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# A NON-LINEAR PARABOLIC PDE WITH A DISTRIBUTIONAL COEFFICIENT AND ITS APPLICATIONS TO STOCHASTIC ANALYSIS

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Abstract. We consider a non-linear parabolic partial differential equation (PDE) on $\mathbb{R}^d$ with a distributional coefficient in the non-linear term. The distribution is an element of a Besov space with negative regularity and the non-linearity is of quadratic type in the gradient of the unknown. Under suitable conditions on the parameters we prove local existence and uniqueness of a mild solution to the PDE, and investigate properties like continuity with respect to the initial condition and blow-up times. We prove a global existence and uniqueness result assuming further properties on the non-linearity. To conclude we consider an application of the PDE to stochastic analysis, in particular to a class of non-linear backward stochastic differential equations with distributional drivers.

## 1. Introduction

In this paper we consider the following non-linear parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + F(\nabla u(t,x))b(t,x), & x \in \mathbb{R}^d, t \in (0,T] \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \tag{1}$$

where $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is the unknown, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a given (generalised) function and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a suitable initial condition. Here the gradient operator $\nabla$ and the Laplacian $\Delta$ refer to the space component. The term $F : \mathbb{R}^d \to \mathbb{R}$ is a non-linear map of quadratic type whose regularity will be specified below (see Assumption A1).

In this paper we are interested in the case when the coefficient $b$ is highly singular in the space component, in particular we will consider bounded functions of time taking values in a suitable class of Schwartz distributions, $b \in L^\infty([0,T]; \mathcal{C}^\beta(\mathbb{R}^d))$ for some $\beta \in (-1/2, 0)$. Here $\mathcal{C}^\beta$ is a Besov space whose exact definition will be recalled later.

The main motivation for looking at this kind of *rough* equations with singular coefficients comes from Physics. In recent years there has been a great interest in the study of stochastic partial differential equations (SPDEs), fuelled by the success of the theories of regularity structures by Hairer [16] and of paracontrolled distributions by Gubinelli and coauthors [11, 12, 14].

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These two theories allowed for the first time to study stochastic PDEs with very singular coefficients (such as the Kardar-Parisi-Zhang equation, see [15]) which posed long standing problems. Amongst the many papers in the area of stochastic PDEs that build on these ideas, we mention a series of recent ones on quasilinear stochastic PDEs [2, 10, 13, 24, 25] that may be of interest to the reader.

Also in the present paper we consider a quasilinear PDE, but a deterministic one, where one of the coefficients is *singular* because it is a distribution. This coefficient however, is *regular* enough to allow for *Young-type* products to be used. This approach is the same in spirit as [17, 18, 19, 22], where the authors look for solutions to linear and non-linear parabolic PDEs for which some of the coefficients are distributions that may arise as realisations of stochastic noises. The aim of these papers, as well as the present work, is to solve such PDEs with *classical* techniques and in particular without using any special properties of the coefficients that derive from the fact that they are the realization of a stochastic noise – hence avoiding to use the machinery mentioned above for SPDEs. This of course will result in restrictions on the (ir)regularity of the distributional coefficient $b$ (which would play the role of the space-time noise in the SPDEs context). In the present paper, the non-linearity $F$ is assumed to be continuously differentiable with Lipschitz partial derivatives, hence allowing for quadratic growth. To the best of our knowledge this is the first time that existence and uniqueness of mild solutions for (1) is studied in the literature. It may be worth emphasizing that the key technical difficulty is that the non-linearity involves the gradient of the unknown (as for example in the Burger's equation). This is different to [22] where the non-linearity involves the solution itself. In both cases, the non-linear term is 'multiplied' by the distributional coefficient.

Our main result is *local existence and uniqueness of a mild solution* in $C([0,T];\mathcal{C}^{\alpha+1})$, where $\alpha > 0$ depends on $\beta$ (see Assumption A2 below). Here local solution means either a solution with an arbitrary initial condition and a sufficiently small time $T$ (see Theorem 3.7) or with an arbitrary time $T$ but a sufficiently small (in norm) initial condition (see Theorem 3.10). Both theorems are proven with a fixed point argument and careful a-priori bounds on the quadratic non-linearity $F$. We also show continuity of the solution with respect to the initial condition (Proposition 3.12) and we start to investigate blow-up times for the solution (see Proposition 3.13).

The quadratic growth of the non-linearity $F$ is the main issue that prevents us from finding a global in time solution. Indeed, if we assume that $F$ is Lipschitz with sub-linear growth (see Assumption A4) then we can show *existence and uniqueness of a global mild solution* in $C^{\varepsilon}([0,T];\mathcal{C}^{\alpha+1})$ for some $\varepsilon > 0$ and for all times $T < \infty$ (see Theorem 4.7).

To conclude the paper we illustrate an application of PDE (1) to stochastic analysis, in particular to a class of non-linear backward stochastic differential equations (BSDEs) with singular coefficients. This example falls in the class of quadratic BSDEs and the novelty is the presence of a distributional coefficient in the so-called *driver* of the BSDE. The study of quadratic BSDEs has been initiated in 2000 by Kobylanski [23], while BSDEs with singular terms (mostly linear) have started gaining attention only recently, see e.g. [6, 7, 20, 21]. To the best of our knowledge, the only paper that deals with *singular* quadratic BSDEs is [8], but the singular term is a linear stochastic integral with respect to a rough function, unlike in the present paper where the singularity appears in the quadratic term.

The paper is organised as follows: In Section 2 we recall known results that will be needed later, including the definition of product between distributions and the definition of the function spaces used. In Section 3 we show useful properties of the integral operator appearing in the mild solution and show all necessary a priori bounds and contraction properties. Using those we prove the main result of local existence and uniqueness of a mild solution (Theorems 3.7 and 3.10). We also investigate continuity with respect to initial condition and blow-up of the solution. In Section 4 we study global existence and uniqueness of a mild solution (see Theorem 4.7) under more restrictive assumptions on the non-linearity. Finally in Section 5 we apply these results to stochastic analysis, and give a meaning and solve a class of non-linear BSDEs with distributional coefficients.

For ease of reading we collect here some of the function spaces used more often in this papers (and point the reader to the precise definition in the section below when needed). We have

- $C_T X := C([0,T]; X)$, that is the space of $X$-valued continuous functions defined on $[0,T]$ for any Banach space $X$, see Section 2
- $L^\infty_T X := L^\infty(0,T;X)$, that is the space of $X$-valued $L^\infty$-functions defined on $[0,T]$ for any Banach space $X$
- $\mathcal{C}^\gamma := B^\gamma_{\infty,\infty}$, where the Besov spaces $B^\gamma_{p,q}$ are defined in (2)
- $C_T\mathcal{C}^{\alpha+1}$ is then a particular case (often used below) and this is the space of continuous functions of time defined on $[0,T]$ taking values in the Besov space $\mathcal{C}^{\alpha+1}$
- $C^\varepsilon_T\mathcal{C}^{\alpha+1} := C^\varepsilon([0,T];\mathcal{C}^{\alpha+1})$ is the space of $\varepsilon$-Hölder continuous functions on $[0,T]$ taking values in the Besov space $\mathcal{C}^{\alpha+1}$, see Section 4

## 2. Preliminaries

### 2.1. **Fractional Sobolev spaces, semigroups and products.**

We start by recalling the definition of Besov spaces $B^\gamma_{p,q}$ on $\mathbb{R}^d$ for $\gamma \in \mathbb{R}$ and $1 < p, q \leq \infty$. For more details see for example Triebel [28, Section 1.1] or Gubinelli [12, Appendix A.1]. Let $\mathcal{S}'$ be the space of real valued Schwartz distributions on $\mathbb{R}^d$. We denote by $|\cdot|_d$ the Euclidean norm in $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$ we write $x \cdot y$ to denote the scalar product in $\mathbb{R}^d$. Let us consider a dyadic partition of unity $\{\phi_j, j \geq 0\}$ with the following properties: the zero-th element is such that

$$\phi_0(x) = 1 \text{ if } |x|_d \leq 1 \quad \text{and} \quad \phi_0(x) = 0 \text{ if } |x|_d \geq \frac{3}{2}$$

and the rest satisfies

$$\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x) \text{ for } x \in \mathbb{R}^d \text{ and } j \in \mathbb{N}.$$

We define

$$B^\gamma_{p,q} := \left\{ f \in \mathcal{S}' \, : \, \|f\|_{B^\gamma_{p,q}} := \left( \sum_{j=0}^\infty 2^{\gamma j q} \|(\phi_j \hat{f})^\vee\|^q_{L^p} \right)^{1/q} < \infty \right\}, \tag{2}$$

where $\hat{\cdot}$ and $()^\vee$ denote the Fourier transform and its inverse, respectively. If $q = \infty$ in (2) we consider the usual modification of the norm as follows

$$\|f\|_{B^\gamma_{p,\infty}} := \sup_j 2^{\gamma j} \|(\phi_j \hat{f})^\vee\|_{L^p}$$

In the special case where both $p = q = \infty$ in (2), we use a different notation for the Besov space, namely $\mathcal{C}^\gamma := B^\gamma_{\infty,\infty}$. The norm in this space will be denoted by $\|\cdot\|_\gamma$. Note that the norm depends on the choice of the dyadic partition of unity $\{\phi_j\}$ but the space $B^\gamma_{p,q}$ does not, and all norms defined with a different $\{\phi_j\}$ are equivalent. In the case when $0 < \gamma < 1$ we will sometimes use yet another equivalent norm in $\mathcal{C}^\gamma$ which is given by

$$\sup_{x \in \mathbb{R}^d} \left( |f(x)| + \sup_{0 < |h|_d \le 1} \frac{|f(x+h) - f(x)|}{|h|^\gamma_d} \right), \tag{3}$$

see [28, equation (1.22) with $m = 1$]. Note moreover that for a non-integer $\gamma > 0$, the space $\mathcal{C}^\gamma$ is the usual space of functions differentiable $m$ times (with $m$ being the highest integer smaller than $\gamma$), with bounded partial derivatives up to order $m$ and whose partial derivatives of order $m$ are $(\gamma - m)$-Hölder continuous (see [1, page 99]). On the other hand, if $\gamma < 0$ then the space $\mathcal{C}^\gamma$ contains distributions. Besov spaces are well suited to give a meaning to multiplication between distributions. Indeed using Bony's estimates (see [4]) one can show that for $f \in \mathcal{C}^\gamma$ and $g \in \mathcal{C}^\delta$ with $\gamma + \delta > 0$ and $\delta < 0$, then $fg$ exists as an element of $\mathcal{C}^\delta$ and

$$\|fg\|_\delta \le c \|f\|_\gamma \|g\|_\delta, \tag{4}$$

for some constant $c > 0$, see [12, Lemma 2.1] for more details and a proof.

