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Characterization of multiplication commutative rings with finitely many minimal prime ideals

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Abstract

The aim of the paper is to give a characterization of a multiplication commutative ring with finitely many minimal prime ideals: *Each such ring is a finite direct product of rings* $\prod_{i=1}^{n} D_i$ *where* D_i *is either a Dedekind domain or an Artinian, local, principal ideal ring and vice versa.* In particular, each such ring is a Noetherian ring. As a corollary subclasses of such rings are described (semiprime, Artinian, semiprime and Artinian, local, domain, etc).

Key Words: a multiplication module, a multiplication ideal, a multiplication ring, a Dedekind domain, an Artinian local principal ideal ring.

Mathematics subject classification 2010: 13C05, 13E05, 13F05,13F10.

Contents

1 Introduction

In this paper, all rings are commutative with 1 and all modules are unital. A ring R is called a *multiplication ring* if I and J are ideals of R such that $J \subseteq I$ then $J = I'I$ for some ideal I' of R. The concept of multiplication ring was introduced by Krull in [8]. In [9], Mott proved that a multiplication ring has finitely many minimal prime ideals iff it is a Noetherian ring.

The next theorem is a description of multiplication rings with finitely many minimal prime ideals.

Theorem 1.1 *Let R be a ring with finitely many minimal prime ideals. Then the ring R is a multiplication ring iff* $R \cong \prod_{i=1}^{n} D_i$ *is a finite direct product of rings where D_i is either a Dedekind domain or an Artinian, local principal ideal ring.*

The next corollary is a description of semiprime multiplication rings with finitely many minimal prime ideals.

Corollary 1.2 *Let R be a semiprime ring with finitely many minimal prime ideals. Then R is a multiplication ring iff* $R \cong \prod_{i=1}^{n} D_i$ *is a finite direct product of rings where D_i is either a Dedekind domain or a field.*

Proof. The corollary follows from Theorem 1.1. $\square$

The next corollary is a description of Artinian multiplication rings.

Corollary 1.3 *Let R be an Artinian ring. Then R is a multiplication ring iff it is a finite direct product of Artinian, local, principal ideal rings.*

Proof. The corollary follows from Theorem 1.1. $\square$

The next corollary is a description of semiprime Artinian multiplication rings.

Corollary 1.4 *Let R be a semiprime Artinian ring. Then R is a multiplication ring iff it is a finite direct product of fields.*

Proof. The corollary follows from Corollary 1.2 and Corollary 1.3. $\square$

The next theorem is a description of multiplication domains.

Theorem 1.5 *Let R be a domain. Then R is a multiplication ring iff R is either a field or a Dedekind domain.*

Proof. The theorem follows from Theorem 1.1. $\square$

Corollary 1.6 *Let R be a ring with finitely many minimal prime ideals. Then*

1. *R is a local multiplication ring iff R is either a local Dedekind ring or an Artinian, local, principal ideal ring.*
2. *R is a local multiplication domain iff R is a local Dedekind ring.*
3. *R is a local, Artinian, multiplication ring iff R is an Artinian, local, principal ideal ring.*

In Theorem 3.10, we give a characterization of multiplication rings containing a unique minimal prime ideal and which is not maximal. In this situation, R is a multiplication ring iff it is a Dedekind domain.

The paper is organized as follows: In Section 2, definitions and some known results are given. In Section 3, we prove some properties of finitely generated prime ideals with zero annihilator of a multiplication ring (Proposition 3.1 and Proposition 3.2). Also, some results about prime ideals of a multiplication domain are proven (Proposition 3.3 and Proposition 3.4). Finally, we study the class of multiplication rings containing only finitely many minimal prime ideals, prove Theorem 1.1 and obtain some corollaries.

2 Preliminaries

In this section, we collect some results on multiplication modules that are used in the proofs of the paper.

An R-module is called a *cyclic* module if it is 1-generated. For an R-module M, let $\mathrm{Cyc}_R(M)$ be the set of its cyclic submodules. For an R-module M, we denote by $\mathrm{ann}_R(M)$ its annihilator. A module is called a *faithful* module if its annihilator is equal to zero. For a submodule N of M, the set $[N : M] := \mathrm{ann}_R(M/N) = \{r \in R \,|\, rM \subseteq N\}$ is an ideal of the ring R that contains the annihilator $\mathrm{ann}_R(M) = [0 : M]$ of the module M. The set $\theta(M) := \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]$ is an ideal of R. Clearly, $\mathrm{ann}_R(M) \subseteq \theta(M)$, and if M is an ideal of R then $M \subseteq \theta(M)$.

