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EXPLICIT DESCRIPTION OF GENERALIZED WEIGHT MODULES OF THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS \mathbb{I}_n

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ABSTRACT. For the algebra $\mathbb{I}_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$ of polynomial integro-differential operators over a field K of characteristic zero, a classification of simple weight and generalized weight (left and right) \mathbb{I}_n -modules is given. It is proven that the category of weight \mathbb{I}_n -modules is semisimple. An explicit description of generalized weight \mathbb{I}_n -modules is given and using it a criterion is obtained for the problem of classification of indecomposable generalized weight \mathbb{I}_n -modules to be of finite representation type, tame or wild. In the tame case, a classification of indecomposable generalized weight \mathbb{I}_n -modules is given. In the wild case ‘natural’ tame subcategories are considered with explicit description of indecomposable modules. It is proven that every generalized weight \mathbb{I}_n -module is a unique sum of absolutely prime modules. For an arbitrary ring R , we introduce the concept of *absolutely prime* R -module (a nonzero R -module M is absolutely prime if all nonzero subfactors of M have the same annihilator). It is shown that every indecomposable generalized weight \mathbb{I}_n -module is equidimensional. A criterion is given for a generalized weight \mathbb{I}_n -module to be finitely generated.

Key Words: the algebra of polynomial integro-differential operators, weight and generalized weight modules, indecomposable module, simple module, finite representation type, tame and wild.

Mathematics subject classification 2000: 16D60, 16D70, 16P50, 16U20.

1. INTRODUCTION

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; $\mathbb{N}_+ := \{1, 2, \dots\}$ and $\mathbb{Z}_{\leq 0} := -\mathbb{N}$; K is a field of characteristic zero and K^* is its group of units; $\otimes = \otimes_K$; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over K ; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K -linear derivations) of P_n ; $\text{End}_K(P_n)$ is the algebra of all K -linear maps from P_n to P_n ; the subalgebras

$$A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle \quad \text{and} \quad \mathbb{I}_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$$

of the algebra $\text{End}_K(P_n)$ are called the n 'th *Weyl algebra* and the *algebra of polynomial integro-differential operators*, respectively.

The Weyl algebras A_n are Noetherian algebras and domains. The algebras \mathbb{I}_n were introduced in [7, 8] and they are neither left nor right Noetherian and not domains. Moreover, they contain infinite direct sums of nonzero left and right ideals [7]. The algebra \mathbb{I}_n contains a polynomial algebra

$$D_n = K[H_1, \dots, H_n], \quad \text{where} \quad H_1 := \partial_1 x_1, \dots, H_n := \partial_n x_n,$$

which is a maximal commutative subalgebra of \mathbb{I}_n , [7]. An \mathbb{I}_n -module M is called a *weight module* if it is a *semisimple* D_n -module provided the field K is an algebraically closed field. For an arbitrary field, a weight \mathbb{I}_n -module is a direct sum of common eigenspaces for the commuting elements H_1, \dots, H_n , by definition. An \mathbb{I}_n -module M is called a *generalized weight module* if for all elements $m \in M$, $\dim_K(D_n m) < \infty$, i.e., M is a *locally finite dimensional* D -module provided the field K is an algebraically closed field. Every weight module is a generalized weight module but not vice versa. The \mathbb{I}_n -module P_n is a simple weight module. Introduction of (generalized) weight modules for the algebras \mathbb{I}_n was inspired by a similar concept for semisimple finite dimensional Lie algebras. For a semisimple finite dimensional Lie algebra \mathcal{G} , a classification of *simple* generalized weight modules is known only when $\mathcal{G} = \mathfrak{sl}_2$. Furthermore, a classification of *indecomposable* generalized weight \mathfrak{sl}_2 -modules was done in [5] and the problem of classification turned out to be

tame provided the Casimir element acts as a scalar. In fact, in [5] in the tame case a classification of indecomposable generalized weight modules was obtained for a large class of algebras, the, so-called, *generalized Weyl algebras* (the universal enveloping algebra $U(\mathfrak{sl}_2)$ and the Weyl algebras are examples of generalized Weyl algebras as well as many other quantum groups are). Recently, for some non-semisimple Lie algebras \mathcal{G} and their quantum analogues, classifications of simple weight modules are given: the Schrödinger algebra, [15]; $\mathfrak{sl}_2 \ltimes V_2$, [16], where V_2 is the simple 2-dimensional \mathfrak{sl}_2 -module; the enveloping algebra of the Euclidean algebra, [14]; $\mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$, [17]; the quantum spatial ageing algebra, [13].

In the paper, explicit descriptions of weight and generalized weight \mathbb{I}_n -modules are given (Theorem 3.2, Theorem 3.6 and (9), (19)). They are too technical to explain them in the Introduction. Classifications of simple weight and simple generalized weight \mathbb{I}_n -modules are obtained (Theorem 2.3). It is proven that the category of weight \mathbb{I}_n -modules is a semisimple category (Theorem 2.5). In [6], a classification of indecomposable generalized weight \mathbb{I}_1 -modules of finite length is given. This classification is used in the present paper in order to obtain the general case. Usually, the case $n = 1$ serves as the base of induction for the case $n > 1$. Using Theorem 2.3, a criterion is given to decide whether the problem of classification of indecomposable generalized weight \mathbb{I}_n -modules is of finite representation type, tame or wild (Theorem 3.7). In the case, $n = 1$ the problem is tame, [6]. In the tame case, a classification of indecomposable generalized weight \mathbb{I}_n -modules is given. It is shown that every indecomposable generalized weight \mathbb{I}_n -module is equidimensional (Corollary 3.8). A criterion is given for a generalized weight \mathbb{I}_n -module to be finitely generated (Corollary 3.9). In the wild case, ‘natural’ tame subcategories are considered with explicit description of indecomposable modules, see the end of Section 2. In particular, descriptions of categories $\text{ind}(D_2, \mathfrak{m}^2)$, $\text{ind}_f(\Gamma)$ and $\text{ind}_f(A)$ are obtained. In Section 4, similar results are proven for generalized weight *right* \mathbb{I}_n -modules.

Properties of the algebras \mathbb{I}_n are studied in [7, 9, 10]. In the case $n = 1$, for a more general setting see also [25]. The simple \mathbb{I}_1 -modules are classified in [9]. Simple A_1 -modules were classified in [19] (see also [2, 3] for some generalized Weyl algebras including A_1). The automorphism groups $\text{Aut}_{K\text{-alg}}(\mathbb{I}_n)$ are found in [8]. The weak homological dimension of the algebra \mathbb{I}_n is n , [7]. Furthermore, the weak homological dimension is n for all the factor algebras of \mathbb{I}_n , [12].

Finite dimensionality of Ext-groups of simple modules over the (first) Weyl algebra A_1 was proven in [30]. Finite dimensionality of Ext-groups of simple modules over the generalized Weyl algebras was proven in [1]. Simple modules over certain generalized Weyl algebras were classified in [2]. In [6], Ext-groups are described between indecomposable generalized weight \mathbb{I}_1 -modules, it is shown that they are finite dimensional vector spaces. In [10], it is proven that the algebra \mathbb{I}_n is a left coherent algebra iff the algebra \mathbb{I}_n is a right coherent iff $n = 1$; the algebra \mathbb{I}_n is a maximal left (resp., right) order in the largest left (resp., right) quotient ring $Q_l(\mathbb{I}_n)$ (resp., $Q_r(\mathbb{I}_n)$) of \mathbb{I}_n . The (left and right) global dimension of the algebra \mathbb{I}_n and all prime factor algebras of \mathbb{I}_n is equal to n , [12].

Classifications of (various classes of) simple weight modules over algebras that are close to the (generalized) Weyl algebras are given in [28, 18, 26, 32, 27, 22, 29, 23, 11].

2. CLASSIFICATION OF SIMPLE (GENERALIZED) WEIGHT \mathbb{I}_n -MODULES

In this section, a classification of simple generalized weight and simple weight \mathbb{I}_n -modules is given (Theorem 2.3). It is proven that the category of weight \mathbb{I}_n -modules is a semisimple category (Theorem 2.5). At the beginning of the section, we collect some results about the algebras \mathbb{I}_n that are used in the paper. In the case when $n = 1$, we drop the subscript ‘1’ in order to simplify the notation.

As an abstract algebra, the algebra \mathbb{I}_1 is generated by the elements ∂ , $H := \partial x$ and \int (since $x = \int H$) that satisfy the defining relations, [7, Proposition 2.2] (where $[a, b] := ab - ba$):

$$\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial. \quad (1)$$

Since $\mathbb{I}_n = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_1$ (n times), defining relations of the algebra \mathbb{I}_n is the union of the defining relations (1) for each index $i = 1, \dots, n$ and the relations $a_i a_j = a_j a_i$ for all $i \neq j$ where $a_i \in \{\partial_i, H_i, \int_i\}$. The elements of the algebra \mathbb{I}_1 ,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \quad (2)$$

satisfy the relations $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00} \partial^j$. The matrices of the linear maps $e_{ij} \in \text{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \text{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js} x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij}, \quad (3)$$

$Ke_{ij} = KE_{ij}$, and $F := \bigoplus_{i,j \geq 0} Ke_{ij} = \bigoplus_{i,j \geq 0} KE_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices. The algebra $\mathbb{I}_n = \mathbb{I}_1(1) \otimes \cdots \otimes \mathbb{I}_1(n) \simeq \mathbb{I}_1^{\otimes n}$ where $\mathbb{I}_1(i) = K\langle \partial_i, H_i, \int_i \rangle$ for $i = 1, \dots, n$.

\mathbb{Z}^n -grading on the algebra \mathbb{I}_n and the canonical form of an integro-differential operator, [7]. The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a \mathbb{Z} -graded algebra ($\mathbb{I}_{1,i} \mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases} \quad (4)$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of \mathbb{I}_1 , $He_{ii} = e_{ii}H = (i+1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of D_1);

$$\left(\int^i D_1 \right)_{D_1} \simeq D_1, \quad \int^i d \mapsto d; \quad D_1(D_1 \partial^i) \simeq D_1, \quad d \partial^i \mapsto d \text{ for all } i \geq 0$$

since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \rightarrow D_1 \int^i$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \rightarrow \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$. The algebra

$$\mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha}$$

is a \mathbb{Z}^n -graded algebra ($\mathbb{I}_{n,\alpha} \mathbb{I}_{n,\beta} \subseteq \mathbb{I}_{n,\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$) where $\mathbb{I}_{n,\alpha} = \bigotimes_{i=1}^n \mathbb{I}_1(i)_{\alpha_i}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$.

Each element a of the algebra \mathbb{I}_1 is a unique finite sum

$$a = \sum_{i>0} a_{-i} \partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (5)$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator, [7]. Let

$$v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then $\mathbb{I}_{1,i} = D_1 v_i = v_i D_1$ and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (6)$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^j \partial^i, H^j, \int^i H^j, e_{st} \mid i \geq 1; j, s, t \geq 0\}$ is a K -basis for the algebra \mathbb{I}_1 . The tensor product of these bases is a basis for the algebra \mathbb{I}_n . The multiplication in the algebra \mathbb{I}_1 is given by the rule:

$$\begin{aligned} \int H &= (H - 1) \int, \quad H \partial = \partial(H - 1), \quad \int e_{ij} = e_{i+1,j}, \\ e_{ij} \int &= e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij} \partial = \partial e_{i,j+1}, \\ H e_{ii} &= e_{ii} H = (i + 1) e_{ii}, \quad i \in \mathbb{N}, \end{aligned}$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The algebra \mathbb{I}_1 has the only proper ideal $F = \bigoplus_{i,j \in \mathbb{N}} K e_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra \mathbb{I}_1/F is canonically isomorphic to the skew Laurent polynomial algebra

$$B_1 := K[H][\partial, \partial^{-1}; \tau], \quad \tau(H) = H + 1, \quad \text{via } \partial \mapsto \partial, \quad \int \mapsto \partial^{-1}, \quad H \mapsto H$$

(where $\partial^{\pm 1} \alpha = \tau^{\pm 1}(\alpha) \partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1} H$).

Recall that the algebra of polynomial integro-differential operators

$$\mathbb{I}_n = K \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$$

over a field K of characteristic zero $\mathbb{I}_n = \mathbb{I}_1(1) \otimes \dots \otimes \mathbb{I}_1(n)$ is the tensor product of algebras $\mathbb{I}_1(i) = K \langle x_i, \partial_i, \int_i \rangle \simeq \mathbb{I}_1$. Each algebra $\mathbb{I}_1(i)$ contains a unique proper ideal

$$F(i) = \bigoplus_{s,t \in \mathbb{N}} K e_{st}(i) \quad \text{where } e_{st} = \int_i^s \partial_i^t - \int_i^{s+1} \partial_i^{t+1},$$

$F(i)^2 = F(i)$ and $B_1(i) := \mathbb{I}(i)/F(i) \simeq K[H_i][\partial_i, \partial_i^{-1}; \sigma_i^{-1}]$ where $\sigma_i(H_i) = H_i - 1$. The algebra \mathbb{I}_n is a local algebra where the unique maximal ideal \mathfrak{a}_n is generated by $F(1), \dots, F(n)$ and the factor algebra $\mathbb{I}_n/\mathfrak{a}_n$ is isomorphic to the skew Laurent polynomial algebra

$$B_n = D_n[\partial_1^{\pm 1}, \dots, \partial_n^{\pm 1}; \sigma_1^{-1}, \dots, \sigma_n^{-1}] \quad \text{where } \sigma_i(H_j) = H_j - \delta_{ij},$$

see [7]. Furthermore, the algebra B_n is the only left/right Noetherian factor algebra of \mathbb{I}_n , [7].

