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Initial Stress Symmetry and Applications in Elasticity

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Abstract

An initial stress within a solid can arise to support external loads or from processes such as thermal expansion in inert matter or growth and remodelling in living materials. For this reason it is useful to develop a mechanical framework of initially stressed solids irrespective of how this stress formed. An ideal way to do this is to write the free energy density $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ in terms of initial stress $\boldsymbol{\tau}$ and the elastic deformation gradient $\boldsymbol{F}$. In this paper we present a new constitutive condition for initially stressed materials, which we call the initial stress symmetry (ISS). We focus on two consequences of this symmetry. First we examine how ISS restricts the free energy density $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ and present two examples of $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ that satisfy ISS. Second we show that the initial stress can be derived from the Cauchy stress and the elastic deformation gradient. To illustrate we take an example from biomechanics and calculate the optimal Cauchy stress within an artery subjected to internal pressure. We then use ISS to derive the optimal target residual stress for the material to achieve after remodelling.

Keywords: residual stress, initial stress, biomechanics, elasticity, constitutive equations

1 Introduction

When all loads are removed a body can still hold a significant amount of internal stress, called the *residual stress*. In manufacturing, residual stress has long been noted to be detrimental to, or enhance, the performance of a material. For biological tissues, residual stress is used to self-regulate stress and strain, and ultimately preserve ideal mechanical conditions for the tissue (Fung, 1991; Holzapfel and Ogden, 2003). In geophysics, due to gravity the Earth has developed high initial stress within, which greatly influence the propagation of elastic waves.

Here we use the term *initial stress* to broadly mean the internal stress of some reference configuration, irrespective of how the stress was formed or the boundary conditions. In this sense residual stress is a form of initial stress.

The initial stress felt by any region of a material is due to the push and pull of the surrounding regions. If any region were to be cut out from the material, the stress on its

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newly formed boundary would be zero, thus reducing the potential energy in the bulk. Based on this concept Hoger developed constitutive laws for residually stressed materials, see for example Hoger (1986); Hoger (1993); Johnson and Hoger (1995). Hoger showed that by taking this idea to its limit, and cutting the material into possibly an infinite number of disconnected regions, the material may be relieved of all of its internal stress in a configuration called the *virtual stress-free state*. From that configuration, a hyperelastic energy can be defined as a function of the strain from the virtual state to the current configuration.

Though the use of this virtual state is technically sound, it leads to challenging calculations even for simple deformations and rarely yields analytic results, unless great simplifications are assumed. Moreover, the experimentally identification of the virtual state requires cutting the material, which is not always suitable, especially for living organisms. However, using a virtual stress-free reference is routinely seen as the only viable alternative, quoting Chuong and Fung (1986): "To characterize the arterial wall or any other biological soft tissue, we need a stress-free state". Conversely, We believe that by developing tools to work directly with initially stressed reference configurations, without the need of a stress-free state, will be very useful, specially in biomechanics.

An ideal way to account for the initial stress would be to have a free energy density function $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ written explicitly in terms of the deformation gradient $\boldsymbol{F}$ and the initial stress $\boldsymbol{\tau}$, without any a priori restrictions. For the development of constitutive laws there is no need to distinguish between residual stresses and initial stresses, a view which is shared with Merodio et al. (2013). The initial stress $\boldsymbol{\tau}$ could then be determined from elastic wave speeds (Shams and Ogden, 2014; Destrade and Ogden, 2012; Man and Lu, 1987) or by solving the linear equations of momentum balance.

Shams et al. (2011) and Shams (2010) worked towards a general framework for initially stressed solids, while others have investigated the mechanics for some examples of $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ (Merodio et al., 2013; Merodio and Ogden, 2015). For a more complex geometry, Wang et al. (2014) found that including one residual stress invariant in the free energy density was a simple way to model the effects of residual stress on the myocardium. However, in general a major obstacle still remains: how to write $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ in terms of the combined invariants of $\boldsymbol{F}$ and $\boldsymbol{\tau}$? If the only source of anisotropy is due to the residual stress, then the free energy still depends on ten independent invariants.

Johnson and Hoger (1995) developed representations for the Cauchy stress response $\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau})$ in terms of $\boldsymbol{F}$ and $\boldsymbol{\tau}$ by assuming a stress free virtual state and numerically inverting the residual stress-strain equation. However, this approach often requires solving numerical nonlinear implicit equations. Johnson and Hoger (1995); Johnson and Hoger (1998) exemplified this approach for a material with virtual state composed by a Mooney-Rivlin strain energy.

In this work, we introduce a new constitutive requirement on $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ called the Initial Stress Symmetry (ISS). ISS restricts the constitutive form of the stress $\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau})$ by providing a constitutive equation for the initial stress $\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma})$ that must hold for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$. To our knowledge this symmetry has never been discussed before.

The ISS also helps answer an important question: how much can the Cauchy stress be altered by adjusting the initial stress? From a modelling perspective, initial stress has been used to make the material more or less compliant (Johnson and Hoger, 1998), to control the Poynting effect (Merodio et al., 2013) or to maintain an ideal internal stress (Fung, 1991). Given a $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ that satisfies ISS, then $\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma})$ suggests that for any choice of $\boldsymbol{\sigma}$, there will exist an initial stress $\boldsymbol{\tau}$ that supports $\boldsymbol{\sigma}$.

The basic equations for an elastic material subject to initial stress are summarized in Section 2, and then the initial stress symmetry is first presented in Section 2.1. ISS is satisfied automatically if the initial stress is due to an elastic deformation of a stress free configuration, which we demonstrate in Appendix A. For this reason we develop in Section 3 an example for $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$, that satisfies ISS, by deforming an incompressbile neo-Hookean material from a stress free virtual state.

In Section 4 we express ISS, in all generality, as nine scalar equations for an incompressible material, written in terms of Ψ; two undetermined scalars p and p_τ; and the invariants of $\boldsymbol{F}$ and $\boldsymbol{\tau}$. This form of ISS makes it easier to select representations for $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$. For example, in Section 4.1 we show, with a minor adjustament to the equations, how to use the scalar equations of ISS to deduce an example of $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ for a compressibly material.

It is commonly thought that arteries attempt to maintain a homogeneous stress gradient within their walls (Taber and Eggers, 1996). In Section 5 we calculate this optimal Cauchy stress for a simplified arterial wall, and then show how by using ISS we can calculate the residual stress that exactly supports this optimal Cauchy stress in Section 5.3. We finally compare these results against the commonly used opening-angle method (Chuong and Fung, 1986).

2 Initially stressed elastic materials

A common approach to model the effect of *residual stress* is to consider a *virtual stress free configuration* $\tilde{\mathcal{B}}$, from which the material is deformed and "glued" together to produce a residually stressed equilibrium state $\overset{\circ}{\mathcal{B}}$. See Figure 1 for a diagram of all the configurations. An elastic stored energy density Ψ can then be defined as a function of the *deformation gradient* $\tilde{\boldsymbol{F}}$ from $\tilde{\mathcal{B}}$ to the current configuration $\mathcal{B}$ so that $\Psi = \Psi(\tilde{\boldsymbol{F}})$.

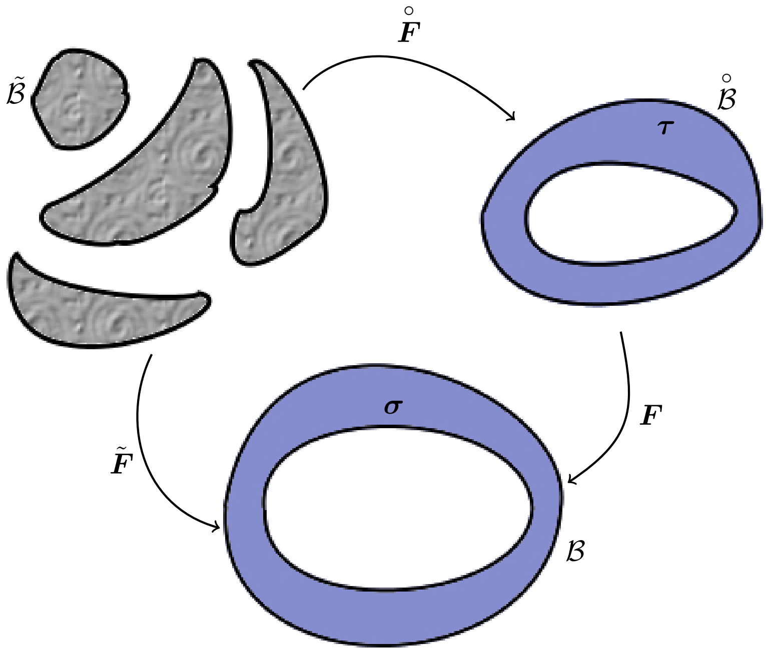

Figure 1: $\mathcal{B}$ is the current configuration with internal stress $\boldsymbol{\sigma}$, while $\overset{\circ}{\mathcal{B}}$ is a reference configuration with internal stress $\boldsymbol{\tau}$. The virtual stress-free state $\tilde{\mathcal{B}}$ is a collection of configurations where the body is stress-free.

In this paper we want to write the free energy density Ψ as a function of the initial stress $\boldsymbol{\tau}$ and of the deformation gradient $\boldsymbol{F} : \overset{\circ}{\mathcal{B}} \to \mathcal{B}$, so that $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$, and $\overset{\circ}{\mathcal{B}}$ will not necessarily be an unloaded configuration. We feel that it is natural to consider that initial stress contributes to the potential energy stored by a material. One extreme example is the wapa tree, which has been know to burst open once cut, possibly causing injury, due to its immense level of residual stress (Détienne and Thiel, 1988).

We assume that $\boldsymbol{F}$ is a purely elastic deformation, but we do not make any assumptions about the origins of the initial stress $\boldsymbol{\tau}$, except that $\boldsymbol{\tau}$ affects the stored energy density Ψ. Assuming that the body is incompressible, i.e. $J = \det \boldsymbol{F} = 1$ at all times, the Cauchy stress tensor $\boldsymbol{\sigma}$ reads (Guillou and Ogden, 2006):

$$\boldsymbol{\sigma} = \boldsymbol{F}\frac{\partial \Psi}{\partial \boldsymbol{F}}(\boldsymbol{F}, \boldsymbol{\tau}) - p\boldsymbol{I}, \tag{1}$$

where p is the Lagrange multiplier associated with the constraint of incompressibility, $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{\tau}$ is the initial stress. If the material is compressible then p is replaced by $-2I_3\partial\Psi/\partial I_3$. Note we have and will omit the possible dependence of Ψ on the position $X \in \overset{\circ}{\mathcal{B}}$ for the sake of simplicity. For the body in the configuration $\overset{\circ}{\mathcal{B}}$ we have that $\boldsymbol{F} = \boldsymbol{I}$ and $\boldsymbol{\sigma} = \boldsymbol{\tau}$; thus we require that

$$\boldsymbol{\tau} = \frac{\partial \Psi}{\partial \boldsymbol{F}}(\boldsymbol{I}, \boldsymbol{\tau}) - \overset{\circ}{p}\boldsymbol{I}, \tag{2}$$

where $\overset{\circ}{p}$ is the value of p when $\boldsymbol{F} = \boldsymbol{I}$. We call the above equation the *residual stress compatibility.*