An R-module M is called a *multiplication module* if every submodule of M is equal to IM for some ideal I of the ring R. In addition, if M is an ideal of R then M is called a *multiplication ideal*. It is easy to show that *M is a multiplication R-module iff for every submodule N of M, $N = [N : M]M$.* The set of all multiplication R-modules is denoted by $\mathrm{Mod}_m(R)$. The set $\mathrm{Mod}_m(R)$ contains R, all cyclic R-modules and all invertible ideals of R. Let $\mathcal{I}(R)$ be the set of ideals of the ring R. The set $(\mathcal{I}(R), \subseteq)$ is a partially ordered set (a poset, for short). For an R-module M, let $\mathrm{Sub}_R(M)$ be the set of its submodules. The set $(\mathrm{Sub}_R(M), \subseteq)$ is a poset. The map $\mu_M : \mathcal{I}(R) \to \mathrm{Sub}_R(M)$, $I \mapsto IM$ respects inclusion, i.e., $I \subseteq J$ implies $IM \subseteq JM$, i.e., the map μ_M is a homomorphism of posets. *An R-module M is a multiplication module iff the map μ_M is a surjection.* A ring R is called a *multiplication ring* if every ideal of R is a multiplication module, i.e., $\mathcal{I}(R) \subseteq \mathrm{Mod}_m(R)$. Examples of multiplication rings are Dedekind domains, principal ideal domains and rings all ideals of which are idempotent ideals.

A non-empty subset S of a ring R is called a *multiplicatively closed subset* iff $SS \subseteq S$, $1 \in S$ and $0 \notin S$.

Lemma 2.1 *([6, Lemma 2]) Let S be a multiplicatively closed subset of a ring R. If M is multiplication R-module then $S^{-1}M$ is multiplication $S^{-1}R$-module.*

Lemma 2.2 *([7, Corollary 1.4]) Let I be a multiplication ideal of a ring R and M be a multiplication R-module. Then IM is multiplication module.*

The ideal $\theta(M)$ is very useful in studying of multiplication modules. The next results are about some characteristics of a multiplication R-module M in terms of the ideal $\theta(M)$.

Lemma 2.3

1. *([4, Lemma 1.1]) Let M be a multiplication R-module and N be a submodule of M. Then $M = \theta(M)M$ and $N = \theta(M)N$.*
2. *([1, Lemma 1.3]) Let M be an R-module. Then M is a multiplication module iff $\theta(M) + \mathrm{ann}_R(m) = R$ for all $m \in M$ iff $Rm = \theta(M)m$ for all $m \in M$.*

The next lemma is a criterion for a multiplication module to be finitely generated in terms of the ideal $\theta(M)$.

Lemma 2.4 *([4, Corollary 2.2]) Let M be a multiplication R-module. Then the following statements are equivalent.*

1. *The R-module M is finitely generated.*
2. *$\theta(M) = R$.*
3. *The R-module $\theta(M)$ is finitely generated.*

In particular, every multiplication module over Noetherian ring is finitely generated.

Lemma 2.5 *([4, Lemma 2.1]) Let I be a finitely generated ideal of R and M be a multiplication R-module. If $I \subseteq \theta(M)$ then IM is finitely generated.*

Y. Alshaniafi and S. Singh provide a cancelation law of a faithful multiplication module as follows:

Lemma 2.6 *([2, Theorem 1.4]) Let M be a faithful multiplication R-module. If I and J are two ideals of R that are contained in $\theta(M)$ then $IM = JM$ iff $I = J$.*

By Lemma 2.4 and Lemma 2.6, if M is a finitely generated faithful multiplication R-module and I and J are two ideals of R then $IM = JM$ iff $I = J$.

The next lemma provides a criterion for a faithful multiplication module to be finitely generated.