A classification of all ideals (including prime ideals) of the algebra \mathbb{I}_n is obtained in [7]. There are precisely n height 1 prime ideals:

$$\mathfrak{p}_1 = F \otimes \mathbb{I}_{n-1}, \quad \mathfrak{p}_2 = \mathbb{I}_1 \otimes F \otimes \mathbb{I}_{n-2}, \dots, \mathfrak{p}_n = \mathbb{I}_{n-1} \otimes F,$$

see [7]. The algebra \mathbb{I}_n is a prime algebra (0 is a prime ideal of \mathbb{I}_n). Every nonzero prime ideal \mathfrak{p} is a unique sum

$$\mathfrak{p}_I = \sum_{i \in I} \mathfrak{p}_i$$

of height 1 prime ideals where $I \subseteq \{1, \dots, n\}$ is a unique set for the ideal $\mathfrak{p} = \mathfrak{p}_I$, and $\text{ht}(\mathfrak{p}) = |I|$ where $\text{ht}(\mathfrak{p})$ is the height of the ideal \mathfrak{p} . Every ideal of \mathbb{I}_n is an idempotent ideal ($\mathfrak{a}^2 = \mathfrak{a}$), ideals of \mathbb{I}_n commute ($\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$) and the ideal $\mathfrak{a}_n = \mathfrak{p}_1 + \dots + \mathfrak{p}_n$ is the only maximal ideal of the algebra \mathbb{I}_n .

Generalized weight \mathbb{I}_n -modules. The group \mathbb{Z}^n acts on the vector space K^n by addition. For an element $\lambda = (\lambda_1, \dots, \lambda_n) \in K^n$, $\mathcal{O}(\lambda) = \lambda + \mathbb{Z}^n$ is its orbit. The set of all \mathbb{Z}^n -orbits is isomorphic to the factor group K^n/\mathbb{Z}^n , $\mathcal{O}(\lambda) \leftrightarrow \lambda + \mathbb{Z}^n$. In particular, two orbits are equal, $\mathcal{O}(\lambda) = \mathcal{O}(\lambda')$ iff $\lambda - \lambda' \in \mathbb{Z}^n$.

Let e_1, \dots, e_n be the standart basis for the vector space K^n . Then $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z} e_i \subseteq K^n = \bigoplus_{i=1}^n K e_i$.

The K -automorphisms $\sigma_1, \dots, \sigma_n$ of the polynomial algebra D_n commute, $\sigma_i \sigma_j = \sigma_j \sigma_i$, since $\sigma_i(H_j) = H_j - \delta_{ij}$ where δ_{ij} is the Kronecker delta. The subgroup $G = \langle \sigma_1, \dots, \sigma_n \rangle$ of the group of K -algebra automorphisms $\text{Aut}_K(D_n)$ generated by the automorphisms $\sigma_1, \dots, \sigma_n$ is an abelian group isomorphic to \mathbb{Z}^n via isomorphism $G \rightarrow \mathbb{Z}^n$, $\sigma_1 \mapsto e_1, \dots, \sigma_n \mapsto e_n$.

Let \mathcal{M}_n be the set of all maximal ideals of the algebra \mathbb{I}_n of the type

$$\mathfrak{m} = \mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n) \text{ where } \lambda = (\lambda_1, \dots, \lambda_n) \in K^n.$$

The group $\text{Aut}_K(D_n)$ acts on \mathcal{M}_n , $(\sigma, \mathfrak{m}) \mapsto \sigma(\mathfrak{m})$. Recall that the group \mathbb{Z}^n acts on K^n in the obvious way: $\mathbb{Z}^n \times K^n \rightarrow K^n$, $(i, \lambda) \mapsto i + \lambda$. In a similar way, the group G acts on \mathcal{M}_n , $(\sigma_1^{i_1} \cdots \sigma_n^{i_n}, \mathfrak{m}) \mapsto \sigma_1^{i_1} \cdots \sigma_n^{i_n}(\mathfrak{m})$ where $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$. When we identify the set \mathcal{M}_n with K^n via the bijection

$$\mathcal{M}_n \rightarrow K^n, \quad \mathfrak{m}_\lambda \mapsto \lambda$$

and the group G with \mathbb{Z}^n via the group isomorphism $G \rightarrow \mathbb{Z}^n$, $\sigma_i \mapsto e_i$ ($i = 1, \dots, n$), then the set of G -orbits \mathcal{M}_n/G is identified with the factor group K^n/\mathbb{Z}^n via the bijection

$$\mathcal{M}_n/G \rightarrow K^n/\mathbb{Z}^n, \quad G\mathfrak{m}_\lambda \mapsto \lambda + \mathbb{Z}^n$$

and the action $\sigma_1^{i_1} \cdots \sigma_n^{i_n}(\mathfrak{m}_\lambda)$ is identified with the action $\lambda + i$ where $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$.

The polynomial algebra $D_n = K[H_1, \dots, H_n]$ is a maximal commutative subalgebra of \mathbb{I}_n . For each $\lambda = (\lambda_1, \dots, \lambda_n) \in K^n$, the field

$$K_\mathfrak{m} := D_n/\mathfrak{m}_\lambda = D_n/(H_1 - \lambda_1, \dots, H_n - \lambda_n) = K_\lambda \simeq K$$

is a unique simple D_n -module that is annihilated by the maximal ideal $\mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n)$. Let $\text{Max}(D_n)$ be the maximum spectrum of D_n (the set of maximal ideals of D_n). Notice that if K is an algebraically closed field then $\mathcal{M}_n = \text{Max}(D_n)$.

An \mathbb{I}_n -module M is called a *weight module* if

$$M = \bigoplus_{\mathfrak{m} \in \mathcal{M}_n} M_\mathfrak{m} \text{ where } M_\mathfrak{m} = \text{ann}_M(\mathfrak{m}) := \{m \in M \mid \mathfrak{m}m = 0\}.$$

If $\mathfrak{m} = \mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n)$ for some $\lambda \in K^n$ then

$$M_\lambda := M_{\mathfrak{m}_\lambda} = \{m \in M \mid H_1 m = \lambda_1 m, \dots, H_n m = \lambda_n m\}.$$

The set $\text{Supp}(M) = \{\mathfrak{m} \in \mathcal{M}_n \mid M_\mathfrak{m} \neq 0\}$ is called the *support* of the weight \mathbb{I}_n -module M . So, an \mathbb{I}_n -module M is weight iff it is a (direct) sum of common eigen-spaces for the commuting elements H_1, \dots, H_n of the algebra \mathbb{I}_n .

An \mathbb{I}_n -module M is called a *generalized weight module* if

$$M = \bigoplus_{\mathfrak{m} \in \mathcal{M}_n} M^\mathfrak{m} \text{ where } M^\mathfrak{m} = \{m \in M \mid \mathfrak{m}^i m = 0 \text{ for some } i \geq 0\} = \bigcup_{i \geq 0} \text{ann}_M(\mathfrak{m}^i).$$

If $\mathfrak{m} = \mathfrak{m}_\lambda$ for some $\lambda \in K^n$ then

$$M^\lambda := M_{\mathfrak{m}_\lambda}^\lambda = \{m \in M \mid (H_1 - \lambda_1)^i m = 0, \dots, (H_n - \lambda_n)^i m = 0 \text{ for some } i \geq 1\}.$$

The set $\text{Supp}(M) = \{\mathfrak{m} \in \mathcal{M}_n \mid M^\mathfrak{m} \neq 0\}$ is called the *support* of the generalized weight \mathbb{I}_n -module M . Recall that we identified $(G, \mathcal{M}_n, \mathcal{M}_n/G)$ with $(\mathbb{Z}^n, K^n, K^n/\mathbb{Z}^n)$. Therefore, \mathfrak{m}_λ is identified with λ , $M_\lambda := M_{\mathfrak{m}_\lambda}$ and $M^\lambda := M^{\mathfrak{m}_\lambda}$. So,

$$\text{Supp}(M) = \{\lambda \in K^n \mid M^\lambda \neq 0\}.$$

If, in addition, the field K is an algebraically closed then the set $\text{Max}(D_n)$ of maximal ideals of the ring D_n is equal to \mathcal{M}_n , every weight \mathbb{I}_n -module is an \mathbb{I}_n -module which is a semisimple D_n -module (and vice versa), and every generalized weight \mathbb{I}_n -module is an \mathbb{I}_n -module which is *locally finite dimensional* D_n -module (and vice versa).

We denote by $\text{W}(\mathbb{I}_n)$ (resp., $\text{GW}(\mathbb{I}_n)$) the category of weight (resp., generalized weight) \mathbb{I}_n -modules. Clearly,

$$\text{W}(\mathbb{I}_n) \subseteq \text{GW}(\mathbb{I}_n) \subseteq \mathbb{I}_n\text{-Mod}$$

are inclusions of categories where $\mathbb{I}_n\text{-Mod}$ is the category of all left \mathbb{I}_n -modules. The category $\text{GW}(\mathbb{I}_n)$ is a full subcategory of $\mathbb{I}_n\text{-Mod}$, it is closed under arbitrary direct sums, extensions, submodules and factor modules. The category $\text{W}(\mathbb{I}_n)$ is closed under direct sums, submodules and factor modules but not under extensions, see [6, Theorem 2.5].

Let M be a generalized weight \mathbb{I}_n -module. It follows from the defining relations of the algebra \mathbb{I}_n , [7, Proposition 2.2] or (1), that for all $\mathfrak{m} \in \mathcal{M}_n$ and $i = 1, \dots, n$,

$$\begin{aligned} x_i M^{\mathfrak{m}} &\subseteq M^{\sigma_i(\mathfrak{m})}, \quad \int_i M^{\mathfrak{m}} \subseteq M^{\sigma_i(\mathfrak{m})}, \quad \partial_i M^{\mathfrak{m}} \subseteq M^{\sigma_i^{-1}(\mathfrak{m})}, \\ x_i M_{\mathfrak{m}} &\subseteq M_{\sigma_i(\mathfrak{m})}, \quad \int_i M_{\mathfrak{m}} \subseteq M_{\sigma_i(\mathfrak{m})}, \quad \partial_i M_{\mathfrak{m}} \subseteq M_{\sigma_i^{-1}(\mathfrak{m})}. \end{aligned}$$

So, the generalized weight \mathbb{I}_n -module M is a direct sum of its generalized weight submodules $M^{\mathcal{O}}$,

$$M = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} M^{\mathcal{O}} \quad \text{where} \quad M^{\mathcal{O}} = \bigoplus_{\mathfrak{m} \in \mathcal{O}} M^{\mathfrak{m}}. \quad (7)$$

Similarly, a weight \mathbb{I}_n -module M is a direct sum of its weight submodules $M_{\mathcal{O}}$,

$$M = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} M_{\mathcal{O}} \quad \text{where} \quad M_{\mathcal{O}} = \bigoplus_{\mathfrak{m} \in \mathcal{O}} M_{\mathfrak{m}}. \quad (8)$$

For each orbit $\mathcal{O} \in \mathcal{M}_n/G$, let $\mathbb{W}(\mathbb{I}_n, \mathcal{O})$ (resp., $\mathbb{GW}(\mathbb{I}_n, \mathcal{O})$) be the subcategory of weight (resp., generalized weight) \mathbb{I}_n -modules M with $\text{Supp}(M) \subseteq \mathcal{O}$. By (7) and (8),

$$\mathbb{W}(\mathbb{I}_n) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} \mathbb{W}(\mathbb{I}_n, \mathcal{O}) \quad \text{and} \quad \mathbb{GW}(\mathbb{I}_n) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} \mathbb{GW}(\mathbb{I}_n, \mathcal{O}), \quad (9)$$

direct sum of full subcategories of $\mathbb{W}(\mathbb{I}_n)$ and $\mathbb{GW}(\mathbb{I}_n)$, respectively.

So, the problem of classification of indecomposable weight or generalized weight \mathbb{I}_n -modules is reduced to the case when the support of a module belongs to a single orbit.

Let $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ be a short exact sequence of \mathbb{I}_n -modules. Then M is a generalized weight module iff so are the modules N and L , and in this case,

$$\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(L). \quad (10)$$

The simple weight \mathbb{I}_n -module P_n . By the definition, the algebra \mathbb{I}_n is a subalgebra of the algebra $\text{End}_K(P_n)$ of all K -endomorphism of the vector space P_n . So, the polynomial algebra P_n is a (left) \mathbb{I}_n -module. Since $A_n \subset \mathbb{I}_n$, the A_n -module P_n is a simple faithful A_n -module. Hence, the \mathbb{I}_n -module P_n is also simple and faithful. The action of the elements x_i, ∂_i, H_i and \int_i on P_n is given by the rule: For all elements $p \in P_n$,

$$\partial_i p = \frac{\partial p}{\partial x_i}, \quad H_i p = \frac{\partial}{\partial x_i}(x_i p), \quad \int_i p = \int_0^{x_i} p dx_i \quad \text{and} \quad x_i p = x_i \cdot p \quad (\text{multiplication by } x_i).$$

For all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $i = 1, \dots, n$, $H_i x^\alpha = (\alpha_i + 1)x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Therefore, the \mathbb{I}_n -module P_n is a weight module, $\text{Supp}(P_n) = \mathbb{N}_+^n$. The polynomial algebra $P_n = K[x_1] \otimes \cdots \otimes K[x_n]$ is a tensor product of polynomial algebras $K[x_i]$. Furthermore, it is a tensor product of simple weight $\mathbb{I}_1(i)$ -modules $K[x_i]$ where $\mathbb{I}_1(i) = K\langle \partial_i, H_i, \int_i \rangle$.