The presence of residual stress generally leads to an anisotropy response of the material in reference to $\overset{\circ}{\mathcal{B}}$. Here we assume no other source of intrinsic anisotropic so that Ψ can be written as a function of all the independent invariants generated by $\boldsymbol{\tau}$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, the right Cauchy-Green deformation tensor. Following Shams et al.(2011), we take the following complete set of 10 independent invariants

$$I_1 = \operatorname{tr}\boldsymbol{C}, \quad I_2 = \frac{1}{2}[(I_1^2 - \operatorname{tr}(\boldsymbol{C}^2)], \quad I_3 = \det\boldsymbol{C}, \tag{3}$$

$$I_{\tau_1} = \operatorname{tr}\boldsymbol{\tau}, \quad I_{\tau_2} = \frac{1}{2}[(I_{\tau_1}^2 - \operatorname{tr}(\boldsymbol{\tau}^2)], \quad I_{\tau_3} = \det\boldsymbol{\tau}, \tag{4}$$

$$J_1 = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{C}), \quad J_2 = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{C}^2), \quad J_3 = \operatorname{tr}(\boldsymbol{\tau}^2\boldsymbol{C}), \quad J_4 = \operatorname{tr}(\boldsymbol{\tau}^2\boldsymbol{C}^2). \tag{5}$$

The Cauchy stress Eq. (1) can then be written as (Shams et al., 2011)

$$\begin{aligned}\boldsymbol{\sigma} = 2\Psi_{I_1}\boldsymbol{B} + 2\Psi_{I_2}(I_1\boldsymbol{B} - \boldsymbol{B}^2) - p\boldsymbol{I} + 2\Psi_{J_1}\boldsymbol{F}\boldsymbol{\tau}\boldsymbol{F}^T + 2\Psi_{J_2}\boldsymbol{F}(\boldsymbol{\tau}\boldsymbol{C} + \boldsymbol{C}\boldsymbol{\tau})\boldsymbol{F}^T \\ + 2\Psi_{J_3}\boldsymbol{F}\boldsymbol{\tau}^2\boldsymbol{F}^T + 2\Psi_{J_4}\boldsymbol{F}(\boldsymbol{\tau}^2\boldsymbol{C} + \boldsymbol{C}\boldsymbol{\tau}^2)\boldsymbol{F}^T\end{aligned} \tag{6}$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, and $\Psi_{I_1}, \Psi_{I_2}, \Psi_{J_1}, \Psi_{J_2}, \Psi_{J_3}, \Psi_{J_4}$ are the partial derivatives of Ψ with respect to $I_1, I_2, J_1, J_2, J_3, J_4$ respectively. There are no partial derivative of Ψ with respect to I_{τ_1}, I_{τ_2} and I_{τ_3} appearing in (6) because $\boldsymbol{\tau}$ does not depend on $\boldsymbol{F}$.

The residual stress compatibility Eq. (2) becomes

$$2\frac{\partial \Psi}{\partial I_1} + 4\frac{\partial \Psi}{\partial I_2} - \overset{\circ}{p} = 0, \quad 2\frac{\partial \Psi}{\partial J_1} + 4\frac{\partial \Psi}{\partial J_2} = 1, \quad \frac{\partial \Psi}{\partial J_3} + 2\frac{\partial \Psi}{\partial J_4} = 0. \tag{7}$$

Another important physical restriction that can be imposed is the *strong-ellipticity* condition, which is satisfied when the fourth-order tensor

$$\mathcal{A}_{0piqj} = J^{-1}F_{p\alpha}F_{q\beta}\frac{\partial^2\Psi}{\partial F_{i\alpha}\partial F_{j\beta}},$$

satisfies

$$\mathcal{A}_{0piqj} n_p n_q m_i m_j > 0 \quad \text{for every} \;\; \boldsymbol{n}, \boldsymbol{m} \in \mathbb{R}^3, \tag{8}$$

for compressible materials, while for incompressible materials the above need only hold for $\boldsymbol{n} \cdot \boldsymbol{m} = 0$. Imposing SE implies that plane waves may propagate in every direction with a real valued speed (Truesdell, 1966), and other physically expected behaviour (Walton and Wilber, 2003). For a representation of $\mathcal{A}_{0piqj}$ in terms of the invariants of $\boldsymbol{\tau}$ and $\boldsymbol{F}$ see Shams et al. (2011).

The issue we address now is how to write Ψ explicitly in terms of the invariants (3), (4) and (5)? We advocate three criteria. The free energy density Ψ should satisfy the initial stress compatibility (7). Second, it should satisfy strong-ellipticity (8) for all deformations in which the material is expected to be stable (Merodio and Ogden, 2003), and the third criterion we call the *initial stress symmetry* (ISS).

2.1 Initial Stress Symmetry

For convenience let the response function $\hat{\varsigma}$ be denoted by

$$\hat{\varsigma}(\boldsymbol{F}_1, \boldsymbol{\sigma}_2, p_1) := \boldsymbol{F}_1 \frac{\partial \Psi}{\partial \boldsymbol{F}}(\boldsymbol{F}_1, \boldsymbol{\sigma}_2) - \boldsymbol{I} p_1, \tag{9}$$

for every $\boldsymbol{F}_1$ and $\boldsymbol{\sigma}_2$, where the argument on the right hand side is evaluated by taking the partial derivative of $\Psi(\boldsymbol{F}, \cdot)$ with respect to $\boldsymbol{F}$. The scalar p_1 is undetermined if the material is incompressible and $p_1 = -2I_3 \Psi_{I_3}$ with $\boldsymbol{F}$ replaced with $\boldsymbol{F}_1$ if the material is compressible.

The ISS states that $\hat{\varsigma}$ has no preferred reference configuration. Refering to Figure 1, if we take $\overset{\circ}{\mathcal{B}}$ as the reference configuration, then the Cauchy stress becomes

$$\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau}, p). \tag{10}$$

However, we can also take $\mathcal{B}$ as the reference configuration and $\overset{\circ}{\mathcal{B}}$ as the current configuration and therefore express the initial stress as

$$\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}, p_{\tau}), \tag{11}$$

for some scalar p_{τ}. For a compressible material $p_{\tau} = -2I_3 \Psi_{I_3}$ with $\boldsymbol{F}$ replace with $\boldsymbol{F}^{-1}$. In a more precise form, ISS can be stated as

$$\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau}, p) \quad \text{and} \quad \boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}, p_{\tau}), \tag{12}$$

for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$, such that $\boldsymbol{\tau} = \boldsymbol{\tau}^T$ and $\det \boldsymbol{F} = 1$ for incompressible materials, and p and p_{τ} are respectively given by $p = \hat{p}(\boldsymbol{\sigma}, \boldsymbol{F}, \boldsymbol{\tau})$ and $p_{\tau} = \hat{p}(\boldsymbol{\tau}, \boldsymbol{F}^{-1}, \boldsymbol{\sigma})$ for some scalar function $\hat{p}$. The boundary conditions can determine p through its dependence on $\boldsymbol{\sigma}$, and analogously for p_{τ}. The ISS agrees with the initial stress compatibility, i.e. the condition (2), when we have $\hat{\varsigma}(\boldsymbol{I}, \boldsymbol{\tau}, p) = \boldsymbol{\tau}$ for every $\boldsymbol{\tau}$.

Another way to view the ISS symmetry is to assume that $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are due to the elastic deformation of a virtual stress-free state, which we demonstrate in Appendix A. Hoger emphasized many times that using the virtual stress-free state does not restrict how the residual stress was formed. The same can be said about ISS; only $\boldsymbol{F}$ is an elastic deformation. Moreover, the ISS is not restricted to elasticity and should hold for other constitutive equations such as those encountered in viscoelasticity and plasticity.

One practical outcome from the ISS is to restrict the possible constitutive choices for $\hat{\varsigma}$. In Section 4.1 we show that $\Psi = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}(J_1 - I_{\tau_1})$ proposed by Merodio et al. (2013) does not satisfy ISS, and we also use ISS to deduce an expression for $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ for compressible materials.

A second practical feature arising from ISS is that we can write the residual stress as a function of the Cauchy stress (11). This proves useful when $\boldsymbol{\sigma}$ is known a priori, as it will be discussed in Section 5.2 where we determine $\boldsymbol{\sigma}$ from a homeostasis principal and then use Eq. (11) to derive $\boldsymbol{\tau}$. The alternative of choosing the residual stress $\boldsymbol{\tau}$ first can be far more complicated.

Before further developing the implications of ISS in Section 4, we will introduce below an example of stored energy $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ that satisfies ISS, stress compatibility (2) and strong-ellipticity (8).

3 Initially stressed neo-Hookean material

Here we derive a simple constitutive equation for an initially stressed body by assuming that both $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ arise from deforming an incompressible neo-Hookean material. The result will be an explicit representation of Ψ in terms of $\boldsymbol{F}$ and $\boldsymbol{\tau}$, given by Eq. (23) below, that automatically satisfies ISS, the stress compatibility (7) and strong-ellipticity (8). Both ISS and stress compatibility hold because $\boldsymbol{\tau}$ arises due to an elastic deformation from a stress-free state, see Appendix A, and strong-ellipticity is satisfied because the material is a neo-Hookean solid (Ogden, 1997).

Refering to Figure 1, as the material is stress-free in $\tilde{\mathcal{B}}$, we have

$$\Psi = \frac{\mu}{2}(\operatorname{tr}\tilde{\boldsymbol{C}} - 3), \tag{13}$$

where $\mu > 0$ is the constant shear modulus and $\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}}$. The Cauchy stress (1) then becomes

$$\boldsymbol{\sigma} = \mu\tilde{\boldsymbol{F}}\tilde{\boldsymbol{F}}^T - p\boldsymbol{I}. \tag{14}$$

We can rewrite $\operatorname{tr}\tilde{\boldsymbol{C}}$ by substituting $\tilde{\boldsymbol{F}} = \boldsymbol{F}\mathring{\boldsymbol{F}}$ and using the properties of the trace

$$\operatorname{tr}\tilde{\boldsymbol{C}} = \operatorname{tr}(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}}) = \operatorname{tr}(\mathring{\boldsymbol{F}}^T\boldsymbol{F}^T\boldsymbol{F}\mathring{\boldsymbol{F}}) = \operatorname{tr}(\mathring{\boldsymbol{F}}\mathring{\boldsymbol{F}}^T\boldsymbol{F}^T\boldsymbol{F}) = \operatorname{tr}(\mathring{\boldsymbol{B}}\boldsymbol{C}),$$

where $\mathring{\boldsymbol{B}} = \mathring{\boldsymbol{F}}\mathring{\boldsymbol{F}}^T$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, which leads to

$$\Psi = \frac{\mu}{2}\left[\operatorname{tr}(\mathring{\boldsymbol{B}}\boldsymbol{C}) - 3\right]. \tag{15}$$

We can write $\mathring{\boldsymbol{B}}$ in terms of $\boldsymbol{\tau}$ by evaluating Eq. (14) at $\boldsymbol{F} = \boldsymbol{I}$, $\boldsymbol{\sigma} = \boldsymbol{\tau}$, $p = \mathring{p}$ and rearranging the term as follows

$$\mu\mathring{\boldsymbol{B}} = \boldsymbol{\tau} + \mathring{p}\boldsymbol{I}. \tag{16}$$

