) is a homeomorphism (diffeomorphism, respectively).*

Proof. The first statement follows easily from Propositions 3.4 and 3.6. If $|Z(G)|$ is odd, we get $Z(G)/2Z(G) \cong \{0\}$ and $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \mathcal{N}_0^{\text{good}}$ by Proposition 3.4. Furthermore, since $\{s \in Z(G) : s^2 = e\} = \{e\}$, the covering map in Proposition 3.6 is a bijection. \square

The involution τ also acts on $\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G$. Let

$$\text{Hom}_\tau^{\text{irr}}(\Gamma, G) = \{\phi \in \text{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)\}.$$

By the same idea used in the proof of Propositions 3.4 and 3.6, we get

Corollary 3.8. *If G is simple, there exists a decomposition*

$$(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\text{irr}},$$

where $\mathcal{N}_r^{\text{irr}} = \mathcal{N}_r^{\text{good}} \cap (\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau$. Furthermore, there exists a Galois covering map $R: \text{Hom}_\tau^{\text{irr}}(\Gamma, G)/G \rightarrow \mathcal{N}_0^{\text{irr}}$ with structure group $\{s \in Z(G) : s^2 = e\}$. If $|Z(G)|$ is odd, then there is a bijection from $\text{Hom}_\tau^{\text{irr}}(\Gamma, G)/G$ to $(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^\tau$.

The results in this subsection show parts (1) and (2) of Theorem 1.2.

3.3. The Betti moduli space associated to a non-orientable surface. By subsection 3.2 or parts (1) and (2) of Theorem 1.2, we know that a representation $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ such that $\tau[\phi] = [\phi]$ can be extended to one on Γ if and only if $L([\phi]) = 0$. When applied to $\Gamma = \pi_1(M)$ and $\tilde{\Gamma} = \pi_1(\tilde{M})$, where M is non-orientable and \tilde{M} is its oriented cover, we conclude that a τ -invariant flat bundle over the \tilde{M} corresponding to $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ is the pull-back of a flat bundle over M if and only if $L([\phi]) = 0$. We now consider the example when $M = \Sigma$ is a compact non-orientable surface, in which case we can characterize all the components $\mathcal{N}_r^{\text{good}}$ explicitly. The principal G -bundles on Σ are topologically classified by $H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$ whereas those on the oriented cover $\tilde{\Sigma}$ are classified by $H^2(\tilde{\Sigma}, \pi_1(G)) \cong \pi_1(G)$. The classes in these groups are the obstructions of lifting the structure group G of the bundles to its universal cover group.

A compact non-orientable surface Σ is of the form Σ_k^ℓ ($\ell \geq 0, k = 1, 2$), the connected sum of $2\ell + k$ copies of $\mathbb{R}P^2$. Then $\tilde{\Sigma}$ is a compact surface of genus $2\ell + k - 1$. For $k = 1$, we have

$$\begin{aligned} \pi_1(\Sigma) &= \langle a_i, b_i (1 \leq i \leq \ell), c : c^{-2} \prod_{i=1}^\ell [a_i, b_i] \rangle, \\ \pi_1(\tilde{\Sigma}) &= \langle a_i, b_i, a'_i, b'_i (1 \leq i \leq \ell) : \prod_{i=1}^\ell [a_i, b_i] \prod_{i=1}^\ell [a'_i, b'_i] \rangle. \end{aligned}$$

The inclusion $\pi_1(\tilde{\Sigma}) \rightarrow \pi_1(\Sigma)$ is given by $a_i \mapsto a_i, b_i \mapsto b_i, a'_i \mapsto \text{Ad}_c b_i, b'_i \mapsto \text{Ad}_c a_i$ ($1 \leq i \leq \ell$). For $k = 2$, we have

$$\begin{aligned} \pi_1(\Sigma) &= \langle a_i, b_i (1 \leq i \leq \ell), c, d : d^{-1} c d^{-1} c^{-1} \prod_{i=1}^\ell [a_i, b_i] \rangle, \\ \pi_1(\tilde{\Sigma}) &= \langle a_0, b_0, a_i, b_i, a'_i, b'_i (1 \leq i \leq \ell) : [a_0, b_0] \prod_{i=1}^\ell [a_i, b_i] \prod_{i=1}^\ell [a'_i, b'_i] \rangle. \end{aligned}$$

The inclusion $\pi_1(\tilde{\Sigma}) \rightarrow \pi_1(\Sigma)$ is given by $a_0 \mapsto d^{-1}, b_0 \mapsto c^2, a_i \mapsto a_i, b_i \mapsto b_i, a'_i \mapsto \text{Ad}_{d^{-1}c} b_i, b'_i \mapsto \text{Ad}_{d^{-1}c} a_i$ ($1 \leq i \leq \ell$). In both cases, $c \in \pi_1(\Sigma) \setminus \pi_1(\tilde{\Sigma})$.