For a Banach space $X$, let $C_T X := C([0,T]; X)$ denote the space of $X$-valued continuous functions of time. This is a Banach space endowed with the usual supremum norm

$$\|u\|_{C_T X} := \sup_{t \in [0,T]} \|u(t)\|_X$$

for $u \in C_T X$. On the same space $C_T X$ we consider a family of equivalent norms $\|\cdot\|^{(\rho)}_{C_T X}, \rho \ge 1$ given by

$$\|u\|^{(\rho)}_{C_T X} := \sup_{t \in [0,T]} e^{-\rho t} \|u(t)\|_X \tag{5}$$

for $u \in C_T X$. On the space $L^\infty_T X := L^\infty([0,T]; X)$, where $X$ is a Banach space, we consider the norm $\text{esssup}_{t \in [0,T]} \|f(t)\|_X$ for a function $f : [0,T] \to X$ and we denote it by $\|f\|_{L^\infty_T X}$.

It is useful to rewrite equation (1) as the following abstract Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = \Delta u(t) + F(\nabla u(t)) b(t) & \text{on } \mathbb{R}^d \times (0,T] \\ u(0) = u_0, \end{cases} \tag{6}$$

where now $u$ denotes a function of time with values in an infinite dimensional space that will be specified later. The same notation is applied to the field $b$. We are now ready to introduce explicitly the notion of solution of (1) considered in this paper.

**Definition 2.1.** *We say that $u \in C_T\mathcal{C}^{\alpha+1}$ is a mild solution of* (1) *or equivalently* (6) *if it satisfies the following integral equation*

$$u(t) = P_t u_0 + \int_0^t P_{t-s}\left(F(\nabla u(s))b(s)\right) \mathrm{d}s, \tag{7}$$

*where $\{P_t\}_{t\geq 0}$ is the heat semigroup acting on the product $F(\nabla u(s))b(s)$.*

The generator of $\{P_t\}_{t\geq 0}$ is the Laplacian $\Delta$ and the semigroup acts on $\mathcal{S}'$ but as an operator it can be restricted to $\mathcal{C}^\gamma$ for any $\gamma$. It is known that the heat semigroup $P_t$ enjoys useful properties as a mapping on the $\mathcal{C}^\gamma$-spaces, for example the well-known *Schauder's estimates* (see e.g. [12, Lemma A.8] or [5, Prop. 2.4]) recalled in the following. Let $\theta \geq 0$ and $\gamma \in \mathbb{R}$. For any $g \in \mathcal{C}^\gamma$ and $t > 0$ then $P_t g \in \mathcal{C}^{\gamma+2\theta}$ and

$$\|P_t g\|_{\gamma+2\theta} \leq c t^{-\theta} \|g\|_\gamma \tag{8}$$

and

$$\|(P_t - 1)g\|_{\gamma-2\theta} \leq c|t|^\theta \|g\|_\gamma. \tag{9}$$

2.2. **Assumptions.** We list here the main assumptions that we will use throughout the paper on the non-linear term $F$, on the parameters $\alpha, \beta$ and on the distributional term $b$.

- **A1: Assumption on non-linear term $F$.** *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a $\mathscr{C}^1$-function whose partial derivatives $\frac{\partial}{\partial x_i} F$ are Lipschitz with the same constant $L$ for all $i = 1, \dots, d$.*

Note that from Assumption A1 it follows that there exists a positive constant $l$ such that

$$\left| \frac{\partial F}{\partial x_i}(x) \right| \leq l(1 + |x|_d)$$

for all $i = 1, \dots, d$. The key example we have in mind is the *quadratic non-linearity* $F(x) = x^2$ (in dimension $d = 1$).

Using $F$ we define an operator $\mathrm{F}$ as follows: for any element $f \in \mathcal{C}^\alpha$ for some $\alpha > 0$ we define the function $\mathrm{F}(f)$ on $\mathbb{R}^d$ by

$$\mathrm{F}(f)(\cdot) := F(f(\cdot)). \tag{10}$$

- **A2: Assumption on parameters.** *We choose $0 < \alpha < 1$ and $\beta < 0$ such that $\max\{-\alpha, \alpha - 1\} < \beta$. In particular this implies $-\frac{1}{2} < \beta < 0$.*
- **A3: Assumption on $b$.** *We take $b \in L_T^\infty \mathcal{C}^\beta$.*

## 3. Solving the PDE

3.1. **On the non-linear term.** In this section we prove a technical result that will be key to control the non-linear term in equation (6) when applying a fixed point argument later on. We state and prove the result for the operator $\mathrm{F}$ applied to functions $f$ and $g$ with the same regularity as $\nabla u(s)$ will have.

**Proposition 3.1.** *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a non-linear function that satisfies Assumption A1. Then the operator* $\mathrm{F}$ *defined in* (10) *is a map*

$$\mathrm{F} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$$

*for any $\alpha \in (0,1)$. In particular if $\mathbf{0}$ denotes the zero-function then $\|\mathrm{F}(\mathbf{0})\|_\alpha = |F(0)|$. Moreover for $f, g : \mathbb{R}^d \to \mathbb{R}^d$ elements of $\mathcal{C}^\alpha$ component by component then we have*

$$\|\mathrm{F}(f) - \mathrm{F}(g)\|_\alpha \leq c(1 + \|f\|_\alpha^2 + \|g\|_\alpha^2)^{1/2} \|f - g\|_\alpha \tag{11}$$

*where the constant $c$ depends on $L, l$ and $d$.*

*Proof.* For simplicity of notation we will omit the brackets and sometimes write $\mathrm{F}f - \mathrm{F}g$ instead of $\mathrm{F}(f) - \mathrm{F}(g)$ for $f, g \in \mathcal{C}^\alpha$. We recall that a function is an element of $\mathcal{C}^\alpha$ if its norm is bounded. Moreover for $0 < \alpha < 1$ we can use the equivalent norm (3).

We want to bound

$$\begin{aligned} \|\mathrm{F}f - \mathrm{F}g\|_\alpha := & \sup_{x \in \mathbb{R}^d} |Ff(x) - Fg(x)| \\ & + \sup_{0 < |y|_d \leq 1} \sup_{x \in \mathbb{R}^d} \frac{|Ff(x+y) - Fg(x+y) - Ff(x) + Fg(x)|}{|y|_d^\alpha}. \end{aligned} \tag{12}$$

Using the $\mathscr{C}^1$ assumption on $F$, we have for $a, b \in \mathbb{R}^d$ and $\theta \in [0,1]$ that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} F(\theta a + (1-\theta)b) = \sum_{i=1}^d \frac{\partial}{\partial x_i} F(\theta a + (1-\theta)b)(a_i - b_i),$$

and so integrating from 0 to 1 in $\mathrm{d}\theta$ one has

$$F(a) - F(b) = \int_0^1 \nabla F(\theta a + (1-\theta)b) \, \mathrm{d}\theta \cdot (a - b).$$

Furthermore using the linear growth assumption on each component $\frac{\partial}{\partial x_i} F$ of $\nabla F$ and Jensen's inequality we get

$$\begin{aligned} |F(a) - F(b)| &\leq |a - b|_d \int_0^1 \left( \sum_{i=1}^d \left| \frac{\partial}{\partial x_i} F(\theta a + (1-\theta)b) \right|^2 \right)^{1/2} \mathrm{d}\theta \\ &\leq c|a - b|_d \int_0^1 \left( \sum_{i=1}^d l^2 (1 + |\theta a + (1-\theta)b|_d)^2 \right)^{1/2} \mathrm{d}\theta \\ &\leq c|a - b|_d \int_0^1 \left( \sum_{i=1}^d l^2 (1 + \theta^2 |a|^2 + (1-\theta)^2 |b|_d^2) \right)^{1/2} \mathrm{d}\theta \\ &\leq c\sqrt{d} l |a - b|_d (1 + |a|_d^2 + |b|_d^2)^{1/2}. \end{aligned} \tag{13}$$

Hence for the first term in (12) we get

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} |Ff(x) - Fg(x)| &\leq c \sup_{x \in \mathbb{R}^d} |f(x) - g(x)| (1 + |f(x)|_d^2 + |g(x)|_d^2)^{1/2} \\ &\leq c\|f - g\|_\alpha (1 + \|f\|_\alpha^2 + \|g\|_\alpha^2)^{1/2}. \end{aligned}$$

Let us now focus on the numerator appearing in the second term of (12). Inside the absolute value we use twice a computation similar to the one used above and add and subtract the same quantity to get

$$\begin{aligned}
&|Ff(x+y)-Ff(x)-Fg(x+y)+Fg(x)|\\
=&\Big|\int_0^1 \nabla F(\theta f(x+y)+(1-\theta)f(x))\mathrm{d}\theta\cdot(f(x+y)-f(x))\\
&-\int_0^1 \nabla F(\theta g(x+y)+(1-\theta)g(x))\mathrm{d}\theta\cdot(g(x+y)-g(x))\Big|_d\\
\leq&\int_0^1 |\nabla F(\theta f(x+y)+(1-\theta)f(x))|_d\,\mathrm{d}\theta\\
&|f(x+y)-f(x)-g(x+y)+g(x)|_d\\
&+\Big|\int_0^1 [\nabla F(\theta f(x+y)+(1-\theta)f(x))-\nabla F(\theta g(x+y)+(1-\theta)g(x))]\,\mathrm{d}\theta\\
&\cdot(g(x+y)-g(x))\Big|
\end{aligned}$$

The first term can be bounded similarly as in (13) by

$$c(1+\|f\|_\alpha^2)^{1/2}|f(x+y)-f(x)-g(x+y)+g(x)|_d.$$

For the second term above, we first observe that since $\frac{\partial}{\partial x_i}F:\mathbb{R}^d\to\mathbb{R}$ is Lipschitz by assumption for all $i$, then $\nabla F:\mathbb{R}^d\to\mathbb{R}^d$ is Lipschitz with constant $L\sqrt{d}$. Thus we get the upper bound

$$\begin{aligned}
&|g(x+y)-g(x)|_d\sqrt{d}L\\
&\int_0^1 |\theta f(x+y)+(1-\theta)f(x)-\theta g(x+y)-(1-\theta)g(x)|_d\,\mathrm{d}\theta\\
\leq& c|g(x+y)-g(x)|_d\|f-g\|_\alpha. \qquad (14)
\end{aligned}$$

Putting everything together for both terms in (12) we get the bound

$$\begin{aligned}
&\|\mathrm{F}f-\mathrm{F}g\|_\alpha\\
\leq& c\sup_{0<|y|_d\leq 1}\sup_{x\in\mathbb{R}^d}\Big[(1+\|f\|_\alpha^2)^{1/2}\frac{|f(x+y)-f(x)-g(x+y)+g(x)|_d}{|y|_d^\alpha}\\
&+\|f-g\|_\alpha\frac{|g(x+h)-g(x)|_d}{|y|_d^\alpha}\Big]\\
\leq& c(1+\|f\|_\alpha^2)^{1/2}\|f-g\|_\alpha+\|f-g\|_\alpha\|g\|_\alpha\\
\leq& c\|f-g\|_\alpha(1+\|f\|_\alpha^2+\|g\|_\alpha^2)^{1/2}
\end{aligned}$$

having used again the equivalent norm (3). This shows (11) and in particular that $\mathrm{F}f-\mathrm{F}g\in\mathcal{C}^\alpha$.