To write $\mathring{p}$ as a function of $\boldsymbol{\tau}$ we require that $\mathring{\boldsymbol{F}}$ be isochoric, so that $\det(\mu\mathring{\boldsymbol{B}}) = \det(\boldsymbol{\tau} + \mathring{p}\boldsymbol{I}) = \mu^3$, which results in a cubic equation for $\mathring{p}$

$$\mathring{p}^3 + \mathring{p}^2 I_{\tau_1} + \mathring{p} I_{\tau_2} + I_{\tau_3} - \mu^3 = 0. \tag{17}$$

The real roots for $\overset{\circ}{p}$ are given by,

$$\overset{\circ}{p} = \begin{cases} \frac{1}{3}\left[T_3 + \frac{T_1}{T_3} - I_{\tau_1}\right], & T_2 \le T_1^{3/2}, \\ \frac{1}{3}\left[c_1 T_3 + c_1^* \frac{T_1}{T_3} - I_{\tau_1}\right], & -T_1^{3/2} \le T_2, \\ \frac{1}{3}\left[c_1^* T_3 + c_1 \frac{T_1}{T_3} - I_{\tau_1}\right], & -T_1^{3/2} \le T_2 \le T_1^{3/2}, \end{cases} \tag{18}$$

while if $T_1 = 0$ then $\overset{\circ}{p} = \mu - I_{\tau_1}/3$, where

$$T_1 = I_{\tau_1}^2 - 3I_{\tau_2}, \quad T_2 = I_{\tau_1}^3 - \frac{9}{2} I_{\tau_1} I_{\tau_2} + \frac{27}{2}(I_{\tau_3} - \mu^3), \tag{19}$$

$$T_3 = \sqrt[3]{\sqrt{T_2^2 - T_1^3} - T_2}, \quad c_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}. \tag{20}$$

In terms of the eigenvalues τ_1, τ_2 and τ_3 of $\boldsymbol{\tau}$, we can write

$$T_2 = -\frac{27}{2}\mu^3 + \frac{1}{2}\left[(\tau_1 - \tau_2) - (\tau_3 - \tau_1)\right]\left[(\tau_2 - \tau_3) - (\tau_1 - \tau_2)\right]\left[(\tau_3 - \tau_1) - (\tau_2 - \tau_3)\right],$$
$$T_1 = \frac{1}{2}(\tau_1 - \tau_2)^2 + \frac{1}{2}(\tau_1 - \tau_3)^2 + \frac{1}{2}(\tau_2 - \tau_3)^2, \tag{21}$$

so that when $T_1 = 0$ we have that $\tau_1 = \tau_2 = \tau_3$. From the above we see that $T_1 \ge 0$ and $T_1^{3/2} \ge |T_2 + \mu^3 27/2|$ for any* $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$. So if $T_2 < 0$ then clearly $T_2 < T_1^{3/2}$, else if $T_2 \ge 0$ then $T_2 < |T_2 + \mu^3 27/2| \le T_1^{3/2}$. Therefore $T_2 < T_1^{3/2}$, and so the condition for the first case for $\overset{\circ}{p}$ in Eq. (18) is always satisfied. We discard the second and third case for $\overset{\circ}{p}$ because we expect Ψ, and therefore $\overset{\circ}{p}$, to be continuous for every $\boldsymbol{\tau} \in \mathbb{R}^3$. When $\tau_1 = \tau_2 = \tau_3$ the only viable solution for $\overset{\circ}{p}$ is $(18)_1$. If $\boldsymbol{\tau}$ moves into a region where $(18)_2$ or $(18)_3$ becomes real, then $\overset{\circ}{p}$ can not change from $(18)_1$ to $(18)_2$ or $(18)_3$ because it can be shown that $(18)_1$ does not equal $(18)_2$ or $(18)_3$ for any $\boldsymbol{\tau} \in \mathbb{R}^3$.

To represent $\mathrm{tr}(\overset{\circ}{\boldsymbol{B}}\boldsymbol{C})$ in terms of the invariants of $\boldsymbol{\tau}$ and $\boldsymbol{C}$, we multiply each side of Eq. (16) on the right with $\boldsymbol{C}$ and take the trace to get

$$\mu\,\mathrm{tr}(\overset{\circ}{\boldsymbol{B}}\boldsymbol{C}) = \mathrm{tr}(\boldsymbol{\tau}\boldsymbol{C}) + \overset{\circ}{p}\,\mathrm{tr}\,\boldsymbol{C}, \tag{22}$$

which we use to write the free-energy density Eq.(15) as

$$\Psi = \frac{1}{2}\left(\overset{\circ}{p} I_1 + J_1 - 3\mu\right) \tag{23}$$

with $\overset{\circ}{p}$ given by Eq. $(18)_1$. Note that in the absence of residual stress $\boldsymbol{\tau} = 0$, $\overset{\circ}{p} = \mu$ by (17), $J_1 = I_1$, and then Ψ reduces to the classical neo-Hookean model, as expected. Equation (23) represents the general extension of the neo-Hookean strain energy function to a residually stressed material, resulting in a function of only five of the nine independent invariants of $\boldsymbol{C}$ and $\boldsymbol{\tau}$.

Substituting (23) in the Cauchy stress Eq. (1) we arrive at the constitutive relation

$$\boldsymbol{\sigma} = \overset{\circ}{p}\boldsymbol{B} + \boldsymbol{F}\boldsymbol{\tau}\boldsymbol{F}^T - p\boldsymbol{I}. \tag{24}$$

*http://math.stackexchange.com/questions/1255883/prove-that-a2b2c2-geq-2a-b2b-c2a-c21-3/1255907#1255907

Taking $\mathcal{B}$ as the reference configuration, $\overset{\circ}{\mathcal{B}}$ as the current configuration and leaving $\tilde{\mathcal{B}}$ as the stress-free configuration, see Figure 1, the initial stress becomes $\boldsymbol{\sigma}$, the Cauchy stress becomes $\boldsymbol{\tau}$, and Eq. (24) becomes

$$\boldsymbol{\tau} = \overset{\circ}{p}_\tau \boldsymbol{C}^{-1} + \boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} - p_\tau \boldsymbol{I}, \tag{25}$$

where $\overset{\circ}{p}_\tau$ is given by replacing $\boldsymbol{\tau}$ for $\boldsymbol{\sigma}$ in Eq. $(18)_1$, and p_τ is an undetermined scalar.

Substituting (24) in (25) we find the connection

$$(\overset{\circ}{p}_\tau - p)\boldsymbol{C}^{-1} = (p_\tau - \overset{\circ}{p})\boldsymbol{I}. \tag{26}$$

As this equation must hold for every $\boldsymbol{C}$ we conclude that

$$\overset{\circ}{p}_\tau = p \quad \text{and} \quad p_\tau = \overset{\circ}{p}. \tag{27}$$

Note that, as is expected of a neo-Hookean material, the above equations do not determine p in terms of $\boldsymbol{\tau}$ or $\boldsymbol{F}$.

3.1 Plane Strain

The free energy density (23) is simplified when the residual strain has only planar components. Accordingly, let us assume $\overset{\circ}{B}_{13} = \overset{\circ}{B}_{23} = 0$ and $\overset{\circ}{B}_{33} = 1$, which substituted in Eq. (16) results in $\tau_{13} = \tau_{23} = 0$ and $\tau_{33} = \mu - \overset{\circ}{p}$. We let $\boldsymbol{B}_P$, $\boldsymbol{C}_P$, $\overset{\circ}{\boldsymbol{B}}_P$, $\boldsymbol{I}_P$, $\boldsymbol{\sigma}_P$ and $\boldsymbol{\tau}_P$ be $\boldsymbol{B}$, $\boldsymbol{C}$, $\overset{\circ}{\boldsymbol{B}}$, $\boldsymbol{I}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ restricted to the (x_1, x_2) plane respectively. From equation (16) we get

$$\mu\overset{\circ}{\boldsymbol{B}}_P = \boldsymbol{\tau}_P + \overset{\circ}{p}\boldsymbol{I}_P. \tag{28}$$

To obtain $\overset{\circ}{p}$ in terms of $\boldsymbol{\tau}_P$ we take the determinant of each side of Eq. (28) giving

$$\det\left(\mu\overset{\circ}{\boldsymbol{B}}_P\right) = \det(\boldsymbol{\tau}_P + \overset{\circ}{p}\boldsymbol{I}_P) \implies \mu^2 = \overset{\circ}{p}^2 + \overset{\circ}{p}\,\mathrm{tr}\,\boldsymbol{\tau}_P + \det\boldsymbol{\tau}_P \tag{29}$$

where we used $\det(\mu\overset{\circ}{\boldsymbol{B}}) = \mu^2$. We solve the above for $\overset{\circ}{p}$ to get

$$\overset{\circ}{p}_\pm = -\frac{\mathrm{tr}\,\boldsymbol{\tau}}{2} \pm \frac{1}{2}\sqrt{-4\det\boldsymbol{\tau} + (\mathrm{tr}\,\boldsymbol{\tau})^2 + 4\mu^2}, \tag{30}$$

where $-4\det\boldsymbol{\tau} + (\mathrm{tr}\,\boldsymbol{\tau})^2 + 4\mu^2 = (\tau_1 - \tau_2)^2 + 4\mu^2$ in terms of the eigenvalues of $\boldsymbol{\tau}$. The solution $\overset{\circ}{p} = \overset{\circ}{p}_-$ should be discarded due to the following: $\overset{\circ}{p}_- = -\mu$ when $\boldsymbol{\tau}_P = \boldsymbol{0}$, and Eq. (28) shows that $\overset{\circ}{p}$ should be positive when $\boldsymbol{\tau}_P = \boldsymbol{0}$, so for $\boldsymbol{\tau}_P = \boldsymbol{0}$ the only viable solution is $\overset{\circ}{p} = \overset{\circ}{p}_+$. Further, as we expect $\overset{\circ}{p}$ to be continuous in $\boldsymbol{\tau}_P$ for every $\boldsymbol{\tau}_P \in \mathbb{R}^2$, and $\overset{\circ}{p}_- \neq \overset{\circ}{p}_+$ for every $\boldsymbol{\tau}_P$, we should discard the solution $\overset{\circ}{p}_-$.

For simplicity we assume $C_{32} = C_{31} = 0$ and then Eq. (15) reads

$$\Psi = \frac{\mu}{2}\left(\mathrm{tr}(\overset{\circ}{\boldsymbol{B}}_P\boldsymbol{C}_P) - 2\right) + \frac{\mu}{2}(C_{33} - 1), \tag{31}$$

and substituting $\overset{\circ}{\boldsymbol{B}}_P$ from Eq. (28) we get

$$\Psi = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau}_P\boldsymbol{C}_P) + \frac{1}{2}\overset{\circ}{p}\,\mathrm{tr}\,\boldsymbol{C}_P - \mu + \frac{\mu}{2}(C_{33} - 1), \tag{32}$$

where $\overset{\circ}{p} = \overset{\circ}{p}_+$ given by Eq. (30). The Cauchy stress Eq. (1) becomes,

$$\boldsymbol{\sigma}_P = \overset{\circ}{p}\boldsymbol{B}_P - p\boldsymbol{I}_P + \boldsymbol{F}_P\boldsymbol{\tau}_P\boldsymbol{F}_P^T, \tag{33}$$

and $\sigma_{33} = \overset{\circ}{p} - p + \tau_{33} = \mu - p$, where we have used that $\tau_{33} = \mu - \overset{\circ}{p}$. For $\boldsymbol{F} = \boldsymbol{I}$ we have $\boldsymbol{\sigma} = \boldsymbol{\tau}$ and $\overset{\circ}{p} = p$.