The indecomposable \mathbb{I}_1 -modules $M(s, \lambda)$, [6]. For $\lambda \in K$ and a natural number $s \geq 1$, consider the B_1 -module

$$M(s, \lambda) := B_1 \otimes_{K[H]} K[H]/(H - \lambda)^s. \quad (11)$$

Clearly,

$$M(s, \lambda) \simeq B_1/B_1(H - \lambda)^s \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^s). \quad (12)$$

The \mathbb{I}_1 -module/ B_1 -module $M(s, \lambda)$ is a generalized weight module with $\text{Supp} M(s, \lambda) = \lambda + \mathbb{Z}$,

$$M(s, \lambda) = \bigoplus_{i \in \mathbb{Z}} M(s, \lambda)^{\lambda+i} \quad \text{and} \quad \dim M(s, \lambda)^{\lambda+i} = s \quad \text{for all } i \in \mathbb{Z}. \quad (13)$$

Lemma 2.1. (1) *Each simple generalized weight \mathbb{I}_1 -module is a simple weight \mathbb{I}_1 -module, and vice versa.*

- (2) Each simple generalized weight \mathbb{I}_1 -module is isomorphic to one of the modules: $K[x]$ or $M(1, \lambda)$, $\lambda \in \Lambda$ (where Λ is any fixed subset of K such that the map $\Lambda \rightarrow K/\mathbb{Z}$, $\lambda \mapsto \lambda + \mathbb{Z}$ is a bijection), they are pairwise non-isomorphic.
- (3) $\text{End}_{\mathbb{I}_1}(M) \simeq K$ for all simple weight \mathbb{I}_1 -modules.

Proof. 1 and 2. Statements 1 and 2 follow from [6, Theorem 2.5].

3. $\text{Kid}_{K[x]} \subseteq \text{End}_{\mathbb{I}_1}(K[x]) \subseteq \text{End}_{A_1}(K[x]) \simeq \text{Ker}_{K[x]}(\partial \cdot) \simeq K$ where $\partial \cdot : K[x] \rightarrow K[x], p \mapsto \frac{dp}{dx}$ and $A_1 = K\langle x, \partial \mid \partial x - x\partial = 1 \rangle$ is the first Weyl algebra. Therefore, $\text{End}_{\mathbb{I}_1}(K[x]) \simeq K$. Similarly,

$$\text{End}_{\mathbb{I}_1}(M(1, \lambda)) \simeq \text{End}_{\mathbb{I}_1}(\mathbb{I}_1/(\mathbb{I}_1(H - \lambda))) \simeq \text{Ker}_{M(1, \lambda)}((H - \lambda) \cdot) \simeq K. \quad \square$$

Lemma 2.2. (1) [7, Proposition 6.1.(1)] *The \mathbb{I}_1 -module $K[x]$ is isomorphic to $\mathbb{I}_1/\mathbb{I}_1\partial$.*
 (2) [6, Eq. (12)] *For all elements $\lambda \in K \setminus \mathbb{N}_+$, the \mathbb{I}_1 -module $M(1, \lambda)$ is isomorphic to the \mathbb{I}_1 -module $\mathbb{I}_1/\mathbb{I}_1(H - \lambda)$.*

For all $n \in \mathbb{N}_+$, $F + \mathbb{I}_1(H_1 - n) = E_{*, n-1} \oplus \mathbb{I}_1(H_1 - n)$, where $E_{*, n-1} := \bigoplus_{i \geq 0} K e_{i, n-1} \simeq \mathbb{I}_1 K[x_i]$. Hence, $M(1, n) = \mathbb{I}_1/(E_{*, n-1} \oplus \mathbb{I}_1(H_1 - n))$ and there is a short split sequence of \mathbb{I}_1 -modules

$$0 \rightarrow K[x_1] \simeq E_{*, n-1} \rightarrow \mathbb{I}_1/\mathbb{I}_1(H_1 - n) \rightarrow M(1, n) \rightarrow 0. \quad (14)$$

In fact, it splits (Theorem 2.5).

Let A be an algebra and $A\text{-Mod}$ be the category of left A -modules. A subcategory \mathcal{C} of $A\text{-Mod}$ is called a *semisimple* category if every module of \mathcal{C} is a direct sum of its simple modules in \mathcal{C} . The category $\text{GW}(\mathbb{I}_n)$ of generalized weight \mathbb{I}_n -modules is a subcategory of the category $\mathbb{I}_n\text{-Mod}$ of all left \mathbb{I}_n -modules. The category $\text{W}(\mathbb{I}_n)$ of weight \mathbb{I}_n -modules is a subcategory of $\text{GW}(\mathbb{I}_n)$. For an A -module M , we denote by $\text{End}_A(M)$ its algebra of endomorphisms.

Classification of simple weight \mathbb{I}_n -modules. We denote by $\widehat{\mathbb{I}}_n(\text{weight})$ (resp., $\widehat{\mathbb{I}}_n(\text{gen. weight})$) the set of isomorphism classes of simple weight (resp., generalized weight) \mathbb{I}_n -modules. The next theorem classifies (up to isomorphism) all the simple weight \mathbb{I}_n -modules.

Theorem 2.3. (1) $\widehat{\mathbb{I}}_n(\text{gen. weight}) = \widehat{\mathbb{I}}_n(\text{weight}) = \widehat{\mathbb{I}}_1(\text{weight})^{\otimes n}$, i.e., any simple generalized weight \mathbb{I}_n -module is a simple weight \mathbb{I}_n -module, and vice versa; any simple weight \mathbb{I}_n -module M is isomorphic to the tensor product $M_1 \otimes \cdots \otimes M_n$ of simple weight \mathbb{I}_1 -modules and two such modules are isomorphic over \mathbb{I}_n , $M_1 \otimes \cdots \otimes M_n \simeq M'_1 \otimes \cdots \otimes M'_n$, iff for each $i = 1, \dots, n$, the \mathbb{I}_1 -modules M_i and M'_i are isomorphic.
 (2) For each simple weight \mathbb{I}_n -module $M = \bigotimes_{i=1}^n M_i$, $\text{Supp}(M) = \prod_{i=1}^n \text{Supp}(M_i)$.

Proof. To prove the theorem we use induction on n . The case $n = 1$ is true, see [6, Theorem 2.5].

Suppose that $n > 1$ and that theorem is true for all $n' < n$. Let M be a simple generalized weight \mathbb{I}_n -module. By (7), $\text{Supp}(M) \subseteq \mathcal{O}$ for some orbit $\mathcal{O} \in K^n/\mathbb{Z}^n$. Since $\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1}$, the \mathbb{I}_1 -module M is a generalized weight \mathbb{I}_1 -module with $\text{Supp}_{\mathbb{I}_1}(M) \subseteq \lambda_1 + \mathbb{Z}$ where for some $\lambda_1 \in K$ such that $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Supp}_{\mathbb{I}_n}(M)$ for some $\lambda_2, \dots, \lambda_n \in K$. Fix a nonzero vector $v \in M^\lambda$ such that $(H_1 - \lambda_1)v = 0$. The \mathbb{I}_1 -submodule $\mathbb{I}_1 v$ of M is an epimorphic image of the \mathbb{I}_1 -module $N = \mathbb{I}_1/\mathbb{I}_1(H_1 - \lambda_1)$. By [9], the \mathbb{I}_1 -module N has finite length. Hence, so is the \mathbb{I}_1 -module $\mathbb{I}_1 v$. Changing the element v , if necessary, we can assume that the \mathbb{I}_1 -module $M_1 = \mathbb{I}_1 v$ is a simple weight \mathbb{I}_1 -module. The \mathbb{I}_n -module M is a simple module. The \mathbb{I}_n -module homomorphism

$$N \otimes \mathbb{I}_{n-1} = \mathbb{I}_1/\mathbb{I}_1(H_1 - \lambda_1) \otimes \mathbb{I}_{n-1} \longrightarrow M, \quad \bar{1} \otimes 1 \mapsto v \quad (\text{where } \bar{1} = 1 + \mathbb{I}_1(H_1 - \lambda_1))$$

is an epimorphism. By Lemma 2.1.(3), $\text{End}_{\mathbb{I}_1}(M) \simeq K$. By [4], $M \simeq N \otimes M'$ for some simple \mathbb{I}_{n-1} -module M' . The \mathbb{I}_n -module M is a generalized weight \mathbb{I}_n -module. Hence, the \mathbb{I}_{n-1} -module M' is a generalized weight \mathbb{I}_{n-1} -module. Now, the result follows by induction on n . \square

The category $\text{W}(\mathbb{I}_n)$ of weight \mathbb{I}_n -modules is a semisimple category. Let $n = 1$. For each orbit $\mathcal{O} \in K/\mathbb{Z}$, we fix an element $\lambda_{\mathcal{O}} \in K$ such that $\lambda_{\mathcal{O}} \in \mathcal{O}$. In particular, $\lambda_{\mathcal{O}} + \mathbb{Z} = \mathcal{O}$. For $\mathcal{O} = \mathbb{Z}$ let $\lambda_{\mathbb{Z}} := 0$. Let $n > 1$. for each orbit $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \in K^n/\mathbb{Z}^n$, let $\lambda_{\mathcal{O}} :=$

$(\lambda_{\mathcal{O}_1}, \dots, \lambda_{\mathcal{O}_n}) \in K^n$. In particular $\lambda_{\mathcal{O}} + \mathbb{Z}^n = \mathcal{O}$. The map $\mathcal{O} \mapsto \lambda_{\mathcal{O}}$ is a bijection, by definition. An orbit $\mathcal{O} = \mathcal{O}_1 \times \dots \times \mathcal{O}_n \in K^n/\mathbb{Z}^n$ is a direct product of orbits $\mathcal{O}_i \in K/\mathbb{Z}$. For the orbit \mathcal{O} , let

$$\mathbb{D}_{\mathcal{O}} := \{i \in \{1, \dots, n\} \mid \mathcal{O}_i = \mathbb{Z}\}.$$

Then $\{1, \dots, n\} \setminus \mathbb{D}_{\mathcal{O}} = \{j \in \{1, \dots, n\} \mid \mathcal{O}_j \neq \mathbb{Z}\}$. Let $\mathcal{D}_{\mathcal{O}}$ be any subset of $\mathbb{D}_{\mathcal{O}}$ (eg., $\mathcal{D}_{\mathcal{O}} = \emptyset$). Then

$$\{1, \dots, n\} = \mathcal{D}_{\mathcal{O}} \sqcup \mathcal{N}_{\mathcal{O}} \quad (15)$$

is a disjoint union where $\mathcal{N}_{\mathcal{O}} = \{1, \dots, n\} \setminus \mathcal{D}_{\mathcal{O}}$. For each pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$, let us define the \mathbb{I}_n -module

$$M(\mathcal{D}_{\mathcal{O}}) := \bigotimes_{i=1}^n M(\mathcal{D}_{\mathcal{O}_i}) \quad \text{where} \quad M(\mathcal{D}_{\mathcal{O}_i}) = \begin{cases} K[x_i] & \text{if } i \in \mathcal{D}_{\mathcal{O}}, \\ M(1, \lambda_{\mathcal{O}_i}) & \text{if } i \notin \mathcal{D}_{\mathcal{O}}. \end{cases} \quad (16)$$

For each choice of the set $\mathcal{D}_{\mathcal{O}}$, $\mathcal{O} = \mathcal{O}_{\text{deg}} \times \mathcal{O}_{\text{ndeg}}$ where $\mathcal{O}_{\text{deg}} := \prod_{i \in \mathcal{D}_{\mathcal{O}}} \mathcal{O}_i \simeq \mathbb{N}_+^{|\mathcal{D}_{\mathcal{O}}|}$ is called the *degenerate part* of the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$ and $\mathcal{O}_{\text{ndeg}} := \prod_{j \in \mathcal{N}_{\mathcal{O}}} \mathcal{O}_j$ is called the *non-degenerate part* of the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$. The elements of the set $\mathcal{D}_{\mathcal{O}}$ (resp., $\mathcal{N}_{\mathcal{O}}$) are called the *degenerate* (resp., *non-degenerate*) *indices* with respect to the pair $(\mathcal{O}, \mathcal{D}_{\mathcal{O}})$, or, simply, the *$\mathcal{D}_{\mathcal{O}}$ -degenerate* (resp., *$\mathcal{D}_{\mathcal{O}}$ -non-degenerate*) *indices*.

For each subset I of $\{1, \dots, n\}$, let $\mathfrak{a}_n(I)$ be the ideal of \mathbb{I}_n generated by the ideals $F(i)$ of $\mathbb{I}_1(i)$ where $i \in I$. If $I = \emptyset$ we set $\mathfrak{a}_n(\emptyset) = 0$. Clearly, $\mathfrak{a}_n(I) = \sum_{i \in I} \mathfrak{p}_i$. If $I = \{1, \dots, n\}$ then $\mathfrak{a}_n(I) = \mathfrak{a}_n$ is the maximal ideal of \mathbb{I}_n . The factor algebra

$$\mathbb{I}_n(\mathcal{D}_{\mathcal{O}}) := \mathbb{I}_n/\mathfrak{a}_n(\mathcal{D}_{\mathcal{O}}) \simeq B(\mathcal{O}_{\text{deg}}) \otimes \mathbb{I}_l(\mathcal{O}_{\text{ndeg}})$$

is a tensor product of algebras where

$$B(\mathcal{O}_{\text{deg}}) := \bigotimes_{i \in \mathcal{D}_{\mathcal{O}}} B_1(i), \quad \mathbb{I}_l(\mathcal{O}_{\text{ndeg}}) := \bigotimes_{i \in \mathcal{N}_{\mathcal{O}}} \mathbb{I}_1(i) \quad \text{and} \quad l = |\mathcal{N}_{\mathcal{O}}|.$$

For an algebra A and an A -module M , $\text{ann}_A(M) := \{a \in A \mid aM = 0\}$ is the *annihilator* of the A -module M . The A -module M is called a *faithfull* A -module if $\text{ann}_A(M) = 0$.