While a flat G -bundle over Σ may be non-trivial, its pull-back to $\tilde{\Sigma}$ is always trivial topologically [19]. We assume that G is semi-simple, simply connected and denote $PG = G/Z(G)$. Then $\pi_1(PG) = Z(G)$ and we have $H^2(\Sigma, \pi_1(PG)) \cong Z(G)/2Z(G)$. The map

$$O: \text{Hom}(\pi_1(\Sigma), PG)/PG \rightarrow Z(G)/2Z(G)$$

that gives the obstruction class can be explicitly described as follows [18]. Let $\phi \in \text{Hom}(\pi_1(\Sigma), PG)$. For $k = 1$, let $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}, \widetilde{\phi(c)} \in G$ be any lifts of $\phi(a_i), \phi(b_i), \phi(c) \in PG$, respectively. Then $O([\phi])$ is the element in $Z(G)/2Z(G)$ represented by $\widetilde{\phi(c)}^2 (\prod_{i=1}^\ell [\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}])^{-1} \in Z(G)$. (It is easy to check that the class in $Z(G)/2Z(G)$ is independent of the lifts.) The description of the case $k = 2$ is similar. Consequently, there is a decomposition

$$\text{Hom}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r,$$

where $\mathcal{M}_r = O^{-1}(r)$.

Let $G \rightarrow PG, g \mapsto \bar{g}$ be the quotient map. Denote the induced map by $\text{Hom}(\pi_1(\Sigma), G) \rightarrow \text{Hom}(\pi_1(\Sigma), PG), \phi \mapsto \bar{\phi}$. In this section, we need to be restricted to Ad-irreducible representations. The reason is that ϕ is Ad-irreducible

if and only if $\bar{\phi}$ is so, whereas if ϕ is good, $\bar{\phi}$ is not necessarily so and its stabilizer may be larger than $Z(G)$. We have

$$\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r^{\mathrm{irr}},$$

where $\mathcal{M}_r^{\mathrm{irr}} = \mathcal{M}_r \cap (\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG)$.

Lemma 3.9. *There is a natural map*

$$\Psi: (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

satisfying $L = O \circ \Psi$. Consequently, Ψ maps $\mathcal{N}_r^{\mathrm{irr}}$ to $\mathcal{M}_r^{\mathrm{irr}}$ for each $r \in Z(G)/2Z(G)$.

Proof. Given $[\phi] \in (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$, there exists $g \in G$ (which is unique up to $Z(G)$ since $G_\phi = Z(G)$) such that $\mathrm{Ad}_g \circ \phi = \phi \circ \mathrm{Ad}_c$. We define $\check{\phi} \in \mathrm{Hom}(\pi_1(\Sigma), PG)$ by $\check{\phi}|_{\pi_1(\tilde{\Sigma})} = \bar{\phi}$ and $\check{\phi}(c) = \bar{g}$. The representation $\check{\phi}$ is a homomorphism because $\check{\phi}(c)^2 = \bar{g}^2 = \bar{\phi}(c^2)$, which follows from the result $z = g^2\phi(c^2)^{-1} \in Z(G)$ in Proposition 3.4. Since $\bar{\phi} \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)$, we have $\check{\phi} \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)$. We define Ψ by $\Psi([\phi]) = [\check{\phi}]$. To show that $O([\check{\phi}]) = L([\phi]) = [z]$, we work in the case $k = 1$. By using the respective lifts $\phi(a_i)$, $\phi(b_i)$, $g \in G$ of $\check{\phi}(a_i)$, $\check{\phi}(b_i)$, $\check{\phi}(c) \in PG$, we get

$$O([\check{\phi}]) = [g^2(\prod_{i=1}^{\ell} [\phi(a_i), \phi(b_i)])^{-1}] = [g^2\phi(c^2)^{-1}] = [z],$$

where we have used the relation $\prod_{i=1}^{\ell} [\phi(a_i), \phi(b_i)] = c^2$ in $\pi_1(\tilde{\Sigma})$. The case $k = 2$ is similar. \square

Proposition 3.10. *The map*

$$\Psi: (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

is surjective. Consequently, $\Psi: \mathcal{N}_r^{\mathrm{irr}} \rightarrow \mathcal{M}_r^{\mathrm{irr}}$ is surjective for each $r \in Z(G)/2Z(G)$.