4 The Initial Stress Symmetry Equations

In this Section we explained how to express the ISS condition (12) as a set of scalar equations that relates the free energy density Ψ; p and p_τ from Eq. (12); and the invariants of $\boldsymbol{\tau}$ and $\boldsymbol{C}$. Let us first consider a simple example, namely assuming $\Psi = I_1 J_1/2$. One can check that this strain energy satisfies the compatibilty equations (2) with $\overset{\circ}{p} = \operatorname{tr}\boldsymbol{\tau}$, but does it satisfy the ISS (12)? We can use the stress in the form (1) to write

$$\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau}, p) = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{C})\boldsymbol{B} - p\boldsymbol{I} + \operatorname{tr}(\boldsymbol{C})\boldsymbol{F}\boldsymbol{\tau}\boldsymbol{F}^T, \tag{34}$$

and then from ISS we have that

$$\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}, p_\tau) = \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{B}^{-1})\boldsymbol{C}^{-1} - p_\tau\boldsymbol{I} + \operatorname{tr}\left(\boldsymbol{B}^{-1}\right)\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}. \tag{35}$$

To check that both Eqs. (34) and (35) hold for every $\boldsymbol{\tau}$ and $\boldsymbol{F}$ we substitute $\boldsymbol{\sigma}$ into Eq. (35), which after some rearranging becomes

$$(1 - I_1 I_2)\boldsymbol{\tau} = (3J_1 - 2pI_2 + I_1 I_{\tau 1})\boldsymbol{C}^{-1} + (I_2 J_1 - p_\tau)\boldsymbol{I}, \tag{36}$$

where we have used that $\operatorname{tr}(\boldsymbol{B}^{-1}) = I_2$, which can be shown by applying the Cayley-Hamilton theorem to $\boldsymbol{B}$ with $I_3 = 1$ (due to incompressibility). In order to satisfy Eq. (36) for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$, the coefficients of $\boldsymbol{\tau}$, $\boldsymbol{C}^{-1}$ and $\boldsymbol{I}$ must all be identically zero (this is shown rigorously in Appendix B). For the coefficient of $\boldsymbol{\tau}$ to be zero, the identity $I_1 I_2 = 1$ must hold for every $\boldsymbol{F}$, which is obviously impossible: so we conclude that $\Psi = I_1 J_1/2$ does not correctly furnish the Cauchy stress for every reference configuration.

In Appendix B we reduce ISS to nine scalar equations, following a procedure similar to the one above. They are compactly written as

$$\boldsymbol{b} + \boldsymbol{P}^{\{1\}}\Psi^\sigma_{J_1} + \boldsymbol{P}^{\{2\}}\Psi^\sigma_{J_2} + \Psi_{J_n}\left(\boldsymbol{P}^{\{3\}}_{\{n\}}\Psi^\sigma_{J_3} + \boldsymbol{P}^{\{4\}}_{\{n\}}\Psi^\sigma_{J_4}\right) = 0, \tag{37}$$

$$\Psi_{J_n}\left(\boldsymbol{Q}^{\{1\}}_{\{n\}}\Psi^\sigma_{J_1} + \boldsymbol{Q}^{\{2\}}_{\{n\}}\Psi^\sigma_{J_2} + \boldsymbol{Q}^{\{3\}}_{\{n\}}\Psi^\sigma_{J_3} + \boldsymbol{Q}^{\{4\}}_{\{n\}}\Psi^\sigma_{J_4}\right) = 0, \tag{38}$$

where n sums over $0, 1, 2, 3, 4$, $\Psi_{J_0} := 1$,

$$\Psi_{I_k} = \Psi_{I_k}(\boldsymbol{F}, \boldsymbol{\tau}), \quad \Psi_{J_m} = \Psi_{J_m}(\boldsymbol{F}, \boldsymbol{\tau}), \tag{39}$$

$$\Psi^\sigma_{I_k} := \Psi_{I_k}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}),\ \Psi^\sigma_{J_m} := \Psi_{J_m}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}), \tag{40}$$

for $k \in \{1, 2\}$ and $m \in \{1, 2, 3, 4\}$ with $\boldsymbol{\sigma}$ given by (6),

$$\boldsymbol{b} = \begin{bmatrix} 2\Psi^\sigma_{I_2} \\ -2\Psi^\sigma_{I_1} \\ p_\tau \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \boldsymbol{P}^{\{1\}} = 4\begin{bmatrix} \Psi_{I_2} \\ p/2 \\ -\Psi_{I_1} \\ -2\Psi_{J_2} \\ 2I_2\Psi_{J_2} \\ -\Psi_{J_1} - 2I_1\Psi_{J_2} \end{bmatrix},\ \boldsymbol{P}^{\{2\}} = 4\begin{bmatrix} p \\ I_2 p - 2\Psi_{I_1} - 2I_1\Psi_{I_2} \\ 2\Psi_{I_2} \\ 0 \\ -2\Psi_{J_1} \\ -4\Psi_{J_2} \end{bmatrix}, \tag{41}$$

$$\boldsymbol{Q}^{\{1\}}_{\{1\}} = \boldsymbol{Q}^{\{1\}}_{\{2\}} = \boldsymbol{Q}^{\{2\}}_{\{1\}} = \boldsymbol{Q}^{\{2\}}_{\{2\}} = \boldsymbol{0}, \tag{42}$$

while all over matrices are given in Appendix C. Eqs (37) and (38) represent 6 and 3 scalar equations, respectively, with the unknowns being the functions Ψ, p and p_τ.

We now look at special cases of Ψ which simplify the ISS equations and lead to more practical representations for free energy density.

4.1 Free energy independent of J_3 and J_4

When Ψ is independent of J_3 and J_4, Eq. (38) is identically zero and only the first three terms of Eq. (37) are non-zero. The fourth scalar equation of (37) becomes $-8\Psi_{J_2}\Psi^{\sigma}_{J_1} = 0$ which is true, for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$, only when Ψ does not depend on J_2 or does not depend on J_1. We will investigate the first case in more detail below.

Assuming Ψ independent of J_2

If Ψ does not depend on J_2, J_3 and J_4 then Eq. (37) reduces to

$$4\Psi_{J_1}\Psi^{\boldsymbol{\sigma}}_{J_1} = 1, \quad \frac{\Psi^{\boldsymbol{\sigma}}_{I_2}}{\sqrt{\Psi^{\boldsymbol{\sigma}}_{J_1}}} + \frac{\Psi_{I_2}}{\sqrt{\Psi_{J_1}}} = 0, \quad p\Psi^{\boldsymbol{\sigma}}_{J_1} = \Psi^{\boldsymbol{\sigma}}_{I_1}, \quad p_{\boldsymbol{\tau}}\Psi_{J_1} = \Psi_{I_1}, \tag{43}$$

where we have used Eq. $(43)_1$ to derive the other three equations. We can check if these equations are consistent with the compatibility Eqs. (7) by letting $\boldsymbol{F} = \boldsymbol{I}$, $\boldsymbol{\sigma} = \boldsymbol{\tau}$ and $p_\tau = p = \mathring{p}$. This results in $\Psi_{J_1} = \pm 1/2$, $2\Psi_{I_1} = \pm\mathring{p}$ and $\Psi_{I_2}(1 \pm 1) = 0$, which only satisfies Eqs. (7) if $\Psi_{J_1} = 1/2$, $2\Psi_{I_1} = \mathring{p}$ and $\Psi_{I_2} = 0$ for $\boldsymbol{F} = \boldsymbol{I}$.

Let us apply Eqs. (43) to the following example

$$\Psi = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}(J_1 - I_{\tau_1}), \tag{44}$$

used by Merodio et al.(2013). In this case, Eqs. $(43)_1$ and $(43)_2$ are satisfied while Eqs. $(43)_3$ and $(43)_4$ become $p = p_\tau = \mu$. However, restricting p to be constant, which is normally determined through boundary conditions, would result in physically unexpected behaviour. This is best seen with an example.

Example 1 *For the free energy density expressed in Eq.* (44) *the Cauchy stress from Eq.*(6) *becomes*

$$\boldsymbol{\sigma} = \mu\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}p + \boldsymbol{F}^T\boldsymbol{\tau}\boldsymbol{F}. \tag{45}$$

Let us consider a cube in the reference configuration, subject to the residual stress $\tau_{ij} = \tau_i\delta_{ij}$, aligned with the axes of the cube. For the current configuration we impose two clamped conditions that fix $F_{11} = \lambda_1$ and $F_{33} = \lambda_3$, then $F_{22} = \lambda_2 = (\lambda_1\lambda_3)^{-1}$ due to incompressibility, with all the other components of $\boldsymbol{F}$ being zero. With this imposed deformation the Cauchy stress is diagonal with components $\sigma_{ij} = \delta_{ij}\sigma_i$. We can also prescribe the stress σ_2, as this will not alter λ_1 and λ_3 because they are kept fixed. We choose $\sigma_2 = \tau_2$, to ensure compatibility with the reference configuration when $\lambda_1 = \lambda_3 = 1$, which results in

$$\tau_2 = \mu\lambda_2^2 - p + \lambda_2^2\tau_2.$$

Clearly it is not possible for p to satisfy the above and be fixed at $p = \mu$ so that ISS is satisfied. This means that the free energy density Eq. (44) *either results in non-physically behaviour, or it does not satisfy ISS; which implies that the constitutive relation* (45) *does not hold for every reference configuration.*

The choice of Ψ should satisfy Eqs. (43) without restricting the deformation $\boldsymbol{F}$, initial stress $\boldsymbol{\tau}$ or p, otherwise the material will likely exhibit non-physical behaviour. Below we deduce a a free energy density Ψ for a compressible model, inspired by the neo-Hookean example in Section 3, that satisfies ISS without any unphysical restrictions.

For a compressible material we substitute p with $-2I_3\Psi_{I_3}$ and p_τ with $-2I_3^{-1}\Psi^{\boldsymbol{\sigma}}_{I_3}$ in the ISS Eqs. (43). We will find under what conditions does the following free energy density

$$\Psi = \frac{1}{2}g(\boldsymbol{\tau}, I_3)I_1 I_3^{-1/3} + \frac{1}{2}f(\boldsymbol{\tau}, I_3) + \frac{J_1}{2}I_3^{-1/3} \tag{46}$$

satisfy ISS (43) and stress compatibility (7), where f and g are arbitrary scalar functions.

The Cauchy stress (6) becomes

$$\boldsymbol{\sigma} = g(\boldsymbol{\tau}, I_3)I_3^{-1/3}\boldsymbol{B} + 2I_3\Psi_{I_3}\boldsymbol{I} + I_3^{-1/3}\boldsymbol{F}^T\boldsymbol{\tau}\boldsymbol{F}, \tag{47}$$

with

$$\Psi_{I_3} = \frac{1}{2}f_{I_3}(\boldsymbol{\tau}, I_3) + \frac{1}{2}g_{I_3}(\boldsymbol{\tau}, I_3)I_1 I_3^{-1/3} - \frac{1}{6}I_3^{-4/3}\left[g(\boldsymbol{\tau}, I_3)I_1 + J_1\right]. \tag{48}$$

The ISS Eq. $(43)_1$ is satisfied as $\Psi_{J_1}\Psi^{\boldsymbol{\sigma}}_{J_1} = (1/2)I_3^{-1/3}(1/2)I_3^{1/3} = 1/4$ and stress compatibility $(7)_2$ is satisfied as $2\Psi_{J_1} = 1$ when $\boldsymbol{F} = \boldsymbol{I}$. By substituting p for $-2I_3\Psi_{I_3}$, the remaining compatibility Eq. $(7)_1$ becomes

$$g(\boldsymbol{\tau}, I_3) = -2\Psi_{I_3} \quad \text{evaluated at} \quad \boldsymbol{F} = \boldsymbol{I}. \tag{49}$$

Eq. (49) is satisfied if we set $f_{I_3}(\boldsymbol{\tau}, 1) = \operatorname{tr}\boldsymbol{\tau}/3 - 3g_{I_3}(\boldsymbol{\tau}, 1)$, then Eq. (49) and Eq. (47) together imply that $\boldsymbol{\sigma} = \boldsymbol{\tau}$ when $\boldsymbol{F} = \boldsymbol{I}$.