Corollary 2.4. *Let $\mathcal{O} \in K^n/\mathbb{Z}^n$. Then the set $\text{GW}(\mathbb{I}_n, \widehat{\mathcal{O}})$ of isomorphism classes of simple (generalized) weight \mathbb{I}_n -modules in the category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is equal to $\{M(\mathcal{D}_{\mathcal{O}}) \mid \mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}\}$ and the number of elements in this set is $2^{|\mathbb{D}_{\mathcal{O}}|}$.*

- (1) *The \mathbb{I}_n -module $M(\mathcal{D}_{\mathcal{O}})$ is faithfull iff $\mathcal{D}_{\mathcal{O}} = \{1, \dots, n\}$.*
- (2) *If $\mathcal{D}_{\mathcal{O}} \neq \{1, \dots, n\}$ then $\text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \mathfrak{a}_n(\mathcal{N}_{\mathcal{O}})$.*
- (3) *The elements in $\text{GW}(\mathbb{I}_n, \widehat{\mathcal{O}})$ are uniquely determined by their annihilators, i.e., the map $M(\mathcal{D}_{\mathcal{O}}) \mapsto \text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}}))$ is a bijection.*

Proof. Since $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ and $M(\mathcal{D}_{\mathcal{O}}) = \bigotimes_{i=1}^n M(\mathcal{D}_{\mathcal{O}_i})$, we have $\text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \mathfrak{a}_n(\mathcal{N}_{\mathcal{O}})$ and statements 1 and 2 follow. Then statement 2 implies statement 3, and the rest follows. \square

Clearly,

$$S(\mathcal{D}_{\mathcal{O}}) := \text{Supp}(M(\mathcal{D}_{\mathcal{O}})) = \mathbb{N}_+^{|\mathcal{D}_{\mathcal{O}}|} \times \prod_{i \in C\mathcal{D}_{\mathcal{O}}} \mathcal{O}_i \quad (17)$$

where $\mathcal{O} = \prod_{i=1}^n \mathcal{O}_i$ and $C\mathcal{D}_{\mathcal{O}} = \{1, \dots, n\} \setminus \mathcal{D}_{\mathcal{O}}$ is the complement of the set $\mathcal{D}_{\mathcal{O}}$ in $\{1, \dots, n\}$.

Theorem 2.5. *Every weight \mathbb{I}_n -module is a direct sum of simple weight \mathbb{I}_n -modules. In particular, the category $\text{W}(\mathbb{I}_n)$ of weight \mathbb{I}_n -modules is a semisimple category.*

Proof. In view of (9), it suffices to show that for each orbit $\mathcal{O} \in K^n/\mathbb{Z}^n$ and for any two weight simple \mathbb{I}_n -modules M and \overline{M} with $\text{Supp}(M), \text{Supp}(\overline{M}) \subseteq \mathcal{O}$ the short exact sequence

$$0 \rightarrow M \rightarrow M' \xrightarrow{f} \overline{M} \rightarrow 0$$

splits where M' is a weight \mathbb{I}_n -module. Notice that $\text{Supp}(M') = \text{Supp}(M) \cup \text{Supp}(\overline{M}) \subseteq \mathcal{O}$. By Corollary 2.4, $M = M(\mathcal{D}_{\mathcal{O}})$ and $\overline{M} = M(\overline{\mathcal{D}}_{\mathcal{O}})$ for some subsets $\mathcal{D}_{\mathcal{O}}$ and $\overline{\mathcal{D}}_{\mathcal{O}}$ of the set $\mathbb{D}_{\mathcal{O}}$.

For each element $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Supp}(\overline{M}) \subseteq \mathcal{O}$, we have the short exact sequence of $K_{\mathfrak{m}_\lambda}$ -modules where $K_{\mathfrak{m}_\lambda} := D_n/\mathfrak{m}_\lambda$ and $\mathfrak{m}_\lambda = (H_1 - \lambda_1, \dots, H_n - \lambda_n)$,

$$0 \rightarrow M_\lambda \rightarrow M'_\lambda \xrightarrow{f} \overline{M}_\lambda \rightarrow 0.$$

Notice that $\dim_K(M_\lambda) = \dim_K(\overline{M}_\lambda) = 1$. Since the algebra $K_{\mathfrak{m}_\lambda}$ is a field, the short exact sequence above splits. Let $\overline{e} = \overline{e}_\lambda$ be a nonzero element of \overline{M}_λ and $e = e_\lambda$ be an element of M'_λ such that $f_\lambda(e) = \overline{e}$. To finish the proof it suffices to show that the \mathbb{I}_n -submodule of M' , $N := \mathbb{I}_n e$, is a simple \mathbb{I}_n -module: Indeed, in this case, $M \cap N = 0$ since $f(M) = 0$, $f(N) = \overline{M}$ and the \mathbb{I}_n -modules M and N are simple. Then the \mathbb{I}_n -module M' of length 2 contains the submodule $M \oplus N$ of length 2, and so $M' = M \oplus N$, as required. There are two cases to consider: either $\overline{\mathcal{D}}_{\mathcal{O}} = \emptyset$ or $\overline{\mathcal{D}}_{\mathcal{O}} \neq \emptyset$.

(i) $\overline{\mathcal{D}}_{\mathcal{O}} = \emptyset$: In this case, $\overline{M} = \bigotimes_{i=1}^n \overline{M}_i$ where $\overline{M}_i = \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i)$ and $\lambda_i \in K \setminus \mathbb{Z}$. For each number $i = 1, \dots, n$, we have the $\mathbb{I}_1(i)$ -module epimorphism

$$\overline{M}_i = \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i) \rightarrow \mathbb{I}_1(i)e, \quad 1 + \mathbb{I}_1(i)(H_i - \lambda_i) \mapsto e,$$

which is necessarily an isomorphism since the $\mathbb{I}_1(i)$ -module \overline{M}_i is simple. Since $\mathbb{I}_n = \mathbb{I}_1(1) \otimes \dots \otimes \mathbb{I}_1(n)$, we have the \mathbb{I}_n -epimorphism

$$\overline{M} = \bigotimes_{i=1}^n \overline{M}_i = \bigotimes_{i=1}^n \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i) \rightarrow \mathbb{I}_n e = N$$

which is necessarily an isomorphism since the \mathbb{I}_n -module \overline{M} is simple.

(ii) $\overline{\mathcal{D}}_{\mathcal{O}} \neq \emptyset$: Without loss of generality we can assume that $\overline{\mathcal{D}}_{\mathcal{O}} = \{1, \dots, m\}$ for some natural number m such that $1 \leq m \leq n$. Then

$$\overline{M}_i = \begin{cases} K[x_i] & \text{if } i = 1, \dots, m, \\ M(1, \lambda_i) & \text{otherwise.} \end{cases}$$

Notice that $\text{Supp}_{\mathbb{I}_1(i)}(K[x_i]) = \{1, 2, \dots\}$. Choose $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Supp}(\overline{M})$ such that $\lambda_i = 1$ for all $i = 1, \dots, m$. Then

$$H_i e = \begin{cases} e & \text{if } i = 1, \dots, m, \\ \lambda_i e & \text{otherwise.} \end{cases}$$

For each $i = 1, \dots, m$,

$$0 = (H_i - 1)e = (\partial_i x_i - 1)e = (x_i \partial_i + [\partial_i, x_i] - 1)e = (x_i \partial_i + 1 - 1)e = x_i \partial_i e.$$

Since $\overline{M} = \bigotimes_{j=1}^n \overline{M}_j$ and $\overline{M}_i = K[x_i]$ for $i = 1, \dots, m$, the map $x_i \cdot : \overline{M} \rightarrow \overline{M}$, $m \mapsto x_i m$ is an injection. Therefore, $\partial_i e = 0$ for $i = 1, \dots, m$. So, by Lemma 2.1.(1), we have an $\mathbb{I}_1(i)$ -epimorphism

$$K[x_i] = \mathbb{I}_1(i)/\mathbb{I}_1(i)\partial_i \rightarrow \mathbb{I}_1(i)e, \quad 1 + \mathbb{I}_1(i)\partial_i \mapsto e,$$

which is necessarily an isomorphism since the $\mathbb{I}_1(i)$ -module $K[x_i]$ is simple. Using the same argument as in the case (i), we see that for all $j > m$, $\mathbb{I}_1(j)e \simeq M(1, \lambda_j)$ and the \mathbb{I}_n -epimorphism $\overline{M} = \bigotimes_{i=1}^n \overline{M}_i \rightarrow \mathbb{I}_n e = N$ is an isomorphism since the \mathbb{I}_n -module M is simple. The proof of the theorem is complete. \square

By Theorem 2.5, each weight \mathbb{I}_n -module M is a unique direct sum

$$M = \bigoplus_{\mathcal{O} \in K^n / \mathbb{Z}^n} \bigoplus_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} M(\mathcal{D}_{\mathcal{O}})^{\mu(\mathcal{D}_{\mathcal{O}})} \quad (18)$$

where $\mu(\mathcal{D}_{\mathcal{O}})$ is the multiplicity of $M(\mathcal{D}_{\mathcal{O}})$ (which can be any set).

3. EXPLICIT DESCRIPTION OF INDECOMPOSABLE GENERALIZED WEIGHT \mathbb{I}_n -MODULES

In this section, an explicit description of indecomposable generalized weight \mathbb{I}_n -modules is obtained (Theorem 3.6). One of the key steps is to show that each category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is a direct sum of its subcategories $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ that are generated by the single simple weight \mathbb{I}_n -modules $M(\mathcal{D}_{\mathcal{O}})$, see (19). Using (19) and some results about representations of Artinian rings, a criterion is given for the category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ and its subcategories $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ to be of finite representation type, tame or wild. Explicit classes of indecomposable modules in $\text{GW}(\mathbb{I}_n, \mathcal{O})$ are considered.

Let A be an algebra, \mathfrak{m} be a *co-finite ideal* of A (i.e., $\dim_K(A/\mathfrak{m}) < \infty$). An A -module M is called a *locally finite module* if for each $m \in M$, $\dim_K(Am) < \infty$. An A -module M is called an *\mathfrak{m} -locally finite module* if for each $m \in M$, $\dim_K(Am) < \infty$ and $\text{ann}_A(Am) \subseteq \mathfrak{m}^i$ for some $i \geq 1$ (if, in addition, A is a commutative algebra then the last condition is equivalent to the condition that $\mathfrak{m}^i m = 0$ for some $i \geq 1$). We denote by $\text{LF}_{\mathfrak{m}}(A)$ the category of all \mathfrak{m} -locally finite A -modules. The category $\text{LF}_{\mathfrak{m}}(A)$ is closed under arbitrary direct sums, submodules and factor modules. An \mathbb{I}_n -module $M \in \text{GW}(\mathbb{I}_n, \mathcal{O})$ (resp., $M \in \text{W}(\mathbb{I}_n, \mathcal{O})$) is called *equidimensional* if $\dim_K(M^{\mathfrak{m}}) = \dim_K(M^{\mathfrak{n}})$ (resp., $\dim_K(M_{\mathfrak{m}}) = \dim_K(M_{\mathfrak{n}})$) for all $\mathfrak{m}, \mathfrak{n} \in \text{Supp}(M)$. If the common value of all $\dim_K(M^{\mathfrak{m}})$ (resp., $\dim_K(M_{\mathfrak{m}})$) is d , we say that M is *d -equidimensional*. Let \mathfrak{m} be a maximal ideal of the polynomial algebra D_n and $\mathcal{I}(D_n, \mathfrak{m})$ is the set of all ideals I of D_n such that $\mathfrak{m} \supseteq I \supseteq \mathfrak{m}^i$ for some $i \geq 1$. For all ideals $I \in \mathcal{I}(D_n, \mathfrak{m})$, the factor algebra D_n/I is a local, finite dimensional, commutative algebra with maximal ideal \mathfrak{m}/I .

The category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is a direct sum of subcategories $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$. Let $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ be the full subcategory of $\text{GW}(\mathbb{I}_n, \mathcal{O})$ generated by the simple weight \mathbb{I}_n -module $M(\mathcal{D}_{\mathcal{O}})$. There are precisely $2^{|\mathbb{D}_{\mathcal{O}}|}$ such subcategories in the categories $\text{GW}(\mathbb{I}_n, \mathcal{O})$. They are key objects in the description of all indecomposable generalized weight modules M with $\text{Supp}(M) \subseteq \mathcal{O}$ since

$$\text{GW}(\mathbb{I}_n, \mathcal{O}) = \bigoplus_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}), \quad (19)$$

Theorem 3.2.

Let R be a ring, M be a nonzero R -module and $\mathfrak{p} = \text{ann}_R(M)$. The R -module M is called a *prime R -module* (or a *\mathfrak{p} -prime R -module*) if \mathfrak{p} is a prime ideal of R and $\text{ann}_R(N) = \mathfrak{p}$ for all nonzero R -submodules N of M . The R -module M is called an *absolutely prime R -module* (or an *absolutely \mathfrak{p} -prime R -module*) if \mathfrak{p} is a prime ideal of R and $\text{ann}_R(N) = \mathfrak{p}$ for all nonzero subfactors of M , i.e., $N = M_2/M_1$ for some submodules M_1 and M_2 of M such that $0 \subseteq M_1 \subset M_2 \subseteq M$.

Lemma 3.1 shows that all nonzero modules in each category $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ are absolutely prime \mathbb{I}_n -modules (Corollary 3.4). Lemma 3.1 is one of the key steps in proving that the equality (19) holds.