Proof. Let $[\phi] \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\Sigma, PG)/PG$. Although $\phi(c) \in PG$, $\mathrm{Ad}_{\phi(c)}$ acts on G . We show the case $k = 1$ only. Fix the lifts $\widetilde{\phi(a_i)}$, $\widetilde{\phi(b_i)} \in G$ of $\phi(a_i)$, $\phi(b_i) \in PG$. Define $\tilde{\phi} \in \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)$ by setting $\tilde{\phi}(a_i) = \widetilde{\phi(a_i)}$, $\tilde{\phi}(b_i) = \widetilde{\phi(b_i)}$, $\tilde{\phi}(a'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(b_i)$, $\tilde{\phi}(b'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a_i)$, for $i = 1, \dots, \ell$. This indeed defines a representation because

$$\prod_{i=1}^{\ell} [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \prod_{i=1}^{\ell} [\tilde{\phi}(a'_i), \tilde{\phi}(b'_i)] = \prod_{i=1}^{\ell} [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \mathrm{Ad}_{\phi(c)} \prod_{i=1}^{\ell} [\tilde{\phi}(b_i), \tilde{\phi}(a_i)] = e.$$

The last equality is because $\prod_{i=1}^{\ell} [\tilde{\phi}(a_i), \tilde{\phi}(b_i)] \in G$ projects to $\phi(c)^2 \in PG$. Since ϕ is Ad-irreducible, so is $\tilde{\phi}$. $[\tilde{\phi}]$ is τ -invariant because $\tilde{\phi} \circ \mathrm{Ad}_c = \mathrm{Ad}_{\phi(c)} \circ \tilde{\phi}$, which can be checked on the generators: $\tilde{\phi}(\mathrm{Ad}_c a_i) = \tilde{\phi}(b'_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a_i)$, $\tilde{\phi}(\mathrm{Ad}_c a'_i) = \mathrm{Ad}_{\phi(c)^2} \tilde{\phi}(b_i) = \mathrm{Ad}_{\phi(c)} \tilde{\phi}(a'_i)$, etc. It is then obvious that $\Psi([\tilde{\phi}]) = [\phi]$. \square

For the group PG , since $Z(PG)$ is trivial, $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$ does not decompose according to Proposition 3.4 and the map

$$\bar{R}: \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG \rightarrow (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$$

in Proposition 3.6 is bijective. The map Ψ is in fact the composition of $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \rightarrow (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)/PG)^\tau$ (induced by $G \rightarrow PG$) followed by \bar{R}^{-1} . So for each $r \in Z(G)/2Z(G)$, the component $\mathcal{N}_r^{\mathrm{irr}}$ of the fixed point set $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$ corresponds precisely to the component $\mathcal{M}_r^{\mathrm{irr}}$ of $\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$ which consists of flat PG -bundles over Σ of topological type $r \in Z(G)/2Z(G)$. In particular, $\mathcal{N}_0^{\mathrm{irr}}$ corresponds to the component $\mathcal{M}_0^{\mathrm{irr}}$ of topologically trivial flat PG -bundles over Σ .

The results in subsection shows part (3) of Theorem 1.2.

4. COMPARISON OF REPRESENTATION VARIETY AND GAUGE THEORETICAL CONSTRUCTIONS

Suppose M is a compact non-orientable manifold, $\pi: \tilde{M} \rightarrow M$ is the oriented cover, and $\tau: \tilde{M} \rightarrow \tilde{M}$ is the non-trivial deck transformation. In subsection 2.2, we considered the natural lift of τ on $\tilde{P}^C = \pi^* P^C$, where P^C is a principal G -bundle over M . Such a lift, still denoted by τ , is a G -bundle map satisfying $\tau^2 = \mathrm{id}_{\tilde{P}^C}$ and induces involutions on the space $\mathcal{A}(\tilde{P}^C)$ of connections on \tilde{P}^C and various moduli spaces. Moduli spaces associated to $P^C \rightarrow M$ are then related to the τ -invariant parts of those associated to $\tilde{P}^C \rightarrow \tilde{M}$ (cf. Theorem 1.1, especially part 3). This can also be seen in the language of representation varieties (cf. Lemma 3.5, Proposition 3.6 on $\mathcal{N}_0^{\mathrm{good}}$

and Corollary 3.7). To provide a geometric interpretation of the rest of the results in subsections 3.2 and 3.3 on $\mathcal{N}_r^{\text{good}}$ or $\mathcal{N}_r^{\text{irr}}$ when $r \neq 0$, we will need to generalize the setting in gauge theory.