Let us denote by $k:=F(0)$. Then clearly $\mathrm{F}\mathbf{0}\equiv k$ and

$$\begin{aligned}
\|\mathrm{F}\mathbf{0}\|_\alpha &= \sup_{x\in\mathbb{R}^d}|(\mathrm{F}\mathbf{0})(x)|+\sup_{0<|y|_d\leq 1}\sup_{x\in\mathbb{R}^d}\frac{|(\mathrm{F}\mathbf{0})(x+y)-(\mathrm{F}\mathbf{0})(x)|}{|y|_d^\alpha}\\
&=\sup_{x\in\mathbb{R}^d}|k|+0\\
&=|k|.
\end{aligned}$$

Finally to show that F maps $\mathcal{C}^\alpha$ into itself it is enough to observe that

$$\|\mathrm{F}f\|_\alpha \leq \|\mathrm{F}f - \mathrm{F}\mathbf{0}\|_\alpha + |k|$$

and then the RHS of the above equation is finite by (11) hence $\mathrm{F}f \in \mathcal{C}^\alpha$ for all $f \in \mathcal{C}^\alpha$. □

3.2. **Existence and Uniqueness.** Let us denote by $J_t(u)$ the right-hand side of (7), more precisely

$$J_t(u) := P_t u_0 + I_t(u), \tag{15}$$

where the integral operator $I$ is given by

$$I_t(u) := \int_0^t P_{t-s}\left(\mathrm{F}(\nabla u(s)) b(s)\right) \mathrm{d}s \tag{16}$$

and the semigroup $P_{t-s}$ acts on the whole product $\mathrm{F}(\nabla u(s))b(s)$.

Using Schauder's estimates it is easy to show that $t \mapsto I_t(u)$ is continuous from $[0, T]$ to $\mathcal{C}^{\alpha+1}$. We show the result below for a general $f$ in place of $F(\nabla u(s))b(s)$. Note that the result might look not sharp because one normally gains 2 derivatives in parabolic PDEs when using semigroup theory (and possibly some time regularity too). Here we gain slightly less than 2 derivatives (we go from $\beta$ to $\alpha + 1$ and $\alpha + 1 - \beta < 2$) because we need the time singularities $t^{-\theta}$ and $t^{-\frac{\alpha+1-\beta}{2}}$ to be integrable. We will investigate the time regularity, that is, Hölder continuity in time of small order, later in Section 4.

**Lemma 3.2.** *Let $\alpha, \beta$ satisfy Assumption A2. Let $f \in L_T^\infty \mathcal{C}^\beta$. Then $\mathcal{I}_\cdot(f) \in C_T \mathcal{C}^{\alpha+1}$, where $\mathcal{I}_t(f) := \int_0^t P_{t-s} f(s) \mathrm{d}s$.*

*Proof.* We first observe that for fixed $0 \leq s \leq t \leq T$ then $P_{t-s} f(s) \in \mathcal{C}^{\alpha+1}$ by (8). The singularity in time is still integrable if $\alpha$ and $\beta$ satisfy Assumption A2. To show continuity of $\mathcal{I}$ we take some $\varepsilon > 0$ and we bound $\mathcal{I}_{t+\varepsilon}(f) - \mathcal{I}_t(f)$ in the space $\mathcal{C}^{\alpha+1}$ by

$$\begin{aligned}
&\| \int_0^t P_{t-s}(P_\varepsilon f(s)) \mathrm{d}s + \int_t^{t+\varepsilon} P_{t+\varepsilon-s} f(s) \mathrm{d}s - \int_0^t P_{t-s} f(s) \mathrm{d}s \|_{\alpha+1} \\
\leq &\| \int_0^t P_{t-s}(P_\varepsilon f(s) - f(s)) \mathrm{d}s \|_{\alpha+1} + \| \int_t^{t+\varepsilon} P_{t+\varepsilon-s} f(s) \mathrm{d}s \|_{\alpha+1}.
\end{aligned}$$

Now we use Schauder's estimates (8) and (9) with some $\nu > 0$ such that $\theta := \alpha + 1 - \beta + 2\nu < 2$ (which always exists by Assumption A2) and we get

$$\begin{aligned}
&\|\mathcal{I}_{t+\varepsilon}(f) - \mathcal{I}_t(f)\|_{\alpha+1} \\
&\quad \le c \int_0^t (t-s)^{-\frac{\theta}{2}} \|P_\varepsilon f(s) - f(s)\|_{\beta-2\nu} \mathrm{d}s \\
&\qquad + c \int_t^{t+\varepsilon} (t+\varepsilon-s)^{-\frac{\alpha+1-\beta}{2}} \|f(s)\|_\beta \mathrm{d}s \\
&\quad \le c \int_0^t (t-s)^{-\frac{\theta}{2}} |\varepsilon|^\nu \|f(s)\|_\beta \mathrm{d}s \\
&\qquad + c \int_t^{t+\varepsilon} (t+\varepsilon-s)^{-\frac{\alpha+1-\beta}{2}} \|f(s)\|_\beta \mathrm{d}s \\
&\quad \le c \|f\|_{L^\infty_T \mathcal{C}^\beta} \left( |\varepsilon|^\nu \int_0^t (t-s)^{-\frac{\theta}{2}} \mathrm{d}s + \int_t^{t+\varepsilon} (t+\varepsilon-s)^{-\frac{\alpha+1-\beta}{2}} \mathrm{d}s \right) \\
&\quad \le c \|f\|_{L^\infty_T \mathcal{C}^\beta} \left( |\varepsilon|^\nu t^{-\frac{\theta}{2}+1} + \varepsilon^{\frac{-\alpha+1+\beta}{2}} \right),
\end{aligned}$$

and the latter tends to 0 as $\varepsilon \to 0$ for all $t \in [0, T]$ because $\nu > 0$ and $-\frac{\theta}{2} + 1 > 0$ by construction and $-\alpha + 1 + \beta > 0$ by Assumption A2. □

Next we show an auxiliary result useful later on.

**Proposition 3.3.** *Let Assumptions A1, A2 and A3 hold. Let $u, v \in C_T\mathcal{C}^{\alpha+1}$. Then for all $\rho \ge 1$*

$$\begin{aligned}
\|I(u) - I(v)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \le & c \|b\|_{L^\infty_T \mathcal{C}^\beta} \rho^{\frac{\alpha-1-\beta}{2}} (1 + \|u\|^2_{C_T\mathcal{C}^{\alpha+1}} + \|v\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2} \\
& \|u - v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \qquad (17)
\end{aligned}$$

*where the constant $c$ depends only on $L, l$ and $d$.*

*Proof.* Using the definition of $I$ we have

$$\begin{aligned}
\|I(u) - I(v)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} & \\
&= \sup_{0 \le t \le T} e^{-\rho t} \|I_t(u) - I_t(v)\|_{\alpha+1} \\
&= \sup_{0 \le t \le T} e^{-\rho t} \left\| \int_0^t P_{t-s} \left( [\mathrm{F}(\nabla u(s)) - \mathrm{F}(\nabla v(s))] b(s) \right) \mathrm{d}s \right\|_{\alpha+1}.
\end{aligned}$$

Now using (8) with $\theta = \frac{\alpha+1-\beta}{2}$ (which is positive by Assumption A2) and (4) (again by A2 $\alpha + \beta > 0$) we bound the integrand by

$$(t-s)^{-\frac{\alpha+1-\beta}{2}} \|b\|_{L^\infty_T \mathcal{C}^\beta} \|\mathrm{F}(\nabla u(s)) - \mathrm{F}(\nabla v(s))\|_\alpha$$

and using the result of Proposition 3.1 we further bound it by

$$c(t-s)^{-\frac{\alpha+1-\beta}{2}} \|b\|_{L^\infty_T \mathcal{C}^\beta} \|\nabla u(s) - \nabla v(s)\|_\alpha (1 + \|\nabla u(s)\|^2_\alpha + \|\nabla v(s)\|^2_\alpha)^{1/2},$$

where the constant $c$ depends on $L, l$ and $d$. Substituting the last bound into the equation above we get

$$\begin{aligned}
&\|I(u)-I(v)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}\\
&\quad\leq c\|b\|_{L^\infty_T\mathcal{C}^\beta} \sup_{0\leq t\leq T}\int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}} e^{-\rho(t-s)}\\
&\qquad e^{-\rho s}\|\nabla u(s)-\nabla v(s)\|_\alpha (1+\|\nabla u(s)\|^2_\alpha+\|\nabla v(s)\|^2_\alpha)^{1/2}\mathrm{d}s\\
&\quad\leq c\|b\|_{L^\infty_T\mathcal{C}^\beta} \sup_{0\leq t\leq T}\int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}} e^{-\rho(t-s)}\mathrm{d}s\\
&\qquad \|\nabla u-\nabla v\|^{(\rho)}_{C_T\mathcal{C}^\alpha}(1+\|\nabla u\|^2_{C_T\mathcal{C}^\alpha}+\|\nabla v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}.
\end{aligned}$$

Finally we use the bound $\|\nabla f\|_\alpha \leq c\|f\|_{\alpha+1}$ for $f\in\mathcal{C}^{\alpha+1}$ (which follows from Bernstein inequalities, see e.g. [1, Lemma 2.1]) and we integrate the singularity since $-\frac{\alpha+1-\beta}{2} > -1$ to get

$$c\|b\|_{L^\infty_T\mathcal{C}^\beta}\rho^{\frac{\alpha-1-\beta}{2}}(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}}+\|v\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2}\|u-v\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}},$$

as wanted. □

We remark that the power of $\rho$ in (17) is negative due to Assumption A2 and the idea is to pick $\rho$ large enough so that $I$ is a contraction. However this cannot be done using (17) directly because of the term $(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}}+\|v\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2}$. Indeed we are only able to show existence and uniqueness of a solution for a small time-interval or alternatively for a small initial condition, as we will see later.