Lemma 2.7 *([7, Proposition 3.4]) Let M be a faithful multiplication R-module. Then M is finitely generated iff $PM \neq M$ for every minimal prime ideal P of R.*

Corollary 2.8 *Let R be a domain and M be a faithful multiplication R-module. Then M is finitely generated. In particular, a multiplication domain is a Noetherian ring.*

Proof. As R is a domain, R has only one minimal prime which is 0. By Lemma 2.7, M is finitely generated. □

Lemma 2.9 *([7, Theorem 2.8]) Let R be a ring with only finitely many maximal ideals. If M is a multiplication R-module then M is cyclic. In particular, if R is an Artinian ring then every multiplication R-module is cyclic.*

3 Characterization of multiplication commutative rings

Proposition 3.1 *Let R be a multiplication ring. Then every finitely generated prime ideal with zero annihilator is a maximal ideal. In particular, if R is a multiplication domain then every nonzero prime ideal of R is a maximal ideal.*

Proof. Let $P \in \mathrm{Spec}(R)$ and suppose that $P \subsetneq J \subseteq R$ where J is an ideal of R. Since R is a multiplication ring, $P = IJ$ for some ideal I of R. It follows that $I \subseteq P = IJ \subseteq I$ (since P is a prime ideal and $J \subsetneq P$). Hence, $I = P$. As P is a finitely generated multiplication ideal with zero annihilator and $P = PJ$, by Lemma 2.4 and Lemma 2.6, $J = R$, i.e., P is a maximal ideal of R. The result holds for a multiplication domain because every nonzero ideal is a finitely generated ideal with zero annihilator, by Corollary 2.8. $\square$

Proposition 3.2 *Let R be a multiplication ring. If P is a finitely generated prime ideal of R with zero annihilator then $R \supsetneq P \supsetneq P^2 \supsetneq \cdots \supsetneq P^n \supsetneq \cdots$ is a strictly descending chain of ideals such that all R-modules P^n/P^{n+1} are isomorphic to the simple R-module R/P.*

Proof. Since the ideal P is a finitely generated R-module, so are all its powers P^n, $n \geq 1$.

(i) *All R-modules $\{P^n\}_{n\geq 0}$ have zero annihilator*: Suppose that $rP^n = 0$ for some nonzero element $r \in R$ and $n \geq 0$, we seek a contradiction. Clearly, $n \geq 2$ since $P^0 = R \ni 1$ and $\mathrm{ann}_R(P) = 0$. We assume that n is the least possible. Then $(rP^{n-1})P = 0$, and so $rP^{n-1} = 0$ (since $\mathrm{ann}_R(P) = 0$), a contradiction.

By Proposition 3.1, the ideal P of R is a maximal ideal. In particular, $R = P^0 \neq P$.

(ii) *The ideals $\{P^n\}_{n\geq 0}$ are distinct*: Suppose this is not the case, we seek a contradiction. We can choose the least natural number $n \geq 0$ such that $P^n = P^{n+1}$. Clearly, $n \geq 1$. The R-modules P^n are finitely generated faithful multiplication modules. By Lemma 2.6, the equality $P^n = P^{n+1}$ implies the equality $P = R$, a contradiction.

(iii) *For all $n \geq 0$, the R-modules P^n/P^{n+1} are isomorphic to the simple R-module R/P*: Recall that P is a maximal ideal of the ring R. Hence, the R-module R/P is simple. Clearly, the R-modules P^n/P^{n+1} are R/P-modules and R/P is a field. By the statement (ii), the R-modules P^n/P^{n+1} are nonzero. To prove that the statement (iii) holds it suffices to show that the R-module P^n/P^{n+1} is simple. Given an ideal J of R such that $P^{n+1} \subsetneq J \subseteq P^n$, we have to show that $J = P^n$. The ring R is a multiplication ring. So, the inclusions $P^{n+1} \subseteq J$ and $J \subseteq P^n$ yield the equalities $P^{n+1} = IJ$ and $J = J'P^n$ for some ideals I and J' of R. Therefore, $P^{n+1} = IJ'P^n$, and, by Lemma 2.6, $P = IJ'$. Hence, either $P = I$ or $P = J'$ (since P is a prime ideal). Hence, either $P^{n+1} = PJ$ or $J = P^{n+1}$. The second case is not possible, by the choice of J. So, $P^{n+1} = PJ$. Then, by Lemma 2.6, $J = P^n$, as required. $\square$

Proposition 3.3 *Let R be a multiplication domain and P be a nonzero prime ideal. Then $IP = I \cap P$ for every ideal I of R such that $I \nsubseteq P$.*