Suppose $Q \rightarrow \tilde{M}$ is a principal G -bundle and the non-trivial deck transformation τ on \tilde{M} is lifted to a bundle map τ_Q on Q , which is not necessarily an involution. Let A be an irreducible connection on Q that is invariant under τ_Q up to a gauge transformation, i.e., $\tau_Q^* A = \varphi^* A$ for $\varphi \in \mathcal{G}(Q)$. Since $(\tau_Q \circ \varphi^{-1})^2$ is a gauge transformation on Q which fixes A , it is in the center $Z(G)$. So by modifying τ_Q with a gauge transformation φ , we can assume that τ_Q satisfies $\tau_Q^2 = z \in Z(G)$. In this way, although τ_Q is not strictly an involution, it is so up to a gauge transformation, the right action of z on Q . Since φ and hence τ_Q can be adjusted by an element in $Z(G)$, $z = \tau_Q^2$ is well defined modulo $2Z(G)$. If $z = t^2 \in 2Z(G)$ ($t \in Z(G)$), then z can be absorbed in τ_Q by a redefinition such that τ_Q is an honest involution, and we are back to the situation before. In the general case when $\tau_Q^2 = z \in Z(G)$ is not the identity element, since $Z(G)$ acts trivially on the connections as gauge transformations, the action $\tau_Q^*: \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$ of τ_Q on connections is still an honest involution. So we can define the invariant subspace $\mathcal{A}(Q)^{\tau_Q}$ and much of the analysis in subsections 2.2 and 2.3 applies.

We now consider flat connections and relate this generalized setting to our results on representation varieties. Choose base points $x_0 \in M$ and $\tilde{x}_0 \in \pi^{-1}(x_0) \subset \tilde{M}$, and let $\Gamma = \pi_1(M, x_0)$, $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$. We fix an element $c \in \Gamma \setminus \tilde{\Gamma}$.

Proposition 4.1. *For any $z \in Z(G)$, there is a 1-1 correspondence between the following two sets:*

(1) *isomorphism classes of pairs (Q, A) , where $Q \rightarrow \tilde{M}$ is a principal G -bundle with a G -bundle map τ_Q lifting the deck transformation τ on \tilde{M} satisfying $\tau_Q^2 = z$, A is a τ_Q -invariant flat connection on Q*

and

(2) *equivalence classes of pairs (ϕ, g) under the diagonal adjoint action of G , where $\phi \in \text{Hom}(\tilde{\Gamma}, G)$ and $g \in G$ satisfy $\phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi$ and $g^2 \phi(c^2)^{-1} = z$.*

Proof. Given a bundle Q and a τ_Q -invariant flat connection A , let $T_\alpha: Q_{\alpha(0)} \rightarrow Q_{\alpha(1)}$ be the parallel transport along a path $\alpha: [0, 1] \rightarrow \tilde{M}$. τ_Q -invariance of the connection implies $\tau_Q \circ T_\alpha = T_{\tau \circ \alpha} \circ \tau_Q$ for any path α . Let γ be a path in \tilde{M} from \tilde{x}_0 to $\tau(\tilde{x}_0)$ so that $[\pi \circ \gamma] = c$. Choose $q_0 \in Q_{\tilde{x}_0}$ and let $g \in G$ be defined by $T_\gamma q_0 = \tau_Q(q_0)g^{-1}$. On the other hand, define $\phi \in \text{Hom}(\tilde{\Gamma}, G)$ by $T_\alpha q_0 = q_0 \phi(a)^{-1}$ for any $a \in \tilde{\Gamma}$, where α is a loop in \tilde{M} based at \tilde{x}_0 such that $[\alpha] = a$. To check the conditions on (ϕ, g) , we note that $\tau_Q(T_\alpha q_0) = \tau_Q(q_0)\phi(a)^{-1}$ and

$$T_{\tau \circ \alpha} \tau_Q(q_0) = T_\gamma \circ T_{\gamma \cdot (\tau \circ \alpha) \cdot \gamma^{-1}}(q_0 g) = (T_\gamma q_0) \phi(\text{Ad}_c a) g = \tau_Q(q_0) \text{Ad}_g^{-1} \phi(\text{Ad}_c a).$$