**Proposition 3.4.** *Let Assumptions A1, A2 and A3 hold. Let $u_0\in\mathcal{C}^{\alpha+1}$ be given. Then the operator $J$ maps $C_T\mathcal{C}^{\alpha+1}$ into itself. In particular, for arbitrary $T,\rho$ and $u\in C_T\mathcal{C}^{\alpha+1}$ we have*

$$\begin{aligned}
\|J(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\leq \|u_0\|_{\alpha+1}\\
&\quad + C\rho^{\frac{\alpha-1-\beta}{2}}\left(1+\|u\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2}\right),
\end{aligned} \tag{18}$$

*where $C=c\|b\|_{L^\infty_T\mathcal{C}^\beta}$ is the constant appearing in* (17) *in front of $\rho$ and $c$ depends only on $L,l$ and $d$.*

*Proof.* It is clear that (18) implies that $J$ maps $C_T\mathcal{C}^{\alpha+1}$ into itself. To prove (18) we use the definition of $J$ to get

$$\begin{aligned}
\|J(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &= \|P_\cdot u_0+I(u)\|_{C_T\mathcal{C}^{\alpha+1}}\\
&\leq \|P_\cdot u_0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}+\|I(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}\\
&=: (A)+(B).
\end{aligned}$$

The term (A) is bounded using the contraction property of $P_t$ in $\mathcal{C}^\alpha$ and by the definition of the equivalent norm

$$(A)\leq \|u_0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} = \sup_{0\leq t\leq T} e^{-\rho t}\|u_0\|_{\alpha+1} = \|u_0\|_{\alpha+1}.$$

The term (B) can be bounded similarly as in the proof of Proposition 3.3 and one gets

$$(B) \leq c \sup_{0\leq t\leq T} e^{-\rho t} \int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}} \|\mathrm{F}(\nabla u(s))\|_\alpha \|b(s)\|_\beta \mathrm{d}s.$$

Now we apply Proposition 3.1 with $f = \nabla u(s)$ and $g = 0$ to get

$$\begin{aligned}\|\mathrm{F}(\nabla u(s)) - \mathrm{F}(\mathbf{0}) + \mathrm{F}(\mathbf{0})\|_\alpha &\leq \|\mathrm{F}(\nabla u(s)) - \mathrm{F}(\mathbf{0})\|_\alpha + \|\mathrm{F}(\mathbf{0})\|_\alpha \\ &\leq c + (1 + \|\nabla u(s)\|_\alpha^2)^{1/2} \|\nabla u(s)\|_\alpha \\ &\leq c(1 + \|u(s)\|_{\alpha+1}(1 + \|u(s)\|_{\alpha+1}^2)^{1/2}).\end{aligned}$$

Plugging this into (B) we get

$$\begin{aligned}(B) \leq& c\|b\|_{L_T^\infty \mathcal{C}^\beta} \sup_{0\leq t\leq T} \int_0^t e^{-\rho(t-s)}(t-s)^{-\frac{\alpha+1-\beta}{2}} \mathrm{d}s \\ &\sup_{0\leq s\leq T} e^{-\rho s}\left(1 + \|u(s)\|_{\alpha+1}(1 + \|u(s)\|_{\alpha+1}^2)^{1/2}\right) \\ \leq& c\|b\|_{L_T^\infty \mathcal{C}^\beta} \rho^{\frac{\alpha-1-\beta}{2}} \left(1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}^{(\rho)} (1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}^2)^{1/2}\right)\end{aligned}$$

as wanted. □

Carrying out the same proof in the special case when $F(0) = 0$ we easily obtain the result below.

**Corollary 3.5.** *Under the assumptions of Proposition 3.4 and if moreover $F(0) = 0$ then we have*

$$\|J(u)\|_{C_T\mathcal{C}^{\alpha+1}}^{(\rho)} \leq \|u_0\|_{\alpha+1} + C\rho^{\frac{\alpha-1-\beta}{2}} \|u\|_{C_T\mathcal{C}^{\alpha+1}}^{(\rho)} (1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}^2)^{1/2}. \tag{19}$$

To show that $J$ is a contraction in a suitable (sub)space we introduce a subset of $C_T\mathcal{C}^{\alpha+1}$ which depends on three parameters, $\rho$, $R$ and $T$. We define

$$B_{R,T}^{(\rho)} := \left\{ f \in C_T\mathcal{C}^{\alpha+1} \,:\, \|f\|_{C_T\mathcal{C}^{\alpha+1}}^{(\rho)} \leq 2Re^{-\rho T} \right\}. \tag{20}$$

Now choosing $\rho$, $R$ and $T$ appropriately (depending on the initial condition $u_0$) one can show that $J$ is a contraction by applying Proposition 3.4 as illustrated below.

**Proposition 3.6.** *Let Assumptions A1, A2 and A3 hold. Let $R_0$ be a given arbitrary constant. Then there exists $\rho_0$ large enough depending on $R_0$, and $T_0$ small enough depending on $\rho_0$ such that*

$$J : B_{R_0,T_0}^{(\rho_0)} \to B_{R_0,T_0}^{(\rho_0)},$$

*for any initial condition $u_0 \in \mathcal{C}^{\alpha+1}$ such that $\|u_0\|_{\alpha+1} \leq R_0$. Moreover for each $u, v \in C_{T_0}\mathcal{C}^{\alpha+1}$ then*

$$\|J(u) - J(v)\|_{C_{T_0}\mathcal{C}^{\alpha+1}}^{(\rho_0)} < \|u - v\|_{C_{T_0}\mathcal{C}^{\alpha+1}}^{(\rho_0)}.$$

*Proof.* We begin by taking $u \in B_{R_0,T}^{(\rho)}$ for some arbitrary parameters $T$ and $\rho$. For this $u$ we have the following bounds

$$\|u\|_{C_T\mathcal{C}^{\alpha+1}}^{(\rho)} \leq 2R_0 e^{-\rho T}$$

and

$$\|u\|_{C_T\mathcal{C}^{\alpha+1}} \leq 2R_0 e^{-\rho T} e^{\rho T} = 2R_0. \tag{21}$$

Let $u_0 \in C^{\alpha+1}$ be such that $\|u_0\|_{\alpha+1} \leq R_0$. Then by Proposition 3.4 we obtain

$$\begin{aligned}\|J(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\leq R_0 + C\rho^{\frac{\alpha-1-\beta}{2}}\left(1+2R_0e^{-\rho T}(1+4R_0^2)^{1/2}\right)\\ &= R_0e^{-\rho T}\left(e^{\rho T}+\frac{C}{R_0}\rho^{\frac{\alpha-1-\beta}{2}}e^{\rho T}+2C\rho^{\frac{\alpha-1-\beta}{2}}(1+4R_0^2)^{1/2}\right).\end{aligned}$$

To show that $J(u) \in B^{(\rho)}_{R_0,T}$ we need to pick $\rho_0$ and $T_0$ such that

$$e^{\rho T}+\frac{C}{R_0}\rho^{\frac{\alpha-1-\beta}{2}}e^{\rho T}+2C\rho^{\frac{\alpha-1-\beta}{2}}(1+4R_0^2)^{1/2} \leq 2. \tag{22}$$

This is done as follows. First we pick $\rho_0 \geq 1$ depending on $R_0$ and large enough such that the following three conditions hold

$$2C\rho_0^{\frac{\alpha-1-\beta}{2}}(1+4R_0^2)^{1/2} \leq \frac{1}{4} \tag{23}$$

$$\frac{C}{R_0}\rho_0^{\frac{\alpha-1-\beta}{2}} \leq \frac{1}{4} \tag{24}$$

$$C\rho_0^{\frac{\alpha-1-\beta}{2}}(1+8R_0^2)^{1/2} < 1. \tag{25}$$

This is always possible since $\rho \mapsto \rho^{\frac{\alpha-1-\beta}{2}}$ is decreasing. Moreover this can be done independently of $T$. We also remark that the third bound is not needed to show that $J(u) \in B^{(\rho)}_{R_0,T}$ but will be needed below to show that $J$ is a contraction for the chosen set of parameters $R_0, \rho_0, T_0$.
Next we pick $T_0 > 0$ depending on $\rho_0, R_0$ and small enough such that

$$e^{\rho_0 T_0} \leq 1 + \frac{2}{5}. \tag{26}$$

This is always possible since $T \mapsto e^{\rho_0 T}$ is increasing, continuous and has minimum 1 at 0.
With these parameters, (22) is satisfied under the assumptions (23), (24) and (26). Indeed

$$e^{\rho_0 T_0}+\frac{C}{R_0}\rho^{\frac{\alpha-1-\beta}{2}}e^{\rho_0T_0}+2C\rho_0^{\frac{\alpha-1-\beta}{2}}(1+4R_0^2)^{1/2} \leq 1+\frac{2}{5}+\frac{1}{4}(1+\frac{2}{5})+\frac{1}{4} = 2.$$

It is left to prove that $J$ is a contraction on $B^{(\rho_0)}_{R_0,T_0}$. For this, it is enough to use Proposition 3.3 for $u, v \in B^{(\rho_0)}_{R_0,T_0}$

$$\begin{aligned}\|I(u)-I(v)\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} &\leq C\rho_0^{\frac{\alpha-1-\beta}{2}}(1+2(2R_0)^2)^{1/2}\|u-v\|^{(\rho)}_{C_{T_0}\mathcal{C}^{\alpha+1}}\\ &< \|u-v\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}},\end{aligned}$$

where the last bound is ensured by (25). □

Using the last result we can show that a unique solution exists locally (for small time $T_0$) in the whole space $C_{T_0}\mathcal{C}^{\alpha+1}$.

**Theorem 3.7.** *Let Assumptions A1, A2 and A3 hold. Let $u_0 \in \mathcal{C}^{\alpha+1}$ be given. Then there exists a unique local mild solution $u$ to (7) in $C_{T_0}\mathcal{C}^{\alpha+1}$, where $T_0$ is small enough and it is chosen as in Proposition 3.6 (depending on the norm of $u_0$).*

*Proof.* Let $R_0 = \|u_0\|_{\alpha+1}$ and $\rho_0$ and $T_0$ such that (23)–(26) are satisfied.
*Existence.* By Proposition 3.6 we know that the mapping $J$ is a contraction on $B^{(\rho_0)}_{R_0,T_0}$ and so there exists a solution $u \in B^{(\rho_0)}_{R_0,T_0}$ which is unique in the latter subspace.
*Uniqueness.* Suppose that there are two solutions $u_1$ and $u_2$ in $C_{T_0}\mathcal{C}^{\alpha+1}$. Then obviously $u_i = J(u_i)$ and $\|u_i\|_{C_{T_0}\mathcal{C}^{\alpha+1}} < \infty$ for $i = 1, 2$. We set $r := \max\{\|u_i\|_{C_{T_0}\mathcal{C}^{\alpha+1}}, i = 1, 2\}$ (which only depends on $u_i$ and not on any $\rho$). By Proposition 3.3 for any $\rho \geq 1$ we have that the $\rho$-norm of the difference $u_1 - u_2$ is bounded by

$$\begin{aligned}
\|u_1-u_2\|^{(\rho)}_{C_{T_0}\mathcal{C}^{\alpha+1}} &= \|I(u_1) - I(u_2)\|^{(\rho)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \\
&\leq C\rho^{\frac{\alpha-1-\beta}{2}}(1 + \|u_1\|^2_{C_{T_0}\mathcal{C}^{\alpha+1}} + \|u_2\|^2_{C_{T_0}\mathcal{C}^{\alpha+1}})^{1/2}\|u_1 - u_2\|^{(\rho)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \\
&\leq C\rho^{\frac{\alpha-1-\beta}{2}}(1 + 2r^2)^{1/2}\|u_1 - u_2\|^{(\rho)}_{C_{T_0}\mathcal{C}^{\alpha+1}}.
\end{aligned}$$

Choosing $\rho_0$ large enough such that $1 - C\rho_0^{\frac{\alpha-1-\beta}{2}}(1 + 2r^2)^{1/2} > 0$ implies that $\|u_1 - u_2\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \leq 0$ and hence the difference must be 0 in the space $C_{T_0}\mathcal{C}^{\alpha+1}$, thus $u_1 = u_2$. □

**Remark 3.8.** Note that in the proof of uniqueness of Theorem 3.7 we do not assume anything about the size of time $T_0$. Hence, if a solution to (7) exists up to time $T$ in the space $C_T\mathcal{C}^{\alpha+1}$, then it is unique.