The ISS Eqs. $(43)_3$ and $(43)_4$ now read respectively

$$-2I_3\Psi_{I_3} = g(\boldsymbol{\sigma}, I_3^{-1}) \quad \text{and} \quad -2I_3^{-1}\Psi^{\boldsymbol{\sigma}}_{I_3} = g(\boldsymbol{\tau}, I_3), \tag{50}$$

for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$, where f_{I_3} and g_{I_3} are respectively the partial derivatives of f and g with respect to I_3. Note that the above two equations are equivalent as Eq. $(50)_1$ becomes Eq. $(50)_2$ when we substitute $\boldsymbol{F}$ for $\boldsymbol{F}^{-1}$ and $\boldsymbol{\tau}$ for $\boldsymbol{\sigma}$. For this reason it is sufficient to satisfy Eq. $(50)_1$. If $\boldsymbol{F} = \boldsymbol{I}$, Eq. $(50)_1$ is satisfied as it reduces to Eq. (49). The rest of this section will focus on showing that Eq. $(50)_1$ is satisfied for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$.

By rearranging Eq. (47) and taking the determinant on either side we get

$$\det\left(\boldsymbol{\sigma} - 2I_3\Psi_{I_3}\boldsymbol{I}\right) = \det\left(I_3^{-1/3}\boldsymbol{F}^T\boldsymbol{\tau}\boldsymbol{F} + g(\boldsymbol{\tau}, I_3)I_3^{-1/3}\boldsymbol{B}\right) = I_3^{-1}\det\boldsymbol{B}\det(\boldsymbol{\tau} + g(\boldsymbol{\tau}, I_3)\boldsymbol{I}),$$

and by using incompressibility $\det\boldsymbol{B} = 1$ we find that

$$I_3^{1/2}\det(\boldsymbol{\sigma} - 2I_3\Psi_{I_3}\boldsymbol{I}) = I_3^{-1/2}\det(\boldsymbol{\tau} + g(\boldsymbol{\tau}, I_3)\boldsymbol{I}). \tag{51}$$

If we choose $g(\boldsymbol{\tau}, I_3)$ such that

$$I_3^{-1/2}\det(\boldsymbol{\tau} + g(\boldsymbol{\tau}, I_3)\boldsymbol{I}) = k, \tag{52}$$

where k is a constant, then the right handside of Eq. (51) is also equal to k, thus $\boldsymbol{\sigma}$ must satisfy

$$I_3^{1/2}\det(\boldsymbol{\sigma} - 2I_3\Psi_{I_3}\boldsymbol{I}) = k. \tag{53}$$

By swapping $\boldsymbol{\tau}$ for $\boldsymbol{\sigma}$ and $\boldsymbol{F}$ for $\boldsymbol{F}^{-1}$ Eq. (52) becomes

$$I_3^{1/2}\det(\boldsymbol{\sigma} + g(\boldsymbol{\sigma}, I_3^{-1})\boldsymbol{I}) = k. \tag{54}$$

For $g(\boldsymbol{\tau}, I_3)$ to satisfy Eq. (52) there are three possible solutions given by replacing μ^3 with $kI_3^{1/2}$ and $g(\boldsymbol{\tau}, I_3)$ with $\overset{\circ}{p}$ in $(18)_1$,$(18)_2$ and $(18)_3$, which we respectively denote by $g_1(\boldsymbol{\tau}, I_3)$, $g_2(\boldsymbol{\tau}, I_3)$ and $g_3(\boldsymbol{\tau}, I_3)$. We highlighted in Section 3 that only $g(\boldsymbol{\tau}, I_3) = g_1(\boldsymbol{\tau}, I_3)$ is a physically viable solution. So if we choose $g(\boldsymbol{\tau}, I_3) = g_1(\boldsymbol{\tau}, I_3)$, then for Eq. $(50)_1$ to be satisfied we need

$$g_1(\boldsymbol{\sigma}, I_3^{-1}) = -2I_3\Psi_{I_3} \quad \text{for every } \boldsymbol{F} \text{ and } \boldsymbol{\tau}. \tag{55}$$

We will show that the above is a consequence of Eqs. (53), (54) and stress compatibility (49). Due to Eq. (49), we know that initially for $\boldsymbol{F} = \boldsymbol{I}$ Eq. (55) is true. We can also conclude from Eqs. (53) and (54) that $-2I_3\Psi_{I_3}$ will equal either $g_1(\boldsymbol{\sigma}, I_3^{-1})$, $g_2(\boldsymbol{\sigma}, I_3^{-1})$ or $g_3(\boldsymbol{\sigma}, I_3^{-1})$ for every $\boldsymbol{F}$. Since $-2I_3\Psi_{I_3}$ should be a continuous function of $\boldsymbol{F}$ and $\boldsymbol{\tau}$, it can not change from $g_1(\boldsymbol{\sigma}, I_3^{-1})$ to another solution $g_k(\boldsymbol{\sigma}, I_3^{-1})$ because it can be shown that $g_k(\boldsymbol{\sigma}, I_3^{-1}) \neq g_1(\boldsymbol{\sigma}, I_3^{-1})$ for every $\boldsymbol{\sigma}$ and I_3. For this reason if $g_1(\boldsymbol{\tau}, 1) = -2\Psi_{I_3}$ for $\boldsymbol{F} = \boldsymbol{I}$, then $g_1(\boldsymbol{\sigma}, I_3^{-1}) = -2I_3\Psi_{I_3}$ for every $\boldsymbol{F}$.

In conclusion, the model (46) satisfies ISS as long as $g(\boldsymbol{\tau}, I_3) = g_1(\boldsymbol{\tau}, I_3)$, where $g_1(\boldsymbol{\tau}, I_3)$ is given by replacing μ^3 with $kI_3^{1/2}$ in $(18)_1$.

5 Homogeneous stress gradient in a hollow cylinder

It is now well acknowledged that residual stresses in living materials are vital to maintain ideal mechanical conditions. When an external load changes, growth and remodelling will alter the residual stress to best adapt to the new load. An excellent example is how arteries remodel in response to the internal pressure. The residual stress in arteries is thought to protect the arterial wall against strain concentration (Destrade et al., 2012) or stress concentration (Fung, 1991; Taber and Eggers, 1996). Here we will adopt the most accepted hypothesis, that the residual stress in the artery acts to minimize the stress gradient (Fung, 1991; Cardamone et al., 2009). The reasoning behind this hypothesis is that if a given tissue grows in response to stress, then homoeostasis is only possible if the tissue is under similar stress conditions throughout. So by minimizing the stress gradient we are selecting the most homogeneous stress possible. We will also consider a simplified artery with only one layer and no shear stress applied to the interior of the artery.

Our workflow is to first choose an optimal Cauchy stress field $\boldsymbol{\sigma}$, and then the to derive the residual stress from the ISS equation $\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}, p_{\tau})$. So first we calculate the Cauchy stress with a near homogeneous circumferential and radial components in Section 5.2, then we find the residual stress that supports this optimal Cauchy stress in Section 5.3. In Section 5.2 we also show that the optimal Cauchy stress has a simple asymptotic formula.

We finally compare the results with the corresponding ones obtained using the opening angle method (Chuong and Fung, 1986). Pasquale: cute the rest.

5.1 Plain Strain Cylinder

To describe the arterial wall we use cylindrical coordinates (r, θ, z) with the components of $\boldsymbol{\sigma}$ written in terms of unit basis vectors. We assume that the arterial wall retains its cylindrical symmetry when the internal pressure is removed and that there is no shear stress at the inner wall. These simplifications imply that when a pressure is applied in the cylinder the resulting deformation is an inflation.

Due to radial symmetry, the Cauchy stress $\boldsymbol{\sigma}$ is independent of θ and z, so in the absence of body forces the equilibrium equations reduce to

$$\frac{\partial}{\partial r}\left(r^2\sigma_{\theta r}\right) = 0, \quad \frac{\partial}{\partial r}\left(r\sigma_{zr}\right) = 0, \tag{56}$$

$$\frac{\partial}{\partial r}\left(r\sigma_{rr}\right) - \sigma_{\theta\theta} = 0, \tag{57}$$

with the boundary conditions

$$\sigma_{\theta r} = 0, \ \sigma_{zr} = 0 \ \text{ and } \ \sigma_{rr} = -P \ \text{ for } r = a, \tag{58}$$

$$\sigma_{\theta r} = 0, \ \sigma_{zr} = 0 \ \text{ and } \ \sigma_{rr} = 0 \ \text{ for } r = b, \tag{59}$$

where a and b are the inner and outer radius of the loaded artery, respectively. The equilibrium Eqs. (56) together with the boundary conditions lead to $\sigma_{\theta r} = 0$ and $\sigma_{zr} = 0$ for all r. As neither $\sigma_{z\theta}$ nor σ_{zz} appear in the equations of equilibrium, they are only restricted by the constitutive choice and boundary conditions on the cross-section of the artery. We use this degree of freedom to assume there is no axial torsion $\sigma_{z\theta} = 0$. Note that for the constitutive choice Eq. (6) τ_{ZZ} can be chosen so that σ_{zz} is constant, while for the simpler choice of plain strain Eq.(33) (with $\sigma_{zz} = \sigma_{33}$) σ_{zz} is determined by p. Finally, the remaining equation of equilibrium we need to enforce is Eq. (57).

5.2 Minimal stress gradient

Here we develop a method to minimize the stress gradient fields σ'_{rr} and $\sigma'_{\theta\theta}$, where the prime denotes differentiation in r. We make no assumptions about any reference configuration nor do we make any constitutive choice. We only make use of the assumptions from the section above which result in both σ_{rr} and $\sigma_{\theta\theta}$ being independent of the coordinates θ and z; $\sigma_{\theta r} = \sigma_{zr} = \sigma_{z\theta} = 0$; and for simplicity we will not consider σ_{zz}.

Once $\sigma_{\theta\theta}$ and σ_{rr} are determined, we use ISS Eq.(11) to write the residual stresses $\tau_{\Theta\Theta}$ and τ_{RR} in terms of $\sigma_{\theta\theta}$, σ_{rr} and the deformation gradient in Section 5.3.

