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Extremes of $\alpha(\boldsymbol{t})$-locally Stationary Gaussian Random Fields

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Abstract: The main result of this contribution is the derivation of the exact asymptotic behaviour of the supremum of a class of $\alpha(\boldsymbol{t})$-locally stationary Gaussian random fields. We present two applications of our result; the first one deals with the extremes of aggregate multifractional Brownian motions, whereas the second one establishes the exact asymptotics of the supremum of χ-process generated by multifractional Brownian motions.

1 Introduction and Main Result

The classical Central Limit Theorem and its ramifications show that the Gaussian model is a natural and correct paradigm for building an approximate solution to many otherwise unsolvable problems encountered in various research fields. While the theory of Gaussian processes and Gaussian random fields (GRF's) is well-developed and mature, the range of their applications is constantly growing. Recently, applications in brain mapping, cosmology, quantum chaos, queueing theory, insurance mathematics, number theory and some other fields have been added to its palmares, see e.g., [2, 5, 6, 9, 3, 4, 22, 23, 18]. In applications related to extremes of non-smooth Gaussian processes the fractional Brownian motion (fBm) appears in the definition of the Pickands constant, see e.g., [28, 10, 29, 20, 16]. Numerous research articles have shown the importance of fBm in both theoretical models and applications. For certain applications, the stationarity of increments, which together with the self-similarity property characterizes fBm in the class of Gaussian processes, can be a severe restriction. A natural way to relax the stationarity of increments assumption is to introduce the multifractional Brownian motion (mfBm), see e.g., [7, 8, 31]. By definition, a centered Gaussian process $\{B_{\alpha(t)}(t), t \ge 0\}$ is called a mfBm with parameter $\alpha(t), t \ge 0$, if

$$\mathbb{E}\left(B_{\alpha(t)}(t)B_{\alpha(s)}(s)\right) = \frac{1}{2}D(\alpha(s,t))\left(s^{\alpha(s,t)} + t^{\alpha(s,t)} - |t-s|^{\alpha(s,t)}\right), \quad \alpha(s,t) := \alpha(s)/2 + \alpha(t)/2, s, t \ge 0, \qquad (1.1)$$

where $D(x) = \frac{2\pi}{\Gamma(x+1)\sin(\pi x/2)}$ and $\alpha(\cdot)$ is a Hölder function of exponent $\gamma > 0$ such that $0 < \alpha(t) < 2\min(1, \gamma), t \ge 0$. For $\alpha(t) = \alpha \in (0, 2), t \ge 0$, the mfBm B_α reduces to a (non-standard) fBm.

Inspired by the structure of the mfBm, the recent paper [12] introduced a class of $\alpha(t)$-locally stationary Gaussian processes. Therein the exact asymptotics of the tail behavior of the supremum of the $\alpha(t)$-locally stationary Gaussian process are derived, which can be applied, for instance, in the analysis of the extremes of standardized mfBm.

It is worth noting that this new class of Gaussian processes includes the locally stationary ones discussed in [11, 25, 29]. Let $\{X_i(t), t \in [0, T]\}, i \le k \in \mathbb{N}$, be independent real-valued Gaussian processes, with $T > 0$. A natural GRF

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associated with these processes is the aggregate random field

$$Z(\boldsymbol{t}) = \sum_{i=1}^{k} \theta_i(\boldsymbol{t}) X_i(t_i), \quad \boldsymbol{t} = (t_1, \ldots, t_k) \in [0,T]^k,$$

with $\theta_i(\cdot), 1 \le i \le k$ some deterministic real-valued functions defined on $[0,T]^k$.

Extremes of this GRF can not be analyzed in general by aggregating the corresponding results for processes. Moreover, the analysis of it leads to technical difficulties, see e.g., the excellent monographs [5, 29, 30]. Recently, [1] dealt with multivariate piece-wise linear interpolation of locally stationary random fields, whereas [24] investigated the piece-wise approximation of $\alpha(t)$-locally stationary processes. With motivation from the aforementioned papers and [12], we consider, in this paper, extremes of $\alpha(\boldsymbol{t})$-locally stationary GRF $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0,T]^k\}$ (to be defined below). Specifically, we are interested in the exact asymptotic behavior of

$$\mathbb{P}\left(\sup_{\boldsymbol{t} \in [0,T]^k} X(\boldsymbol{t}) > u \right), \qquad u \to \infty, \tag{1.2}$$

with $T > 0$ a given constant and $k \in \mathbb{N}$ a positive integer.

Let $\mathcal{C}(\boldsymbol{D})$ denote the set of all continuous functions on $\boldsymbol{D} \subset \mathbb{R}^k$. Next, we give a formal definition of the GRF's of interest.

Definition. A real-valued almost surely (a.s.) continuous GRF $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0,T]^k\}$ is said to be $\alpha(\boldsymbol{t})$-locally stationary if the following conditions are satisfied:

D1. $\mathbb{E}(X(\boldsymbol{t})) = 0$ and $Var(X(\boldsymbol{t})) = 1$ for all $\boldsymbol{t} \in [0,T]^k$;

D2. $\alpha_i(t) \in \mathcal{C}([0,T])$ and $\alpha_i(t) \in (0,2]$ for all $t \in [0,T]$, $i = 1, \cdots, k$;

D3. $C_i(\boldsymbol{t}) \in \mathcal{C}([0,T]^k)$ and $0 < \inf\{C_i(\boldsymbol{t}) : \boldsymbol{t} \in [0,T]^k\} \le \sup\{C_i(\boldsymbol{t}) : \boldsymbol{t} \in [0,T]^k\} := C_U^i < \infty$, $i = 1, \cdots, k$;

D4. uniformly with respect to $\boldsymbol{t} \in [0,T]^k$

$$1 - Cov(X(\boldsymbol{t}), X(\boldsymbol{t}+\boldsymbol{s})) = \sum_{i=1}^{k} C_i(\boldsymbol{t}) |s_i|^{\alpha_i(t_i)} + o\left(\sum_{i=1}^{k} C_i(\boldsymbol{t}) |s_i|^{\alpha_i(t_i)} \right) \tag{1.3}$$

as $\boldsymbol{s} \to \boldsymbol{0} := (0, \cdots, 0) \in \mathbb{R}^k$.

A canonical example of $\alpha(\boldsymbol{t})$-locally stationary GRF's is the aggregate mfBm defined by aggregating independent standardized mfBm's, see Section 2.

Similarly to [12] we impose the following conditions on the functions $\alpha_i(\cdot), i = 1, \cdots, k$. Suppose there exists some integer $1 \le k_1 \le k$ such that:

A1. each of $\alpha_i(t), i = 1, \cdots, k_1$, attains its global minimum on $[0,T]$ at a unique point t_i^0 (set $q_{k_1} := \#\{i \in \mathbb{N} : 1 \le i \le k_1, t_i^0 \in (0,T)\}$), and further for any $i = k_1 + 1, \cdots, k$, there is some interval $[a_i, b_i] \subset (0,T)$ such that $\alpha_i(t_i) \equiv \alpha_i$ in $[a_i, b_i]$ which is the global minimum of $\alpha_i(t_i)$ on $[0,T]$;

A2. there exist $M_i, \beta_i > 0$, and $\delta_i > 1$, $i = 1, \cdots, k_1$, such that

$$\alpha_i(t + t_i^0) = \alpha_i(t_i^0) + M_i |t|^{\beta_i} + o(|t|^{\beta_i} |\ln|t||^{-\delta_i}), \quad t \to 0, \tag{1.4}$$

and there exist $M_i, \beta_i, \tilde{M}_i, \tilde{\beta}_i > 0$, and $\tilde{\delta}, \delta_i > 1$, $i = k_1+1, \cdots, k$, such that

$$\alpha_i(b_i + t) = \alpha_i(b_i) + M_i t^{\beta_i} + o(t^{\beta_i}|\ln t|^{-\delta_i}), \quad t \downarrow 0, \tag{1.5}$$

$$\alpha_i(a_i - t) = \alpha_i(a_i) + \tilde{M}_i t^{\tilde{\beta}_i} + o(t^{\tilde{\beta}_i}|\ln t|^{-\tilde{\delta}_i}), \quad t \downarrow 0. \tag{1.6}$$

The assumption A1 is initially suggested in [12], whereas assumption A2 is a weaker version of a similar condition given therein which assumes (1.4-1.6) with $|\ln|t||^{-\delta_i}$ (or $|\ln t|^{-\tilde{\delta}_i}$) replaced by $|t|^{\delta_i}$ (or $t^{\tilde{\delta}_i}$).
For notational simplicity, set

$$\alpha_i := \alpha_i(t_i^0), \quad i = 1, \cdots, k_1,$$

and

$$\int_{\boldsymbol{x} \in \{\boldsymbol{x}_0\} \times \boldsymbol{D}_1} C(\boldsymbol{x}) d\boldsymbol{x} := \int_{\boldsymbol{x} \in \boldsymbol{D}_1} C(\boldsymbol{x}_0, \boldsymbol{x}) d\boldsymbol{x}$$

for all integrable functions $C(\cdot)$. Further, denote by $\Psi(\cdot)$ the survival function of an $N(0,1)$ random variable, and by $\Gamma(\cdot)$ the Euler's Gamma function.
The proof of our main result (Theorem 1.1) relies on the so-called double-sum method that was mainly developed by Pickands [28] and Piterbarg [29, 30]. As expected, the Pickands constant defined by

$$\mathcal{H}_\alpha = \lim_{\mathcal{T} \to \infty} \mathcal{T}^{-1} \mathbb{E}\left\{ \sup_{t \in [0, \mathcal{T}]} e^{\sqrt{2} B_\alpha(t) - t^\alpha} \right\} \in (0, \infty), \quad \alpha \in (0, 2],$$

appears in the asymptotic expansion, where $\{B_\alpha(t), t \ge 0\}$ is an fBm with Hurst index $\alpha/2$. See also [15, 13, 20, 17] for the basic properties of Pickands constant and its generalizations.

Theorem 1.1. *Let $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0,T]^k\}$ be an $\alpha(\boldsymbol{t})$-locally stationary GRF that satisfies*

$$Cov(X(\boldsymbol{t}), X(\boldsymbol{s})) < 1, \quad \forall \boldsymbol{t}, \boldsymbol{s} \in [0,T]^k, \quad \boldsymbol{t} \neq \boldsymbol{s}. \tag{1.7}$$

If both conditions A1 and A2 are satisfied, then we have

$$\mathbb{P}\left(\sup_{\boldsymbol{t} \in [0,T]^k} X(\boldsymbol{t}) > u \right) = \mathcal{C} u^{\alpha} (\ln u)^{\beta} \Psi(u)(1 + o(1)), \quad u \to \infty, \tag{1.8}$$

where where $\alpha = 2\sum_{i=1}^k 1/\alpha_i$, $\beta = -\sum_{i=1}^{k_1} 1/\beta_i$ and

$$\mathcal{C} = 2^{q_{k_1}} \left(\prod_{i=1}^{k_1} \left(\frac{\alpha_i^2}{2M_i} \right)^{1/\beta_i} \Gamma(1/\beta_i + 1) \right) \left(\prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \right) \int_{\boldsymbol{x} \in \boldsymbol{O}} \prod_{i=1}^{k} (C_i(\boldsymbol{x}))^{1/\alpha_i} d\boldsymbol{x} \in (0, \infty), \tag{1.9}$$

with $\boldsymbol{O} = \prod_{i=1}^{k_1} \{t_i^0\} \times \prod_{i=k_1+1}^{k} [a_i, b_i]$.

Remarks: a) Under the conditions of Theorem 1.1, if, for the chosen $k_1 < k$, $\alpha_i(t_i) \equiv \alpha_i, i = k_1+1, \cdots, k$, on some compact set $\boldsymbol{O}_2 \subset \mathbb{R}_+^{k-k_1}$, with positive Lebesgue measure, then (1.8) holds for $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0,T]^{k_1} \times \boldsymbol{O}_2\}$ with $\boldsymbol{O} = \prod_{i=1}^{k_1} \{t_i^0\} \times \boldsymbol{O}_2$. In addition, with the convention that $\sum_{i=1}^0 = 0, \prod_{i=1}^0 = 1$, we have that (1.8) still holds when $k_1 = 0$; see Theorem 7.1 in [29].
b) We see from the proof of Lemma 3.5 that if $k_1 = k$, then case (3) in Lemma 3.5 does not appear and thus condition (1.7) can be removed. That is the reason why a similar condition was not assumed in [12].

Brief outline of the paper: We give two applications of our main result in Section 2. In Section 3 we present some preliminary results. All the proofs are relegated to Section 4 and Appendix.

2 Applications

In this section we apply our result to two interesting examples of $\alpha(\boldsymbol{t})$-locally stationary GRF's, namely, the aggregate mfBm's and the χ-process generated by mfBm's defined below.

Let $\{B_{\alpha(t)}(t), t \geq 0\}$ be a mfBm with parameter $\alpha(t) \in (0, 2], t \geq 0$. We define the standardized mfBm by

$$\overline{B}_{\alpha(t)}(t) = \frac{B_{\alpha(t)}(t)}{\sqrt{Var(B_{\alpha(t)}(t))}}, t \in [T_1, T_2], \quad \text{with } 0 < T_1 < T_2 < \infty.$$

As shown in [12]

$$1 - Cor\Big(B_{\alpha(t)}(t), B_{\alpha(s+t)}(s+t)\Big) = \frac{1}{2} t^{-\alpha(t)} |s|^{\alpha(t)} + o(|s|^{\alpha(t)})$$

uniformly with respect to $t \in [T_1, T_2]$, as $s \to 0$.