An alternative existence and uniqueness result is shown below. A global in time solution is found up to any given time $T$, but in this case we have to restrict the choice of initial conditions $u_0$ to a set with small norm (depending on $T$). Moreover we are able to show this result only under the extra condition that $F(0) = 0$.

**Proposition 3.9.** *Let Assumptions A1, A2 and A3 hold. Assume $F(0) = 0$. Let $T > 0$ be given and arbitrary. Then there exists $\rho_0$ large enough such that for all $u_0 \in B^{(\rho_0)}_{\frac{1}{2},T}$ then*

$$J : B^{(\rho_0)}_{1,T} \to B^{(\rho_0)}_{1,T} \tag{27}$$

*and $J$ is a contraction on $B^{(\rho_0)}_{1,T}$, namely for $u, v \in B^{(\rho_0)}_{1,T}$ we have*

$$\|J(u) - J(v)\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} < \|u - v\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}. \tag{28}$$

*Proof.* We recall that for some given $R, \rho$ and $T$, the assumption $u_0 \in B^{(\rho)}_{R,T}$ means that $\|u_0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq 2Re^{-\rho T}$, see (20). Moreover $u_0$ does not depend on time hence $\|u_0\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} = \|u_0\|_{\alpha+1}$ so $u_0 \in B^{(\rho)}_{\frac{1}{2},T}$ implies

$$\|u_0\|_{\alpha+1} \leq e^{-\rho T}.$$

Using this and Corollary 3.5 we have

$$\begin{aligned}\|J(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\leq \|u_0\|_{\alpha+1} + C\rho^{\frac{\alpha-1-\beta}{2}}\|u\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2}\\ &\leq e^{-\rho T} + C\rho^{\frac{\alpha-1-\beta}{2}}\|u\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}})^{1/2}.\end{aligned}$$

Let $u \in B^{(\rho)}_{1,T}$. Then $\|u\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} \leq 2e^{-\rho T}$ and

$$\|u\|_{C_T\mathcal{C}^{\alpha+1}} \leq 2. \tag{29}$$

Thus the bound above becomes

$$\begin{aligned}\|J(u)\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} &\leq e^{-\rho T} + C\rho^{\frac{\alpha-1-\beta}{2}}2e^{-\rho T}(1+4)^{1/2}\\ &= 2e^{-\rho T}(\frac{1}{2} + C\sqrt{5}\rho^{\frac{\alpha-1-\beta}{2}}).\end{aligned}$$

We choose $\bar\rho_0$ such that $\frac{1}{2} + C\sqrt{5}\bar\rho_0^{\frac{\alpha-1-\beta}{2}} = 1$, and since the function $\rho \mapsto \rho^{\frac{\alpha-1-\beta}{2}}$ is decreasing, for each $\rho_0 \geq \bar\rho_0$ we have

$$\frac{1}{2} + C\sqrt{5}\rho_0^{\frac{\alpha-1-\beta}{2}} \leq 1. \tag{30}$$

Then for $\rho = \rho_0$ we have $\|J(u)\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} \leq 2e^{-\rho_0 T}$ which implies that $J(u) \in B^{(\rho_0)}_{1,T}$ and this shows (27).

To show (28), let $u, v \in B^{(\rho_0)}_{1,T} \subset C_T\mathcal{C}^{\alpha+1}$ with $\rho_0 \geq \bar\rho_0$. Then by Proposition 3.3 and by (29)

$$\begin{aligned}&\|J(u)-J(v)\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}\\ &\quad\leq C\rho_0^{\frac{\alpha-1-\beta}{2}}\left(1+\|u\|^2_{C_T\mathcal{C}^{\alpha+1}}+\|v\|^2_{C_T\mathcal{C}^{\alpha+1}}\right)^{1/2}\|u-v\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}\\ &\quad\leq C\rho_0^{\frac{\alpha-1-\beta}{2}}(1+4+4)^{1/2}\|u-v\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}\\ &\quad\leq 3C\rho_0^{\frac{\alpha-1-\beta}{2}}\|u-v\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned}$$

We now chose $\rho_0 \geq \bar\rho_0$ large enough so that

$$3C\rho_0^{\frac{\alpha-1-\beta}{2}} < 1 \tag{31}$$

and the proof is concluded. □

**Theorem 3.10.** *Let Assumptions A1, A2 and A3 hold. Let $T > 0$ be given and let $F(0) = 0$. Then there exists $\delta > 0$ depending on $T$ such that for each $u_0$ with $\|u_0\|_{\alpha+1} \leq \delta$ there exists a unique solution $u \in C_T\mathcal{C}^{\alpha+1}$ to (7).*

*Proof. Existence.* We choose $\rho_0$ according to (31) and (30). Let $\delta = e^{-\rho_0 T}$. Then the assumption $\|u_0\|_{\alpha+1} \leq \delta$ means $u_0 \in B^{(\rho_0)}_{\frac{1}{2},T}$ and by Proposition 3.9 we know that the mapping $J$ is a contraction on $B^{(\rho_0)}_{1,T}$. Thus there exists a unique fixed point $u$ in $B^{(\rho_0)}_{1,T}$ which is a solution.

*Uniqueness.* This is shown like in the uniqueness proof of Theorem 3.7, with $T$ instead of $T_0$. □

**Remark 3.11.** Note that in the proof of uniqueness of Theorem 3.7 we do not actually use the assumption $\|u_0\|_{\alpha+1} \leq \delta$, so if $F(0) = 0$ then uniqueness holds for any initial condition and any time $T$, when a solution exists.

We now show continuity of the solution $u$ with respect to the initial condition $u_0$. This is done in the following proposition both for the case of existence and uniqueness of a solution $u$ for an arbitrary initial condition and a sufficiently small time $T_0$ (Theorem 3.7) and for the case of existence and uniqueness of a solution $u$ for an arbitrary time $T$ and for a sufficiently small (in norm) initial condition $u_0$ (Theorem 3.10).

**Proposition 3.12.** (i) *Let the assumptions of Theorem 3.7 hold and let $R_0 > 0$ be arbitrary and fixed. Let $u$ be the unique solution found in Theorem 3.7 on $[0, T_0]$ with initial condition $u_0$ such that $\|u_0\| \leq R_0$ and where $T_0$ depends on $R_0$. Then $u$ is continuous with respect to the initial condition $u_0$, namely*

$$\|u\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \leq 2\|u_0\|_{\alpha+1}$$

*for $\rho_0$ large enough.*

(ii) *Let the assumptions of Theorem 3.10 hold and let $T > 0$ be arbitrary and fixed. Let $u$ be the unique solution found in Theorem 3.10 on $[0, T]$ with initial condition $u_0$ such that $\|u_0\| \leq e^{-\rho_0 T}$ for $\rho_0$ large enough. Then the unique solution $u$ is continuous with respect to the initial condition $u_0$, namely*

$$\|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} \leq 2\|u_0\|_{\alpha+1}.$$

*Proof. (i)* Let $\rho_0$ be chosen according to (23)–(25) and $T_0$ according to (26). Take $u_0$ such that $\|u_0\|_{\alpha+1} \leq R_0$. Then by Proposition 3.6 we have $J : B^{(\rho_0)}_{R_0,T_0} \to B^{(\rho_0)}_{R_0,T_0}$ and so by (21) the unique solution $u$ given in Theorem 3.7 satisfies $\|u\|_{C_{T_0}\mathcal{C}^{\alpha+1}} \leq 2R_0$ for any initial conditions $u_0$ with $\|u_0\|_{\alpha+1} \leq R_0$. Using this and Corollary 3.5 we have

$$\begin{aligned}\|u\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} &= \|J(u)\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \\ &\leq \|u_0\|_{\alpha+1} + C\rho_0^{\frac{\alpha-1-\beta}{2}} \|u\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} (1 + \|u\|_{C_{T_0}\mathcal{C}^{\alpha+1}})^{1/2} \\ &\leq \|u_0\|_{\alpha+1} + \sqrt{1 + 4R_0^2}C\rho_0^{\frac{\alpha-1-\beta}{2}} \|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned}$$

By the choice of $\rho_0$ according to (23) we have $2\sqrt{1 + 4R_0^2}C\rho_0^{\frac{\alpha-1-\beta}{2}} \leq \frac{1}{4}$ hence

$$\|u\|^{(\rho_0)}_{C_{T_0}\mathcal{C}^{\alpha+1}} \leq \|u_0\|_{\alpha+1} + \frac{1}{2}\|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}},$$

and rearranging terms we conclude.

*(ii)* Let $\rho_0$ be chosen according to (30). Then for all $u_0 \in B^{(\rho_0)}_{\frac{1}{2},T}$ (that is for $\|u_0\|_{\alpha+1} \leq e^{-\rho_0 T}$) we have $J : B^{(\rho_0)}_{1,T} \to B^{(\rho_0)}_{1,T}$ by Proposition 3.9. In particular, the unique solution $u$ given in Theorem 3.10 belongs to $B^{(\rho_0)}_{1,T}$,

and (29) holds, that is $\|u\|_{C_T\mathcal{C}^{\alpha+1}} \leq 2$. Using this and Corollary 3.5 we have

$$\begin{aligned}\|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} &= \|J(u)\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} \\ &\leq \|u_0\|_{\alpha+1} + C\rho_0^{\frac{\alpha-1-\beta}{2}} \|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}} (1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}})^{1/2} \\ &\leq \|u_0\|_{\alpha+1} + \sqrt{5}C\rho_0^{\frac{\alpha-1-\beta}{2}} \|u\|^{(\rho_0)}_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned}$$

By the choice of $\rho_0$ according to (30) we have $\sqrt{5}C\rho_0^{\frac{\alpha-1-\beta}{2}} \leq \frac{1}{2}$ and we conclude as in part (i). $\square$

Finally we conclude this section by investigating the blow-up for the solution $u$ to the PDE. It is still an open problem to show whether the solution $u$ blows up or not, but we have the following result that states that if blow-up occurs, then it does so in finite time.