Proof. Since R is a multiplication domain, every ideal of R is finitely generated, by Corollary 2.8. Since $I \cap P \subseteq I$ and I is a multiplication ideal of R, $I \cap P = I'I$ for some ideal I' of R. Also, there is an ideal $I^\star$ of R such that $IP = I^\star(I \cap P)$ (since $IP \subseteq I \cap P$ and $I \cap P$ is a multiplication ideal). So, $PI = I^\star I'I$. By Lemma 2.4 and Lemma 2.6, $P = I^\star I'$. Since P is a prime ideal then either $I^\star = P$ or $I' = P$. If $I^\star = P$ then $IP = (I \cap P)P$, and, by Lemma 2.4 and Lemma 2.6, $I = I \cap P$, and so $I \subseteq P$ (a contradiction). Therefore $I' = P$, and hence $I \cap P = IP$. $\square$

A proper ideal Q of a ring R is called *primary* if whenever $ab \in Q$ for $a, b \in R$ then either $a \in Q$ or $b \in \sqrt{Q} := \{r \in R \,|\, r^n \in Q \text{ for some } n \in \mathbb{N}\}$. If Q is a primary ideal then $P := \sqrt{Q}$ is necessarily a prime ideal. It is called *the associated prime ideal of Q*. In this case, Q is called a P-primary ideal.

Proposition 3.4 *Let R be a multiplication domain. Then for each $P \in \mathrm{Spec}(R)$, P^n is a P-primary ideal for any $n \in \mathbb{N}$.*

Proof. As R is a multiplication domain, every ideal is finitely generated, by Corollary 2.8. Suppose that $IJ \subseteq P^n$ and $I \nsubseteq P^n$. We have to show that $J \subseteq P$. Since R is a multiplication ring, $IJ = KP^n$ for some ideal K of R. Now, since $I \nsubseteq P^n$, there exists a natural number n' such that $n' < n$ and $I \subseteq P^{n'}$ and $I \nsubseteq P^{n'+1}$ (notice that $P^0 = R$). So there exists an ideal $I^\star$ of R such that $I = I^\star P^{n'}$. As $IJ = KP^n = I^\star P^{n'} J$ and $P^{n'}$ is a finitely generated multiplication ideal with zero annihilator, $KP^{n-n'} = I^\star J \subseteq P$, by Lemma 2.6. As $I^\star \subsetneq P$ and P is a prime ideal, $J \subseteq P$. Hence, P^n is P-primary. $\square$

The following three lemmas are obvious.

Lemma 3.5 *Let $R = \prod_{i\in I} R_i$ be a direct product of rings. Then R is a multiplication ring iff all rings R_i are multiplication rings.*

Lemma 3.6 *Let R be a multiplication ring and I be an ideal of R. Then R/I is a multiplication ring.*

Lemma 3.7 *Let R be a multiplication ring. Then $S^{-1}R$ is a multiplication ring where S is a multiplicatively closed subset of R.*

The next theorem is a description of local multiplication rings with nilpotent maximal ideal which are not fields.

Theorem 3.8 *Let $(R, \mathfrak{m})$ be a local ring where $\mathfrak{m}$ is a nilpotent ideal. Then the ring R is a multiplication ring iff it is an Artinian, principal ideal ring. If so, then $\mathfrak{m} = (x)$ for some element x of R and $\{(x^i) \,|\, i = 0, 1, \dots, \nu+1\}$ are the only distinct ideals of R where $\mathfrak{m}^{\nu+1} = 0$ and $\mathfrak{m}^\nu \neq 0$, and $\mathrm{Ass}(R) = \{\mathfrak{m}\}$.*

Proof. ($\Rightarrow$) Let $K = R/\mathfrak{m}$ (the residue field of $\mathfrak{m}$). Then $V = \mathfrak{m}/\mathfrak{m}^2$ is a vector space over K.

(i) $\dim_K(V) = 1$: Given a nonzero subspace U of V. We have to show that $U = V$. Clearly, $U = I/\mathfrak{m}^2$ for some ideal I such that $\mathfrak{m}^2 \subsetneq I \subseteq \mathfrak{m}$. Since the ring R is a multiplication ring, $I = J\mathfrak{m}$ for some ideal J of R which is necessarily equal to R (since $\mathfrak{m}^2 \subsetneq I$ and $(R, \mathfrak{m})$ is a local ring), i.e., $I = \mathfrak{m}$, and so $U = V$, as required.