<u>**Aggregate multifractional Brownian motions**</u>: Let $\{\overline{B}_{\alpha_i(t)}(t), t \in [T_1, T_2]\}, i = 1, \cdots, k$, be independent standardized mfBm's, with parameters $\alpha_i(t) \in (0, 2], t \geq 0, i = 1, \cdots, k$, respectively. Assume, for any fixed $i = 1, \cdots, k$, that $\alpha_i(t)$ attains its minimum at the unique point $t_i^0 \in (T_1, T_2)$, and that there exist some positive M_i, β_i, and $\delta_i > 1, i = 1, \cdots, k$, such that condition A2 is satisfied. Set $X(\boldsymbol{t}) = \frac{1}{\sqrt{k}} \left(\overline{B}_{\alpha_1(t_1)}(t_1) + \cdots + \overline{B}_{\alpha_k(t_k)}(t_k) \right), \boldsymbol{t} \in [T_1, T_2]^k$ and recall that we set $\alpha_i := \alpha_i(t_i^0)$. It follows that as $\boldsymbol{s} \to \boldsymbol{0}$

$$1 - Cov(X(\boldsymbol{t}), X(\boldsymbol{t} + \boldsymbol{s})) = \frac{1}{2k} \sum_{i=1}^{k} t_i^{-\alpha_i(t_i)} |s_i|^{\alpha_i(t_i)} (1 + o(1))$$

uniformly with respect to $\boldsymbol{t} \in [T_1, T_2]^k$. Thus, the conditions in Theorem 1.1 are satisfied by X. Then

$$\begin{aligned} & \mathbb{P}\left(\sup_{\boldsymbol{t} \in [T_1, T_2]^k} X(\boldsymbol{t}) > u \right) \\ & = 2^k (2k)^{-\sum_{i=1}^k \frac{1}{\alpha_i}} \left(\prod_{i=1}^{k} \frac{\mathcal{H}_{\alpha_i} \Gamma(1/\beta_i + 1)}{t_i^0} \left(\frac{\alpha_i}{\sqrt{2M_i}} \right)^{2/\beta_i} \right) \frac{u^{\sum_{i=1}^k \frac{2}{\alpha_i}}}{(\ln u)^{\sum_{i=1}^k 1/\beta_i}} \Psi(u)(1 + o(1)) \end{aligned} \quad (2.10)$$

as $u \to \infty$. We note in passing that in view of the fact

$$\mathbb{P}\left(\sup_{\boldsymbol{t} \in [T_1, T_2]^k} X(\boldsymbol{t}) > u \right) = \mathbb{P}\left(\sum_{i=1}^{k} \sup_{t \in [T_1, T_2]} \overline{B}_{\alpha_i(t)}(t) > \sqrt{k} u \right)$$

the claim in (2.10) also follows from Theorem 2.1 in [12] and Theorem 2.2 in [21].

<u>**χ-process**</u>: Let $\{\overline{B}_{i,\alpha(t)}(t), t \in [T_1, T_2]\}, i = 1, \cdots, k$, be independent copies of $\{\overline{B}_{\alpha(t)}(t), t \in [T_1, T_2]\}$. Assume that $\alpha(t)$ attains its minimum at the unique point $t^0 \in (T_1, T_2)$, and that there exist some positive M, β, and $\delta > 1$, such that

$$\alpha(t + t^0) = \alpha(t^0) + M|t|^\beta + o(|t|^\beta |\ln|t||^{-\delta}), \quad \text{as } t \to 0.$$

Consider a χ-process defined by

$$\chi(t) = \sqrt{\overline{B}^2_{1,\alpha(t)}(t) + \cdots + \overline{B}^2_{k,\alpha(t)}(t)}, \quad t \in [T_1, T_2].$$

Further, we introduce a GRF

$$Y(t, \boldsymbol{u}) = \overline{B}_{1,\alpha(t)}(t) u_1 + \cdots + \overline{B}_{k,\alpha(t)}(t) u_k, \quad \boldsymbol{u} = (u_1, \cdots, u_k)$$

defined on the cylinder $\mathcal{G}_T = [T_1, T_2] \times \mathcal{S}_{k-1}$, with $\mathcal{S}_{k-1}$ being the unit sphere in $\mathbb{R}^k$ (with respect to L_2-norm). In the light of [29]

$$\sup_{t\in[T_1,T_2]} \chi(t) = \sup_{(t,\boldsymbol{u})\in\mathcal{G}_T} Y(t,\boldsymbol{u}).$$

Further we have as $(s,\boldsymbol{v}) \to (0,\boldsymbol{0})$

$$1 - Cov(Y(t,\boldsymbol{u}), Y(t+s,\boldsymbol{u}+\boldsymbol{v})) = \frac{1}{2}t^{-\alpha(t)}|s|^{\alpha(t)} + \frac{1}{2}\sum_{i=1}^{k-1}|v_i|^2 + o\left(|s|^{\alpha(t)} + \sum_{i=1}^{k-1}|v_i|^2\right)$$

uniformly with respect to $(t,\boldsymbol{u}) \in \mathcal{G}_T$. Therefore, the conditions in Theorem 1.1 are satisfied by Y, and thus we have as $u \to \infty$ (set $\alpha_0 := \alpha(t^0)$)

$$\mathbb{P}\left(\sup_{t\in[T_1,T_2]} \chi(t) > u\right) = 2^{\frac{5}{2}-\frac{k}{2}-\frac{1}{\beta}-\frac{1}{\alpha_0}} \frac{\mathcal{H}_{\alpha_0}\alpha_0^{\frac{2}{\beta}}\Gamma(\frac{1}{\beta}+1)}{M^{1/\beta}t^0\Gamma((k-1)/2)} \frac{u^{k-1+\frac{2}{\alpha_0}}}{(\ln u)^{1/\beta}}\Psi(u)(1+o(1)).$$

3 Preliminary Lemmas

This section is concerned with some preliminary lemmas used for the proof of Theorem 1.1. We assume, without loss of generality, that $1 \le k_1 < k$ and $M_i = 1, i = 1, \cdots, k_1$. As pointed out in [12], for the asymptotics of the original process, we have to replace $C_i(\cdot)$ with $(M_i)^{-\alpha_i/\beta_i}C_i(\cdot)$, $i = 1, \cdots, k_1$. We may further assume that $t_i^0 = 0, i = 1, \cdots, k_1$, and thus the final general result should be multiplied by $2^{q_{k_1}}$. Hereafter, consider $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0,T]^k\}$ to be an $\alpha(\boldsymbol{t})$-locally stationary GRF with the above simplification (called *simplified* $\alpha(\boldsymbol{t})$*-locally stationary GRF*). Set next (recall $\alpha_i = \alpha_i(t_i^0)$)

$$t_u^i = \left(\frac{\alpha_i^2}{\beta_i}\frac{\ln\ln u}{\ln u}\right)^{\frac{1}{\beta_i}}, \quad i = 1, \cdots, k_1.$$

Clearly

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]} X(\boldsymbol{t}) > u\right) \le \mathbb{P}\left(\sup_{\boldsymbol{t}\in[0,T]^k} X(\boldsymbol{t}) > u\right)$$
$$\le \mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]} X(\boldsymbol{t}) > u\right) + \mathbb{P}\left(\sup_{\boldsymbol{t}\in\left([0,T]^k\setminus\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]\right)} X(\boldsymbol{t}) > u\right). \quad (3.11)$$

There are two steps in the proof of Theorem 1.1. In step 1, we focus on the asymptotics of

$$\pi(u) := \mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]} X(\boldsymbol{t}) > u\right), \quad u \to \infty, \quad (3.12)$$

which is the main part of our proof. In step 2, we shall show that (see Lemma 3.7 below)

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\left([0,T]^k\setminus\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]\right)} X(\boldsymbol{t}) > u\right) = o(\pi(u)), \quad u \to \infty. \quad (3.13)$$

The idea of finding the asymptotics of $\pi(u)$ is based on the so-called double-sum method; see e.g., [28] or [29]. Before going to the detail of the proof, let us recall the brief outline of the double-sum method. First of all, we need to find a suitable partition, say with cubes $\{W_u^i\}$, of the set $\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]$. Then using the well-known Bonferroni's inequality we find upper and lower bounds for $\pi(u)$, i.e.,

$$\sum_i \mathbb{P}\left(\sup_{\boldsymbol{t}\in W_u^i} X(\boldsymbol{t})>u\right)\geq \pi(u)\geq \sum_i \mathbb{P}\left(\sup_{\boldsymbol{t}\in W_u^i} X(\boldsymbol{t})>u\right)-\sum\sum_{i<j}\mathbb{P}\left(\sup_{\boldsymbol{t}\in W_u^i} X(\boldsymbol{t})>u,\ \sup_{\boldsymbol{t}\in W_u^j} X(\boldsymbol{t})>u\right).$$

Finally, we show that the single-sum terms on both sides are asymptotically equivalent and the double-sum term is relatively asymptotically negligible. In what follows, we shall first introduce the cubes that are used as the partition, followed then by some preliminary results (Lemmas 3.1-3.5) concerning the estimation for the summands of both single-sum and double-sum terms in the above formula. For any $p_i\in\mathbb{Z}_+$, $i=1,\cdots,k_1$, define

$$c_{p_i}^i=c_{p_i}^i(u):=\left(\frac{p_i}{\ln u(\ln\ln u)^{1/\beta_i}}\right)^{1/\beta_i},\quad A_{p_i}^i=A_{p_i}^i(u):=[c_{p_i}^i,c_{p_i+1}^i],$$

and let $m_i=m_i(u):=\lfloor\frac{(\alpha_i)^2}{\beta_i}(\ln\ln u)^{1+1/\beta_i}\rfloor$, where $\lfloor x\rfloor$ denotes the integer part of x. Further, let $S>1$ be any fixed integer; by dividing each $A_{p_i}^i$ into subintervals of length $S/u^{2/(\alpha_i(c_{p_i+1}^i))}$ (recall functions $\alpha_i()$ in (1.3)), we define

$$B_{j_i,p_i}^i=B_{j_i,p_i}^i(u):=\left[c_{p_i}^i+\frac{j_iS}{u^{2/(\alpha_i(c_{p_i+1}^i))}},c_{p_i}^i+\frac{(j_i+1)S}{u^{2/(\alpha_i(c_{p_i+1}^i))}}\right]$$

for $j_i=0,1,\cdots,n_{i,p_i}=n_{i,p_i}(u):=\lfloor\frac{c_{p_i+1}^i-c_{p_i}^i}{S}u^{2/(\alpha_i(c_{p_i+1}^i))}\rfloor$.

Moreover, let $k_2:=k-k_1$, $\boldsymbol{a}=(a_{k_1+1},\cdots,a_k)$, and let $\boldsymbol{k}=(K_1,\cdots,K_{k_2})\in\mathbb{Z}^{k_2}$ be a vector with integer coordinates. For $\delta>0$, we denote

$$\delta_{\boldsymbol{k}}=(\boldsymbol{a}+\delta\boldsymbol{k}+[0,\delta]^k)\cap\prod_{i=k_1+1}^{k}[a_i,b_i],$$

where $\boldsymbol{k}\in\mathcal{B}$ with

$$\mathcal{B}=\{\boldsymbol{k}\in\mathbb{Z}^{k_2}:\delta_{\boldsymbol{k}}\neq\emptyset\}.$$

Define an operator g_u on $\mathbb{R}^{k_2}$ as in [29], i.e., for $\boldsymbol{t}=(t_{k_1+1},\cdots,t_k)\in\mathbb{R}^{k_2}$

$$g_u\boldsymbol{t}=\left(u^{-\frac{2}{\alpha_{k_1+1}}}t_{k_1+1},\cdots,u^{-\frac{2}{\alpha_k}}t_k\right).\tag{3.14}$$

Denote $\triangle_0=g_u[0,1]^{k_2}$, and, for fixed $\boldsymbol{k}\in\mathcal{B}$, $\triangle_{\boldsymbol{I}_{\boldsymbol{k}}}=\triangle_{\boldsymbol{I}_{\boldsymbol{k}}}(u):=g_uS\boldsymbol{I}_{\boldsymbol{k}}+\triangle_0S$ with $\boldsymbol{I}_{\boldsymbol{k}}=(I_1^{\boldsymbol{k}},\cdots,I_{k_2}^{\boldsymbol{k}})\in\mathbb{Z}^{k_2}$ being a vector with integer coordinates. Further, let $V_{\boldsymbol{I}_{\boldsymbol{k}},\boldsymbol{k}}:=\boldsymbol{a}+\delta\boldsymbol{k}+\triangle_{\boldsymbol{I}_{\boldsymbol{k}}}$, where $\boldsymbol{I}_{\boldsymbol{k}}\in\mathcal{A}_{\boldsymbol{k}}$ with

$$\mathcal{A}_{\boldsymbol{k}}=\{\boldsymbol{I}_{\boldsymbol{k}}\in\mathbb{Z}^{k_2}:V_{\boldsymbol{I}_{\boldsymbol{k}},\boldsymbol{k}}\cap\delta_{\boldsymbol{k}}\neq\emptyset\}.$$