Lemma 3.1. *Let $0 \neq M \in \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$. Then*

- (1) $\text{Supp}(M) = \text{Supp}(M(\mathcal{D}_{\mathcal{O}}))$.
- (2) $\text{ann}_{\mathbb{I}_n}(M) = \mathfrak{a}(\mathcal{D}_{\mathcal{O}})$ where $\mathfrak{a}(\mathcal{D}_{\mathcal{O}}) = \text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \sum_{i \in C\mathcal{D}_{\mathcal{O}}} \mathfrak{p}_i \in \text{Spec}(\mathbb{I}_n)$ where $C\mathcal{D}_{\mathcal{O}} = \{1, \dots, n\} \setminus \mathcal{D}_{\mathcal{O}}$. So, all nonzero modules in the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ are absolutely $\mathfrak{a}(\mathcal{D}_{\mathcal{O}})$ -prime \mathbb{I}_n -modules.

Proof. 1. By the definition, the category of generalized weight \mathbb{I}_n -modules $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$ is generated by the simple weight \mathbb{I}_n -module $M(\mathcal{D}_{\mathcal{O}})$, and statement 1 follows, by (10).

2. The ideal

$$\mathfrak{a}(\mathcal{D}_{\mathcal{O}}) = \text{ann}_{\mathbb{I}_n}(M(\mathcal{D}_{\mathcal{O}})) = \sum_{i \in C\mathcal{D}_{\mathcal{O}}} \mathfrak{p}_i$$

is a prime ideal of \mathbb{I}_n . Let $N = \text{ann}_M(\mathfrak{a}) = \{m \in M \mid \mathfrak{a}m = 0\}$ where $\mathfrak{a} = \mathfrak{a}(\mathcal{D}_{\mathcal{O}})$. We have to show that $M = N$. Suppose not, we seek a contradiction. Then $S := \text{soc}_{\mathbb{I}_n}(M/N) \neq 0$ since $0 \neq M/N \in \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$. Now, $S = L/N$ for some submodule L of M such that $N \subsetneq L$. Recall that $\mathfrak{a}^2 = \mathfrak{a}$. So, $0 \neq \mathfrak{a}L = \mathfrak{a}^2L \subseteq \mathfrak{a}N = 0$, a contradiction. \square

Theorem 3.2. *For all $\mathcal{O} \in K^n/\mathbb{Z}^n$ and all $\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}$, the equality (19) holds.*

Proof. CLAIM. $\sum_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}) = \bigoplus_{\mathcal{D}_{\mathcal{O}} \subseteq \mathbb{D}_{\mathcal{O}}} \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

Suppose that the Claim does not hold, i.e., $M_1 + \cdots + M_s \neq M_1 \oplus \cdots \oplus M_s$ for some modules $M_i \in \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}_i})$ such that the sets $\mathcal{D}_{\mathcal{O}_1}, \dots, \mathcal{D}_{\mathcal{O}_s}$ are distinct. Then $s \geq 2$. By Corollary 2.4, the prime ideals

$$\mathfrak{a}_1 = \text{ann}_{\mathbb{I}_1}(M_1), \dots, \mathfrak{a}_s = \text{ann}_{\mathbb{I}_1}(M_s)$$

are *distinct*. Up to order, we may assume that the ideal \mathfrak{a}_1 a minimal element (with respect to \subseteq) of the set $\{\mathfrak{a}_1, \dots, \mathfrak{a}_s\}$. By replacing the module M_1 , by a (possibly) smaller nonzero submodule, say $M'_1 \subseteq M_1$, we may assume that $M'_1 \subseteq M_2 + \cdots + M_s$. By Lemma 3.1.(2), $\text{ann}_{\mathbb{I}_1}(M'_1) = \mathfrak{a}_1$. Let $\mathfrak{a} = \mathfrak{a}_2 \cdots \mathfrak{a}_s$. Then $\mathfrak{a}M'_1 \subseteq \mathfrak{a}(M_2 + \cdots + M_s) = 0$, and so $\mathfrak{a}_1 \cdots \mathfrak{a}_s \subseteq \text{ann}_{\mathbb{I}_1}(M'_1) = \mathfrak{a}_1$. Recall that the ideal \mathfrak{a}_1 is a prime ideal (Lemma 3.1.(2)), hence $\mathfrak{a}_i \subseteq \mathfrak{a}_1$ for some i such that $2 \leq i \leq s$. This contradicts the minimality of the ideal \mathfrak{a}_1 . Now, the theorem follows from Proposition 3.3. \square

Proposition 3.3. *Let $\mathcal{O} \in K^n/\mathbb{Z}^n$. Suppose that $\mathcal{D}_{\mathcal{O}}$ and $\mathcal{D}'_{\mathcal{O}}$ are distinct subsets of the set $\mathbb{D}_{\mathcal{O}}$. Let $M = M(\mathcal{D}_{\mathcal{O}})$ and $M' = M(\mathcal{D}'_{\mathcal{O}})$. Then $\text{Ext}_{\mathbb{I}_n}^1(M', M) = 0$.*

Proof. By Lemma 3.1.(2), the ideals $\mathfrak{a} = \text{ann}_{\mathbb{I}_n}(M)$ and $\mathfrak{a}' = \text{ann}_{\mathbb{I}_n}(M')$ are *distinct* prime ideals of the algebra \mathbb{I}_n . Therefore, either $\mathfrak{a} \not\subseteq \mathfrak{a}'$ or otherwise $\mathfrak{a} \subset \mathfrak{a}'$ (a proper inclusion since $\mathfrak{a} \neq \mathfrak{a}'$). Let $0 \rightarrow M \rightarrow N \rightarrow M' \rightarrow 0$ be a short exact sequence of \mathbb{I}_n -modules. To finish the proof it suffices to show that the short exact sequence splits.

(i) *Suppose that $\mathfrak{a} \not\subseteq \mathfrak{a}'$:* Then $\mathfrak{a}M' \neq 0$ (since otherwise $\mathfrak{a}M' = 0$, and so $\mathfrak{a} \subseteq \text{ann}_{\mathbb{I}_n}(M') = \mathfrak{a}'$, a contradiction). In particular, $\mathfrak{a}M' = M'$ since the \mathbb{I}_n -module M' is simple. Now,

$$N \supseteq \mathfrak{a}N = \mathfrak{a}(N/M) = \mathfrak{a}M' = M'.$$

Therefore, the \mathbb{I}_n -submodule $\mathfrak{a}N$ of N is isomorphic to the simple \mathbb{I}_n -module M' . Hence, $N \supseteq M + \mathfrak{a}N = M \oplus \mathfrak{a}N \simeq M \oplus M'$, i.e., $N = M \oplus \mathfrak{a}N \simeq M \oplus M'$, i.e., the short exact sequence splits.

(ii) *Suppose that $\mathfrak{a} \subset \mathfrak{a}'$:* Let $\mathcal{D} = \mathcal{D}_{\mathcal{O}}$ and $\mathcal{D}' = \mathcal{D}'_{\mathcal{O}}$. By Lemma 3.1.(2), $\mathfrak{a} = \sum_{i \in C_{\mathcal{D}}} \mathfrak{p}_i \subset \mathfrak{a}' = \sum_{j \in C_{\mathcal{D}'}} \mathfrak{p}_j$, and so $C_{\mathcal{D}} \subset C_{\mathcal{D}'}$ or, equivalently, $\mathcal{D} \supset \mathcal{D}'$. Up to order, let $\mathbb{D}_{\mathcal{O}} = \{1, \dots, m\}$, $\mathcal{D} = \{1, \dots, l\} \amalg \mathcal{D}'$ and $\mathcal{D}' = \{l+1, \dots, k\}$ provided $\mathcal{D}' \neq \emptyset$. Notice that $1 \leq l \leq m$ and $k \leq m$, $\mathcal{O} = \mathbb{Z}^m \times (\lambda_{m+1} + \mathbb{Z}) \times \cdots \times (\lambda_n + \mathbb{Z})$ where $\lambda_i \notin \mathbb{Z}$ for all i such that $m+1 \leq i \leq n$.

Clearly,

$$\begin{aligned} \text{Supp}(M) &= \mathbb{N}_+^k \times \mathbb{Z}^{m-k} \times \prod_{i=m+1}^n (\lambda_i + \mathbb{Z}), \\ \text{Supp}(M') &= \mathbb{Z}^l \times \mathbb{N}_+^{k-l} \times \mathbb{Z}^{m-k} \times \prod_{i=m+1}^n (\lambda_i + \mathbb{Z}) \end{aligned}$$

since

$$\begin{aligned} M &= K[x_1, \dots, x_k] \otimes \bigotimes_{j=k+1}^m \mathbb{I}_1(j)/\mathbb{I}_1(j)H_j \otimes \bigotimes_{i=m+1}^n \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i), \\ M' &= \bigotimes_{s=1}^l \mathbb{I}_1(s)/\mathbb{I}_1(s)H_s \otimes K[x_{l+1}, \dots, x_k] \otimes \bigotimes_{j=k+1}^m \mathbb{I}_1(j)/\mathbb{I}_1(j)H_j \otimes \bigotimes_{i=m+1}^n \mathbb{I}_1(i)/\mathbb{I}_1(i)(H_i - \lambda_i). \end{aligned}$$

Let $\lambda = (1, \dots, 1, \lambda_{m+1}, \dots, \lambda_n)$. Then $\dim_K(M^\lambda) = \dim_K(M'^\lambda) = 1$ and $\dim_K(N^\lambda) = 2$ since there is a short exact sequence $0 \rightarrow M^\lambda \rightarrow N^\lambda \rightarrow M'^\lambda \rightarrow 0$ of K -modules. Fix an element $v \in N^\lambda \setminus M^\lambda$. Then $v \neq 0$. Let $\theta := (1 - e)v$ where $e = e_{00}(1) = 1 - \int_1 \partial_1$. Then $\theta \equiv v \pmod{M}$ since $eM' = 0$. In particular, $\theta \neq 0$.

(iii) $\int_1 \partial_1 \theta = \theta$: Notice that $\int_1 \partial_1 = 1 - e$ is an idempotent and the result follows:

$$\int_1 \partial_1 \theta = (1 - e)\theta = (1 - e)(1 - e)v = (1 - e)v = \theta.$$

(iv) $\partial_1 \theta \neq 0$: Since $\theta \neq 0$, the statement (iv) follows from the statement (iii).

(v) $H_1\partial_1\theta = 0$: Since $(H_1 - 1)M^\lambda = 0$, we must have $(H_1 - 1)N^\lambda \subseteq M^\lambda$. Now,

$$H_1\partial_1\theta = \partial_1(H_1 - 1)\theta \in \partial_1(H_1 - 1)N^\lambda \subseteq \partial_1M^\lambda = 0.$$

The $\mathbb{I}_1(1)$ -module $\mathbb{I}_1(1)/\mathbb{I}_1(1)H_1$ is a simple weight module. By the statement (v), the $\mathbb{I}_1(1)$ -submodule $L_1 = \mathbb{I}_1(1)\partial_1\theta$ of N is isomorphic to the $\mathbb{I}_1(1)$ -module $\mathbb{I}_1(1)/\mathbb{I}_1(1)H_1$. Recall that $\text{End}_{\mathbb{I}_1(1)}(L_1) = K$ and $\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1}$. By [4], the \mathbb{I}_n -submodule $\mathbb{I}_n\partial_1\theta$ of N is isomorphic to the tensor product $L_1 \otimes L$ of the \mathbb{I}_1 -module L_1 and an \mathbb{I}_{n-1} -module L .

(vi) $N = M \oplus L_1 \otimes L$: Since the map $\partial_1 \cdot : L_1 \otimes L \rightarrow L_1 \otimes L$, $u \mapsto \partial_1 u$ is a bijection (since the map $\partial_1 \cdot : L_1 \rightarrow L_1$, $w \mapsto \partial_1 w$ is so) and the map $\partial_1 \cdot : M \rightarrow M$, $p \mapsto \partial_1 p$ has nonzero kernel (since $\partial_1 M^\lambda = 0$ and $M^\lambda \neq 0$), the simple \mathbb{I}_n -module M is not a submodule of $L_1 \otimes L$. Hence, $M \cap L_1 \otimes L = 0$, and so $M \oplus L_1 \otimes L \subseteq N$, and the statement (vi) follows since the length of the \mathbb{I}_n -module N is 2.

Now, the proposition follows from the statement (vi). \square

Corollary 3.4. (1) *Every module $M \in \text{GW}(\mathbb{I}_n, \mathcal{O})$ is a unique direct sum of absolutely prime generalized weight \mathbb{I}_n -modules, and this direct sum is $M = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} M_{\mathcal{D}_\mathcal{O}}$ where $M_{\mathcal{D}_\mathcal{O}} \in \text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.*

(2) *Every generalized weight module is a unique sum of absolutely prime (generalized weight) \mathbb{I}_n -modules.*

Proof. 1. Statement 1 follows from Corollary 2.4 and Theorem 3.2.

2. Statement 2 follows from Statement 1 and (9). \square

The next proposition shows that there are plenty of indecomposable generalized weight \mathbb{I}_n -modules with support from a single orbit.