Using the equilibrium Eq. (57) we write $\sigma_{\theta\theta}$ in terms of σ_{rr} as

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma'_{rr} \implies \sigma'_{\theta\theta} = 2\sigma'_{rr} + r\sigma''_{rr}. \tag{60}$$

As there are aortas and veins of many different sizes, for the sake of generality let us introduce the following dimensionless variables:

$$\varrho = \frac{r-a}{b-a} \quad \text{and} \quad \varsigma(\rho) = \frac{\sigma_{rr}(r)}{P}, \tag{61}$$

which result in

$$\sigma'_{rr} = \varsigma'\frac{P}{b-a}, \quad \sigma''_{rr} = \varsigma''\frac{P}{(b-a)^2}, \tag{62}$$

and

$$\sigma'_{\theta\theta} = \frac{P}{b-a}\left(2\varsigma' + (\rho+\alpha)\varsigma''\right). \tag{63}$$

Our aim is to minimize the stress gradient density

$$\frac{1}{b-a}\int_a^b \left[\left(\sigma'_{rr}\right)^2 + (\sigma'_{\theta\theta})^2\right] dr = \frac{P^2}{a^2}\alpha^2\int_0^1 \left[\left(\varsigma'\right)^2 + \left(2\varsigma' + (\varrho+\alpha)\,\varsigma''\right)^2\right] d\varrho \tag{64}$$

with the constraint

$$\varsigma(1) = 0 \ \text{ and } \ \varsigma(0) = -1 \implies \int_0^1 \varsigma' d\varrho = 1, \tag{65}$$

where we have introduced the quantity $\alpha = a(b-a)^{-1}$.

Using the calculus of variations (Gregory and Lin, 1992) we find that ς' must satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial \varsigma'} - \frac{d}{d\varrho}\frac{\partial f}{\partial \varsigma''} = \Lambda \;\text{ for } \varrho \in (0,1) \quad \text{and} \quad \frac{\partial f}{\partial \varsigma''} = 0 \;\text{ for } \varrho = 0, 1, \tag{66}$$

where

$$f = \left(\varsigma'\right)^2 + \left(2\varsigma' + (\varrho + \alpha)\,\varsigma''\right)^2, \tag{67}$$

and Λ is a Lagrange multiplier due to the constraint (65). The solution to (66) is given by

$$\varsigma' = \frac{\Lambda}{6} - \frac{\Lambda}{6}(\sqrt{13}-3)\frac{\alpha^{\frac{\sqrt{13}}{2}+\frac{1}{2}} - (1+\alpha)^{\frac{\sqrt{13}}{2}+\frac{1}{2}}}{\alpha^{\sqrt{13}} - (1+\alpha)^{\sqrt{13}}}(\alpha+\varrho)^{\frac{\sqrt{13}}{2}+\frac{1}{2}} \\ + \frac{2\Lambda}{3(\sqrt{13}-3)}\frac{\alpha^{\frac{\sqrt{13}}{2}-\frac{1}{2}} - (1+\alpha)^{\frac{\sqrt{13}}{2}-1\frac{1}{2}}}{\alpha^{\sqrt{13}} - (1+\alpha)^{\sqrt{13}}}\left(\frac{\alpha(1+\alpha)}{\alpha+\varrho}\right)^{\frac{\sqrt{13}}{2}+\frac{1}{2}}. \tag{68}$$

We determine Λ from the constraint (65) to be

$$\frac{\Lambda}{6} = \Bigg(\int_0^1 1 - (\sqrt{13}-3)\frac{\alpha^{\frac{\sqrt{13}}{2}+\frac{1}{2}} - (1+\alpha)^{\frac{\sqrt{13}}{2}+\frac{1}{2}}}{\alpha^{\sqrt{13}} - (1+\alpha)^{\sqrt{13}}}(\alpha+\varrho)^{\frac{\sqrt{13}}{2}+\frac{1}{2}} \\ + \frac{4}{\sqrt{13}-3}\frac{\alpha^{\frac{\sqrt{13}}{2}-\frac{1}{2}} - (1+\alpha)^{\frac{\sqrt{13}}{2}-1\frac{1}{2}}}{\alpha^{\sqrt{13}} - (1+\alpha)^{\sqrt{13}}}\left(\frac{\alpha(1+\alpha)}{\alpha+\varrho}\right)^{\frac{\sqrt{13}}{2}+\frac{1}{2}} d\varrho\Bigg)^{-1}. \tag{69}$$

Realistic values for $\alpha = a(b-a)^{-1}$ can be obtained from in vivo measurements of the lumen radius (inner radius) a and total artery thickness $b-a$. For the descending thoracic aorta of healthy men around 51 years old the mean lumen radius is 20 mm (Stefanadis et al., 1995) and the mean thickness is 1.4 mm (Mensel et al., 2014). These estimates give $\alpha^{-1} \approx 0.07$ making it appropriate to expand ς' and Λ as a power series in α^{-1}, which gives

$$\varsigma' = \frac{\Lambda}{2}\left(1 + (1-2\rho)\alpha^{-1} + \frac{1}{3}\left(9\rho^2 - 6\rho - 1\right)\alpha^{-2}\right) + O\big(\alpha^{-3}\big), \tag{70}$$

$$\Lambda = 2 + \frac{2}{3}\alpha^{-2} + O\big(\alpha^{-3}\big), \tag{71}$$

or simply

$$\varsigma' = 1 + (1-2\rho)\alpha^{-1} + \rho(3\rho-2)\alpha^{-2} + O\big(\alpha^{-3}\big). \tag{72}$$

We can substitute ς' in Eqs. (60), (61) and (63) to obtain

$$\sigma_{rr} = -P(1-\rho)(1 - \rho\alpha^{-1} + \rho^2\alpha^{-2}) + O\big(\alpha^{-3}\big), \tag{73}$$

$$\sigma_{\theta\theta} = P\alpha(1 + \alpha^{-3}\rho^2(4\rho-3)) + O\big(\alpha^{-3}\big). \tag{74}$$

We plot the above Cauchy stress for α^{-1} from 0.05 to 0.1, which is within a physiological range, in Figure 2. A graph for σ_{rr}/P would show all the curves bunched along the same straight line, so instead we have depicted the curves for $\sigma_{rr}/P - \rho$. Figure 2 reveals that as the relative thickness of the arterial wall increases (shading from blue towards red), the circumferential stress $\sigma_{\theta\theta}$ decreases while the radial stress σ_{rr} both increases and becomes less homogeneous.

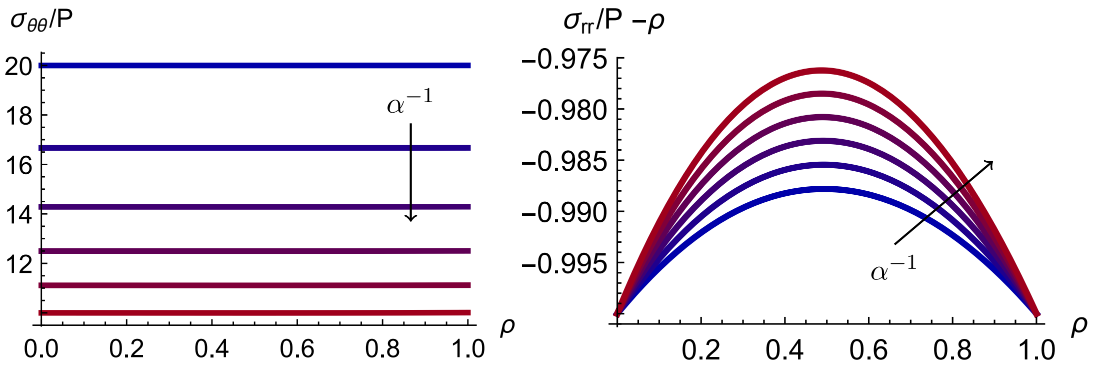

Figure 2: The plots of the dimensionless Cauchy stress against the dimensionaless radius ρ, where the color shades from blue to red as α^{-1} (the wall thickness divided by the inner radius) goes from 0.05 to 0.1.

A well established method to quantify the residual stresses within arteries is the opening angle method (Chuong and Fung, 1986). For a neo-Hookean material (13) we will use the opening angle method to minimize the circumferential stress component

$$\frac{1}{b-a}\int_a^b \left(\frac{d\sigma_{\theta\theta}}{dr}\right)^2 dr, \tag{75}$$

in terms of the opening angle ϕ, restricted to the boundary conditions (58) and (59). The only two parameters with a unit of time, and mass, are P and μ, so by rewriting $P = P_0\mu$ the stress will not depend on μ. The results for $P_0 = 0.05$, $P_0 = 0.2$ and $P_0 = 0.35$ are compared with the optimal stress Eqs. (73) and (74) with parameters for the descending thoracic aorta in Figure 3. We can see that $\sigma_{\theta\theta}$ for the opening angle method converges to the optimal stress $\sigma_{\theta\theta}$ as P/μ tends to zero, while the plots for σ_{rr} all overlap. Note however that σ_{zz} for the opening angle method is not necessarily homogeneous in r.

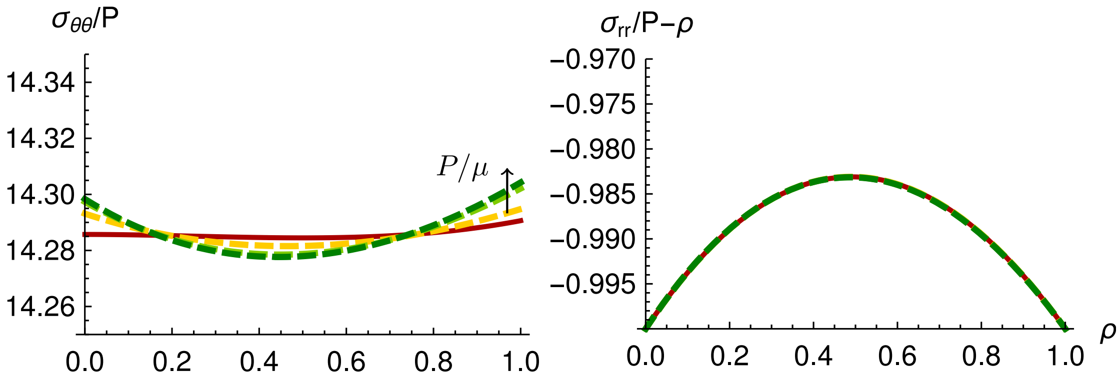

Figure 3: The graphs of the dimensionless Cauchy stress against the dimensionless radius ρ for the opening angle method (dashed curves) and the optimal stress Eqs. (73) and (74) (red solid curves), for which we used parameters for the descending thoracic aorta: $a = 20\,\text{mm}$, $b = 21.4\,\text{mm}$, $\alpha^{-1} = 0.07$ and $\lambda_z = 1$. As the dashed curves shade from yellow to green P/μ goes through the values 0.05, 0.2 and 0.35 with optimal opening angle $\phi = 65.9°$, $204.2°$ and $261.723°$; stress free reference inner radius 20.3 mm, 25.2 mm and 31.1 mm; and stress free reference outer radius 22 mm, 27.8 mm and 34.4 mm respectively.

We can now invoke Finite Elasticity to derive the residual stress $\boldsymbol{\tau}$ in the unloaded state necessary to sustain the optimal homogeneous Cauchy stress, and then compare our results with the residual stress predicted by the opening angle method. To do so, we let Ψ be a function of both stress and strain and use the initial stress symmetry (ISS), in the form of Eq. (11), to determine $\boldsymbol{\tau}$ as a function of $\boldsymbol{\sigma}$ and $\boldsymbol{F}$.