Denote

$$N_{\boldsymbol{k}}^+=\#\{\boldsymbol{I}_{\boldsymbol{k}}\in\mathbb{Z}^{k_2}:V_{\boldsymbol{I}_{\boldsymbol{k}},\boldsymbol{k}}\cap\delta_{\boldsymbol{k}}\neq\emptyset\}\quad\text{and}\quad N_i=\left\lfloor\frac{\delta}{S}u^{2/\alpha_{k_1+i}}\right\rfloor,i=1,\cdots,k_2.$$

Moreover, let, for $i=1,\cdots,k_1$,

$$\begin{aligned}\mathcal{L}_1^i&=\{(j_i,p_i):j_i,p_i\in\mathbb{Z},0\leq p_i\leq m_i-1,0\leq j_i\leq n_{i,p_i}-1\},\\ \mathcal{U}_1^i&=\{(j_i,p_i):j_i,p_i\in\mathbb{Z},0\leq p_i\leq m_i,0\leq j_i\leq n_{i,p_i}\},\end{aligned}$$

and

$$\mathcal{L}_2=\{(\boldsymbol{I\!I_k},\boldsymbol{k}):\boldsymbol{k}\in\mathcal{B},V_{\boldsymbol{I\!I_k},\boldsymbol{k}}\subset\delta_{\boldsymbol{k}}\},\quad \mathcal{U}_2=\{(\boldsymbol{I\!I_k},\boldsymbol{k}):\boldsymbol{k}\in\mathcal{B},\boldsymbol{I\!I_k}\in\mathcal{A}_{\boldsymbol{k}}\}.$$

We have

$$\bigcup_{\substack{(j_i,p_i)\in\mathcal{L}_1^i,i=1,\cdots,k_1\\(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{L}_2}}\prod_{i=1}^{k_1}B^i_{j_i,p_i}\times V_{\boldsymbol{I\!I_k},\boldsymbol{k}}\ \subset\ \prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]\ \subset\ \bigcup_{\substack{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1\\(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2}}\prod_{i=1}^{k_1}B^i_{j_i,p_i}\times V_{\boldsymbol{I\!I_k},\boldsymbol{k}}.$$

In order to specify the 'distance' between cubes of the type $\prod_{i=1}^{k_1}B^i_{j_i,p_i}\times V_{\boldsymbol{I\!I_k},\boldsymbol{k}}$, we introduce the following order relation: for any $(j,p),(j',p')\in\mathbb{Z}^2$, we write

$$(j,p)\prec(j',p')\quad\text{iff}\quad\{p<p'\}\text{ or }\{p=p'\text{ and }j<j'\}.$$

Further, for $\boldsymbol{j},\boldsymbol{p},\boldsymbol{j}',\boldsymbol{p}'\in\mathbb{Z}^{k_1}$ with $(j_i,p_i),(j_i',p_i')\in\mathcal{L}_1^i, i=1,\cdots,k_1$,

$$(\boldsymbol{j},\boldsymbol{p})\prec(\boldsymbol{j}',\boldsymbol{p}')\quad\text{iff}\quad(j_i,p_i)\prec(j_i',p_i')\quad\text{for some } i=1,\cdots,k_1,\text{and }(j_l,p_l)=(j_l',p_l')\text{ for } l=1,\cdots,i-1,$$

and, for $(\boldsymbol{I\!I_k},\boldsymbol{k}),(\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}')\in\mathcal{L}_2$,

$$(\boldsymbol{I\!I_k},\boldsymbol{k})\prec(\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}')\quad\text{iff } (I_i^{\boldsymbol{k}},K_i)\prec(I_i'^{\boldsymbol{k}'},K_i')\text{ for some } i=1,\cdots,k_2,\text{and }(I_l^{\boldsymbol{k}},K_l)=(I_l'^{\boldsymbol{k}'},K_l')\text{ for } l=1,\cdots,i-1.$$

Moreover, define, for $j,p,j',p'\in\mathbb{Z}$,

$$N_{j,p}^{j',p'}:=\#\{(j'',p'')\in\mathbb{Z}^2:(j,p)\prec(j'',p'')\prec(j',p')\}.$$

In the sequel, for fixed $j_i,p_i,\boldsymbol{I\!I_k},\boldsymbol{k}$ such that $(j_i,p_i)\in\mathcal{U}_1^i, i=1,2,\cdots,k_1$ and $(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2$, we consider the GRF $X(\boldsymbol{v}):=X(v_1,\cdots,v_k)$ on

$$A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}:=\prod_{i=1}^{k_1}B^i_{j_i,p_i}\times V_{\boldsymbol{I\!I_k},\boldsymbol{k}}.$$

In order to obtain the estimates of the tail probabilities of the supremum of X on $A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}$ (see Lemmas 3.1 and 3.3 below), we introduce the following stationary GRF's, for a fixed (marked) point $\boldsymbol{v}^0=(v_1^0,\cdots,v_k^0):=\boldsymbol{v}^0_{\boldsymbol{j},\boldsymbol{p},\boldsymbol{I\!I_k},\boldsymbol{k}}$ in $A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}$:

— $\{Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}),\boldsymbol{\nu}\in[0,S]^k\}$ is a family of centered stationary GRF's with

$$Cov(Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}),Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}+\boldsymbol{x}))\ =\ e^{-(1-\varepsilon)\left(\sum_{i=1}^{k_1}C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i+2(t_u^i)^{\beta_i}}+\sum_{i=k_1+1}^{k}C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}\right)}$$

for $\varepsilon\in(0,1)$, $u>0$ such that $\alpha_i+2(t_u^i)^{\beta_i}\le 2$, $i=1,\cdots,k_1$, and $\boldsymbol{\nu},\boldsymbol{\nu}+\boldsymbol{x}\in[0,S]^k$.

— $\{Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}),\boldsymbol{\nu}\in[0,S]^k\}$ is a family of centered stationary GRF's with

$$Cov(Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}),Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}+\boldsymbol{x}))\ =\ e^{-(1+\varepsilon)\left(\sum_{i=1}^{k}C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}\right)}\tag{3.15}$$

for $\varepsilon>0$, $u>0$ and $\boldsymbol{\nu},\boldsymbol{\nu}+\boldsymbol{x}\in[0,S]^k$.

Lemma 3.1. *For any $\varepsilon \in (0,1)$, there exists $u_\varepsilon > 0$ such that for $u > u_\varepsilon$,*

$$(i)\ \mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I_k},\boldsymbol{k}}} X(\boldsymbol{v}) > u\right) \ \geq\ \mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^k} Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}) > u\right),$$

$$(ii)\ \mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I_k},\boldsymbol{k}}} X(\boldsymbol{v}) > u\right) \ \leq\ \mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^k} Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}) > u\right). \tag{3.16}$$

Remark: Due to continuity of the functions $C_i(\cdot), i = 1, \cdots, k$, the point $\boldsymbol{v}^0$ can also be chosen as a fixed (marked) point in $\prod_{i=1}^{k_1} A_{p_i}^i \times \delta_{\boldsymbol{k}}$ when δ is sufficiently small and u is sufficiently large. In the sequel, we chose $\boldsymbol{v}^0$ in this way. Actually $\boldsymbol{v}^0$ depends on $\boldsymbol{p}$,$\boldsymbol{k}$, but, if no confusion is caused, for notational simplicity we still write $\boldsymbol{v}^0$.

Next we introduce a *structural modulus* on $\mathbb{R}^k$ by

$$|\boldsymbol{s}|_{\boldsymbol{\alpha}} = \sum_{i=1}^{k} |s_i|^{\alpha_i}, \quad \boldsymbol{s} \in \mathbb{R}^k, \quad \text{with } \alpha_i \in (0,2], 1 \leq i \leq k.$$

The following result inspired by Lemma 7 of [26] is crucial for our investigation; its proof is relegated to Appendix.

Lemma 3.2. *For any compact set $\boldsymbol{D} \subset \mathbb{R}_+^k$ with $\boldsymbol{0} \in \boldsymbol{D}$, let $\{X_u(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{D}\}$, $u > 0$, be a family of a.s. continuous GRF's , with $\mathbb{E}(X_u(\boldsymbol{t})) \equiv 0$, $\mathbb{E}\left((X_u(\boldsymbol{t}))^2\right) \equiv 1$ for all u, and with correlation function $r_u(\boldsymbol{t},\boldsymbol{s}) = \mathbb{E}(X_u(\boldsymbol{t})X_u(\boldsymbol{s}))$. If*

$$\lim_{u\to\infty} u^2(1 - r_u(\boldsymbol{t},\boldsymbol{s})) = |\boldsymbol{t} - \boldsymbol{s}|_{\boldsymbol{\alpha}} \tag{3.17}$$

uniformly with respect to $\boldsymbol{t}, \boldsymbol{s} \in \boldsymbol{D}$, then

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} X_u(\boldsymbol{t}) > u\right) = \mathcal{H}_{(k,\boldsymbol{\alpha})}[\boldsymbol{D}]\Psi(u)(1 + o(1))$$

as $u \to \infty$, where

$$\mathcal{H}_{(k,\boldsymbol{\alpha})}[\boldsymbol{D}] = \mathbb{E}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} e^{\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t}) - |\boldsymbol{t}|_{\boldsymbol{\alpha}}}\right) \in (0,\infty) \tag{3.18}$$

as defined in [29], with $\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t}) = \sqrt{2}\sum_{i=1}^k B_{\alpha_i}^{(i)}(t_i)$ and $B_{\alpha_i}^{(i)}, 1 \leq i \leq k$, being independent fBm's with Hurst indexes $\alpha_i/2 \in (0,1]$, respectively.

Lemma 3.3. *For any $S > 1$ and $\varepsilon \in (0,1)$, we have, as $u \to \infty$*

$$(i)\ \ \mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^k} Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}) > u\right) = \prod_{i=1}^k \mathcal{H}_{\alpha_i}\left[0, (C_i(\boldsymbol{v}^0)(1-\varepsilon))^{1/\alpha_i} S\right] \Psi(u)(1 + o(1)),$$

$$(ii)\ \ \mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^k} Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}) > u\right) = \prod_{i=1}^k \mathcal{H}_{\alpha_i}\left[0, (C_i(\boldsymbol{v}^0)(1+\varepsilon))^{1/\alpha_i} S\right] \Psi(u)(1 + o(1)),$$

where (recall (3.18)*) we set $\mathcal{H}_{\alpha_i}[0,S] := \mathcal{H}_{(1,\alpha_i)}[[0,S]], \quad i = 1, 2, \cdots, k$.*

In order to estimate the double-sum term in the derivation of (3.12), we need the following two lemmas.

Lemma 3.4. *Let GRF $\{\widetilde{Z}_{\boldsymbol{\varepsilon},u}^{\boldsymbol{w}^0}(\nu); \boldsymbol{\nu} \in [0,S]^k\}$, having covariance structure* (3.15) *with $\boldsymbol{v}^0$ replaced by $\boldsymbol{w}^0$, be independent of $\{Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}); \boldsymbol{\nu} \in [0,S]^k\}$, with $\varepsilon > 0$. Then there exists some positive constant F_ε, for u large enough, we have*

$$\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\mu}\in[0,S]^k} \frac{1}{\sqrt{2}}\left(Z_{\varepsilon,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu}) + \widetilde{Z}_{\varepsilon,u}^{\boldsymbol{w}^0}(\boldsymbol{\mu})\right) > u\right) \leq F_\varepsilon S^{2k}\Psi(u).$$

Next, we introduce a distance of two sets $\boldsymbol{D}_1, \boldsymbol{D}_2 \subset \mathbb{R}_+^k$ by

$$dist(\boldsymbol{D}_1, \boldsymbol{D}_2) = \inf_{\boldsymbol{t}\in \boldsymbol{D}_1, \boldsymbol{s}\in \boldsymbol{D}_2} |\boldsymbol{t}-\boldsymbol{s}|_{\boldsymbol{\alpha}}.$$

Further, we fix some sufficiently small $\gamma_0 > 0$ in the following way: uniformly with respect to $\boldsymbol{t} \in [0,T]^k$,

$$1 - Cov(X(\boldsymbol{t}), X(\boldsymbol{t}+\boldsymbol{s})) < \eta_0 \in [0, 1/2) \tag{3.19}$$

for $|\boldsymbol{s}|_{\boldsymbol{\alpha}} < \gamma_0$ (recall (1.3)).