**Proposition 3.13.** *Let $u_0 \in \mathcal{C}^{\alpha+1}$ and $T > 0$ be given. Then one of the following statements holds:*

(a) *There exists a time $t^* \in [0, T]$ such that $\lim_{s\to t^*} \|u(s)\|_{\alpha+1} = \infty$; Or*
(b) *there exists a solution $u$ for all $t \in [0, T]$.*

*Proof.* Assume that $\limsup_{s\to t^*} \|u(s)\|_{\alpha+1} = \infty$ for some $t^* \in [0, T]$. Suppose moreover by contradiction that $\liminf_{s\to t^*} \|u(s)\|_{\alpha+1} < \infty$. Then we can find $R_0 > 0$ and a sequence $t_k \to t^*$ such that $\|u(t_k)\|_{\alpha+1} < R_0$ for all $k$. Let us now restart the PDE from $u(t_k)$ and apply Theorem 3.7: We know that there exists a solution for the interval $[t_k, t_k + T_0]$, where $T_0 > 0$ depends on $R_0$ but not on $k$. Thus we are able to extend the solution $u$ past $t^*$ because as $k \to \infty$ we have $t_k + T_0 \to t^* + T_0$. Thus it cannot be that $\limsup_{s\to t^*} \|u(s)\|_{\alpha+1} = \infty$ and $\liminf_{s\to t^*} \|u(s)\|_{\alpha+1} < \infty$ for some $t^* \in [0, T]$. This means that if $\limsup_{s\to t^*} \|u(s)\|_{\alpha+1} = \infty$ for some $t^* \in [0, T]$ then actually also $\lim_{s\to t^*} \|u(s)\|_{\alpha+1} = \infty$, which is case (a). Otherwise, if $\limsup_{s\to t^*} \|u(s)\|_{\alpha+1} < \infty$ for all $t^* \in [0, T]$ then a global solution on $[0, T]$ must exists, which is case (b). $\square$

Further research is needed to show either global in time solution or the existence of a finite blow-up time. The difficulty here is due to the quadratic non-linearity and the fact that this term is multiplied by the distributional coefficient. This prevents us to apply classical techniques such as the Cole-Hopf transformation which would be used in the special case $F(x) = x^2$ and $b \equiv 1$ to linearise the equation.

## 4. A global existence result

In this section we provide a global result on existence and uniqueness of a solution upon imposing further assumptions on the non-linearity $F$. In particular, we will exclude the quadratic case but still allow for a rich class of non-linear functions.

**A4: Further assumption on non-linear term $F$.** *Let $F : \mathbb{R}^d \to \mathbb{R}$ be globally Lipschitz, i.e., there exists a positive constant $L$ such that for all $x, y \in \mathbb{R}^d$ we have*

$$|F(x) - F(y)| \leq \tilde{L}|x - y|_d.$$

Assumption A4 implies that $F$ has sub-linear growth, that is, there exists a positive constant $\tilde{l}$ such that for all $x \in \mathbb{R}^d$

$$|F(x)| \leq \tilde{l}(1 + |x|_d).$$

Moreover also the operator $\mathrm{F} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ has sub-linear growth in $\mathcal{C}^\alpha$, namely there exists $c > 0$ such that for all $f \in \mathcal{C}^\alpha$ we have

$$\|\mathrm{F}(f)\|_\alpha \leq c(1 + \|f\|_\alpha). \tag{32}$$

Indeed

$$\begin{aligned}
\|\mathrm{F}(f)\|_\alpha &= \sup_{x\in\mathbb{R}^d} |Ff(x)| + \sup_{x\in\mathbb{R}^d} \sup_{|y|_d\leq 1} \frac{|Ff(x+y) - Ff(x)|}{|y|_d^\alpha} \\
&\leq \sup_{x\in\mathbb{R}^d} \tilde{l}(1 + |f(x)) + \sup_{x\in\mathbb{R}^d} \sup_{|y|_d\leq 1} \frac{\tilde{L}|f(x+y) - f(x)|}{|y|_d^\alpha} \\
&\leq c(1 + \|f\|_\alpha).
\end{aligned}$$

This extra assumption allows us to find a priori bounds on the solution, as follows.

**Proposition 4.1** (A priori bounds)**.** *Let Assumptions A1, A2, A3 and A4 hold. Let $T < \infty$ be an arbitrary time and $u_0 \in \mathcal{C}^{\alpha+1}$. If there exists $u \in C_T\mathcal{C}^{\alpha+1}$ such that*

$$u(t) = \lambda P_t u_0 + \lambda \int_0^t P_{t-r}(F(\nabla u(r))b(r))\mathrm{d}r \tag{33}$$

*where $\lambda \in [0, 1]$ is fixed, then for all $t \in [0, T]$ it must hold*

$$\|u(t)\|_{\alpha+1} \leq K$$

*for some finite constant $K$ which depends only on $T, b$ and $u_0$. In particular, $\|u\|_{C_T\mathcal{C}^{\alpha+1}} \leq K$.*

Note that when $\lambda = 1$ then (33) reduces to (7). By slight abuse of notation, in this result we use $u$ for the solution of (33) for $\lambda \in [0, 1]$.

*Proof.* Let $u \in C_T\mathcal{C}^{\alpha+1}$ be a solution of (33), that is

$$u(t) = \lambda P_t u_0 + \lambda I_t(u). \tag{34}$$

Note that $\mathrm{F}(\nabla u) \in C_T\mathcal{C}^{\alpha+1} \subset L_T^\infty\mathcal{C}^{\alpha+1}$ and so $I(u) \in C_T\mathcal{C}^{\alpha+1}$ by Lemma 3.2 and by Assumption A3. Now we apply (8) and assumption A4 to get

$$\begin{aligned}
\|I_t(u)\|_{\alpha+1} &\leq \int_0^t \|P_{t-s}(F(\nabla u(s))b(s)\|_{\alpha+1}\mathrm{d}s \\
&\leq c\|b\|_{L_T^\infty\mathcal{C}^\beta} \int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}}(1 + \|\nabla u(s)\|_\alpha)\mathrm{d}s \\
&\leq c\|b\|_{L_T^\infty\mathcal{C}^\beta} \int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}}(1 + \|u(s)\|_{\alpha+1})\mathrm{d}s.
\end{aligned}$$

Taking the $\mathcal{C}^{\alpha+1}$ norm of (34) and plugging the above estimate in, we obtain

$$\begin{aligned}\|u(t)\|_{\alpha+1} \leq & \lambda\|P_t u_0\|_{\alpha+1} + \lambda\|I_t(u)\|_{\alpha+1}\\ \leq & c\|u_0\|_{\alpha+1} + c\|b\|_{L^\infty_T\mathcal{C}^\beta} T^{\frac{-\alpha+1+\beta}{2}}\\ & + c\|b\|_{L^\infty_T\mathcal{C}^\beta}\int_0^t (t-s)^{-\frac{\alpha+1-\beta}{2}}\|u(s)\|_{\alpha+1}\mathrm{d}s.\end{aligned}$$

Now an application of Gronwall's lemma and the evaluation of the supremum over $t\in[0,T]$ allows to conclude. □

Our strategy to show global existence of a solution of (7) is to apply Schaefer's fixed point theorem. To this aim, for $\varepsilon>0$ let us define the space $C^\varepsilon_T\mathcal{C}^{\alpha+1}$ as the collection of all functions $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ with finite $\|\cdot\|_{\varepsilon,\alpha+1}$ norm, where the latter is given by

$$\|f\|_{\varepsilon,\alpha+1} := \sup_{0\leq t\leq T}\|f(t)\|_{\alpha+1} + \sup_{0\leq s<t\leq T}\frac{\|f(t)-f(s)\|_{\alpha+1}}{(t-s)^\varepsilon}.$$

In order to apply Schaefer's fixed point theorem, it is convenient to work in $C^\varepsilon_T\mathcal{C}^{\alpha+1}$ rather than $C_T\mathcal{C}^{\alpha+1}$, the reason being that balls in $C^{\varepsilon'}_T\mathcal{C}^{\alpha'+1}$ are pre-compact sets in $C^\varepsilon_T\mathcal{C}^{\alpha+1}$ for $\varepsilon'>\varepsilon$ and $\alpha'>\alpha$.

For ease of reading we set

$$G_r(u) := \mathrm{F}(\nabla u(r))b(r).$$

Using Assumption A4 and (4) we have that for $u(r)\in\mathcal{C}^{\alpha+1}$ then

$$\|G_r(u)\|_\beta \leq c(1+\|u(r)\|_{\alpha+1}), \tag{35}$$

where $c$ depends on $b$ and $\tilde{l}$. Moreover by Proposition 3.1 we have that for $u(r),v(r)\in\mathcal{C}^{\alpha+1}$ then

$$\|G_r(u)-G_r(v)\|_\beta \leq c(1+\|u(r)\|^2_{\alpha+1}+\|v(r)\|^2_{\alpha+1})^{1/2}\|u(r)-v(r)\|_{\alpha+1}, \tag{36}$$

where $c$ depend on $b,l,L$ and $d$.

We now state and prove three preparatory results that are the keys steps needed to apply Schaefer's fixed point theorem.

**Lemma 4.2.** *Let Assumptions A1, A2 and A3 hold and fix $\varepsilon>0$ such that $\alpha-1-\beta+\varepsilon<0$. Let $u_0\in\mathcal{C}^{\alpha+1+2\varepsilon+\nu}$ for some small $\nu>0$. If $u\in C_T\mathcal{C}^{\alpha+1}$ then $J(u)\in C^{\varepsilon'}_T\mathcal{C}^{\alpha'+1}$ for some $\varepsilon'>\varepsilon$ and $\alpha'>\alpha$, and*

$$\|J(u)\|_{\varepsilon',\alpha'+1} \leq c\|u_0\|_{\alpha+1+2\varepsilon+\nu} + cT^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}}(1+\|u\|_{C_T\mathcal{C}^{\alpha+1}}). \tag{37}$$

**Remark 4.3.** Note that the parameter $\varepsilon$ in Lemma 4.2 could in principle betaken equal zero, in which case we would only need $u_0\in\mathcal{C}^{\alpha+1+\nu}$ and $\varepsilon'>0$. Later on however, $\varepsilon$ will be chosen strictly greater than zero, hence we state and prove the result for $\varepsilon>0$.

*Proof of Lemma 4.2.* First we note that it is always possible to pick $\varepsilon>0$ such that $\alpha-1-\beta+\varepsilon<0$, because $\alpha-1-\beta<0$ by assumption A2. Let $u\in C_T\mathcal{C}^{\alpha+1}$. Moreover let us pick any $\alpha'>\alpha$ small enough such that

$\alpha' - 1 - \beta < 0$ and $\alpha' + 1 < \alpha + 1 + \nu$. Then we can easily see that for all $t \in [0, T]$ we have $J_t(u) \in \mathcal{C}^{\alpha'+1}$ as follows.

$$\begin{aligned}\|J_t(u)\|_{\alpha'+1} =& \|P_t u_0 + \int_0^t P_{t-r} G_r(u) \mathrm{d}r\|_{\alpha'+1} \\ \leq& \|P_t u_0\|_{\alpha'+1} + \int_0^t \|P_{t-r} G_r(u)\|_{\alpha'+1} \mathrm{d}r \\ \leq& c\|u_0\|_{\alpha'+1} + \int_0^t (t-r)^{-\frac{\alpha'+1-\beta}{2}} \|G_r(u)\|_\beta \mathrm{d}r \\ \leq& c\|u_0\|_{\alpha'+1} + cT^{\frac{-\alpha'+1+\beta}{2}} (1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}),\end{aligned}$$

where we have used (8) and (9) in the second inequality, and (35) in the last inequality. Note that $-\alpha'+1+\beta > 0$ by construction, and $u_0 \in \mathcal{C}^{\alpha+1+2\varepsilon+\nu} \subset \mathcal{C}^{\alpha'+1}$.