5.3 Residual stress

Here we use Finite Elasticity to connect the current state with the unloaded state. Since the cylindrical symmetry of the artery is maintained when the internal pressure is removed, we describe the unloaded state with the cylindrical coordinates (R, Θ, Z), then the deformation gradient for the unit basis vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$, $\boldsymbol{E}_R$, $\boldsymbol{E}_\Theta$ and $\boldsymbol{E}_Z$ becomes

$$\boldsymbol{F} = \frac{\partial r}{\partial R}\boldsymbol{e}_r \otimes \boldsymbol{E}_R + \frac{r}{R}\boldsymbol{e}_\theta \otimes \boldsymbol{E}_\Theta + \lambda_z \boldsymbol{e}_z \otimes \boldsymbol{E}_Z, \tag{76}$$

where we let λ_z be a constant. Assuming the material is incompressible we get

$$\det \boldsymbol{F} = 1 \implies \frac{\partial r}{\partial R}\frac{r}{R}\lambda_z = 1. \tag{77}$$

Let the reference configuration be a hollow cylinder with inner and outer radius A and B, respectively, so that

$$r = \sqrt{(R^2 - A^2)\lambda_z^{-1} + a^2} \implies \frac{\partial r}{\partial R} = \frac{R}{r}\lambda_z^{-1}, \tag{78}$$

which we will use to replace $\partial r/\partial R$ wherever it appears. We will assume that the residual stress $\boldsymbol{\tau}$ is homogeneous in Θ and Z, as the deformation,the Cauchy stress from Eqs. (73) and (74), the boundary conditions (80) and (81) are all homogeneous in Θ and Z.

The equilibrium equations now reduce to

$$\tau_{\Theta\Theta} = (R\tau_{RR})' \quad \text{for } A < R < B, \tag{79}$$

with the zero-traction boundary conditions

$$\tau_{RR} = 0 \ \text{ for } R = A, \tag{80}$$

$$\tau_{RR} = 0 \ \text{ for } R = B. \tag{81}$$

After making a constitutive choice for Ψ, the residual stresses τ_{RR} and $\tau_{\Theta\Theta}$ will be completely determined from $\boldsymbol{F}$, $\boldsymbol{\sigma}$ and p_τ due to ISS (11).

Ψ independent of I_2, J_2, J_3 and J_4

To simplify we assume Ψ independent of I_2, J_2, J_3 and J_4, so the Cauchy stress (6) becomes

$$\boldsymbol{\sigma} = 2\Psi_{I_1}\boldsymbol{F}\boldsymbol{F}^T - p\boldsymbol{I} + 2\Psi_{J_1}\boldsymbol{F}\boldsymbol{\tau}\boldsymbol{F}^T, \tag{82}$$

and, from ISS (12), we can swap $\boldsymbol{\tau}$, $\boldsymbol{F}$ and p respectively with $\boldsymbol{\sigma}$, $\boldsymbol{F}^{-1}$ and p_τ to get

$$\boldsymbol{\tau} = 2\Psi^\sigma_{I_1}\boldsymbol{F}^{-1}\boldsymbol{F}^{-T} - p_\tau\boldsymbol{I} + 2\Psi^\sigma_{J_1}\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}. \tag{83}$$

Substituting $\boldsymbol{F}$ from Eq. (76) into the above equation we arrive at

$$\tau_{RR} = 2\lambda_z^2\frac{r^2}{R^2}\left(\Psi^\sigma_{I_1} + \Psi^\sigma_{J_1}\sigma_{rr}\right) - p_\tau, \tag{84}$$

$$\tau_{\Theta\Theta} = 2\frac{R^2}{r^2}(\Psi^\sigma_{I_1} + \Psi^\sigma_{J_1}\sigma_{\theta\theta}) - p_\tau. \tag{85}$$

We remark that τ_{ZZ} has not been derived since we have not taken into consideration σ_{zz} in Section 5.2.

To compare unloaded arteries of different sizes we write r and R in terms of the dimensionless radius ρ,

$$r = (b-a)\rho + a \quad \text{and} \quad R = \sqrt{A^2 + \lambda_z \left[(a + (b-a)\rho)^2 - a^2\right]}, \tag{86}$$

which we use to rewrite the ODE (79) in the form

$$R\frac{d\tau_{RR}}{dR} + \tau_{RR} - \tau_{\Theta\Theta} = R\frac{d\tau_{RR}}{d\rho}\frac{d\rho}{dR} + \tau_{RR} - \tau_{\Theta\Theta} \tag{87}$$

$$= \frac{d\tau_{RR}}{d\rho} R \left(\frac{dR}{d\rho}\right)^{-1} + \tau_{RR} - \tau_{\Theta\Theta} = 0, \tag{88}$$

which implies that

$$\frac{d\tau_{RR}}{d\rho} + \frac{1}{R}\frac{dR}{d\rho}\left(\tau_{RR} - \tau_{\Theta\Theta}\right) = 0. \tag{89}$$

Integrating both sides in ρ and using the unloaded boundary condition (80) and (81) we reach

$$\tau_{RR} = \int_0^R \frac{\tau_{\Theta\Theta} - \tau_{RR}}{R}\frac{dR}{d\rho} d\rho = 0 \quad \text{and} \quad \int_0^1 \frac{\tau_{\Theta\Theta} - \tau_{RR}}{R}\frac{dR}{d\rho} d\rho = 0. \tag{90}$$

Eq. $(90)_2$ is independent of p_τ, and can be used to solve for one of the parameters P, a, b, A, λ_z or μ, note that B depends on the other parameters because Eq. (78) evaluated at $R = B$ gives $B^2 = A^2 + \lambda_z(b^2 - a^2)$. The only two parameters with a unit of time or mass are P and μ, so by rewriting $P = P_0\mu$ the Eqs.(90) will not depend on μ. To illustrate, we adopt the neo-Hookean material model (32), which results in $\Psi^\sigma_{J_1} = 1/2$ and $2\Psi^\sigma_{I_1} = \overset{\circ}{p}(\boldsymbol{\sigma})$ where $\overset{\circ}{p}$ is given by substituting $\boldsymbol{\tau}$ for $\boldsymbol{\sigma}$ restricted to (r, θ) in $\overset{\circ}{p}_+$ of Eq. (30).

To calculate the residual stress that supports the optimal Cauchy stress, we substitute $\sigma_{\theta\theta}$ and σ_{rr} Eqs. (73) and (74) into $\tau_{\Theta\Theta}$ and τ_{RR} Eqs. (84) and (85). We can then use Eq. $(90)_2$ to determine the unloaded geometry from the loaded geometry. To illustrate we take the parameters for the descending thoracic aorta, as used in the previous section, $a = 20\,\text{mm}$, $b = 1.4\,\text{mm}$, and $\lambda_z = 1$. We can then determine the unloaded inner radius A for different values of P_0, which is shown in Figure 4. Surprisingly the unloaded geometry given by the opening angle method is approximately the same as shown in Figure 4.

We can also investigate the residual stress in the unloaded state. Once A is determined from Eq. $(90)_2$ we can use Eq. $(90)_1$ to determine τ_{RR}, and then $\tau_{\Theta\Theta}$ from Eq.(79). To compare with the opening angle method we choose $P/\mu = 0.05$, 0.2 and 0.35, which all give a reasonable unloaded geometry, see Figure 4. The results are shown in Figure 5.

Biological tissues are very energy efficient. It is a common line of reasoning that biological systems adapt so as to minimize their potential energy. So it is possible that biological materials remodel their residual stress to lower their potential energy. Figure 6a below shows the free energy density Ψ for the opening angle method and for the model (32) with the optimal stresses Eqs. (73) and (74). In all cases Ψ is approximately a straight line that descreases as ρ increases away from the pressured boundary $\rho = 0$ $(r = a)$. Figure 6b shows the difference between Ψ/P from the opening angle method minus Ψ/P from the model (32), which we denote as $\Delta\Psi/P$. The integral of $\Delta\Psi/P$ over $\rho \in [0, 1]$ is positive if the opening angle method has on average a larger free energy density than the model (32), with the optimal stresses Eqs. (73) and (74). We found that as P/μ increased, so did the total free energy in the cylinder cross-section $\int_0^1 \Delta\Psi/P(b-a)(\rho(b-a)+a)d\rho$; for

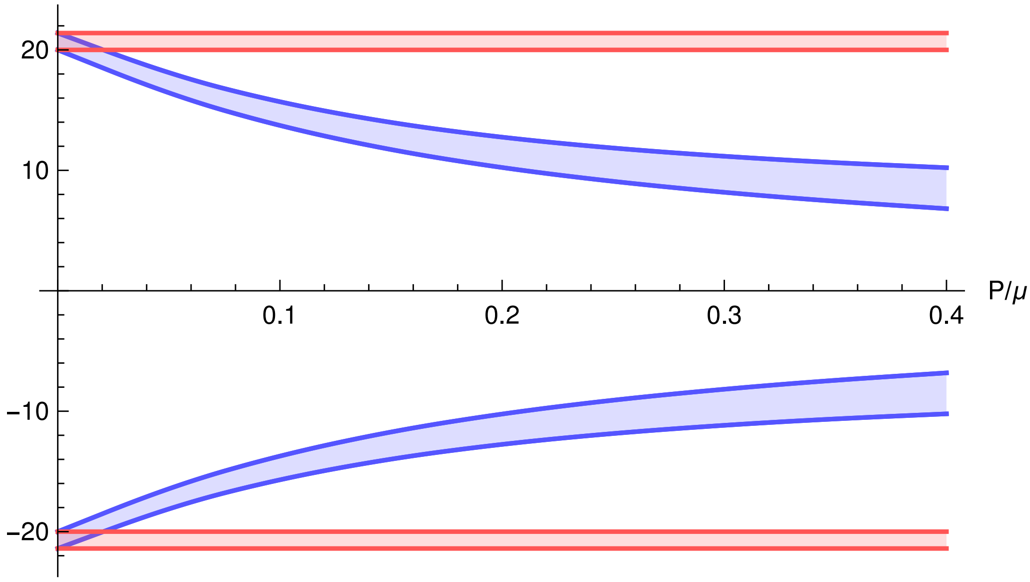

Figure 4: The loaded artery wall is illustrated by the red curves, while the unloaded artery wall is illustrated by the blue curves. The y–axis illustrates the position of the loaded and unloaded walls measured from the center of the artery. The stress in the loaded geometry is taken to be the optimal Cauchy stress (73) and (74), and we assume a residually stressed neo-Hookean material (32).

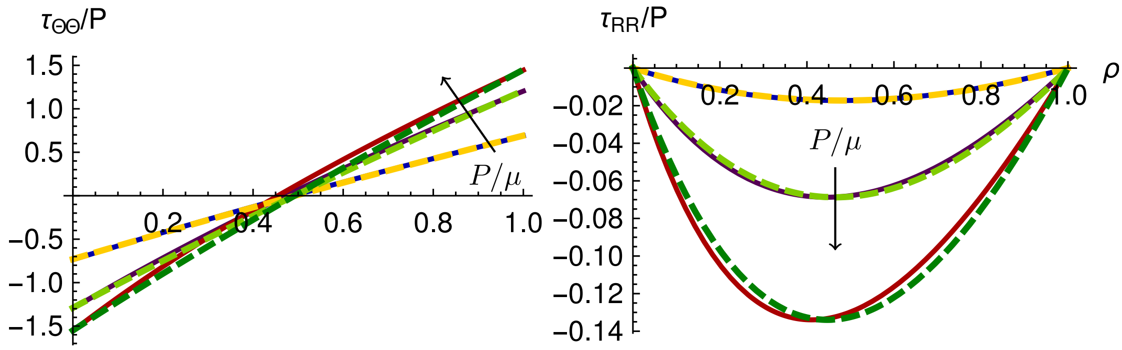

Figure 5: shows the dimensionless residual stress against the dimensionless radius ρ. The dashed curves are from the opening angle method and the solid curves are from Eq. (79) together with the optimal stresses Eqs. (73) and (74). As P/μ goes through the values 0.05, 0.2 and 0.35, the dashed curves shade from yellow to green and the solid curves shade from blue to red. The unloaded geometry for both methods is approximately the same and shown in Figure 4. The other parameters are $a = 20\,\text{mm}$, $b = 21.4\,\text{mm}$, $\alpha^{-1} = 0.07$ and $\lambda_z = 1$.

instance, this integral evaluates to -1.6410^{-4} for $P/\mu = 0.05$, 2.3710^{-3} for $P/\mu = 0.2$ and 4.2510^{-3} for $P/\mu = 0.35$.