Lemma 3.5. *There exist positive constants $\mathbb{C}, \mathbb{C}_1$ such that for sufficiently large u we have:*
(1) *For $(j_i, p_i), (j_i', p_i') \in \mathcal{L}_1^i, i = 1, \cdots, k_1, (\boldsymbol{I\!I}_{\boldsymbol{k}}, \boldsymbol{k}), (\boldsymbol{I\!I}'_{\boldsymbol{k}'}, \boldsymbol{k}') \in \mathcal{L}_2$ satisfying*

$$dist\left(A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I}_{\boldsymbol{k}},\boldsymbol{k}}, A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}\right) < \gamma_0 \tag{3.20}$$

and

$$N_{j_i,p_i}^{j_i',p_i'} > 0 \text{ for some } i = 1, \cdots, k_1, \text{ or } N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'} > 0 \text{ for some } i = 1, \cdots, k_2,$$

we have

$$\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I}_{\boldsymbol{k}},\boldsymbol{k}}} X(\boldsymbol{v}) > u, \sup_{\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}} X(\boldsymbol{v}') > u\right)$$
$$\le \mathbb{C}S^{2k}\exp\left(-\mathbb{C}_1\left(\sum_{i=1}^{k_1}\left(\sqrt{N_{j_i,p_i}^{j_i',p_i'}}S\right)^{\alpha_i} + \sum_{i=1}^{k_2}\left(N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'}S\right)^{\alpha_{k_1+i}}\right)\right)\Psi(u). \tag{3.21}$$

(2) *Let $(j_i, p_i), (j_i', p_i') \in \mathcal{L}_1^i, i = 1, \cdots, k_1, (\boldsymbol{I\!I}_{\boldsymbol{k}}, \boldsymbol{k}), (\boldsymbol{I\!I}'_{\boldsymbol{k}'}, \boldsymbol{k}') \in \mathcal{L}_2$ satisfy*

$$N_{j_i,p_i}^{j_i',p_i'} = 0 \text{ for all } i = 1, \cdots, k_1, \text{ and } N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'} = 0 \text{ for all } i = 1, \cdots, k_2.$$

If $(\boldsymbol{j}, \boldsymbol{p}) \prec (\boldsymbol{j}', \boldsymbol{p}')$, then the following number κ can be defined:

$$\kappa = \begin{cases} i_1^1 := \inf\{1 \le i \le k_1 : p_i = p_i', j_i' = j_i + 1\}, & \text{if the defining set is nonempty,} \\ i_2^1 := \inf\{1 \le i \le k_1 : p_i' = p_i + 1, j_i = n_{i,p_i}, j_i' = 0\}, & \text{if } i_1^1 \text{ does not exist.} \end{cases}$$

Similarly, if $(\boldsymbol{j}, \boldsymbol{p}) = (\boldsymbol{j}', \boldsymbol{p}')$ and $(\boldsymbol{I\!I}_{\boldsymbol{k}}, \boldsymbol{k}) \prec (\boldsymbol{I\!I}'_{\boldsymbol{k}}, \boldsymbol{k}')$, then we can define κ as

$$\kappa = \begin{cases} i_1^2 := k_1 + \inf\{1 \le i \le k_2 : K_i = K_i', I_i'^{\boldsymbol{k}'} = I_i^{\boldsymbol{k}} + 1\}, & \text{if the defining set is nonempty,} \\ i_2^2 := k_1 + \inf\{1 \le i \le k_2 : K_i' = K_i + 1, I_i^{\boldsymbol{k}} = N_i, I_i'^{\boldsymbol{k}'} = 0\}, & \text{if } i_1^2 \text{ does not exist.} \end{cases}$$

Assume, without loss of generality, that $\kappa = i_1^1$ exists. We have

$$\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I}_{\boldsymbol{k}},\boldsymbol{k}}} X(\boldsymbol{v}) > u, \sup_{\boldsymbol{v}'\in A_\kappa''} X(\boldsymbol{v}') > u\right) \le \mathbb{C}S^{2k}\exp\left(-\mathbb{C}_1 S^{\alpha_\kappa/2}\right)\Psi(u), \tag{3.22}$$

where

$$A_\kappa'' = \prod_{i=1}^{\kappa-1} B_{j_i',p_i'}^i \times \left[c_{p_\kappa}^\kappa + \frac{(j_\kappa+1)S+\sqrt{S}}{u^{2/(\alpha_\kappa(c_{p_\kappa+1}^\kappa))}}, c_{p_\kappa}^\kappa + \frac{(j_\kappa+2)S}{u^{2/(\alpha_\kappa(c_{p_\kappa+1}^\kappa))}}\right] \times \prod_{i=\kappa+1}^{k_1} B_{j_i',p_i'}^i \times V_{\boldsymbol{I\!I}'_{\boldsymbol{k}},\boldsymbol{k}'}.$$

(3) *If* $(j_i, p_i), (j_i', p_i') \in \mathcal{L}_1^i, i = 1, \cdots, k_1, (\boldsymbol{I\!I_k}, \boldsymbol{k}), (\boldsymbol{I\!I'_{k'}}, \boldsymbol{k'}) \in \mathcal{L}_2$ *satisfy*

$$dist\left(A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}, A_{\boldsymbol{j'},\boldsymbol{p'}}^{\boldsymbol{I\!I'_{k'}},\boldsymbol{k'}}\right) \geq \gamma_0, \tag{3.23}$$

then there exist some constants (independent of u*)* $h > 0$ *and* $\lambda \in (0, 1)$ *such that*

$$\mathbb{P}\left(\sup_{\boldsymbol{v} \in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}} X(\boldsymbol{v}) > u, \sup_{\boldsymbol{v'} \in A_{\boldsymbol{j'},\boldsymbol{p'}}^{\boldsymbol{I\!I'_{k'}},\boldsymbol{k'}}} X(\boldsymbol{v'}) > u\right) \leq 2\Psi\left(\frac{u - h/2}{\sqrt{1 - \lambda/2}}\right). \tag{3.24}$$

The next lemma gives the asymptotics of (3.12), which is the main part of the proof of Theorem 1.1.

Lemma 3.6. *If* $\{X(\boldsymbol{t}), \boldsymbol{t} \in [0, T]^k\}$ *is a simplified* $\alpha(\boldsymbol{t})$*-locally stationary GRF, then we have*

$$\begin{aligned}\pi(u) &= \left(\prod_{i=1}^{k_1}\left(\frac{\alpha_i^2}{2}\right)^{1/\beta_i}\Gamma(1/\beta_i + 1)\right)\left(\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}\right)\int_{\boldsymbol{x} \in \prod_{i=1}^{k_1}\{t_i^0\} \times \prod_{i=k_1+1}^{k}[a_i, b_i]}\prod_{i=1}^{k}(C_i(\boldsymbol{x}))^{1/\alpha_i}d\boldsymbol{x}\\ &\quad \times u^{\alpha}(\ln u)^{\beta}\Psi(u)(1 + o(1)), \quad u \to \infty,\end{aligned}$$

where α, β *are the same as in Theorem 1.1.*

Lemma 3.7. *Under the assumptions of Lemma 3.6 the claim in* (3.13) *holds.*

4 Proofs

Proof of Theorem 1.1: Taking into account of the (simplification) statement in the beginning of Section 3, we conclude that the claim follows directly from (3.11) and Lemmas 3.6 and 3.7. □

Proof of Lemma 3.1: Set

$$X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu}) = X\left(c_{p_1}^1 + \frac{j_1 S + \nu_1}{u^{2/(\alpha_1(c_{p_1}^1+1))}}, \cdots, c_{p_{k_1}}^{k_1} + \frac{j_{k_1} S + \nu_{k_1}}{u^{2/(\alpha_{k_1}(c_{p_{k_1}}^{k_1}+1))}}, \boldsymbol{a} + \delta\boldsymbol{k} + g_u S\boldsymbol{I\!I_k} + \triangle_0^{\boldsymbol{\nu}}\right),$$

with $\triangle_0^{\boldsymbol{\nu}} = g_u \prod_{i=k_1+1}^{k}[0, \nu_i]$. It follows that

$$\sup_{\boldsymbol{v} \in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}} X(\boldsymbol{v}) \stackrel{d}{=} \sup_{\boldsymbol{\nu} \in [0,S]^k} X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu}). \tag{4.25}$$

Furthermore, we derive, for the fixed point $\boldsymbol{v}^0$ in $A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}$, and u sufficiently large

$$\begin{aligned}&1 - Cov\left(X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu}), X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu} + \boldsymbol{x})\right)\\ \geq\ &(1 - \varepsilon/4)^{1/3}\left(\sum_{i=1}^{k_1} C_i(\boldsymbol{v})|u^{-2/(\alpha_i(c_{p_i}^i+1))}x_i|^{\alpha_i\left(c_{p_i}^i + \frac{j_i S + \nu_i}{u^{2/(\alpha_i(c_{p_i}^i+1))}}\right)} + \sum_{i=k_1+1}^{k} C_i(\boldsymbol{v})u^{-2}|x_i|^{\alpha_i}\right)\\ \geq\ &(1 - \varepsilon/2)^{1/3}\left(\sum_{i=1}^{k_1} C_i(\boldsymbol{v}^0)|u^{-2/(\alpha_i(c_{p_i}^i+1))}x_i|^{\alpha_i\left(c_{p_i}^i + \frac{j_i S + \nu_i}{u^{2/(\alpha_i(c_{p_i}^i+1))}}\right)} + \sum_{i=k_1+1}^{k} C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}\right)\end{aligned}$$

uniformly with respect to $\boldsymbol{\nu}, \boldsymbol{\nu} + \boldsymbol{x} \in [0, S]^k$, where we used the fact that $C_i(\cdot), i = 1, \cdots, k$, are continuous functions. In view of the proof of Lemma 4.1 of [12] for sufficiently large u we obtain

$$1 - Cov\left(X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu}), X_{\boldsymbol{j},\boldsymbol{p},,u}^{\boldsymbol{I\!I_k},\boldsymbol{k}}(\boldsymbol{\nu} + \boldsymbol{x})\right)$$

$$\geq \quad (1-\varepsilon/2)\left(\sum_{i=1}^{k_1} C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i+2(t_u^i)^{\beta_i}} + \sum_{i=k_1+1}^{k} C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}\right) \tag{4.26}$$

uniformly with respect to $\boldsymbol{\nu}, \boldsymbol{\nu}+\boldsymbol{x} \in [0,S]^k$. Similarly, for sufficiently large u

$$1 - Cov\left(X^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p},,u}(\boldsymbol{\nu}), X^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p},,u}(\boldsymbol{\nu}+\boldsymbol{x})\right) \leq (1+\varepsilon/2)\left(\sum_{i=1}^{k} C_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}\right), \tag{4.27}$$

uniformly with respect to $\boldsymbol{\nu}, \boldsymbol{\nu}+\boldsymbol{x} \in [0,S]^k$. The claim follows now by the Slepian's inequality (see e.g., Theorem C.1 of [29]). □

Proof of Lemma 3.3: Since the proofs of (i) and (ii) are similar, we present below only the proof of (i). Note that

$$\lim_{u\to\infty} u^2(1 - Cov(Y^{\boldsymbol{v}^0}_{\varepsilon,u}(\boldsymbol{t}), Y^{\boldsymbol{v}^0}_{\varepsilon,u}(\boldsymbol{s}))) = (1-\varepsilon)\sum_{i=1}^{k} C_i(\boldsymbol{v}^0)|t_i - s_i|^{\alpha_i}$$

uniformly with respect to $\boldsymbol{s}, \boldsymbol{t} \in [0,S]^k$. Hence (i) follows from Lemma 3.2. □

Proof of Lemma 3.4: Let

$$W_{\varepsilon,u}(\boldsymbol{\nu}, \boldsymbol{\nu'}) := \frac{1}{\sqrt{2}}\left(Z^{\boldsymbol{v}^0}_{\varepsilon,u}(\boldsymbol{\nu}) + \widetilde{Z}^{\boldsymbol{w}^0}_{\varepsilon,u}(\boldsymbol{\nu'})\right), \quad \boldsymbol{\nu}, \boldsymbol{\nu'} \in [0,S]^k.$$

Since $W_{\varepsilon,u}$ is a centered GRF with unit variance and uniformly with respect to $\boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\nu'}, \boldsymbol{\mu'} \in [0,S]^k$

$$\lim_{u\to\infty} u^2(1 - Cov(W_{\varepsilon,u}(\boldsymbol{\nu}, \boldsymbol{\nu'}), W_{\varepsilon,u}(\boldsymbol{\mu}, \boldsymbol{\mu'}))) = (1+\varepsilon)\left(\sum_{i=1}^{k} C_i(\boldsymbol{v}^0)|\nu_i - \mu_i|^{\alpha_i} + \sum_{i=1}^{k} C_i(\boldsymbol{w}^0)|\nu_i' - \mu_i'|^{\alpha_i}\right),$$

then by Lemma 3.2

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\mu}\in[0,S]^k} \frac{1}{\sqrt{2}}\left(Z^{\boldsymbol{v}^0}_{\varepsilon,u}(\boldsymbol{\nu}) + \widetilde{Z}^{\boldsymbol{w}^0}_{\varepsilon,u}(\boldsymbol{\mu})\right) > u\right)\\
&= \left(\prod_{i=1}^{k} \mathcal{H}_{\alpha_i}\left[0, (C_i(\boldsymbol{v}^0)(1+\varepsilon))^{1/\alpha_i} S\right]\right)\left(\prod_{i=1}^{k} \mathcal{H}_{\alpha_i}\left[0, C_i(\boldsymbol{w}^0)(1+\varepsilon))^{1/\alpha_i} S\right]\right)\Psi(u)(1+o(1))\\
&\leq \left(\prod_{i=1}^{k} \mathcal{H}_{\alpha_i}[0,1](\lfloor (C_U^i(1+\varepsilon))^{1/\alpha_i}\rfloor + 1)\right)^2 S^{2k}\Psi(u)(1+o(1)),
\end{aligned}$$

where in the last inequality we used the fact that $\mathcal{H}_{\alpha_i}[0,R] \leq \mathcal{H}_{\alpha_i}[0,1](\lfloor R\rfloor + 1)$ (cf. [29]), hence the proof is complete. □