(ii) $\mathfrak{m} = (x)$ for some $x \in R$: Fix an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. By statement (i), $\mathfrak{m} = Rx + \mathfrak{m}^2$. Then $\mathfrak{m} = Rx + (Rx + \mathfrak{m})^2 = Rx + \mathfrak{m}^3 = \cdots = Rx + Rx^\nu + \mathfrak{m}^{\nu+1} = Rx$ since $\mathfrak{m}^{\nu+1} = 0$.

(iii) *The ring R is Artinian*: By the statement (ii), the length $\ell(R)$ of the R-module R is equal to $\sum_{i=0}^{\nu+1} \ell(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \nu + 1 < \infty$, and the statement (iii) follows.

(iv) $\{(x^i) \,|\, i = 0, 1, \dots, \nu+1\}$ *are the only distinct ideals of R; in particular, R is a principal ideal ring*: Let I be a nonzero ideal of R. We may assume that $I \neq R$, i.e., $I \subseteq \mathfrak{m}$. Then there exists a unique natural number i such that $I \subseteq \mathfrak{m}^i$ but $I \nsubseteq \mathfrak{m}^{i+1}$. We claim that $I = \mathfrak{m}^i$. Fix an element $y \in I$ such that $y \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. Since $\mathfrak{m}^i = (x^i)$, $y = x^i u$ for some element $u \in R$ such that $u \notin \mathfrak{m}$ (since $y \notin \mathfrak{m}^{i+1}$), i.e., u is a unit of R. Then $\mathfrak{m}^i = (x^i) = (x^i u) = (y) \subseteq I \subseteq \mathfrak{m}^i$, and so $I = \mathfrak{m}^i = (x^i)$. If $(x^s) = (x^t)$ for some natural numbers s and t such that $0 \leqslant s \leqslant t \leqslant \nu + 1$ then $s = t$, by the Nakayama Lemma, and the statements (iv) follows.

($\Leftarrow$) Since the ring R is a principal ideal ring, it is a multiplication ring. $\square$

The following result is used in the proof of Theorem 3.10.

Lemma 3.9 *([3, Theorem 1.1]) Let R be a local ring. Then all multiplication R-modules are cyclic. In particular, if R is a multiplication ring then all ideals of R are cyclic.*

For a ring R, we denote by $\mathrm{Min}(R)$ and $\mathrm{Max}(R)$ the sets of minimal prime and maximal ideals of R, respectively.

The next theorem is a description of multiplication rings that have a unique minimal prime ideal which is not maximal. The theorem is used in the proof of Theorem 1.1.

Theorem 3.10 *Let R be a ring such that $\mathrm{Min}(R) = \{\mathfrak{p}\}$ and $\mathfrak{p}$ is not a maximal ideal. Then R is a multiplication ring iff R is a Dedekind domain. If so, then $\mathfrak{p} = 0$.*

Proof. ($\Rightarrow$) (i) $\mathfrak{p} = 0$: Since $\mathfrak{p}$ is a unique minimal prime ideal of the ring R and it is not maximal, it is properly contained in every maximal ideal of R. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\mathfrak{p} \subsetneq \mathfrak{m}$ and $\mathfrak{p} = \mathfrak{a}\mathfrak{m}$ for some ideal $\mathfrak{a}$ of R such that $\mathfrak{p} \subseteq \mathfrak{a}$ (since R is a multiplication ring). Hence $\mathfrak{a} = \mathfrak{p}$ (since $\mathfrak{p}$ is a prime ideal and $\mathfrak{p} \subsetneq \mathfrak{m}$), i.e., $\mathfrak{p} = \mathfrak{p}\mathfrak{m}$. Then localizing at $\mathfrak{m}$, we have the equality of $R_\mathfrak{m}$-modules, $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}_\mathfrak{m}$. The ring R is a multiplication ring hence so is the local ring $(R_\mathfrak{m}, \mathfrak{m}_\mathfrak{m})$. By Lemma 3.9, the $R_\mathfrak{m}$-module $\mathfrak{p}_\mathfrak{m}$ is cyclic. By applying the Nakayama Lemma to the equality $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}_\mathfrak{m}$, we must have $\mathfrak{p}_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \mathrm{Max}(R)$. Therefore, $\mathfrak{p} = 0$.

(ii) *R is a domain* (by the statement (i)).