In order to show that $J(u) \in C_T^{\varepsilon'}\mathcal{C}^{\alpha'+1}$ we need to control the $\varepsilon'$-Hölder semi-norm. We now choose $\varepsilon' > \varepsilon$ small enough such that $\alpha'-1-\beta+2\varepsilon' < 0$ and $\alpha'+1+2\varepsilon' < \alpha+1+2\varepsilon+\nu$, which is always possible. Then $u_0 \in \mathcal{C}^{\alpha'+1+2\varepsilon'}$ and we express the difference $J_t(u) - J_s(u)$ for all $0 \leq s < t \leq T$ as

$$\begin{aligned}\|J_t(u) - J_s(u)\|_{\alpha'+1} \leq& \|(P_{t-s} - I)(P_s u_0)\|_{\alpha'+1} \\ &+ \|\int_0^s (P_{t-s} - I)(P_{s-r} G_r(u)) \mathrm{d}r\|_{\alpha'+1} \\ &+ \|\int_s^t P_{t-r} G_r(u) \mathrm{d}r\|_{\alpha'+1} \\ =:& M_1 + M_2 + M_3.\end{aligned} \tag{38}$$

Using (8) we get for the first term

$$M_1 \leq (t-s)^{\varepsilon'} \|P_s u_0\|_{\alpha'+1+2\varepsilon'} \leq c(t-s)^{\varepsilon'} \|u_0\|_{\alpha'+1+2\varepsilon'},$$

and $u_0 \in \mathcal{C}^{\alpha+1+2\varepsilon+\nu} \subset \mathcal{C}^{\alpha'+1+2\varepsilon'}$ by choice of $\alpha'$ and $\varepsilon'$.
The second term can be bounded using (8), (9) and (35), and produces a singularity integrable in time by choice of the parameters. We get

$$\begin{aligned}M_2 \leq& \int_0^s (t-s)^{\varepsilon'} \|P_{s-r} G_r(u)\|_{\alpha'+1+2\varepsilon'} \mathrm{d}r \\ \leq& (t-s)^{\varepsilon'} s^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}} c(1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}) \\ \leq& (t-s)^{\varepsilon'} T^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}} c(1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}).\end{aligned}$$

The third term is similar, and using (8) and (35) we obtain

$$\begin{aligned}M_3 \leq& \int_s^t (t-r)^{-\frac{\alpha'+1-\beta}{2}} \|G_r(u)\|_{\alpha'+1} \mathrm{d}r \\ \leq& \int_s^t (t-r)^{-\frac{\alpha'+1-\beta}{2}} \|G_r(u)\|_{\alpha'+1} \mathrm{d}r \\ \leq& (t-s)^{\varepsilon'} (t-s)^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}} c(1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}) \\ \leq& (t-s)^{\varepsilon'} T^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}} c(1 + \|u\|_{C_T\mathcal{C}^{\alpha+1}}).\end{aligned}$$

Putting everything together we get

$$\begin{aligned}\|J(u)\|_{\varepsilon',\alpha'+1} =& \sup_{0\le t\le T} \|J_t(u)\|_{\alpha'+1} + \sup_{0\le s<t\le T} \frac{\|J_t(u)-J_s(u)\|_{\alpha'+1}}{(t-s)^{\varepsilon'}}\\ \le& c\|u_0\|_{\alpha'+1} + cT^{\frac{-\alpha'+1+\beta}{2}}(1+\|u\|_{C_T\mathcal{C}^{\alpha+1}})\\ &+ c\|u_0\|_{\alpha'+1+2\varepsilon'} + 2T^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}}c(1+\|u\|_{C_T\mathcal{C}^{\alpha+1}})\\ \le& c\|u_0\|_{\alpha+1+2\varepsilon+\nu} + cT^{\frac{-\alpha'+1+\beta-2\varepsilon'}{2}}(1+\|u\|_{C_T\mathcal{C}^{\alpha+1}}),\end{aligned}$$

and the proof is complete. □

**Remark 4.4.** Applying Lemma 4.2 to the unique local solution $u \in C_T\mathcal{C}^{\alpha+1}$ found in Theorem 3.7 and in Theorem 3.10 we obtain that the unique mild solution is not only continuous in time, but it is actually smoother, more precisely $u \in C_T^{\varepsilon'}\mathcal{C}^{\alpha'+1}$, provided that $u_0 \in \mathcal{C}^{\alpha+1+2\varepsilon+\nu}$ for some small $\nu > 0$ and $\varepsilon > 0$ chosen as in Lemma 4.2.

**Lemma 4.5.** *Let Assumptions A1, A2 and A3 hold and let us choose $\varepsilon > 0$ according to Lemma 4.2. Then the operator $J : C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \to C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$ is continuous.*

*Proof.* From Lemma 4.2, the fact that $\varepsilon' > \varepsilon$ and $\alpha' > \alpha$ and the embeddings $C_T^{\varepsilon'}\mathcal{C}^{\alpha'+1} \subset C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \subset C_T\mathcal{C}^{\alpha+1}$ we have that $J : C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \to C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$. To show continuity we take $u, v \in C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$ and bound the sup norm and the Hölder semi-norm of the difference $J(u) - J(v)$.

The sup norm of $J(u)-J(v)$ is bounded by Propositions 3.3 (with $\rho = 1$) together with the fact that the embedding $C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \subset C_T\mathcal{C}^{\alpha+1}$ is continuous. Then one has

$$\sup_{0\le t\le T} \|J_t(u) - J_t(v)\|_{\alpha+1} \le c(1 + \|u\|^2_{\varepsilon,\alpha+1} + \|v\|^2_{\varepsilon,\alpha+1})^{1/2}\|u - v\|_{\varepsilon,\alpha+1}.$$

The Hölder semi-norm of $J(u) - J(v)$ is bounded by splitting the integral similarly to what was done in (38). One obtains

$$\begin{aligned}\|J_t(u)-&J_t(v) - J_s(u) + J_s(v)\|_{\alpha+1}\\ \le&\| \int_0^s (P_{t-s} - I)(P_{s-r}(G_r(u) - G_r(v)))\mathrm{d}r\|_{\alpha+1}\\ &+ \| \int_s^t P_{t-r}(G_r(u) - G_r(v))\,\mathrm{d}r\|_{\alpha+1}.\end{aligned}$$

Then we proceed similarly as for the bounds of $M_2$ and $M_3$ in the proof of Lemma 4.2, but using (36) instead of (35), and with $\varepsilon, \alpha$ instead of $\varepsilon', \alpha'$, to obtain

$$\begin{aligned}\|J_t(u)-&J_t(v) - J_s(u) + J_s(v)\|_{\alpha+1}\\ \le&(t-s)^{\varepsilon}\left(s^{\frac{-\alpha+1+\beta}{2}} + (t-s)^{\frac{-\alpha+1+\beta-2\varepsilon}{2}}\right) \times\\ &\times c(1 + \|u\|_{\varepsilon,\alpha+1} + \|v\|_{\varepsilon,\alpha+1})^{1/2}\|u - v\|_{\varepsilon,\alpha+1}.\end{aligned}$$

Thus

$$\sup_{0\le s<t\le T} \frac{\|J_t(u)-J_t(v)-J_s(u)+J_s(v)\|_{\alpha+1}}{(t-s)^{\varepsilon}}$$
$$\le cT^{\frac{-\alpha+1+\beta-2\varepsilon}{2}}(1+\|u\|_{\varepsilon,\alpha+1}+\|v\|_{\varepsilon,\alpha+1})^{1/2}\|u-v\|_{\varepsilon,\alpha+1}$$

and the proof is complete. □

**Lemma 4.6.** *Let Assumptions A1, A2, A3 and A4 hold and let* $\varepsilon$ *be chosen as in Lemma 4.2. Let* $u_0 \in \mathcal{C}^{\alpha+1+2\varepsilon+\nu}$ *for some small* $\nu > 0$*. Then the set*

$$\Lambda := \{u \in C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \text{ such that } u = \lambda J(u) \text{ for some } \lambda \in [0,1]\}$$

*is bounded in* $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$.

*Proof.* Let $u^* \in \Lambda$, that is $u^* = \lambda J(u^*)$ for some $\lambda \in [0,1]$. Applying Lemma 4.2 and Proposition 4.1 we get

$$\begin{aligned}\|u^*\|_{\varepsilon,\alpha+1} \le & \|J(u^*)\|_{\varepsilon,\alpha+1} \\ \le & c\|u_0\|_{\alpha+1+2\varepsilon+\nu} + cT^{\frac{-\alpha'+1+\beta-2\varepsilon}{2}}(1+\|u^*\|_{C_T\mathcal{C}^{\alpha+1}}) \\ \le & c\|u_0\|_{\alpha+1+2\varepsilon+\nu} + cT^{\frac{-\alpha'+1+\beta-2\varepsilon}{2}}(1+K),\end{aligned}$$

where the constant on the right hand side is finite and independent of $u^*$. □

**Theorem 4.7.** *Let Assumptions A1, A2, A3 and A4 hold and let* $\varepsilon > 0$ *be chosen according to Lemma 4.2. If* $u_0 \in \mathcal{C}^{\alpha+1+2\varepsilon+\nu}$ *for some small* $\nu > 0$*, then there exists a global mild solution* $u$ *of* (6) *in* $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$ *which is unique in* $C_T\mathcal{C}^{\alpha+1}$.

*Proof. Existence.* By Lemma 4.2 we have that

$$J : C_T^{\varepsilon}\mathcal{C}^{\alpha+1} \to C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$$

and by Lemma 4.5 we know that $J$ is also continuous. Moreover using Lemma 4.2 again we have that the operator $J$ maps balls of $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$ into balls of $C_T^{\varepsilon'}\mathcal{C}^{\alpha'+1}$ for some $\varepsilon' > \varepsilon$ and $\alpha' > \alpha$, which are pre-compact sets in $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$. Thus $J$ is compact. We conclude that $J$ has a fixed point $u^*$ in $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$ by Schauder's fixed point theorem and by Lemma 4.6. The fixed point $u^*$ is a mild solution of (6) in $C_T^{\varepsilon}\mathcal{C}^{\alpha+1}$.
*Uniqueness.* Clearly $u^* \in C_T\mathcal{C}^{\alpha+1}$. This solution is unique in the latter space by Remark 3.8. □

## 5. Applications to stochastic analysis

In this section we illustrate an application of non-linear singular PDEs to stochastic analysis, in particular to a class of non-linear backward stochastic differential equations (BSDEs) with distributional coefficients. The class of BSDEs that we consider here has not been studied previously in the BSDEs literature.