Proof of Lemma 3.5: Since the proof of (1) and (2) are similar, we present next only the proof of (1). Let

$$Y_u(\boldsymbol{\nu}, \boldsymbol{\nu'}) = X_{1,u}(\boldsymbol{\nu}) + X_{2,u}(\boldsymbol{\nu'}),$$

where

$$X_{1,u}(\boldsymbol{\nu}) = X\left(c^1_{p_1} + \frac{j_1 S + \nu_1}{u^{2/(\alpha_1(c^1_{p_1+1}))}}, \cdots, c^{k_1}_{p_{k_1}} + \frac{j_{k_1} S + \nu_{k_1}}{u^{2/(\alpha_{k_1}(c^{k_1}_{p_{k_1}+1}))}}, \boldsymbol{a} + \delta\boldsymbol{k} + g_u S\boldsymbol{I\!I_k} + \triangle_0^{\boldsymbol{\nu}}\right)$$

and

$$X_{2,u}(\boldsymbol{\nu'}) = X\left(c^1_{p_1'} + \frac{j_1' S + \nu_1'}{u^{2/(\alpha_1(c^1_{p_1'+1}))}}, \cdots, c^{k_1}_{p_{k_1}'} + \frac{j_{k_1}' S + \nu_{k_1}'}{u^{2/(\alpha_{k_1}(c^{k_1}_{p_{k_1}'+1}))}}, \boldsymbol{a} + \delta\boldsymbol{k'} + g_u S\boldsymbol{I\!I'_{k'}} + \triangle_0^{\boldsymbol{\nu'}}\right),$$

with $\triangle_0^{\boldsymbol{\nu}'}=g_u\prod_{i=k_1+1}^{k}[0,\nu_i']$. For any $u>0$, we have

$$\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u,\ \sup_{\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}}X(\boldsymbol{v}')>u\right)\le\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in[0,S]^k}Y_u(\boldsymbol{\nu},\boldsymbol{\nu}')>2u\right).$$

We see from (3.19) and (3.20) that, for sufficiently large u

$$Var(Y_u(\boldsymbol{\nu},\boldsymbol{\nu}'))=4-2(1-Cov(X_{1,u}(\boldsymbol{\nu}),X_{2,u}(\boldsymbol{\nu}')))>2.$$

It follows, for fixed $i=1,\cdots,k_1$, and $v_i\in B^i_{j_i,p_i}$, $v_i'\in B^i_{j_i',p_i'}$, that $|v_i-v_i'|\ge N_{j_i,p_i}^{j_i',p_i'}\frac{S}{u^{2/(\alpha_i(c^i_{p_i}+1))}}$. Further, we have, for fixed $i=1,\cdots,k_2$, $v_{k_1+i}\in\left[K_i\delta+\frac{I_i^{\boldsymbol{k}}S}{u^{2/\alpha_{k_1+i}}},K_i\delta+\frac{(I_i^{\boldsymbol{k}}+1)S}{u^{2/\alpha_{k_1+i}}}\right]$ and $v'_{k_1+i}\in\left[K_i'\delta+\frac{I_i'^{\boldsymbol{k}'}S}{u^{2/\alpha_{k_1+i}}},K_i'\delta+\frac{(I_i'^{\boldsymbol{k}'}+1)S}{u^{2/\alpha_{k_1+i}}}\right]$ that $|v_{k_1+i}-v'_{k_1+i}|\ge\ N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'}\frac{S}{u^{2/\alpha_{k_1+i}}}$. Therefore, there exists some $\mathbb{C}_2>0$ such that for sufficiently large u

$$Var(Y_u(\boldsymbol{\nu},\boldsymbol{\nu}'))\le4-\mathbb{C}_2\left(\sum_{i=1}^{k_1}\left(N_{j_i,p_i}^{j_i',p_i'}\frac{S}{u^{2/\alpha_i(c^i_{p_i}+1)}}\right)^{\alpha_i(c^i_{p_i'}+1)}+\sum_{i=1}^{k_2}\left(\ N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'}\frac{S}{u^{2/\alpha_{k_1+i}}}\right)^{\alpha_{k_1+i}}\right).$$

In view of Lemma 4.4 in [12] for some $\mathbb{C}_3>0$

$$Var(Y_u(\boldsymbol{\nu},\boldsymbol{\nu}'))\le4-\mathbb{C}_3\left(\sum_{i=1}^{k_1}\left(\sqrt{N_{j_i,p_i}^{j_i',p_i'}}S\right)^{\alpha_i}+\sum_{i=1}^{k_2}\left(N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'}S\right)^{\alpha_{k_1+i}}\right)u^{-2}=:H(S,u).$$

Consequently,

$$\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in[0,S]^k}Y_u(\boldsymbol{\nu},\boldsymbol{\nu}')>2u\right)\le\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in[0,S]^k}\overline{Y}_u(\boldsymbol{\nu},\boldsymbol{\nu}')>\frac{2u}{\sqrt{H(S,u)}}\right),$$

where $\overline{Y}_u(\boldsymbol{\nu},\boldsymbol{\nu}')=Y_u(\boldsymbol{\nu},\boldsymbol{\nu}')/\sqrt{Var(Y_u(\boldsymbol{\nu},\boldsymbol{\nu}'))}$. Furthermore, borrowing the arguments of the proof of Lemma 6.3 in [29] (see alternatively the proof of Lemma 4.5 in [12]), for $\boldsymbol{\nu},\boldsymbol{\nu}',\boldsymbol{\mu},\boldsymbol{\mu}'\in[0,S]^k$

$$\begin{aligned}Var(\overline{Y}_u(\boldsymbol{\nu},\boldsymbol{\nu}')-\overline{Y}_u(\boldsymbol{\mu},\boldsymbol{\mu}'))&\le4\left(Var(X_{1,u}(\boldsymbol{\nu})-X_{1,u}(\boldsymbol{\mu}))+Var(X_{2,u}(\boldsymbol{\nu}')-X_{2,u}(\boldsymbol{\mu}'))\right)\\&\le\frac12\left(Var(Z_{8,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu})-Z_{8,u}^{\boldsymbol{v}^0}(\boldsymbol{\mu}))+Var(\widetilde{Z}_{8,u}^{\boldsymbol{v}'^0}(\boldsymbol{\nu}')-\widetilde{Z}_{8,u}^{\boldsymbol{v}'^0}(\boldsymbol{\mu}'))\right),\end{aligned}$$

where the GRF $\widetilde{Z}_{8,u}^{\boldsymbol{v}'^0}$ is independent of $Z_{8,u}^{\boldsymbol{v}^0}$, and has covariance structure (3.15) with $\boldsymbol{v}^0$ replaced by $\boldsymbol{v}'^0$ (chosen similarly as $\boldsymbol{v}^0$). Next, by Slepian's inequality and Lemma 3.4, we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in[0,S]^k}\overline{Y}_u(\boldsymbol{\nu},\boldsymbol{\nu}')>\frac{2u}{\sqrt{H(S,u)}}\right)&\le\mathbb{P}\left(\sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in[0,S]^k}\frac{1}{\sqrt2}\left(Z_{8,u}^{\boldsymbol{v}^0}(\boldsymbol{\nu})+\widetilde{Z}_{8,u}^{\boldsymbol{v}'^0}(\boldsymbol{\nu}')\right)>\frac{2u}{\sqrt{H(S,u)}}\right)\\&\le F_8S^{2k}\Psi\left(\frac{2u}{\sqrt{H(S,u)}}\right)\\&\le\mathbb{C}S^{2k}\exp\left(-\mathbb{C}_1\left(\sum_{i=1}^{k_1}\left(\sqrt{N_{j_i,p_i}^{j_i',p_i'}}S\right)^{\alpha_i}+\sum_{i=1}^{k_2}\left(N_{I_i^{\boldsymbol{k}},K_i}^{I_i'^{\boldsymbol{k}'},K_i'}S\right)^{\alpha_{k_1+i}}\right)\right)\Psi(u)\end{aligned}$$

for u sufficiently large. In order to prove (3) we apply the Borell theorem (e.g., [29]). By (1.7) and (3.23), we see that

$$\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}},\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}}Var(X(\boldsymbol{v})+X(\boldsymbol{v}'))=4-2\inf_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}},\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}}(1-Cov(X(\boldsymbol{v}),X(\boldsymbol{v}')))<4-2\lambda,$$

with some $\lambda \in (0,1)$. Further, there exists some $h>0$, such that

$$\mathbb{P}\left(\sup_{\boldsymbol{v}\in A^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p}},\boldsymbol{v'}\in A^{\boldsymbol{I\!I'_{k'}},\boldsymbol{k'}}_{\boldsymbol{j'},\boldsymbol{p'}}} (X(\boldsymbol{v})+X(\boldsymbol{v'}))>h\right)\le 2\mathbb{P}\left(\sup_{\boldsymbol{v}\in[0,T]^k} X(\boldsymbol{v})>h/2\right)<\frac{1}{2}.$$

Consequently, utilising Borell theorem, we obtain, for u sufficiently large

$$\begin{aligned}&\mathbb{P}\left(\sup_{\boldsymbol{v}\in A^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p}}} X(\boldsymbol{v})>u,\ \ \sup_{\boldsymbol{v'}\in A^{\boldsymbol{I\!I'_{k'}},\boldsymbol{k'}}_{\boldsymbol{j'},\boldsymbol{p'}}} X(\boldsymbol{v'})>u\right)\\ &\le \mathbb{P}\left(\sup_{\boldsymbol{v}\in A^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p}},\boldsymbol{v'}\in A^{\boldsymbol{I\!I'_{k'}},\boldsymbol{k'}}_{\boldsymbol{j'},\boldsymbol{p'}}} (X(\boldsymbol{v})+X(\boldsymbol{v'}))>2u\right)\le 2\Psi\left(\frac{u-h/2}{\sqrt{1-\lambda/2}}\right)\end{aligned}$$

establishing thus the claim. □

Proof of Lemma 3.6: Let $\varepsilon\in(0,1)$ be an arbitrarily chosen constant, and set $\overline{\varepsilon}:=1+\varepsilon$. We derive next the upper bound. Since $n_{i,p_i}=\lfloor\frac{c^i_{p_i+1}-c^i_{p_i}}{S}u^{2/\alpha_i(c^i_{p_i+1})}\rfloor$, we have as $u\to\infty$

$$\begin{aligned}\pi(u)&\le \sum_{\substack{(j_i,p_i)\in\mathcal{U}^i_1,1\le i\le k_1,\\ (\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2}}\mathbb{P}\left(\sup_{\boldsymbol{v}\in A^{\boldsymbol{I\!I_k},\boldsymbol{k}}_{\boldsymbol{j},\boldsymbol{p}}} X(\boldsymbol{v})>u\right)\le \sum_{(j_i,p_i)\in\mathcal{U}^i_1,1\le i\le k_1}\sum_{\boldsymbol{k}\in\mathcal{B}}\sum_{\boldsymbol{I\!I_k}\in\mathcal{A}_{\boldsymbol{k}}}\mathbb{P}\left(\sup_{\nu\in[0,S]^k} Z^{\boldsymbol{v}^0}_{\varepsilon,u}(\nu)>u\right)\\ &\le \sum_{p_i\le m_i,1\le i\le k_1}\sum_{\boldsymbol{k}\in\mathcal{B}}\left(\prod_{i=1}^{k_1}\left(\frac{c^i_{p_i+1}-c^i_{p_i}}{S}u^{2/\left(\alpha_i(c^i_{p_i+1})\right)}\right)N^+_{\boldsymbol{k}}\left(\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}[0,C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S]\right)\Psi(u)(1+o(1))\right)\\ &= \sum_{p_i\le m_i,1\le i\le k_1}\sum_{\boldsymbol{k}\in\mathcal{B}}\left(\frac{\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}[0,C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S]}{\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S)}\left(\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S)\right)\frac{1}{S^{k_1}}\left(\prod_{i=1}^{k_1}\frac{u^{2/\alpha_i}}{(\ln u)^{1/\beta_i}}\right)\right.\\ &\left.\times\frac{N^+_{\boldsymbol{k}}\left(\prod_{i=k_1+1}^{k}(Su^{-2/\alpha_i})\right)}{\prod_{i=k_1+1}^{k}(Su^{-2/\alpha_i})}\Psi(u)(1+o(1))\prod_{i=1}^{k_1}\left((\ln u)^{1/\beta_i}(c^i_{p_i+1}-c^i_{p_i})e^{\frac{2\left(\alpha_i-\alpha_i(c^i_{p_i+1})\right)}{\alpha_i\alpha_i(c^i_{p_i+1})}\ln u}\right)\right)\\ &\le \sum_{p_i\le m_i,1\le i\le k_1}\sum_{\boldsymbol{k}\in\mathcal{B}}\left(\frac{\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}[0,C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S]}{\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S)}\left(\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i})\right)\left(N^+_{\boldsymbol{k}}\prod_{i=k_1+1}^{k}(Su^{-2/\alpha_i})\right)\right.\\ &\left.\times\prod_{i=1}^{k_1}\left((\ln u)^{1/\beta_i}(c^i_{p_i+1}-c^i_{p_i})e^{-\frac{2(1-\varepsilon)}{\alpha_i^2}\left((\ln u)^{1/\beta_i}c^i_{p_i+1}\right)^{\beta_i}}e^{\frac{2(1-\varepsilon)}{\alpha_i^2}(\ln u)\left(c^i_{m_i+1}\right)^{\beta_i}|\ln(c^i_{m_i+1})|^{-\delta_i}}\right)\right)\\ &\times\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)(1+o(1)),\end{aligned}$$