Proposition 3.5. *Let $\mathcal{O} \in \mathcal{M}_n/G$ and $\mathfrak{m} \in \mathcal{O}$.*

(1) *If $\mathbb{D}_\mathcal{O} = \emptyset$ then $\{V(I) := B_n \otimes_{D_n} D_n/I \mid I \in \mathcal{I}(D_n, \mathfrak{m})\} \subseteq \text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ and each \mathbb{I}_n -module $V(I)$ is an indecomposable, equidimensional, generalized weight \mathbb{I}_n -module of length $d := \dim_K(D_n/I) < \infty$ which is isomorphic to the \mathbb{I}_n -module $B_n/B_n I = \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha D_n/I$ (where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$) and $\dim_K(V(I)^n) = d$ for all $\mathfrak{n} \in \mathcal{O}$, and $\text{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.*

(2) *Suppose that $\mathbb{D}_\mathcal{O} = \{1, \dots, l\}$ (up to order) for some l such that $1 \leq l \leq n$.*

(a) *Suppose that $\mathcal{D}_\mathcal{O} = \emptyset$. Let $\mathfrak{m} = (H_1, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in \text{Supp}(M(\mathcal{D}_\mathcal{O}))$. Then $\{V(I) := B_n \otimes_{D_n} D_n/I \mid I \in \mathcal{I}(D_n, \mathfrak{m})\} \subseteq \text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ and each \mathbb{I}_n -module $V(I)$ is an indecomposable, equidimensional, generalized weight \mathbb{I}_n -module of length $d = \dim_K(D_n/I) < \infty$ which is isomorphic to the \mathbb{I}_n -module $B_n/B_n I = \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha D_n/I$ and $\dim_K(V(I)^n) = d$ for all $\mathfrak{n} \in \text{Supp}(V(I)) = \text{Supp}(M(\mathcal{D}_\mathcal{O}))$, and $\text{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.*

(b) *If $\mathcal{D}_\mathcal{O} \neq \emptyset$ then $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$, up to order, for some m such that $1 \leq m \leq l$. Let $k = n - m$, $D_k = K[H_{m+1}, \dots, H_n]$ and $B_k = \bigotimes_{i=m+1}^n B_1(i)$. Then*

$$\{V(I) := P_m \otimes (B_k \otimes_{D_k} D_k/I) \mid I \in \mathcal{I}(D_k, \mathfrak{m}')\} \subseteq \text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$$

where $\mathfrak{m}' = (H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in S(\mathcal{D}_\mathcal{O})$, and each \mathbb{I}_n -module $V(I)$ is an indecomposable, equidimensional, generalized weight \mathbb{I}_n -module of length $d := \dim_K(D_k/I) < \infty$ which is isomorphic to the \mathbb{I}_n -module

$$P_m \otimes B_k/B_k I \simeq \bigoplus_{\alpha \in \mathbb{Z}^k} P_m \otimes \partial^\alpha D_k/I$$

(where $\partial^\alpha = \partial_{m+1}^{\alpha_1} \cdots \partial_n^{\alpha_k}$) and $\dim_K(V(I)^n) = d$ for all $\mathfrak{n} \in \text{Supp}(V(I)) = \text{Supp}(M(\mathcal{D}_\mathcal{O}))$, and $\text{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n(\mathcal{D}_\mathcal{O})$.

Proof. 1. Let $I \in \mathcal{I}(D_n, \mathfrak{m})$. Notice that $B_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha D_n$. Then

$$V(I) = \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha D_n \otimes_{D_n} D_n/I \simeq \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha \otimes D_n/I = \bigoplus_{\alpha \in \mathbb{Z}^n} \partial^\alpha D_n/I.$$

So, the \mathbb{I}_n -module $V(I)$ is an equidimensional, generalized weight module with $\text{Supp}(V(I)) = \mathcal{O}$ and $\dim_K(V(I)^n) = l = \dim_K(D_n/I) < \infty$. By Theorem 2.3, the simple (generalized) weight

\mathbb{I}_n -module $M(\mathcal{D}_{\mathcal{O}})$ from $\text{GW}(\mathbb{I}_n, \mathcal{O})$ has support \mathcal{O} and is 1-equidimensional. Hence, $l_{\mathbb{I}_n}(V(I)) = \dim_K(D_n/I) = l$.

It remains to show that the \mathbb{I}_n -module $V(I)$ is an indecomposable. The functor

$$B_n \otimes_{D_n} - : D_n\text{-Mod} \rightarrow B_n\text{-Mod}, \quad N \mapsto B_n \otimes_{D_n} N$$

is exact. The commutative algebra D_n/I is a local, commutative, finite dimensional algebra with maximal ideal \mathfrak{m}/I . Since $(D_n/I)/(\mathfrak{m}/I) \simeq D/\mathfrak{m}$ is a field, the D_n -module D_n/I is indecomposable. Hence, so is the induced module $V(I) = B_n \otimes_{D_n} D_n/I$. Clearly, $\mathfrak{a}_n \subseteq \text{ann}_{\mathbb{I}_n}(V(I))$. Since \mathfrak{a}_n is a maximal ideal of \mathbb{I}_n and $V(I) \neq 0$, we must have $\text{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$.

2(a). Repeat the arguments of statement 1.

2(b). The functor

$$P_m \otimes - : \mathbb{I}_k\text{-Mod} \rightarrow \mathbb{I}_n\text{-Mod}, \quad L \mapsto P_m \otimes L$$

is an exact functor. Now, statement 2 follows from statement 2(a). \square

Explicit description of modules in $\text{GW}(\mathbb{I}_n, \mathcal{O})$. In view of Theorem 3.2, Theorem 3.6 below is an explicit description of generalized weight \mathbb{I}_n -modules.

Theorem 3.6. *Let $\mathcal{O} \in \mathcal{M}_n/G$.*

- (1) *Suppose that $\mathcal{D}_{\mathcal{O}} = \emptyset$, $\mathfrak{m} \in \mathcal{O}$ if $\mathbb{D}_{\mathcal{O}} = \emptyset$ and $\mathfrak{m} = (H_1, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in S(\mathcal{D}_{\mathcal{O}})$ if $\mathbb{D}_{\mathcal{O}} = \{1, \dots, l\} \neq \emptyset$, up to order. Then the functor*

$$\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}) \rightarrow \text{LF}_{\mathfrak{m}}(D_n), \quad M \mapsto M^{\mathfrak{m}}$$

is an equivalence of categories with the inverse $N \mapsto B_n \otimes_{D_n} N$, the induced functor.

- (2) *Suppose that $\mathcal{D}_{\mathcal{O}} \neq \emptyset$ and, up to order, $\mathcal{D}_{\mathcal{O}} = \{1, \dots, m\}$, $\mathbb{D}_{\mathcal{O}} = \{1, \dots, l\}$ for some m such that $1 \leq m \leq l \leq n$. Then $\mathbb{I}_n = \mathbb{I}_m \otimes \mathbb{I}_{n-m}$ and $\mathcal{O} = \mathbb{Z}^m \times \mathcal{O}'$ where $\mathcal{O}' = \mathbb{Z}^{l-m} \times \mathcal{O}_1 \times \dots \times \mathcal{O}_{n-l}$ and $\mathcal{O}_i \neq \mathbb{Z}$ for all $i = 1, \dots, n-l$, and*

- (a) $\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}) = P_m \otimes \text{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'}) := \{P_m \otimes M \mid M \in \text{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'})\}$ with $\mathcal{D}'_{\mathcal{O}'} = \emptyset$.

- (b) *Fix $\mathfrak{m} \in \mathcal{O}$ such that $\mathfrak{m} = (H_1 - 1, \dots, H_m - 1, H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n)$. Then $\mathfrak{m}' = (H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in \mathcal{O}'$ and the functor*

$$\text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}) \rightarrow \text{LF}_{\mathfrak{m}'}(D_{n-m}), \quad P_m \otimes M \mapsto M^{\mathfrak{m}'}$$

is an equivalence of categories with the inverse $N \mapsto P_m \otimes (B_{n-m} \otimes_{D_{n-m}} N)$ where $D_{n-m} = K[H_{m+1}, \dots, H_n]$ and $D_0 := K$.

Proof. 1. Let $M \in \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

(i) M is a sum of modules $V(I)$ where $I \in \mathcal{I}(D_n, \mathfrak{m})$, see Proposition 3.5.(1): The statement follows from Proposition 3.5.(1).

(ii) $\text{ann}_{\mathbb{I}_n}(M) = \mathfrak{a}_n$, by Lemma 3.1 (since $\text{ann}_{\mathbb{I}_n}(V(I)) = \mathfrak{a}_n$).

The statement (ii) means that M is a \mathbb{Z}^n -graded B_n -module. By Proposition 3.5.(1), the functor $\text{GW}(\mathbb{I}_n, \mathcal{O}) \rightarrow \text{LF}_{\mathfrak{m}}(D_n)$, $M \mapsto M^{\mathfrak{m}}$ is an equivalence of categories with the inverse $N \mapsto B_n \otimes_{D_n} N$.

2. Recall that $\mathfrak{m} = (H_1 - 1, \dots, H_m - 1, H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n)$. Let $M \in \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})$.

(i) M is a sum of modules $V(I)$ where $I \in \mathcal{I}(D_k, \mathfrak{m}')$, see Proposition 3.5.(2): The statement follows from Proposition 3.5.(2). Since, for all $I \in \mathcal{I}(D_k, \mathfrak{m}')$,

$$V(I)^{\mathfrak{m}} = (P_m \otimes (B_k \otimes_{D_k} D_k/I))^{\mathfrak{m}} \simeq D_k/I,$$

the statement 2(a) follows from Proposition 3.5.(2) and statement 1. Now, the statement 2(b) follows from the statement 2(a) and statement 1. \square

Let A be an algebra and M be an A -module. We denote by $[M]$ the isomorphism class of the A -module M and $A\text{-Mod}/\simeq$ is the set of all the isomorphism classes of A -modules. In particular, $\text{LF}_{\mathfrak{m}}(D_n)/\simeq$ is the set of isomorphism classes of D_n -modules in $\text{LF}_{\mathfrak{m}}(D_n)$. A category of modules is called a *category of finite representation type* if it contains only finitely many indecomposable modules up to isomorphism. Definition of tame and wild category the reader can

find in [21]. Notice that every category of finite representation type is tame but not vice versa.

Criterion for the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ to be of finite representation type, tame or wild. The next theorem is a criterion for the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ to be of finite representation type, tame or wild.

Theorem 3.7. *Let $\mathcal{O} \in \mathcal{M}_n/G$.*

- (1) *The category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is of finite representation type iff $\mathcal{O} = \mathbb{Z}^n$ and $\mathcal{D}_\mathcal{O} = \{1, \dots, n\}$, and in this case the simple \mathbb{I}_n -module $P_n = K[x_1, \dots, x_n]$ is the unique indecomposable \mathbb{I}_n -module in the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.*
- (2) *The category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is tame iff $|\mathcal{D}_\mathcal{O}| = n - 1$, and in this case, up to order, $\mathcal{O} = \mathbb{Z}^{n-1} \times (\lambda_n + \mathbb{Z})$ for some $\lambda_n \in K$, $\mathcal{D}_\mathcal{O} = \{1, \dots, n - 1\}$ and $\{P_{n-1} \otimes M(i, \lambda) \mid i \in \mathbb{N}_+, \text{ where } \lambda = \lambda_n \text{ if } \lambda_n \notin \mathbb{Z} \text{ and } \lambda = 0 \text{ if } \lambda_n \in \mathbb{Z}\}$ is the set of all indecomposable, pairwise non-isomorphic modules in $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.*
- (3) *The category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is wild iff $n \geq 2$ and $m := |\mathcal{D}_\mathcal{O}| < n - 1$, and in this case, up to order, $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$ where $1 \leq m < n - 1$, and $\{P_m \otimes (B_{n-m} \otimes_{D_{n-m}} N) \mid [N] \in \text{LF}_{\mathfrak{m}'}(D_{n-m})/\simeq \text{ and } N \text{ is an indecomposable } D_{n-m}\text{-module}\}$ is a set of indecomposable, pairwise non-isomorphic modules in $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ where $D_{n-m} = K[H_{m+1}, \dots, H_n]$ and $D_0 = K$.*

Proof. If $n \geq 2$ and $|\mathcal{D}_\mathcal{O}| < n - 1$ then by Theorem 3.6 and [20] (see, also [31]), the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is wild. If either $n = 1$ or $n \geq 2$ and $|\mathcal{D}_\mathcal{O}| \geq n - 1$ then by Theorem 3.6 and [20] (see, also [31]), the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is tame. Clearly, the category $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is of finite representation type iff $|\mathcal{D}_\mathcal{O}| = n$. \square

Corollary 3.8. *Let $\mathcal{O} \in \mathcal{M}_n/G$. All modules in $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ and $\text{W}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ are equidimensional and the length of the module is equal the dimension of any (generalized) weight component. In particular, all indecomposable generalized weight \mathbb{I}_n -modules are equidimensional.*

Proof. The corollary follows from Theorem 3.6. \square

The next corollary is a criterion for a generalized weight \mathbb{I}_n -modules to be finitely generated.

Corollary 3.9. *Let M be a generalized weight \mathbb{I}_n -module. The \mathbb{I}_n -module M is finitely generated iff its support is a subset of a union of finitely many orbits in \mathcal{M}_n/G and the dimensions of all generalized weight components are bounded by a natural number.*

Proof. The corollary follows from Theorem 3.6 and Proposition 3.5. \square

Criterion for the category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ to be of finite representation type, tame or wild. Corollary 3.10 is such a criterion.