So τ_Q -invariance implies $\phi(\text{Ad}_c a) = \text{Ad}_g \phi(a)$ for all $a \in \tilde{\Gamma}$. Similar calculations give $\tau_Q(T_\gamma q_0) = \tau_Q(\tau_Q(q_0)g^{-1}) = q_0 z g^{-1}$ and $T_{\tau \circ \gamma}(\tau_Q q_0) = T_{\gamma \cdot (\tau \circ \gamma)}(q_0 g) = q_0 \phi(c^2)^{-1} g$ which imply $g^2 \phi(c^2)^{-1} = z$. If another point $q'_0 = q_0 h \in Q_{\tilde{x}_0}$ is chosen (where $h \in G$), then the resulting pair is $(\phi', g') = (\text{Ad}_{h^{-1}} \circ \phi, \text{Ad}_{h^{-1}} g)$.

Conversely, given a pair (ϕ, g) satisfying the conditions, we want to construct a bundle Q together with a lifting τ_Q of τ such that $\tau_Q^2 = z$ and a τ_Q -invariant flat connection on Q . Let \hat{M} be the universal covering space of \tilde{M} (and of M). Then $\tilde{\Gamma}$ and Γ act on \hat{M} , and $\tilde{M} = \hat{M}/\tilde{\Gamma}$, $M = \hat{M}/\Gamma$. Let $Q = \hat{M} \times_{\tilde{\Gamma}} G$, that is, points in Q are equivalence classes $[(x, h)]$, where $x \in \hat{M}$ and $h \in G$, and $(xa, h) \sim (x, \phi(a)h)$ for any $a \in \tilde{\Gamma}$. Let $\tau_Q: Q \rightarrow Q$ be defined by $\tau_Q: [(x, h)] \mapsto [(xc^{-1}, gh)]$. To check that τ_Q is well-defined, we note that for any $a \in \tilde{\Gamma}$, $(xac^{-1}, gh) \sim (xc^{-1}, \phi(\text{Ad}_c a)gh) = (xc^{-1}, g\phi(a)h)$. Clearly, τ_Q commutes with the right G -action on Q . Furthermore, $\tau_Q^2 = z$ because $\tau_Q^2: [(x, h)] \mapsto [(xc^{-2}, g^2h)] = [(x, \phi(c^{-2})g^2h)] = [(x, h)]z$. It is easy to see that the trivial connection on $\hat{M} \times G$ is $\tilde{\Gamma}$ -invariant and descends to a flat connection on Q . The latter is invariant under τ_Q since the trivial connection on $\hat{M} \times G$ is invariant under $(x, h) \mapsto (xc^{-1}, gh)$. Moreover, this connection induces the pair (ϕ, g) . \square

Remark 4.2. We explain the gauge theoretic perspective of the results in subsections 3.2 and 3.3 using the correspondence in Proposition 4.1.

1. As we noted, the τ is lifted to a G -bundle map τ_Q on $Q \rightarrow \tilde{M}$ such that $\tau_Q^2 = z \in Z(G)$, then z is determined up to $2Z(G)$. Likewise, $z = g^2 \phi(c^2)^{-1}$ is determined also modulo $2Z(G)$ by $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ (Proposition 3.4). If $\tau_Q^2 = t^2$ for some $t \in Z(G)$, then τ_Q can be redefined as $\tau'_Q = \tau_Q t^{-1}$ so that $(\tau'_Q)^2 = \text{id}_Q$. We then have a G -bundle $Q/\tau'_Q \rightarrow M$ over the non-orientable manifold M whose pull-back of to \tilde{M} is Q . If a flat connection is invariant under τ_Q , it is also invariant under τ'_Q and hence descends to a flat connection on Q/τ'_Q .

This is the situation in Lemma 3.5 and Proposition 3.6 (where Q/τ'_Q was $P^{\mathbb{C}}$). In fact, from these results, we see that $[z] \in Z(G)/2Z(G)$ is the obstruction to the existence of a flat G -bundle on M whose pull-back to \tilde{M} is Q .