(iii) *All maximal ideals of R has height* 1: This statement follows from the statement (ii) and Proposition 3.1.

(iv) *For every maximal ideal $\mathfrak{m}$, $R_\mathfrak{m}$ is a discrete valuation ring*: The ring $(R_\mathfrak{m}, \mathfrak{m}' = \mathfrak{m}_\mathfrak{m})$ is a local multiplication domain. By Lemma 3.9, every ideal is 1-generated. In particular, $\mathfrak{m}' = (x)$ for some element $x \in R$. We have to show that every proper ideal I of $R_{\mathfrak{m}'}$ $(I \neq 0, R_\mathfrak{m})$ is equal to $x^i R_\mathfrak{m}$ for some $i \geqslant 1$. There is a unique natural number $i \geqslant 1$ such that $I \subseteq \mathfrak{m}'^i$ but $I \subsetneq \mathfrak{m}'^{i+1}$. Notice that $I = yR$ for some $y \in \mathfrak{m}'^i \setminus \mathfrak{m}'^{i+1}$. Then $y = x^i u$ for some $u \in R_\mathfrak{m} \setminus \mathfrak{m}'$, a unit of $R_\mathfrak{m}$. Hence, $I = yR_\mathfrak{m} = x^i u R_\mathfrak{m} = x^i R_\mathfrak{m}$.

(v) *R is a Dedekind domain*: This follows from the statement (iv) and [5, Theorem 9.3].

($\Leftarrow$) Recall that every nonzero ideal of a Dedekind domain R is a unique finite product of maximal ideals. Hence, every ideal of R is a multiplication module, i.e., R is a multiplication ring. $\square$

Example. Let K be a field. Then the local ring $R = K[x]_x[y,z]/(y^2, yz, z^2) = D \bigoplus Dy \bigoplus Dz$ is not a multiplication ring where $D = K[x]_x$ is a local Dedekind domain, and $\mathfrak{p} = Dy \bigoplus Dz$ is a unique minimal prime which is not a maximal ideal and $\mathfrak{p}^2 = 0$.

Proof. If R were a multiplication ring then, by Theorem 3.10, $\mathfrak{p} = 0$, a contradiction. $\square$

Proof of Theorem 1.1. ($\Rightarrow$) Recall that the set $\mathrm{Min}(R)$ is a finite set then R is a Noetherian ring, by [9, Theorem 11], and so, the prime radical $\mathfrak{n} = \cap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$ is a nilpotent ideal. Let $\mathcal{M} = \mathrm{Min}(R) \cap \mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and $\mathcal{M}' = \mathrm{Min}(R) \setminus \mathrm{Max}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$.

(i) *If $\mathcal{M}' = \emptyset$, i.e.,* $\mathrm{Min}(R) = \mathrm{Max}(R) = \mathcal{M} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ *then $R \cong \prod_{\mathfrak{m}_i \in \mathrm{Min}(R)} R_i$ is a product of Artinian, local principal ideal rings*: Since $\mathrm{Min}(R) = \mathrm{Max}(R)$, the ring R is an Artinian ring. Hence it is a finite direct product of Artinian local rings, say $R = \prod_{i=1}^{n} R_i$. Since R is a multiplication ring, so are the rings R_i. By Theorem 3.8, the rings R_i are Artinian, local, principal ideal ring.

Till the end of the proof we assume that $\mathcal{M}' \neq \emptyset$, i.e., $\overline{\mathcal{M}} := \mathrm{Max}(R) \backslash \mathcal{M} = \mathrm{Max}(R) \backslash \mathrm{Min}(R) \neq \emptyset$.

(ii) *Every maximal ideal $\mathfrak{m} \in \overline{\mathcal{M}}$ contains a unique minimal prime ideal $\mathfrak{p}(\mathfrak{m})$ that necessarily belongs to $\mathcal{M}'$*: The maximal ideal of R contains at least one minimal prime ideal, say, $\mathfrak{p} = \mathfrak{p}(\mathfrak{m})$. Suppose that $\mathfrak{p}'$ is another minimal prime ideal that is contained in $\mathfrak{m}$, we seek a contradiction. The ring $R_\mathfrak{m}$ is a local multiplication ring with the maximal ideal $\mathfrak{m}' = \mathfrak{m}R_\mathfrak{m}$.