Corollary 3.10. *Let $\mathcal{O} \in \mathcal{M}/G$.*

- (1) *The category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is tame iff $n = 1$.*
- (2) *The category $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is wild iff $n \geq 2$.*
- (3) *None of the categories $\text{GW}(\mathbb{I}_n, \mathcal{O})$ is of finite representation type.*

Proof. The corollary follow from Theorem 3.6. \square

Explicit classes of indecomposable \mathbb{I}_n -modules in $\text{GW}(\mathbb{I}_n, \mathcal{O})$. By Theorem 3.6, the problem of classifying indecomposable generalized weight \mathbb{I}_n -modules in $\text{GW}(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is equivalent to the problem of classifying indecomposable modules in $\text{LF}_{\mathfrak{m}}(D_{n'})$ for some $n' \leq n$. The set $\text{ind.LF}_{\mathfrak{m}}(D_n)$ of isomorphism classes of indecomposable modules in $\text{LF}_{\mathfrak{m}}(D_n)$ is the union

$$\text{ind.LF}_{\mathfrak{m}}(D_n) = \bigcup_{i \geq 1} \text{ind}(D_n, \mathfrak{m}^i)$$

where the set $\text{ind}(D_n, \mathfrak{m}^i)$ contains the isomorphism classes of all the indecomposable D_n -modules M with $\mathfrak{m}^i M = 0$. Clearly,

$$\text{ind}(D_n, \mathfrak{m}) = \{D_n/\mathfrak{m}\} \subseteq \text{ind}(D_n, \mathfrak{m}^2) \subseteq \dots$$

By Theorem of Drozd, see [20],

- $\text{ind}(D_n, \mathfrak{m}^i)$ is tame iff either $n = 1$ or $n = 2$ and $m = 1, 2$.

Description of the set $\text{ind}(D_2, \mathfrak{m}^2)$. Let $\Lambda = D_2/\mathfrak{m}^2$, $\mathfrak{m} = (h_1, h_2)$ where $h_1 = H_1 - \lambda_1$ and $h_2 = H_2 - \lambda_2$ for some $\lambda_1, \lambda_2 \in K$ and $M \in \text{ind}(D_2, \mathfrak{m}^2)$. Then $M = M_1 \oplus M_2$ where $M_2 = \mathfrak{m}M$ is a Λ -module and M_1 is any (fixed) complement subspace of the vector space M_2 . Clearly, $M = 0$ iff $M_2 = M$ iff $M_1 = 0$. The Λ -module structure on M is uniquely determined by the linear maps

$$M_1 \begin{array}{c} \xrightarrow{h_1} \\ \xleftarrow{h_2} \end{array} M_2, \quad m_1 \mapsto h_1 m_1, \quad m_1 \mapsto h_2 m_1 \quad (\text{where } m_1 \in M_1).$$

So, the problem of describing the set $\text{ind}(D_2, \mathfrak{m}^2)$ is ‘almost’ equivalent to the problem of classifying indecomposable finite dimensional representations of the *Kronecker quiver*:

$$1 \begin{array}{c} \xrightarrow{h_1} \\ \xleftarrow{h_2} \end{array} 2.$$

More precisely, every indecomposable finite dimensional representation of the Kronecker quiver (M_1, M_2) such that $M_1 \neq 0$ belongs to $\text{ind}(D_2, \mathfrak{m}^2)$, and vice versa. Up to isomorphism there are following 5 series of indecomposable modules (in bracket bases of the vector spaces M_1 and M_2 are given):

- (1) $K = D_2/\mathfrak{m}$.
- (2) For each $n \geq 1$, $M_1 = \langle e_1, \dots, e_n \rangle$, $M_2 = \langle e'_1, \dots, e'_{n+1} \rangle$, $h_1 e_i = e'_i$ and $h_2 e_i = e'_{i+1}$ for $i = 1, \dots, n$.
- (3) For each $n \geq 1$, $M_1 = \langle e_1, \dots, e_{n+1} \rangle$, $M_2 = \langle e'_1, \dots, e'_n \rangle$, $h_1 e_i = e'_i$ and $h_2 e_{i+1} = e'_i$ for $i = 1, \dots, n$, $h_1 e_{n+1} = 0$ and $h_2 e_1 = 0$.
- (4) For each $n \geq 1$, $M_1 = \langle e_1, \dots, e_n \rangle$, $M_2 = \langle e'_1, \dots, e'_n \rangle$, $h_1 e_i = e'_i$ and $(h_2 - \lambda)e_i = e'_{i-1}$ for $i = 1, \dots, n$ where $e'_0 = 0$ and $\lambda \in K$.
- (5) For each $n \geq 1$, $M_1 = \langle e_1, \dots, e_n \rangle$, $M_2 = \langle e'_1, \dots, e'_n \rangle$, $h_1 e_i = e'_{i-1}$ and $h_2 e_i = e'_i$ for $i = 1, \dots, n$ where $e'_0 = 0$.

The set $\text{ind.LF}_{\mathfrak{m}}(D_n)$ is a disjoint union of subsets,

$$\text{ind.LF}_{\mathfrak{m}}(D_n) = \bigsqcup_{I \in \mathcal{I}(D_n, \mathfrak{m})} \text{ind.LF}_{\mathfrak{m}}(D_n, I) \quad (20)$$

where the set $\text{ind.LF}_{\mathfrak{m}}(D_n, I)$ contains all the indecomposable modules $M \in \text{ind.LF}_{\mathfrak{m}}(D_n)$ with $\text{ann}_{D_n}(M) = I$. By Theorem of Drozd, see [20],

- $\text{ind}(D_n, I)$ is tame iff either $n = 1$ or $n = 2$ and I contains a product $h'_1 h'_2$ of elements $h'_1, h'_2 \in \mathfrak{m}$ such that their images in the K -vector space $\mathfrak{m}/\mathfrak{m}^2$ are K -linearly independent (equivalently, are a basis).

In the second case (i.e., $n = 2$), the elements h'_1 and h'_2 are K -algebra generators for the algebra Γ . So, up to change of algebra generators, we can assume that $h_1 h_2 \in I$. Let $\Gamma = D_2/(h_1 h_2) = K[h_1, h_2]/(h_1 h_2)$. Then $\text{ind.LF}_{\mathfrak{m}}(D_2, I) = \{M \in \text{ind}_f(\Gamma) \mid IM = 0\}$ where $\text{ind}_f(\Gamma)$ is the set of isomorphism classes of indecomposable finite dimensional left Γ -modules.

Description of $\text{ind}_f(\Gamma)$. Let $W = \langle h_1, h_2 \rangle$ be a free (noncommutative) semigroup. Each element (word) $w \in W$ is a unique product $w_1 \cdots w_l$ where $w_i \in \{h_1, h_2\}$ and $l = 1, 2, \dots$. The number $l = l(w)$ is called the *length* of the word w and $W = \bigsqcup_{l \geq 1} W_l$, a disjoint union, where W_l is the set of all words of length l . The cyclic group of order l , $\bar{C}_l = \langle \tau_l \rangle = \{\tau_l^i \mid i = 0, \dots, l-1\}$, where $\tau_l = (12 \dots l)$, acts on the set W_l by the rule $\tau_l(w_1 \cdots w_l) = w_{\tau_l(1)} \cdots w_{\tau_l(l)}$. Let W_l/C_l be the set of orbits. We say that two elements w and w' of W are *equivalent*, $w \sim w'$, if they belong

to the same orbit ($w \sim w'$ iff $l(w) = l(w')$ and $w = \tau_l^i(w')$ for some i where $l = l(w)$). An orbit $\mathbb{O} \in W_l/C_l$ is called a *periodic* orbit if it contains an element w such that $w = \theta^i$ for some $\theta \in W$ and $i \geq 2$. We denote by \mathcal{N} the set of all *non-periodic orbits*. The simple module $K = \Gamma/(h_1, h_2)$ belongs to $\text{ind}_f(\Gamma)$. The set of *non-simple* indecomposable finite dimensional Γ -modules consists of two sets of modules: the modules of the first and second type, see [24]:

$$\text{ind}_f(\Gamma) \setminus \{K\} = \text{ind}_1(\Gamma) \bigsqcup \text{ind}_2(\Gamma) \quad (21)$$

where

- (1) $\text{ind}_1(\Gamma) = \{M_w \mid w \in W\}$ and $M_w = \langle e_1, e_2, \dots, e_{l+1} \rangle$ where $l = l(w)$, $w = w_1 \dots w_l$ and $w_i \in \{h_1, h_2\}$,

$$h_1 e_i = \begin{cases} e_{i+1} & \text{if } w_i = h_1, \\ 0 & \text{otherwise,} \end{cases} \quad h_2 e_i = \begin{cases} e_{i-1} & \text{if } w_{i-1} = h_2, i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

- (2) $\text{ind}_2(\Gamma) = \{N(\mathbb{O}, n, \lambda) \mid \mathbb{O} \in \mathcal{N}, n \in \mathbb{N}_+, \lambda \in K^*\}$. Let $w = w_1 \dots w_l \in \mathbb{O}$ where $l = l(w)$. Then

$$N(\mathbb{O}, n, \lambda) = \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} N_i$$

is a direct sum of n -dimensional vector spaces $N_i = K^n$ and the action of the elements h_1 and h_2 is given below. Schematically, it can be represented by the following diagram

$$\begin{array}{ccccccc} N_1 & \xrightarrow{\text{id}} & N_2 & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & N_l, \\ & & & & \searrow & \swarrow & \\ & & & & & & J_n(\lambda) \end{array}$$

$$h_1|_{N_i} : N_i \rightarrow N_{i+1}, \quad h_1|_{N_i} = \begin{cases} \text{id} & \text{if } i \neq l, w_i = h_1, \\ 0 & \text{if } i \neq l, w_i = h_2, \\ J_n(\lambda) & \text{if } i = l, w_l = h_1, \\ 0 & \text{if } i = l, w_l = h_2, \end{cases}$$

$$h_2|_{N_i} : N_i \rightarrow N_{i-1}, \quad h_2|_{N_i} = \begin{cases} 0 & \text{if } i \neq l, w_i = h_1, \\ \text{id} & \text{if } i \neq l, w_i = h_2, \\ 0 & \text{if } i = l, w_l = h_1, \\ J_n(\lambda) & \text{if } i = l, w_l = h_2. \end{cases}$$

Up to isomorphism, the module $N(\mathbb{O}, n, \lambda)$ does not depend on the choice of the representative w of the orbit \mathbb{O} .

Description of $\text{ind}_f(A)$ where $A = K[h_1, h_2]/(h_1^2, h_2^2)$. The field K is an algebraically closed field of characteristic zero. Let $i = \sqrt{-1}$, $h'_1 = h_1 + ih_2$ and $h'_2 = h_1 - ih_2$. Then $h'_1 h'_2 = h_1^2 + h_2^2 \in (h_1^2, h_2^2)$, and so A is tame, by Theorem of Drozd, see [20]. Since the algebra A is an epimorphic image of the algebra $\Lambda = K[h_1, h_2]/(h_1^2, h_1 h_2, h_2^2)$, $\text{ind}_f(A) = \text{ind}_f(\Lambda) \cup \{ {}_A A \}$.

Lemma 3.11. $\text{ind}_f(A) = \text{ind}_f(\Lambda) \cup \{ {}_A A \}$.

Proof. (i) ${}_A A$ is *indecomposable* (since A is local).

(ii) ${}_A A$ is an *injective module*: straightforward.

(iii) *Any finite dimensional A -module M such that $\mathfrak{m}^2 M \neq 0$ contain ${}_A A$ where $\mathfrak{m} = (h_1, h_2)$:* Since $\mathfrak{m}^2 A \neq 0$, we can find a nonzero element $a \in A$ such that $\mathfrak{m}^2 a \neq 0$. Then ${}_A A a \simeq A$, as required (since $\mathfrak{m}^2 = (h_1 h_2)$).

(iv) $\text{ind}_f(A) = \text{ind}_f(\Lambda) \cup \{ {}_A A \}$: Let $M \in \text{ind}_f(A)$. If $\mathfrak{m}^2 M \neq 0$ then $M \simeq_A A$, by the statement (iii). If $\mathfrak{m}^2 M = 0$ then $M \in \text{ind}_f(\Lambda)$ since $\Lambda = A/\mathfrak{m}^2$. \square

4. GENERALIZED WEIGHT RIGHT \mathbb{I}_n -MODULES

In this section, a classification of simple (generalized) weight right \mathbb{I}_n -modules is given (Theorem 4.2). The category of weight right \mathbb{I}_n -modules is a semisimple category (Theorem 4.3). An explicit description of generalized weight \mathbb{I}_n -modules is given (Theorem 4.4).

The algebra \mathbb{I}_n admits an involution $*$ given by the rule, see [12]: For $i = 1, \dots, n$,

$$\partial_i^* = \int_i, \quad \int_i^* = \partial_i \quad \text{and} \quad H_i^* = H_i.$$

Recall that an involution $*$ on \mathbb{I}_n is a K -algebra *anti-isomorphism* of \mathbb{I}_n ($(ab)^* = b^*a^*$) such that $a^{**} = a$ for all elements $a \in \mathbb{I}_n^*$. Clearly, the involution $*$ above acts as the identify map on the algebra D_n .

Every left \mathbb{I}_n -module M can be seen as a right \mathbb{I}_n -module M^* where $M^* = M$, equality of vector spaces, and right \mathbb{I}_n -module structure on M is given by the rule: For all $m \in M$ and $a \in \mathbb{I}_n$, $ma := a^*m$. Similarly, every right \mathbb{I}_n -module N can be seen as a left \mathbb{I}_n -module N^* where $N^* = N$, equality of vector spaces, and, for all $n \in N$ and $a \in \mathbb{I}_n$, $an := na^*$. The functor

$$\mathbb{I}_n\text{-Mod} \rightarrow \text{Mod-}\mathbb{I}_n, \quad M \mapsto M^*$$

is an equivalence of categories with the inverse $N \mapsto N^*$. Clearly, $M^{**} = M$ and $N^{**} = N$.