where

$$\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta}):=\frac{\prod_{i=1}^{k}u^{2/\alpha_i}}{\prod_{i=1}^{k_1}(\ln u)^{1/\beta_i}},$$

with $\prod_{i=m+1}^{m}(\cdot):=1, m\in\mathbb{N}$. Since

$$\lim_{u\to\infty}\prod_{i=1}^{k_1}(\ln u)^{1/\beta_i}(c^i_{p_i+1}-c^i_{p_i})=0,\quad \lim_{u\to\infty}(\ln u)^{1/\beta_i}c^i_{m_i+1}=\infty,$$

it follows that (see also [12])

$$\lim_{S\to\infty}\frac{\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}[0,C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S]}{\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}S)}=\prod_{i=1}^{k}\mathcal{H}_{\alpha_i},\quad \lim_{u\to\infty}e^{\frac{2(1-\varepsilon)}{\alpha_i^2}(\ln u)(c^i_{m_i+1})^{\beta_i}|\ln(c^i_{m_i+1})|^{-\delta_i}}=1,$$

$$\lim_{\substack{u\to\infty\\ \delta\to 0}}\sum_{\boldsymbol{k}\in\mathcal{B}}\prod_{i=1}^{k}(C_i(\boldsymbol{v}^0)\overline{\varepsilon})^{1/\alpha_i}\left(N_{\boldsymbol{k}}^{+}\prod_{i=k_1+1}^{k}(Su^{-2/\alpha_i})\right)=\int_{\boldsymbol{x}\in\prod_{i=1}^{k_1}\{t_i^0\}\times\prod_{i=k_1+1}^{k}[a_i,b_i]}\prod_{i=1}^{k}(C_i(\boldsymbol{x})\overline{\varepsilon})^{1/\alpha_i}d\boldsymbol{x}$$

and

$$\lim_{u\to\infty}\sum_{p_i\le m_i,1\le i\le k_1}\prod_{i=1}^{k_1}\left((\ln u)^{1/\beta_i}(c^i_{p_i+1}-c^i_{p_i})e^{-\frac{2(1-\varepsilon)}{\alpha_i^2}\left((\ln u)^{1/\beta_i}c^i_{p_i+1}\right)^{\beta_i}}\right)=\int_{\mathbb{R}_+^{k_1}}e^{-\sum_{i=1}^{k_1}\frac{2(1-\varepsilon)}{\alpha_i^2}x_i^{\beta_i}}d\boldsymbol{x}$$
$$=\prod_{i=1}^{k_1}\left(\frac{\alpha_i^2}{2(1-\varepsilon)}\right)^{1/\beta_i}\frac{\Gamma(1/\beta_i)}{\beta_i}.$$

Consequently, the upper bound is given by

$$\begin{aligned}\pi(u)\quad\le\quad&\overline{\varepsilon}^{\sum_{i=1}^{k}\frac{1}{\alpha_i}}\left(\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}\right)\left(\int_{\boldsymbol{x}\in\prod_{i=1}^{k_1}\{t_i^0\}\times\prod_{i=k_1+1}^{k}[a_i,b_i]}\prod_{i=1}^{k}(C_i(\boldsymbol{x}))^{1/\alpha_i}d\boldsymbol{x}\right)\\&\times\prod_{i=1}^{k_1}\left(\frac{\alpha_i^2}{2(1-\varepsilon)}\right)^{1/\beta_i}\frac{\Gamma(1/\beta_i)}{\beta_i}\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)(1+o(1))\end{aligned}$$

as $u\to\infty$. Next we derive the lower bound: using Bonferroni's inequality, we have

$$\begin{aligned}\pi(u)\quad\ge\quad&\sum_{\substack{(j_i,p_i)\in\mathcal{L}_1^i,1\le i\le k_1,\\ (\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{L}_2}}\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u\right)\\&-2\sum_{\substack{(j_i,p_i),(j_i',p_i')\in\mathcal{L}_1^i,1\le i\le k_1,(\boldsymbol{I\!I_k},\boldsymbol{k}),(\boldsymbol{I\!I}'_{\boldsymbol{k}},\boldsymbol{k}')\in\mathcal{L}_2\\ (\boldsymbol{j},\boldsymbol{p})\prec(\boldsymbol{j}',\boldsymbol{p}'),\text{or}\\ (\boldsymbol{j},\boldsymbol{p})=(\boldsymbol{j}',\boldsymbol{p}')\text{ and }(\boldsymbol{I\!I_k},\boldsymbol{k})\prec(\boldsymbol{I\!I}'_{\boldsymbol{k}},\boldsymbol{k}')}}\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u,\ \sup_{\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}}X(\boldsymbol{v}')>u\right)\end{aligned}$$

Similar arguments as in the derivation of the upper bound yield, as $u\to\infty$,

$$\begin{aligned}&\lim_{\delta\to0,S\to\infty}\sum_{\substack{(j_i,p_i)\in\mathcal{L}_1^i,1\le i\le k_1,\\ (\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{L}_2}}\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u\right)\\ \ge\quad&\lim_{\delta\to0,S\to\infty}\sum_{(j_i,p_i)\in\mathcal{L}_1^i,1\le i\le k_1}\sum_{(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{L}_2}\mathbb{P}\left(\sup_{\nu\in[0,S]^k}Y_{\varepsilon,u}^{\boldsymbol{v}^0}(\nu)>u\right)\\ \ge\quad&(1-\varepsilon)^{\sum_{i=1}^{k}\frac{1}{\alpha_i}}\left(\prod_{i=1}^{k}\mathcal{H}_{\alpha_i}\right)\left(\int_{\boldsymbol{x}\in\prod_{i=1}^{k_1}\{t_i^0\}\times\prod_{i=k_1+1}^{k}[a_i,b_i]}\prod_{i=1}^{k}(C_i(\boldsymbol{x}))^{1/\alpha_i}d\boldsymbol{x}\right)\\&\times\prod_{i=1}^{k_1}\left(\frac{\alpha_i^2}{2\overline{\varepsilon}}\right)^{1/\beta_i}\frac{\Gamma(1/\beta_i)}{\beta_i}\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)(1+o(1)).\end{aligned}$$

Therefore, by letting $\varepsilon\to0$, in order to complete the proof, it is sufficient to show that

$$\lim_{\delta\to0,S\to\infty}\lim_{u\to\infty}\frac{\sum_{\substack{(j_i,p_i),(j_i',p_i')\in\mathcal{L}_1^i,1\le i\le k_1,(\boldsymbol{I\!I_k},\boldsymbol{k}),(\boldsymbol{I\!I}'_{\boldsymbol{k}},\boldsymbol{k}')\in\mathcal{L}_2\\ (\boldsymbol{j},\boldsymbol{p})\prec(\boldsymbol{j}',\boldsymbol{p}'),\text{or}\\ (\boldsymbol{j},\boldsymbol{p})=(\boldsymbol{j}',\boldsymbol{p}')\text{ and }(\boldsymbol{I\!I_k},\boldsymbol{k})\prec(\boldsymbol{I\!I}'_{\boldsymbol{k}},\boldsymbol{k}')}}\mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u,\ \sup_{\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\boldsymbol{I\!I}'_{\boldsymbol{k}'},\boldsymbol{k}'}}X(\boldsymbol{v}')>u\right)}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)}$$
$$=\sum_{i=1}^{3}\lim_{\delta\to0,S\to\infty}\lim_{u\to\infty}\frac{\Sigma_u^i}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)}=0,\tag{4.28}$$

where

$$\Sigma_u^i := \sum_{((\boldsymbol{j},\boldsymbol{p}),(\boldsymbol{j}',\boldsymbol{p}'),(\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}),(\mathbb{I}'_{\boldsymbol{k}},\boldsymbol{k}'))\in E_i} \mathbb{P}\left(\sup_{\boldsymbol{v}\in A_{\boldsymbol{j},\boldsymbol{p}}^{\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}}} X(\boldsymbol{v})>u, \sup_{\boldsymbol{v}'\in A_{\boldsymbol{j}',\boldsymbol{p}'}^{\mathbb{I}'_{\boldsymbol{k}'},\boldsymbol{k}'}} X(\boldsymbol{v}')>u\right), \ i=1,2,3,$$

with

$$E_\imath = \Big\{((\boldsymbol{j},\boldsymbol{p}),(\boldsymbol{j}',\boldsymbol{p}'),(\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}),(\mathbb{I}'_{\boldsymbol{k}},\boldsymbol{k}')) : \text{ conditions of } (\imath) \text{ in Lemma 3.6 are satisfied, and } (\boldsymbol{j},\boldsymbol{p})\prec(\boldsymbol{j}',\boldsymbol{p}'), \text{or } (\boldsymbol{j},\boldsymbol{p})=(\boldsymbol{j}',\boldsymbol{p}') \text{ and } (\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k})\prec(\mathbb{I}'_{\boldsymbol{k}},\boldsymbol{k}')\Big\}, \ \imath=1,2,3.$$

The calim in (4.28), which follows from Lemma 3.5 is shown in Appendix. □

Proof of Lemma 3.7: It is easy to see that the set $[0,T]^k\setminus\prod_{i=1}^{k_1}[0,t_u^i]\times\prod_{i=k_1+1}^{k}[a_i,b_i]$ is the union of $2^{k_1}3^{k_2}-1$ sets of the form $\prod_{i=1}^{k_1}\Lambda_{i,u}\times\prod_{i=k_1+1}^{k}\Theta_i$, with

$$\Lambda_{i,u}=[0,t_u^i] \text{ or } [t_u^i,T], i=1,\cdots,k_1, \quad \text{and} \quad \Theta_i=[0,a_i] \text{ or } [a_i,b_i] \text{ or } [b_i,T], i=k_1+1,\cdots,k,$$

where at least one of $\{[t_u^i,T], i=1,\cdots,k_1,[0,a_i],[b_i,T], i=k_1+1,\cdots,k\}$ appears. Since the other cases are similar, without loss of generality, it suffices to prove that

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1-1}[0,t_u^i]\times[t_u^{k_1},T]\times\prod_{i=k_1+1}^{k-1}[a_i,b_i]\times[b_k,T]} X(\boldsymbol{t})>u\right)=o(\pi(u)).$$

We see that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1-1}[0,t_u^i]\times[t_u^{k_1},T]\times\prod_{i=k_1+1}^{k-1}[a_i,b_i]\times[b_k,T]} X(\boldsymbol{t})>u\right)\\
&\le\mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1-1}[0,t_u^i]\times[t_u^{k_1},T]\times\prod_{i=k_1+1}^{k-1}[a_i,b_i]\times[b_k,b_k+t_u^k]} X(\boldsymbol{t})>u\right)\\
&+\mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1-1}[0,t_u^i]\times[t_u^{k_1},T]\times\prod_{i=k_1+1}^{k-1}[a_i,b_i]\times[b_k+t_u^k,T]} X(\boldsymbol{t})>u\right)
\end{aligned}$$

It is sufficient to analyze the first probability on the right-hand side of the last inequality since the analysis of the second one is similar. It is derived that

$$\begin{aligned}
\theta(u) &:= \mathbb{P}\left(\sup_{\boldsymbol{t}\in\prod_{i=1}^{k_1-1}[0,t_u^i]\times[t_u^{k_1},T]\times\prod_{i=k_1+1}^{k-1}[a_i,b_i]\times[b_k,b_k+t_u^k]} X(\boldsymbol{t})>u\right)\\
&\le \sum_{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1-1,k,(\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k})\in\mathcal{U}'_2} \mathbb{P}\left(\sup_{\boldsymbol{v}\in\prod_{i=1}^{k_1-1}B_{j_i,p_i}^i\times[t_u^{k_1},T]\times W_{\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}}\times\left(b_k+B_{j_k,p_k}^k\right)} X(\boldsymbol{v})>u\right), \qquad (4.29)
\end{aligned}$$

where $\boldsymbol{k}=(K_1,\cdots,K_{k_2-1})\in\mathbb{Z}^{k_2-1}$, $\mathbb{I}_{\boldsymbol{k}}=(I_1^{\boldsymbol{k}},\cdots,I_{k_2-1}^{\boldsymbol{k}})\in\mathbb{Z}^{k_2-1}$, and B_{j_k,p_k}^k, $\mathcal{U}'_2$ and $W_{\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}}$ are defined similarly as $B_{j_{k_1},p_{k_1}}^{k_1}$, $\mathcal{U}_2$ and $V_{\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}}$, respectively.