The concept of a BSDE was introduced in the early 90s by Pardoux and Peng [26]. Since then, BSDEs have become a popular research field and the literature on this topic is now vast, see for example two recent books [27, 29] and references therein. BSDEs own their success to the many applications they have in other areas of research. The main ones are their use in financial mathematics for pricing and hedging derivatives; their application to stochastic control theory to find the optimal control and the optimal value function; and their use in showing existence and uniqueness of solutions to certain classes of non-linear PDEs by means of a probabilistic representation of their solution (known as non-linear Feynman-Kac formula).

The application that we are going to illustrate below fits in the latter two of these three topics. Indeed, the singular PDE studied above will allow us to define and solve a singular BSDE which is linked to the PDE by an extended Feynman-Kac formula. Moreover this class of BSDEs arises also in stochastic control when looking at problems in Economics where an agent wants to maximise her exponential utility, see for example [3, Chapter 20] and [29, Chapter 7]. This latter class of BSDEs is known as quadratic BSDEs and is linked to the special non-linearity $F(x) = x^2$. Note that in this section we restrict to one space dimension. This restriction and the choice of quadratic $F$ are done to avoid technicalities, but it should be a simple exercise to extend the argument below to a general non-linear $F$ satisfying Assumption A1 and such that $F(0) = 0$. The multidimensional case ($d > 1$) should also be possible to treat, much in the spirit of [20]. Details of this are left to the interested reader and to future work.

Let us start by writing the PDE (6) in one-dimension and backward in time, which is the classical form (Kolmogorov backward equation) when dealing with BSDEs:

(39)
$$\begin{cases} \partial_t u(t,x) + \partial_{xx} u(t,x) + (\partial_x u(t,x))^2 b(t,x) = 0, & \text{for } (t,x) \in [0,T] \times \mathbb{R} \\ u(T,x) = \Phi(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

We observe that (by abuse of notation) we used the same symbol $u$ as in the forward PDE and we denoted by $\Phi$ rather than $u_0$ the final condition. This is done to be in line with classical BSDEs notation. The results of Section 3 and in particular Theorem 3.10 apply to this PDE because the only difference from (6) is the time-change. Indeed it is easy to check that $F(x) = x^2$ satisfies Assumption A1 and moreover $F(0) = 0$.

**Remark 5.1.** Since here we want to work in a given time-interval $[0, T]$ then we must ensure that the terminal condition $\Phi$ is small enough according to Theorem 3.10.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a BSDE of the form

$$Y_r^{t,x} = \Phi(B_T^{t,x}) + \int_r^T b(s, B_s^{t,x})(Z_s^{t,x})^2 \mathrm{d}s - \int_r^T Z_s^{t,x} \mathrm{d}B_s^{t,x}, \tag{40}$$

where $B := (B_r^{t,x})_{t \le r \le T}$ is a Brownian motion starting in $x$ at time $t$ and with quadratic variation $2r$ at time $r \ge t$. This latter non-standard quadratic variation is introduced to account for the fact that the generator of Brownian motion is $\frac{1}{2}\partial_{xx}$ but the operator in the PDE (39) is $\partial_{xx}$. The Brownian motion $B$ generates a filtration $\mathbb{F} := (\mathcal{F}_r)_{t \le r \le T}$. It is known that if $b$ and $\Phi$ are smooth enough functions and satisfy some bounds (see e.g. [29, Theorem 7.3.3]) then the solution to the BSDE exists and it is unique. Note that a solution to (40) is a *couple* of adapted processes $(Y^{t,x}, Z^{t,x})$ that satisfies (40) and some other integrability conditions (like the ones in the second bullet point of Definition 5.2 below). Moreover it is know that, in the classical case, the BSDE and the PDE above are linked via the Feynman-Kac formula, namely $Y_r^{t,x} = u(r, B_r^{t,x})$, and $Z_r^{t,x} = \partial_x u(r, B_r^{t,x})$.[1] In particular for the initial time $t$ one gets the stochastic representation for the solution of the PDE (39) in terms of the solution of the BSDE (40), namely

$$u(t,x) = Y_t^{t,x}.$$

In the remaining of this section we are going to use the results on the singular parabolic PDE to solve the singular BSDE (40) when $b \in L_T^\infty \mathcal{C}^\beta$. One of the delicate points here is to give a meaning to the term $\int_r^T b(s, B_s) Z_s^2 \mathrm{d}s$, which we do by using the *Itô trick*. The Itô trick has been used in the past to treat other SDEs and BSDEs with distributional coefficients, see e.g. [9, 20]. This trick makes use of the following auxiliary PDE

$$\begin{cases} \partial_t w(t,x) + \partial_{xx} w(t,x) = (\partial_x u(t,x))^2 b(t,x), & \text{for } (t,x) \in [0,T] \times \mathbb{R} \\ w(T,x) = 0, & \text{for } x \in \mathbb{R}, \end{cases} \tag{41}$$

where the function $u$ appearing on the right-hand side is the solution to (39). The mild form of this PDE is given by

$$w(t) = -\int_t^T P_{s-t}\left((\partial_x u(s))^2 b(s)\right) \mathrm{d}s.$$

Let us now do some *heuristic* reasoning. If $b$ was smooth, then applying Itô's formula to $w(r, B_r^{t,x})$ would give

$$\begin{aligned} \int_r^T \mathrm{d}w(s, B_s^{t,x}) &= \int_r^T \partial_t w(s, B_s^{t,x}) \mathrm{d}s + \int_r^T \partial_x w(s, B_s^{t,x}) \mathrm{d}B_s^{t,x} \\ &\quad + \frac{1}{2} \int_r^T \partial_{xx} w(s, B_s^{t,x}) 2 \mathrm{d}s \\ &= \int_r^T \partial_x w(s, B_s^{t,x}) \mathrm{d}B_s^{t,x} + \int_r^T (\partial_x u(s, B_s^{t,x}))^2 b(s, B_s^{t,x}) \mathrm{d}s. \end{aligned}$$

Moreover, if $b$ was smooth, then the classical theory on BSDEs ensures that $Z_r = \partial_x u(r, B_r^{t,x})$, so integrating the above equation one has

$$w(T, B_T) - w(r, B_r^{t,x}) = \int_r^T \partial_x w(s, B_s^{t,x}) \mathrm{d}B_s^{t,x} + \int_r^T (Z_s^{t,x})^2 b(s, B_s^{t,x}) \mathrm{d}s.$$

Thus we can express the singular term including $b$ in terms of quantities that are well defined and do not depend on $b$ explicitly, namely

$$\int_r^T (Z_s^{t,x})^2 b(s, B_s^{t,x}) \mathrm{d}s = -w(r, B_r^{t,x}) - \int_r^T \partial_x w(s, B_s^{t,x}) \mathrm{d}B_s^{t,x}. \tag{42}$$

We note that even in the singular case when $b \in L_T^\infty \mathcal{C}^\beta$ we have that all terms on the right hand side of (42) are well defined. Indeed using the regularity of $u$, $b$ and their product (see (4)) together with Lemma 3.2 one has that

[1] One side of the Feynman-Kac formula can be easily checked, namely that the couple $(u(r, B_r^{t,x}), \partial_x u(r, B_r^{t,x}))$ is a solution of the BSDE. This is done by applying Itô's formula to $u(r, B_r^{t,x})$.

$w \in C_T\mathcal{C}^{\alpha+1}$ and therefore $w$ is differentiable (in the classical sense) once in $x$, so $\partial_x w(s,x)$ is well defined.

The idea of the Itô trick is to "replace" the singular integral term with the right-hand side of (42), which is the motivation for the following definition. Note that we drop the superscript $\cdot^{t,x}$ for ease of notation.

**Definition 5.2.** *A couple* $(Y,Z)$ *is called* virtual solution *of* (40) *if*

- $Y$ *is continuous and* $\mathbb{F}$*-adapted and* $Z$ *is* $\mathbb{F}$*-progressively measurable;*
- $E\left[\sup_{r\in[t,T]}|Y_r|^2\right] < \infty$ *and* $E\left[\int_t^T |Z_r|^2 \mathrm{d}r\right] < \infty$*;*
- *for all* $r \in [t,T]$*, the couple satisfies the following backward SDE*

$$Y_r = \Phi(B_T) - w(r,B_r) - \int_r^T (Z_s + \partial_x w(s,B_s))\mathrm{d}B_s \tag{43}$$

$\mathbb{P}$*-almost surely.*

We now observe that BSDE (43) can be transformed into a classical BSDE by setting $\hat{Y}_r := Y_r + w(r,B_r)$ and $\hat{Z}_r := Z_r + \partial_x w(r,B_r)$. One has that (43) is equivalent to

$$\hat{Y}_r = \Phi(B_T) - \int_r^T \hat{Z}_s \mathrm{d}B_s, \tag{44}$$

thus the $\hat{Y}$ component in (44) is given explicitly by $\hat{Y}_r = \mathbb{E}\left[\Phi(B_T)|\mathcal{F}_r\right]$. Moreover by the martingale representation theorem (see e.g. [29, Theorem 2.5.2]) there exists a unique predictable process $\hat{Z}$ such that $\hat{Y}_r = \hat{Y}_t + \int_t^r \hat{Z}_s \mathrm{d}B_s$ and so $\hat{Y}_r = \hat{Y}_T - \int_r^T \hat{Z}_s \mathrm{d}B_s$. Therefore given the transformation $w$, we can find explicitly the virtual solution of (40) by

$$Y_r = \mathbb{E}\left[\Phi(B_T)|\mathcal{F}_r\right] - w(r,B_r), \text{ and } Z_r = \hat{Z}_r - \partial_x w(r,B_r). \tag{45}$$

What we explained above can be summarised in the following theorem.

**Theorem 5.3.** *If* $b \in L_T^\infty \mathcal{C}^\beta$*, then there exists a unique virtual solution* $(Y,Z)$ *of* (40) *given by* (45)*.*

**Remark 5.4.** It is easy to check that the notion of virtual solution coincides with the classical solution when $b$ is smooth, because the heuristic argument explained above to motivate (42) is actually rigorous. Indeed this is the case if $b \in L_T^\infty \mathcal{C}^\beta$ is also a function smooth enough so that $u \in C^{1,2}$ and so that the BSDE can be solved with classical theorems (see e.g. [29, Chapter 7]).

The notion of virtual solution for BSDEs has been previously used in [20] for the linear case when $F(x) = x$. There the authors show existence and uniqueness of a virtual solution for the corresponding BSDE similarly as what has been done here but for a slightly different class of drifts that live in Triebel-Lizorkin spaces rather than Besov spaces. Moreover for the linear case $F(x) = x$ it has been shown in [21] that the virtual solution introduced in [20] indeed coincides with a solution to the BSDE defined directly (hence by giving a meaning to the singular term instead of replacing it with known terms via the Itô trick). This was achieved with the introduction of an integral operator $A$ to represent the singular integral.

It will be objective of future research to investigate the existence of an integral operator $A$ related to the non-linear term $F(x)$ analogously to the integral operator introduced in [21], and give a meaning to the BSDE directly rather than via the Itô trick as done here.

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