Example 4.1. Recall that the polynomial algebra P_n is a left \mathbb{I}_n -module isomorphic to the factor module $\mathbb{I}_n/\mathbb{I}_n(\partial_1, \dots, \partial_n) \simeq K[\int_1, \dots, \int_n]\bar{1}$ where $\bar{1} = 1 + \mathbb{I}_n(\partial_1, \dots, \partial_n)$ (since for all $\alpha \in \mathbb{N}^n$, $(\alpha!)^{-1} \int^\alpha 1 = x^\alpha$). Hence,

$$(P_n^*)_{\mathbb{I}_n} \simeq \mathbb{I}_n/(\partial_1, \dots, \partial_n)\mathbb{I}_n \simeq \mathbb{I}_n/(\int_1, \dots, \int_n)\mathbb{I}_n \simeq \tilde{1}K[\int_1^*, \dots, \int_n^*] \simeq \tilde{1}K[\partial_1, \dots, \partial_n] = \tilde{1}\mathcal{D}_n \simeq \mathcal{D}_n \quad (22)$$

where $\tilde{1} = 1 + (\int_1, \dots, \int_n)\mathbb{I}_n$ and $\mathcal{D}_n = K[\partial_1, \dots, \partial_n]$ is a polynomial algebra. The algebra \mathcal{D}_n is a maximal commutative subalgebra of \mathbb{I}_n . Let $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ be the standard basis of the free abelian group $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. The right \mathbb{I}_n -module $\mathcal{D}_n = (P_n^*)_{\mathbb{I}_n}$ is simple (since $\mathbb{I}_n P_n$ is simple) and $\{\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is a K -basis of \mathcal{D}_n . The right action of the generators H_i, ∂_i, \int_i ($i = 1, \dots, n$) of the algebra \mathbb{I}_n on ∂^α are given below:

$$\partial^\alpha H_i = \partial^\alpha(\alpha_i + 1), \quad \partial^\alpha \partial_i = \partial^{\alpha+e_i} \quad \text{and} \quad \partial^\alpha \int_i = \begin{cases} \partial^{\alpha-e_i} & \text{if } \alpha_i \geq 1, \\ 0 & \text{if } \alpha_i = 0. \end{cases}$$

The definition of generalized weight right \mathbb{I}_n -modules is given in the same way as their left counterparts. We add the subscript ‘r’ to all the notation introduced for generalized weight left modules to indicate that we deal with *right* modules.

Since the involution $*$ acts as the identity map on the polynomial algebra $D_n = K[H_1, \dots, H_n]$, we have, for each orbit $\mathcal{O} \in \mathcal{M}_n/G$,

$$W(\mathbb{I}_n, \mathcal{O})^* = W_r(\mathbb{I}_n, \mathcal{O}), \quad \text{GW}(\mathbb{I}_n, \mathcal{O})^* = \text{GW}_r(\mathbb{I}_n, \mathcal{O}), \quad (23)$$

$$W_r(\mathbb{I}_n, \mathcal{O})^* = W(\mathbb{I}_n, \mathcal{O}), \quad \text{GW}_r(\mathbb{I}_n, \mathcal{O})^* = \text{GW}(\mathbb{I}_n, \mathcal{O}), \quad (24)$$

$$W_r(\mathbb{I}_n) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} W_r(\mathbb{I}_n, \mathcal{O}) \quad \text{and} \quad \text{GW}_r(\mathbb{I}_n) = \bigoplus_{\mathcal{O} \in \mathcal{M}_n/G} \text{GW}_r(\mathbb{I}_n, \mathcal{O}). \quad (25)$$

So, for each $M \in \text{GW}(\mathbb{I}_n, \mathcal{O})$, $\text{Supp}(M^*) = \text{Supp}(M)$ and $(M^*)^{\mathfrak{m}} = M^{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Supp}(M)$. For each $\mathcal{D}_{\mathcal{O}}$, $M(\mathcal{D}_{\mathcal{O}})_r := M(\mathcal{D}_{\mathcal{O}})^*$ is a simple right \mathbb{I}_n -module with

$$\text{Supp}(M(\mathcal{D}_{\mathcal{O}})_r) = \text{Supp}(M(\mathcal{D}_{\mathcal{O}})) = S(\mathcal{D}_{\mathcal{O}}),$$

see (17). Then $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}}) := \text{GW}(\mathbb{I}_n, \mathcal{D}_{\mathcal{O}})^*$ is the full subcategory of $\text{GW}_r(\mathbb{I}_n, \mathcal{O})$ generated by the simple right weight \mathbb{I}_n -module $M(\mathcal{D}_{\mathcal{O}})_r$. There are precisely $2^{|\mathcal{D}_{\mathcal{O}}|}$ such subcategories in the

category $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$, and

$$\text{GW}_r(\mathbb{I}_n, \mathcal{O}) = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} \text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O}), \quad (26)$$

by Theorem 3.2 (apply $*$ to (19)).

Description of simple weight modules. We denote by $\widehat{\mathbb{I}}_n(\text{weight})_r$ (resp., $\widehat{\mathbb{I}}_n(\text{gen. weight})_r$) the set of isomorphism classes of simple weight right (resp., generalized weight right) \mathbb{I}_n -modules. The next theorem classifies (up to isomorphism) all the simple weight right \mathbb{I}_n -modules.

Theorem 4.2. (1) $\widehat{\mathbb{I}}_n(\text{gen. weight})_r = \widehat{\mathbb{I}}_n(\text{weight})_r = \widehat{\mathbb{I}}_1(\text{weight})_r^{\otimes n}$, i.e., any simple generalized weight right \mathbb{I}_n -module is a simple weight right \mathbb{I}_n -module, and vice versa; any simple weight right \mathbb{I}_n -module M is isomorphic to the tensor product $M_1 \otimes \cdots \otimes M_n$ of simple weight right \mathbb{I}_1 -modules and two such modules are isomorphic over \mathbb{I}_n , $M_1 \otimes \cdots \otimes M_n \simeq M'_1 \otimes \cdots \otimes M'_n$, iff for each $i = 1, \dots, n$, the \mathbb{I}_1 -modules M_i and M'_i are isomorphic. Furthermore, $\widehat{\mathbb{I}}_1(\text{weight})_r = \{P_1^* \simeq \mathbb{I}_1 / \int \mathbb{I}_1 \simeq K[\partial], M(1, \lambda)^* \simeq B_1 / (H - \lambda)B_1 \mid \lambda \in K\}$, $\text{Supp}(K[\partial]) = \mathbb{N}_+$ and $\text{Supp}(M(1, \lambda)^*) = \lambda + \mathbb{Z}$.

(2) For each simple weight right \mathbb{I}_n -module $M = \bigotimes_{i=1}^n M_i$, $\text{Supp}(M) = \prod_{i=1}^n \text{Supp}(M_i)$.

Proof. The theorem follows at once from Theorem 2.3, (23), (24) and (25). \square

Theorem 4.3. Every weight right \mathbb{I}_n -module is a direct sum of simple weight right \mathbb{I}_n -modules. In particular, the category $\text{W}_r(\mathbb{I}_n)$ of weight right \mathbb{I}_n -modules is a semisimple category.

Proof. The theorem follows from Theorem 2.5, (23), (24) and (25). \square

Explicit description of modules in $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.

Theorem 4.4. Let $\mathcal{O} \in \mathcal{M}_n/G$.

- (1) Suppose that $\mathcal{D}_\mathcal{O} = \emptyset$, $\mathfrak{m} \in \mathcal{O}$ if $\mathbb{D}_\mathcal{O} = \emptyset$ and $\mathfrak{m} = (H_1, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in S(\mathcal{D}_\mathcal{O})$ if $\mathbb{D}_\mathcal{O} = \{1, \dots, l\} \neq \emptyset$, up to order. Then the functor

$$\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) \rightarrow \text{LF}_\mathfrak{m}(D_n), \quad M \mapsto M^\mathfrak{m}$$

is an equivalence of categories with the inverse $N \mapsto N \otimes_{D_n} B_n$, the induced functor.

- (2) Suppose that $\mathcal{D}_\mathcal{O} \neq \emptyset$ and, up to order, $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$, $\mathbb{D}_\mathcal{O} = \{1, \dots, l\}$ for some m such that $1 \leq m \leq l \leq n$. Then $\mathbb{I}_n = \mathbb{I}_m \otimes \mathbb{I}_{n-m}$ and $\mathcal{O} = \mathbb{Z}^m \times \mathcal{O}'$ where $\mathcal{O}' = \mathbb{Z}^{l-m} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_{n-l}$ and $\mathcal{O}_i \neq \mathbb{Z}$ for all $i = 1, \dots, n-l$, and

- (a) $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) = P_m^* \otimes \text{GW}_r(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'}) := \{P_m^* \otimes M \mid M \in \text{GW}(\mathbb{I}_{n-m}, \mathcal{D}'_{\mathcal{O}'})\}$ with $\mathcal{D}'_{\mathcal{O}'} = \emptyset$.
- (b) Fix $\mathfrak{m} \in \mathcal{O}$ such that $\mathfrak{m} = (H_1 - 1, \dots, H_m - 1, H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n)$. Then $\mathfrak{m}' = (H_{m+1}, \dots, H_l, H_{l+1} - \lambda_{l+1}, \dots, H_n - \lambda_n) \in \mathcal{O}'$ and the functor

$$\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O}) \rightarrow \text{LF}_{\mathfrak{m}'}(D_{n-m}), \quad P_m^* \otimes M \mapsto M^{\mathfrak{m}'}$$

is an equivalence of categories with the inverse $N \mapsto P_m^* \otimes (N \otimes_{D_{n-m}} B_{n-m})$ where $D_{n-m} = K[H_{m+1}, \dots, H_n]$ and $D_0 := K$.

Proof. The theorem follows from Theorem 3.6 by applying $*$. \square

Criterion for the category $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ to be of finite representation type, tame or wild. Theorem 4.5 is such a criterion.

Theorem 4.5. Let $\mathcal{O} \in \mathcal{M}_n/G$.

- (1) The category $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is of finite representation type iff $\mathcal{O} = \mathbb{Z}^n$ and $\mathcal{D}_\mathcal{O} = \{1, \dots, n\}$, and in this case the simple right \mathbb{I}_n -module P_n^* is the unique indecomposable \mathbb{I}_n -module in the category $\text{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.

- (2) The category $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is tame iff $|\mathcal{D}_\mathcal{O}| = n - 1$, and in this case, up to order, $\mathcal{O} = \mathbb{Z}^{n-1} \times (\lambda_n + \mathbb{Z})$ for some $\lambda_n \in K$, $\mathcal{D}_\mathcal{O} = \{1, \dots, n - 1\}$ and $\{P_{n-1}^* \otimes M(i, \lambda) \mid i \in \mathbb{N}_+, \text{ where } \lambda = \lambda_n \text{ if } \lambda_n \notin \mathbb{Z} \text{ and } \lambda = 0 \text{ if } \lambda_n \in \mathbb{Z}\}$ is the set of all indecomposable, pairwise non-isomorphic modules in $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.
- (3) The category $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ is wild iff $n \geq 2$ and $m := |\mathcal{D}_\mathcal{O}| < n - 1$, and in this case, up to order, $\mathcal{D}_\mathcal{O} = \{1, \dots, m\}$ where $1 \leq m < n - 1$, and $\{P_m^* \otimes (N \otimes_{D_{n-m}} B_{n-m}) \mid [N] \in \mathrm{LF}_m(D_{n-m})/\simeq \text{ and } N \text{ is an indecomposable } D_{n-m}\text{-module}\}$ is a set of indecomposable, pairwise non-isomorphic modules in $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ where $D_{n-m} = K[H_{m+1}, \dots, H_n]$ and $D_0 = K$.

Proof. The theorem follows from Theorem 3.7 by applying *. \square

Corollary 4.6. *Let $\mathcal{O} \in \mathcal{M}_n/G$. All modules in $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ and $\mathrm{W}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$ are equidimensional and the length of the module is equal the dimension of any of (generalized) weight component. In particular, all indecomposable right generalized weight \mathbb{I}_n -modules are equidimensional.*

Proof. The corollary follows from Corollary 3.8. \square

The next corollary is a criterion for a generalized weight right \mathbb{I}_n -modules to be finitely generated.

Corollary 4.7. *Let M be a generalized weight right \mathbb{I}_n -module. The \mathbb{I}_n -module M is finitely generated iff its support is a subset of a union of finitely many orbits in \mathcal{M}_n/G and the dimension of all generalized weight components are restricted by a natural number.*

Proof. The corollary follows from Corollary 3.9. \square

- Corollary 4.8.** (1) *Every module $M \in \mathrm{GW}_r(\mathbb{I}_n, \mathcal{O})$ is a unique direct sum of absolutely prime generalized weight right \mathbb{I}_n -modules, and this direct sum is $M = \bigoplus_{\mathcal{D}_\mathcal{O} \subseteq \mathbb{D}_\mathcal{O}} M_{\mathcal{D}_\mathcal{O}}$ where $M_{\mathcal{D}_\mathcal{O}} \in \mathrm{GW}_r(\mathbb{I}_n, \mathcal{D}_\mathcal{O})$.*
- (2) *Every generalized weight right \mathbb{I}_n -module is a unique sum of absolutely prime generalized weight right \mathbb{I}_n -modules.*

Proof. The corollary follows from Corollary 3.4 by applying *. \square

Criterion for the category $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{O})$ to be of finite representation type, tame or wild. Corollary 4.9 is such a criterion.

Corollary 4.9. *Let $\mathcal{O} \in \mathcal{M}/G$.*

- (1) *The category $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{O})$ is tame iff $n = 1$.*
- (2) *The category $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{O})$ is wild iff $n \geq 2$.*
- (3) *None of the categories $\mathrm{GW}_r(\mathbb{I}_n, \mathcal{O})$ is of finite representation type.*

Proof. The corollary follow from Corollary 3.10 by applying *. \square

Using the involution, we can consider right analogues of indecomposable \mathbb{I}_n -modules considered at the end of Section 2. We leave this to the interested reader.

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