For any fixed $j_i,p_i,i=1,\cdots,k_1,k,\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}$ such that $(j_i,p_i)\in\mathcal{U}_1^i, i=1,2,\cdots,k_1-1,k$ and $(\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k})\in\mathcal{U}'_2$, consider the GRF $X(\boldsymbol{v}):=X(v_1,\cdots,v_k)$ on the set

$$\mathcal{A}_{\boldsymbol{j}\boldsymbol{p},\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}} := \prod_{i=1}^{k_1-1} B_{j_i,p_i}^i\times[t_u^{k_1},T]\times W_{\mathbb{I}_{\boldsymbol{k}},\boldsymbol{k}}\times\left(b_k+B_{j_k,p_k}^k\right).$$

For notational simplicity write next $X_{k_1,u}(\boldsymbol{\nu})$ instead of

$$X\left(c_{p_1}^1+\frac{j_1S+\nu_1}{u^{2/(\alpha_1(c_{p_1+1}^1))}},\cdots,c_{p_{k_1-1}}^{k_1-1}+\frac{j_{k_1-1}S+\nu_{k_1-1}}{u^{2/(\alpha_{k_1-1}(c_{p_{k_1-1}+1}^{k_1-1}))}},\nu_{k_1},\boldsymbol{a'}+\delta\boldsymbol{k}+g_u'S\boldsymbol{I\!I_k}+\triangle'^{\boldsymbol{\nu}}_0,b_k+c_{p_k}^k+\frac{j_kS+\nu_k}{u^{2/(\alpha_k(c_{p_k+1}^k))}}\right),$$

where $\triangle'^{\boldsymbol{\nu}}_0=g_u'\prod_{i=k_1+1}^{k-1}[0,\nu_i]$, $\boldsymbol{a'}=(a_{k_1+1},\cdots,a_{k-1})$ and g_u' is defined in a similar way as g_u (see (3.14)). It follows that

$$\sup_{\boldsymbol{v}\in\mathcal{A}_{\boldsymbol{jp},\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})\overset{d}{=}\sup_{\boldsymbol{\nu}\in[0,S]^{k_1-1}\times[t_u^{k_1},T]\times[0,S]^{k_2}}X_{k_1,u}(\boldsymbol{\nu}).\tag{4.30}$$

Let $b_{k_1,u}=u^{-2/\left(\alpha_{k_1}+\frac{3}{4}(t_u^{k_1})^{\beta_{k_1}}\right)}$, and fix $\boldsymbol{v}^0\in\prod_{i=1}^{k_1-1}A_{p_i}^i\times[0,T]\times\delta_{\boldsymbol{k}}\times(b_k+A_{p_k}^k)$ with $A_{p_i}^i,\delta_{\boldsymbol{k}}$ defined similarly as before (the only difference is the dimension). In view of the proof of (3.16), there exists a constant $\mathbb{C}_0$ such that, for sufficiently large u

$$1-Cov(X_{k_1,u}(\boldsymbol{\nu}),X_{k_1,u}(\boldsymbol{\nu}+\boldsymbol{x}))\le 1-e^{-\frac{3}{2}\sum_{i=1,i\neq k_1}^kC_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}-\mathbb{C}_0|x_{k_1}|^{\alpha_{k_1}+\frac{3}{4}(t_u^{k_1})^{\beta_{k_1}}}}$$

uniformly with respect to $\boldsymbol{\nu},\boldsymbol{\nu}+\boldsymbol{x}\in[0,S]^{k_1-1}\times[t_u^{k_1},T]\times[0,S]^{k_2}$ such that $|x_{k_1}|\le b_{k_1,u}$. Let $\{\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{t}),\boldsymbol{t}\in[0,S]^{k_1-1}\times[t_u^{k_1},T]\times[0,S]^{k_2}\}$, $u>0$, be a family of centered stationary GRF's such that

$$Cov(\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{\nu}),\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{\nu}+\boldsymbol{x}))=e^{-\frac{3}{2}\sum_{i=1,i\neq k_1}^kC_i(\boldsymbol{v}^0)u^{-2}|x_i|^{\alpha_i}-\mathbb{C}_0|x_{k_1}|^{\alpha_{k_1}+\frac{3}{4}(t_u^{k_1})^{\beta_{k_1}}}}$$

for u such that $\alpha_{k_1}+\frac{3}{4}(t_u^{k_1})^{\beta_{k_1}}\le 2$, and $\boldsymbol{\nu},\boldsymbol{\nu}+\boldsymbol{x}\in[0,S]^{k_1-1}\times[t_u^{k_1},T]\times[0,S]^{k_2}$. In view of the Slepian's inequality, continuing (4.29) we get, as $u\to\infty$

$$\begin{aligned}\theta(u)&\le\sum_{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1-1,k,(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2'}\mathbb{P}\left(\sup_{\boldsymbol{v}\in\mathcal{A}_{\boldsymbol{jp},\boldsymbol{I\!I_k},\boldsymbol{k}}}X(\boldsymbol{v})>u\right)\\&\le\sum_{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1-1,k(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2'}\sum_{l=0}^{\lfloor T(b_{k_1,u})^{-1}\rfloor+1}\mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^{k_1-1}\times[lb_{k_1,u},(l+1)b_{k_1,u}]\times[0,S]^{k_2}}X_{k_1,u}(\boldsymbol{\nu})>u\right)\\&\le(\lfloor T(b_{k_1,u})^{-1}\rfloor+2)\sum_{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1-1,k(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2'}\mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^{k_1-1}\times[0,b_{k_1,u}]\times[0,S]^{k_2}}\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{\nu})>u\right)\\&\le\left(u^{2/\alpha_{k_1}}(\ln u)^{-\frac{4}{3\beta_{k_1}}}T+2\right)\sum_{(j_i,p_i)\in\mathcal{U}_1^i,i=1,\cdots,k_1-1,k(\boldsymbol{I\!I_k},\boldsymbol{k})\in\mathcal{U}_2'}\mathbb{P}\left(\sup_{\boldsymbol{\nu}\in[0,S]^{k_1-1}\times[0,b_{k_1,u}]\times[0,S]^{k_2}}\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{\nu})>u\right),\end{aligned}$$

where in the last inequality we used that $(b_{k_1,u})^{-1}\le u^{2/\alpha_{k_1}}(\ln u)^{-\frac{4}{3\beta_{k_1}}}$ given in [12]. Furthermore, it follows from Lemma 3.2 that, as $u\to\infty$

$$\begin{aligned}&\mathbb{P}\left(\sup_{\nu\in[0,S]^{k_1-1}\times[0,b_{k_1,u}]\times[0,S]^{k_2}}\widetilde{Z}_u^{\boldsymbol{v}^0}(\boldsymbol{\nu})>u\right)\\&=\left(\prod_{i=1}^{k_1-1}\mathcal{H}_{\alpha_i}\left[0,\left(\frac{3}{2}C_i(\boldsymbol{v}^0)\right)^{1/\alpha_i}S\right]\times\mathcal{H}_{\alpha_{k_1}}[0,\mathbb{C}_0^{1/\alpha_{k_1}}]\times\prod_{i=k_1+1}^{k}\mathcal{H}_{\alpha_i}\left[0,\left(\frac{3}{2}C_i(\boldsymbol{v}^0)\right)^{1/\alpha_i}S\right]\right)\Psi(u)(1+o(1))\\&\le\mathbb{C}_3\prod_{i=1}^k\mathcal{H}_{\alpha_i}[0,1]S^{k-1}\Psi(u)(1+o(1))\end{aligned}$$

for some positive constant $\mathbb{C}_3$. Consequently, similar arguments as in the proof of the upper bound in Theorem 1.1 imply

$$\begin{aligned}\theta(u) &\leq \mathbb{C}_4 T\left(\prod_{i=k_1+1}^{k-1}(b_i-a_i)\right)\left(\prod_{i=1}^{k_1-1}\frac{(\alpha_i)^{2/\beta_i}\Gamma(1/\beta_i)}{\beta_i}\frac{(\alpha_k)^{2/\beta_k}\Gamma(1/\beta_k)}{\beta_k}\right)\left(\frac{\prod_{i=1}^{k}u^{2/\alpha_i}}{\prod_{i=1}^{k_1-1}(\ln u)^{1/\beta_i}}\right)(\ln u)^{-\frac{4}{3\beta_{k_1}}-\frac{1}{\beta_k}}\Psi(u)\\ &= o(\pi(u))\end{aligned}$$

as $u\to\infty$, and thus the proof is complete. □

5 Appendix

Proof of Lemma 3.2: Borrowing Piterbarg's idea to utilise continuous mapping theorem for the conditional Gaussian random field (see [29] for details) we write for any $u>0$

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}X_u(\boldsymbol{t})>u\right) = \frac{1}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}\int_{-\infty}^{\infty}e^{z-\frac{z^2}{2u^2}}\mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}X_u(\boldsymbol{t})>u|X_u(\boldsymbol{0})=u-\frac{z}{u}\right)dz. \tag{5.31}$$

Clearly, for any $u>0$

$$\left\{X_u(\boldsymbol{t})|X_u(\boldsymbol{0})=u-\frac{z}{u},\boldsymbol{t}\in\boldsymbol{D}\right\} \text{ and } \left\{X_u(\boldsymbol{t})-r_u(\boldsymbol{t},\boldsymbol{0})X_u(\boldsymbol{0})+r_u(\boldsymbol{t},\boldsymbol{0})\left(u-\frac{z}{u}\right),\boldsymbol{t}\in\boldsymbol{D}\right\}$$

have the same finite-dimension distribution, which implies

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}X_u(\boldsymbol{t})>u|X_u(\boldsymbol{0})=u-\frac{z}{u}\right)=\mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}\left(\zeta_u(\boldsymbol{t})-u^2(1-r_u(\boldsymbol{t},\boldsymbol{0}))+z(1-r_u(\boldsymbol{t},\boldsymbol{0}))\right)>z\right),$$

with $\{\zeta_u(\boldsymbol{t})=u(X_u(\boldsymbol{t})-r_u(\boldsymbol{t},\boldsymbol{0})X_u(\boldsymbol{0})),\boldsymbol{t}\in\boldsymbol{D}\}$. By (3.17)

$$\lim_{u\to\infty}(u^2(1-r_u(\boldsymbol{t},\boldsymbol{0}))-z(1-r_u(\boldsymbol{t},\boldsymbol{0})))=|\boldsymbol{t}|_{\boldsymbol{\alpha}}$$

uniformly with respect to $\boldsymbol{t}\in\boldsymbol{D}$ for any $z\in\mathbb{R}$.

Next we prove that $\zeta_u, u>0$ converges weakly to $\widetilde{B}_{\boldsymbol{\alpha}}$ in $\mathcal{C}(\boldsymbol{D})$ as $u\to\infty$. To this end, we need to show (e.g., [27, 32]):

i) finite-dimensional distributions of ζ_u converge to those of $\widetilde{B}_{\boldsymbol{\alpha}}$ as $u\to\infty$

ii) tightness, i.e., for any $\eta>0$

$$\lim_{\delta\to 0}\limsup_{u\to\infty}\mathbb{P}\left(\sup_{\substack{\boldsymbol{s},\boldsymbol{t}\in\boldsymbol{D}\\ \max_{1\le i\le k}|s_i-t_i|<\delta}}|\zeta_u(\boldsymbol{t})-\zeta_u(\boldsymbol{s})|>\eta\right)=0.$$

First note that the increments of the centered GRF $\{\zeta_u(\boldsymbol{t}),\boldsymbol{t}\in\boldsymbol{D}\}$ have the following property

$$\begin{aligned}\lim_{u\to\infty}Var(\zeta_u(\boldsymbol{t})-\zeta_u(\boldsymbol{s})) &= \lim_{u\to\infty}(2u^2(1-r_u(\boldsymbol{t},\boldsymbol{s}))-u^2(r_u(\boldsymbol{t},\boldsymbol{0})-r_u(\boldsymbol{s},\boldsymbol{0}))^2)\\ &= 2|\boldsymbol{t}-\boldsymbol{s}|_{\alpha}=Var(\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t})-\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{s}))\end{aligned} \tag{5.32}$$

establishing i). Furthermore, the above holds uniformly with respect to $\boldsymbol{t}, \boldsymbol{s} \in \boldsymbol{D}$. In order to prove the tightness, we use a similar approach as in [19, 14]. We start by defining, for fixed $u > 0$, a semi-metric d_u on $\mathbb{R}_+^k$ as

$$d_u(\boldsymbol{s}, \boldsymbol{t}) = \sqrt{Var(\zeta_u(\boldsymbol{t}) - \zeta_u(\boldsymbol{s}))}.$$

Further write

$$B_{d_u}(\boldsymbol{t}, u, \vartheta) := \{\boldsymbol{s} \in \mathbb{R}_+^k : d_u(\boldsymbol{s}, \boldsymbol{t}) \le \vartheta\}$$

for the d_u-ball centered at $\boldsymbol{t} \in \mathbb{R}_+^k$ and of radius ϑ, and let

$$H_{d_u}(\boldsymbol{D}', u, \vartheta) := \ln(N'(\boldsymbol{D}', u, \vartheta)),$$

with $N'(\boldsymbol{D}', u, \vartheta)$ being the smallest number of such balls that cover $\boldsymbol{D}'$, a compact set in $\mathbb{R}_+^k$. Here $H_{d_u}(\boldsymbol{D}', u, \vartheta)$ is called *(metric) entropy* for $\boldsymbol{D}'$ induced by d_u. We refer to [5] for more details on entropy.