2. In general, $\tau_Q^2 \neq \text{id}_Q$ and the quotient of Q by the subgroup generated by τ_Q is a bundle over M with a fibre smaller than G . However, the PG -bundle $\bar{Q} := Q/Z(G)$ over \tilde{M} does have an honest involution $\tau_{\bar{Q}}$. So \bar{Q} descends to a PG -bundle $\bar{Q}/\tau_{\bar{Q}}$ over M . Moreover, a τ_Q -invariant flat connection on Q descends to a $\tau_{\bar{Q}}$ -invariant flat connection on \bar{Q} and hence to a flat PG -connection on $\bar{Q}/\tau_{\bar{Q}}$. The bundle $\bar{Q}/\tau_{\bar{Q}} \rightarrow M$ is usually non-trivial as its structure group can not be lifted to G . (Otherwise, Q would be its pull-back to \tilde{M} and would admit a lift τ_Q of τ so that $\tau_Q^2 = \text{id}_Q$.) Proposition 3.10 shows that when G is simply connected and when $M = \Sigma$ is a non-orientable surface, the topological type, i.e., the obstruction to lifting the PG -bundle $\bar{Q}/\tau_{\bar{Q}}$ to a G -bundle over M is precisely $[z] \in Z(G)/2Z(G)$.

Remark 4.3. 1. We can use $\tilde{x}_1 = \tau(\tilde{x}_0)$ as another base point of the fundamental group of \tilde{M} so that \tilde{x}_0 and \tilde{x}_1 play symmetric roles. The image of $\pi_1(\tilde{\Sigma}, \tilde{x}_1)$ under π_* can be identified with $\tilde{\Gamma} \subset \Gamma$. The isomorphism $\tau_*: \tilde{\Gamma} \rightarrow \pi_1(\tilde{\Sigma}, \tilde{x}_1) \cong \tilde{\Gamma}$ is then $a \mapsto \text{Ad}_c^{-1} a$. Having chosen $q_0 \in Q_{\tilde{x}_0}$, let $q_1 = \tau_Q(q_0) \in Q_{\tilde{x}_1}$ and define $\phi_1: \pi_1(\tilde{\Sigma}, \tilde{x}_1) \rightarrow G$ by $T_\alpha q_1 = q_1 \phi_1([\alpha])^{-1}$, where α is a loop in $\tilde{\Sigma}$ based at \tilde{x}_1 . Using the identity $\tau_Q \circ T_{\tau \circ \alpha} = T_\alpha \circ \tau_Q$, we obtain $\phi_1([\alpha]) = \phi([\tau \circ \alpha])$. Since $\tau_Q^2 = z$, we also have the identity $T_\gamma z = \tau_Q \circ T_{\tau \circ \gamma} \circ \tau_Q$. So upon the identification of $Q_{\tilde{x}_0}$ and $Q_{\tilde{x}_1}$ by τ_Q , the parallel transports along γ and $\tau \circ \gamma$ differ by z .

2. When $M = \Sigma$ is a non-orientable surface, the approach of double base points was taken in [17, 19]. Consider for example the case $M = \Sigma_1^f$. Let α_i, β_i ($1 \leq i \leq \ell$) be loops in the oriented cover $\tilde{\Sigma}$ based at \tilde{x}_0 and let γ be a path in from \tilde{x}_0 to \tilde{x}_1 so that $[\pi \circ \alpha_i] = a_i$, $[\pi \circ \beta_i] = b_i$, $[\pi \circ \gamma] = c$. Then an element in \mathcal{N}_r ($r = [z] \in Z(G)/2Z(G)$) can be represented by $(A_i, B_i, C; A'_i, B'_i, C') \in G^{4\ell+2}$ satisfying $A'_i = A_i$, $B'_i = B_i$, $C' = Cz$, where $A_i, B_i, C, A'_i, B'_i, C'$ are the holonomies along the loops or paths $\alpha_i, \beta_i, \gamma, \tau \circ \alpha_i, \tau \circ \beta_i, \tau \circ \gamma$, ($1 \leq i \leq \ell$), respectively. By the above discussion, we have the pattern $A_i = \phi([\alpha_i]) = \phi_1([\tau \circ \alpha_i]) = A'_i$, $B_i = \phi([\beta_i]) = \phi_1([\tau \circ \beta_i]) = B'_i$, ($1 \leq i \leq \ell$), $C' = Cz$ as in [17, 19].

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