Claim: $\mathfrak{p}_\mathfrak{m} = 0$ and $\mathfrak{p}'_\mathfrak{m} = 0$.

By Lemma 3.9, every ideal of the ring $R_\mathfrak{m}$ is 1-generated. In particular, $\mathfrak{m}' = (x)$ and $\mathfrak{p}_\mathfrak{m} = (x')$ for some elements $x, x' \in R$. Since $\mathfrak{p}_\mathfrak{m} \subseteq \mathfrak{m}'$ and $R_\mathfrak{m}$ is a multiplication ring, we must have $\mathfrak{p}_\mathfrak{m} = \mathfrak{a}\mathfrak{m}'$ for some ideal $\mathfrak{a}$ of $R_\mathfrak{m}$ that contains $\mathfrak{p}_\mathfrak{m}$. Since $\mathfrak{m}' \nsubseteq \mathfrak{p}_\mathfrak{m}$ and $\mathfrak{p}_\mathfrak{m}$ is a prime ideal, we must have $\mathfrak{a} \subseteq \mathfrak{p}_\mathfrak{m}$, i.e., $\mathfrak{a} = \mathfrak{p}_\mathfrak{m}$, and so $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}'$. Since $\mathfrak{p}_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module and $(R_\mathfrak{m}, \mathfrak{m}')$ is a local ring, $\mathfrak{p}_\mathfrak{m} = 0$, by the Nakayama Lemma. The proof of the claim is complete.

Since $\mathfrak{p} \neq \mathfrak{p}'$ and $\mathfrak{p}, \mathfrak{p}' \subseteq \mathfrak{m}$, we must have $\mathfrak{p}_\mathfrak{m} \neq \mathfrak{p}'_\mathfrak{m}$ which contradicts to the fact $\mathfrak{p}_\mathfrak{m} = 0 = \mathfrak{p}'_\mathfrak{m}$, by the Claim.

For each $\mathfrak{p}_i \in \mathcal{M}'$, let $\mathcal{V}(\mathfrak{p}_i) = \{\mathfrak{m} \in \mathrm{Max}(R) \,|\, \mathfrak{p}_i \subseteq \mathfrak{m}\} = \{\mathfrak{m} \in \overline{\mathcal{M}} \,|\, \mathfrak{p}_i \subseteq \mathfrak{m}\}$.

(iii) $\mathrm{Max}(R) = \mathcal{M} \coprod \mathcal{V}(\mathfrak{p}_1) \coprod \cdots \coprod \mathcal{V}(\mathfrak{p}_s)$, *a disjoint union*: The statement (iii) follows from the statement (ii).

(iv) *All minimal prime ideals of R are co-prime ideals*: Recall that $\text{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s, \mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ are maximal ideals. So, it suffices to show that $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all $i \neq j$, but this follows from the statement (iii). In more detail, if $\mathfrak{p}_i + \mathfrak{p}_j \neq R$ then there is a maximal ideal that contains both $\mathfrak{p}_i$ and $\mathfrak{p}_j$, a contradiction (see the statement (iii)).

(v) $R/\mathfrak{n} \cong \prod_{\mathfrak{p} \in \text{Min}(R)} R/\mathfrak{p}$: This fact follows from the statement (iv).

Let $1 = \sum_{\mathfrak{p} \in \text{Min}(R)} e_{\mathfrak{p}}$ be the corresponding sum of orthogonal primitive idempotents. Since the set of minimal prime is a finite set, the ring R is a Noetherian ring, by [9, Theorem 11]. Hence, $\mathfrak{n}$ is a nilpotent ideal. So, we can lift the decomposition above to $1 = \sum_{\mathfrak{p} \in \text{Min}(R)} e'_{\mathfrak{p}}$, a sum of primitive orthogonal idempotents in R. So,

$$R \cong \prod_{\mathfrak{p} \in \text{Min}(R)} R(\mathfrak{p})$$

where $R(\mathfrak{p}) := e'_{\mathfrak{p}} R$ are local rings with unique minimal prime ideals by the statement (ii). Since R is a multiplication ring, the rings $R(\mathfrak{p})$ are also multiplication rings, by Lemma 3.5. Now, the implication ($\Rightarrow$) follows from Theorem 3.8 and Theorem 3.10.

($\Leftarrow$) This implication follows from Lemma 3.5, Theorem 3.8 and Theorem 3.10. $\square$

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