We see from (5.32) that, for u sufficiently large, there exists some constant $\mathbb{C}_0$ such that

$$d_u(\boldsymbol{s}, \boldsymbol{t}) \le \mathbb{C}_0 \sqrt{|\boldsymbol{s} - \boldsymbol{t}|_\alpha} \le k\mathbb{C}_0 \delta^{\frac{\alpha}{2}}, \tag{5.33}$$

if $\max_{1 \le i \le k} |s_i - t_i| < \delta < 1$, where $\alpha := \min_{1 \le i \le k} \alpha_i$. By Corollary 1.3.4 in [5] there exists some constant $Q_0 > 0$ such that for any $0 < \delta < 1$

$$\begin{aligned} \mathbb{P}\left(\sup_{\substack{\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{D} \\ \max_{1 \le i \le k} |s_i - t_i| < \delta}} |\zeta_u(\boldsymbol{t}) - \zeta_u(\boldsymbol{s})| > \eta \right) &\le \mathbb{P}\left(\sup_{\substack{\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{D} \\ d_u(\boldsymbol{s}, \boldsymbol{t}) < k\mathbb{C}_0 \delta^{\frac{\alpha}{2}}}} |\zeta_u(\boldsymbol{t}) - \zeta_u(\boldsymbol{s})| > \eta \right) \\ &\le \frac{Q_0}{\eta} \int_0^{k\mathbb{C}_0 \delta^{\frac{\alpha}{2}}} \sqrt{H_{d_u}([0, R]^k, u, \vartheta)} d\vartheta, \end{aligned}$$

with $R < \infty$ being a sufficiently large. Define, for $\boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}_+^k$, a semi-metric

$$\tilde{d}(\boldsymbol{t}, \boldsymbol{s}) = \mathbb{C}_0 \sqrt{|\boldsymbol{s} - \boldsymbol{t}|_\alpha}.$$

By (5.33) for sufficiently large u and small ϑ

$$H_{d_u}([0, R]^k, u, \vartheta) \le H_{\tilde{d}}([0, R]^k, u, \vartheta) \le k \ln \left(\frac{R}{\left(\frac{\vartheta^2}{k\mathbb{C}_0^2} \right)^{\frac{1}{\alpha}}} + 1 \right) \le \mathbb{C}_1 \ln \left(\frac{1}{\vartheta} \right),$$

for some positive constant $\mathbb{C}_1$, with $H_{\tilde{d}}([0, R]^k, u, \vartheta)$ being the entropy induced by $\tilde{d}$. Consequently,

$$\lim_{\delta \to 0} \limsup_{u \to \infty} \mathbb{P}\left(\sup_{\substack{\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{D} \\ \max_{1 \le i \le k} |s_i - t_i| < \delta}} |\zeta_u(\boldsymbol{t}) - \zeta_u(\boldsymbol{s})| > \eta \right) \le \lim_{\delta \to 0} \frac{Q_0 \sqrt{\mathbb{C}_1}}{\eta} \int_{\frac{1}{k\mathbb{C}_0} \delta^{-\frac{\alpha}{2}}}^{\infty} \frac{\sqrt{\ln \vartheta}}{\vartheta^2} d\vartheta = 0,$$

establishing ii). Moreover, since the functional $\sup_{\boldsymbol{t} \in \boldsymbol{D}} f(\boldsymbol{t})$ is continuous on $\mathcal{C}(\boldsymbol{D})$ for any $z \in \mathbb{R}$

$$\lim_{u \to \infty} \mathbb{P}\left(\sup_{\boldsymbol{t} \in \boldsymbol{D}} X_u(\boldsymbol{t}) > u | X_u(\boldsymbol{0}) = u - \frac{z}{u} \right) = \mathbb{P}\left(\sup_{\boldsymbol{t} \in \boldsymbol{D}} (\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t}) - |\boldsymbol{t}|_{\boldsymbol{\alpha}}) > z \right).$$

In order to apply the dominated convergence theorem to the integral in (5.31) when taking limit in u, we need a uniform (in u large enough) upper bound of

$$P_u(z) := \mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} \left(\zeta_u(\boldsymbol{t}) - u^2(1-r_u(\boldsymbol{t},\boldsymbol{0})) + z(1-r_u(\boldsymbol{t},\boldsymbol{0}))\right) > z\right)$$

for $z > 0$ sufficiently large. It follows that, for u sufficiently large,

$$\begin{aligned} P_u(z) &\leq \mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}\zeta_u(\boldsymbol{t}) + \sup_{\boldsymbol{t}\in\boldsymbol{D}}(1-r_u(\boldsymbol{t},\boldsymbol{0}))z > z\right) \\ &\leq \mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}\zeta_u(\boldsymbol{t}) > (1-\varrho_0)z\right) \end{aligned} \tag{5.34}$$

for some $\varrho_0 \in (0,1)$. Further, we see from (5.32) that, for sufficiently large u, there exists some positive constant $\mathbb{C}_2$ such that

$$Var(\zeta_u(\boldsymbol{t}) - \zeta_u(\boldsymbol{s})) \leq \mathbb{C}_2 Var(\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t}) - \widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{s}))$$

for all $\boldsymbol{s},\boldsymbol{t} \in \boldsymbol{D}$. Hence by Sudakov-Fernique inequality (e.g., [5])

$$\mathbb{E}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} \zeta_u(\boldsymbol{t})\right) \leq \sqrt{\mathbb{C}_2}\,\mathbb{E}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} \widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t})\right) := U_0. \tag{5.35}$$

The constant U_0 is finite, which follows from Theorem 2.1.1 in [5]. Moreover,

$$\sup_{\boldsymbol{t}\in\boldsymbol{D}} Var(\zeta_u(\boldsymbol{t})) \leq \sigma^2_{\boldsymbol{D}} := \mathbb{C}_2 \sup_{\boldsymbol{t}\in\boldsymbol{D}} Var(\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t})) = 2\mathbb{C}_2 \sup_{\boldsymbol{t}\in\boldsymbol{D}} |\boldsymbol{t}|_{\boldsymbol{\alpha}} < \infty. \tag{5.36}$$

With the help of (5.34), (5.35) and (5.36), Borell-TIS inequality (Theorem 2.1.1 in [5]) gives, for any $z > \frac{U_0}{1-\varrho_0}$ and u sufficiently large,

$$P_u(z) \leq \mathbb{P}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}}\zeta_u(\boldsymbol{t}) > (1-\varrho_0)z\right) \leq \exp\left(-\frac{((1-\varrho_0)z-U_0)^2}{2\sigma^2_{\boldsymbol{D}}}\right).$$

Applying the dominated convergence theorem to the integral in (5.31), we conclude that

$$\lim_{u\to\infty} \int_{-\infty}^{\infty} e^{z-\frac{z^2}{2u^2}} P_u(z)dz = \mathbb{E}\left(\sup_{\boldsymbol{t}\in\boldsymbol{D}} e^{\widetilde{B}_{\boldsymbol{\alpha}}(\boldsymbol{t})-|\boldsymbol{t}|_{\boldsymbol{\alpha}}}\right),$$

thus the proof is complete. □

Proof of Eq. (4.28): According to Lemma 3.5, the three parts of the double-sum in (4.28) can be estimated in different ways. It follows from (3.24) that

$$\lim_{u\to\infty} \frac{\Sigma_u^3}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)} \leq \lim_{u\to\infty} \frac{2\Psi\left(\frac{u-h/2}{\sqrt{1-\lambda/2}}\right)\left(\sum_{(\boldsymbol{I_k},\boldsymbol{k})\in\mathcal{L}_2,(j_i,p_i)\in\mathcal{L}_1^i,1\leq i\leq k_1} 1\right)^2}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)} = 0,$$

where the sum in the middle term can be estimated using the same arguments as the upper bound in Theorem 1.1. Next, for sake of simplicity, we only give the estimates of the first two sums for $k_1 = k_2 = 1$, since the general cases (k_1, k_2 are arbitrary integers) follow from similar arguments. For the first sum, we derive, using (3.21) that, for u sufficiently large

$$\Sigma_u^1 \leq \sum_{(I_1^{\boldsymbol{k}},K_1)\in\mathcal{L}_2,(j_1,p_1)\in\mathcal{L}_1^1} \Bigg(4 \sum_{\substack{(j_1',p_1')\in\mathcal{L}_1^1 \\ (j_1,p_1)\prec(j_1',p_1') \text{ and } N_{j_1,p_1}^{j_1',p_1'}>0}} \sum_{\substack{(I_1'^{\boldsymbol{k}'},K_1')\in\mathcal{L}_2 \\ N_{I_1^{\boldsymbol{k}},K_1}^{I_1'^{\boldsymbol{k}'},K_1'}\geq 0}} \mathbb{C}S^4 \exp\Big(-\mathbb{C}_1\Big((N_{j_1,p_1}^{j_1',p_1'})^{\alpha_1/2}S^{\alpha_1}$$

$$+\left(N_{I_1^{\boldsymbol{k}},K_1}^{I_1'^{\boldsymbol{k}'},K_1'}\right)^{\alpha_2} S^{\alpha_2}\Big)\Big)+2\sum_{\substack{(I_1'^{\boldsymbol{k}'},K_1')\in\mathcal{L}_2\\ N_{I_1^{\boldsymbol{k}},K_1}^{I_1'^{\boldsymbol{k}'},K_1'}>0}} \mathbb{C}S^4\exp\left(-\mathbb{C}_1\left(N_{I_1^{\boldsymbol{k}},K_1}^{I_1'^{\boldsymbol{k}'},K_1'}\right)^{\alpha_2} S^{\alpha_2}\right)\Psi(u)$$

$$\le 4\mathbb{C}S^4\sum_{(I_1^{\boldsymbol{k}},K_1)\in\mathcal{L}_2,(j_1,p_1)\in\mathcal{L}_1^1}\left(\left(\sum_{n_1\ge 1} e^{-\mathbb{C}_1(\sqrt{n_1}S)^{\alpha_1}}\right)\left(\sum_{n_2\ge 0} e^{-\mathbb{C}_1(n_2S)^{\alpha_2}}\right)+\left(\sum_{n_3\ge 1} e^{-\mathbb{C}_1(n_3S)^{\alpha_2}}\right)\right)\Psi(u)$$

$$\le \mathbb{C}'S^4\sum_{(I_1^{\boldsymbol{k}},K_1)\in\mathcal{L}_2,(j_1,p_1)\in\mathcal{L}_1^1}\left(e^{-\mathbb{C}_1'S^{\alpha_1}}\left(1+e^{-\mathbb{C}_1''S^{\alpha_1}}\right)+e^{-\mathbb{C}_1'''S^{\alpha_1}}\right)\Psi(u),$$

for suitably chosen positive constants $\mathbb{C}', \mathbb{C}_1', \mathbb{C}_1'', \mathbb{C}_1'''$. This, combined with the estimate of the last sum in the above formula, yields that

$$\lim_{S\to\infty}\lim_{u\to\infty}\frac{\Sigma_u^1}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)}=0. \tag{5.37}$$

Next, we focus on the second sum. According to (2) of Lemma 3.5, the sum Σ_u^2 can be divided into four parts, denoted by $\Sigma^2_{i_1^1,u}, \Sigma^2_{i_2^1,u}, \Sigma^2_{i_1^2,u}$ and $\Sigma^2_{i_2^2,u}$, respectively. Applying (3.22), Lemma 3.1 and Lemma 3.3 we find that, for u large enough,

$$\Sigma^1_{i_1^1,u}\le 8\sum_{(I_1^{\boldsymbol{k}},K_1)\in\mathcal{L}_2,(j_1,p_1)\in\mathcal{L}_1^1}\Bigg[\mathbb{P}\Bigg(\sup_{\left[c_{p_1}^1+\frac{j_1S}{u^{2/\alpha_1}(cp_1+1)},c_{p_1}^1+\frac{(j_1+1)S}{u^{2/\alpha_1}cp_1+1}\right]\times V_{I_1^{\boldsymbol{k}},K_1}} X(u)>u,$$

$$\sup_{\left[c_{p_1}^1+\frac{(j_1+1)S+\sqrt{S}}{u^{2/\alpha_1}(cp_1+1)},c_{p_1}^1+\frac{(j_1+2)S}{u^{2/\alpha_1}cp_1+1}\right]\times V_{I_1'^{\boldsymbol{k}'},K_1'}} X(u)>u\Bigg)+\mathbb{P}\Bigg(\sup_{\left[c_{p_1}^1+\frac{(j_1+1)S}{u^{2/\alpha_1}(cp_1+1)},c_{p_1}^1+\frac{(j_1+1)S+\sqrt{S}}{u^{2/\alpha_1}cp_1+1}\right]\times V_{I_1'^{\boldsymbol{k}'},K_1'}} X(u)>u\Bigg)\Bigg]$$

$$\le \tilde{\mathbb{C}}\sum_{(I_1^{\boldsymbol{k}},K_1)\in\mathcal{L}_2,(j_1,p_1)\in\mathcal{L}_1^1}\left(\mathbb{C}S^4e^{-\mathbb{C}_1S^{\alpha_1/2}}+\prod_{i=1}^2\mathcal{H}_{\alpha_i}[0,1](C_U)^{1/\alpha_i}S^{3/2}\right)\Psi(u)$$

for suitably chosen constant $\tilde{\mathbb{C}}$. Note that in the last formula $V_{I_1'^{\boldsymbol{k}'},K_1'}$ is one of the adjacent sets of $V_{I_1^{\boldsymbol{k}},K_1}$, and the number of it is at most 8. Using the same arguments we can obtain similar upper bounds for $\Sigma^2_{i_1^2,u}, \Sigma^2_{i_2^1,u}$ and $\Sigma^2_{i_2^2,u}$. Consequently, the same reasoning as (5.37) yields

$$\lim_{S\to\infty}\lim_{u\to\infty}\frac{\Sigma_u^2}{\eta(u,k_1,\boldsymbol{\alpha},\boldsymbol{\beta})\Psi(u)}=0,$$

hence the claim follows. □

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