Essentially we can see from the Figures 3 and 6 that the optimal stresses Eqs. (73) and (74) with the neo-Hookean material (32) produce a more homogeneous stress and lower free energy than the opening angle method, though the two methods gave very similar results. The advantage in using the ISS (12) for a homoeostasis hypothesis for the Cauchy stress $\boldsymbol{\sigma}$ is two fold. First is the freedom to choose $\boldsymbol{\sigma}$ so as to satisfy the homoeostasis hypothesis, which with ISS becomes a separate step from choosing a constitutive equation. Second is that it can be relatively straightforward to use ISS (12) to quantify the residual stress needed to support the chosen $\boldsymbol{\sigma}$.

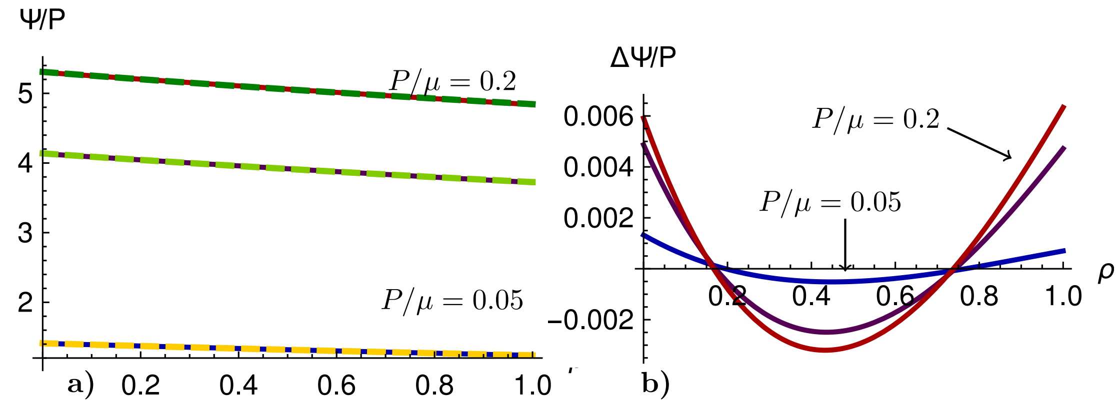

Figure 6: **(a)** compares the dimensionless strain energy density Ψ/P versus the dimensionless radius ρ resulting from the opening angle method, yellow (green) dashed curves, with the model (32) whose residual stress maintains the optimal stress Eqs. (73) and (74), the red (blue) solid curves. The parameters used are given in Figure 5. **(b)** shows Ψ/P from the opening angle method minus Ψ/P from the model (32), which we denote as $\Delta\Psi/P$.

6 Conclusions

In order to quantify the initial stress within a solid using non-destructive experimental techniques, it is convenient to write the free energy density $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ in terms of the deformation gradient $\boldsymbol{F}$ and the initial stress $\boldsymbol{\tau}$.

In this article we presented a new constitutive condition, the initial stress symmetry, that aids in proposing suitable constitutive relations for $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ by providing nine scalar equations, see Eqs. (37) and (38). One immediate result is that guessing a functional dependence for $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$ is not a trivial task. In fact all choices for $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ used in the literature do not satisfy ISS. Conversely, using ISS, we proposed two simple choices for $\Psi = \Psi(\boldsymbol{F}, \boldsymbol{\tau})$, one incompressible Eq. (23) and one compressible Eq. (46), which include a generalization for an initially stressed neo-Hookean material.

One consequence of ISS is that the initial stress can be derived from the Cauchy stress. Furthermore, ISS suggests that it is possible to first choose the Cauchy stress and, second, to make a constitutive choice and then use Eq. (11) to determine the corresponding initial stress. We illustrated this method in Section 5 for a simplified arterial wall, where we chose the ideal Cauchy stress by using a minimal stress gradient hypothesis as the homeostatic condition, and then we calculated the optimal residual stresses of the unloaded remodelled artery.

Since initial stresses are widespread in both inert and living matter, the proposed constitutive theory can be used in many applications towards a non-destructive quantification of initial tensions in solids. Future research include determining of a wider class of material behaviours, e.g. including a natural material anisotropy, and linking the initial stress to elastic wave speeds.

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Appendices

A Stress-free configuration implies ISS

Referring to Figure 1, the virtual stress-free configuration guarantees that for any $\boldsymbol{\tau}$ and $\boldsymbol{F}$ there exists a $\overset{\circ}{\boldsymbol{F}}$ such that

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\vartheta}}(\tilde{\boldsymbol{B}}, p) \quad \text{and} \quad \boldsymbol{\tau} = \hat{\boldsymbol{\vartheta}}(\overset{\circ}{\boldsymbol{B}}, \overset{\circ}{p}). \tag{A.1}$$

where $\overset{\circ}{\boldsymbol{B}} = \boldsymbol{F}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{F}^{-T}$; p and $\overset{\circ}{p}$ are a scalar fields where p depends on the boundary conditions on $\mathcal{B}$ and $\tilde{\boldsymbol{B}}$, while $\overset{\circ}{p}$ depends on the boundary conditions on $\overset{\circ}{\mathcal{B}}$ and $\overset{\circ}{\boldsymbol{B}}$; and

$$\hat{\boldsymbol{\vartheta}}(\boldsymbol{A}\boldsymbol{A}^T, q) := \boldsymbol{A}\frac{\partial \Psi}{\partial(\boldsymbol{A}^T\boldsymbol{A})}\boldsymbol{A}^T - q\boldsymbol{I}.$$

By setting $\boldsymbol{\tau} = 0$ in Eq. (6) we can see that the right side depends only on $\boldsymbol{A}\boldsymbol{A}^T$ and q.

Johnson and Hoger (1995) demonstrated that it is possible to determine $\overset{\circ}{\boldsymbol{B}}$ from any given $\boldsymbol{\tau}$, together with appropriate boundary conditions on $\overset{\circ}{\mathcal{B}}$, in order to reach useful constitutive equations such as

$$\hat{\boldsymbol{\vartheta}}(\boldsymbol{F}\overset{\circ}{\boldsymbol{B}}\boldsymbol{F}^T, p) = \hat{\boldsymbol{\varsigma}}(\boldsymbol{F}, \boldsymbol{\tau}, p) \quad \text{for any } \boldsymbol{F} \text{ and } \overset{\circ}{\boldsymbol{B}}. \tag{A.2}$$

As Eq. $(\text{A.1})_2$ is valid for any $\boldsymbol{F}$ and $\overset{\circ}{\boldsymbol{B}}$, we can substitute

$$\boldsymbol{F} \text{ for } \boldsymbol{F}^{-1}, \quad \overset{\circ}{\boldsymbol{B}} \text{ for } \tilde{\boldsymbol{B}} \tag{A.3}$$

and swap the boundary conditions on $\mathcal{B}$ and $\overset{\circ}{\mathcal{B}}$ (these substitutions effectively swap $\mathcal{B}$ for $\overset{\circ}{\mathcal{B}}$), so that Eq. $(\text{A.1})_2$ becomes $\boldsymbol{\sigma} = \hat{\boldsymbol{\vartheta}}(\tilde{\boldsymbol{B}}, p)$, where we have assumed that the above together with the boundary conditions on $\mathcal{B}$ has $\boldsymbol{\sigma}$ as the unique solution. Analogously, Eq. $(\text{A.1})_1$ becomes $\boldsymbol{\tau} = \hat{\boldsymbol{\vartheta}}(\overset{\circ}{\boldsymbol{B}}, \overset{\circ}{p})$.

This means that when we make the substitutions (A.3) in Eq. (A.2) we should also swap $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, so that

$$\hat{\boldsymbol{\vartheta}}(\overset{\circ}{\boldsymbol{B}}, \overset{\circ}{p}) = \hat{\boldsymbol{\varsigma}}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}, \overset{\circ}{p}), \tag{A.4}$$

The left-hand side is simply $\boldsymbol{\tau}$ and the right-hand side is $\boldsymbol{\tau}$ given by ISS (12). Finally, the above condition is identically true, since this result is valid for every $\boldsymbol{F}$ and $\overset{\circ}{\boldsymbol{B}}$. Moreover, we assumed that $\overset{\circ}{\boldsymbol{B}}$ can be determined from $\boldsymbol{\tau}$, so Eq. (A.4) holds for every $\boldsymbol{F}$ and $\boldsymbol{\tau}$.

B Reducing ISS to nine scalar equations

Deriving the scalar equations equivalent to ISS requires lengthy calculations. First we use the stress expansion (6) and ISS (12) to write the residual stress as

$$\begin{aligned}\boldsymbol{\tau} = -p_{\boldsymbol{\tau}}\boldsymbol{I} + 2\Psi^{\boldsymbol{\sigma}}_{I_1}\boldsymbol{C}^{-1} + 2\Psi^{\boldsymbol{\sigma}}_{I_2}(I_2\boldsymbol{C}^{-1} - \boldsymbol{C}^{-2}) + 2\Psi^{\boldsymbol{\sigma}}_{J_1}\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} + 2\Psi^{\boldsymbol{\sigma}}_{J_3}\boldsymbol{F}^{-1}\boldsymbol{\sigma}^2\boldsymbol{F}^{-T} \\ + 2\Psi^{\boldsymbol{\sigma}}_{J_2}\boldsymbol{F}^{-1}(\boldsymbol{\sigma}\boldsymbol{B}^{-1} + \boldsymbol{B}^{-1}\boldsymbol{\sigma})\boldsymbol{F}^{-T} + 2\Psi^{\boldsymbol{\sigma}}_{J_4}\boldsymbol{F}^{-1}(\boldsymbol{\sigma}^2\boldsymbol{B}^{-1} + \boldsymbol{B}^{-1}\boldsymbol{\sigma}^2)\boldsymbol{F}^{-T}\end{aligned} \tag{B.5}$$

where $I_2 = \text{tr}(\boldsymbol{C}^{-1})$ from Cayley-Hamilton with $\det \boldsymbol{C} = 1$. The above, together with (6) for $\boldsymbol{\sigma}$, must hold for every $\boldsymbol{\tau}$ and $\boldsymbol{C}$. To make use of this restriction, we substitute $\boldsymbol{\sigma}$ from (6) into Eq. (B.5) to write an equation in terms of only $\boldsymbol{\tau}$, $\boldsymbol{C}$, p and $p_{\boldsymbol{\tau}}$, compactly written as

$$\alpha_{ijkmn}\boldsymbol{C}^i\boldsymbol{\tau}^j\boldsymbol{C}^k\boldsymbol{\tau}^m\boldsymbol{C}^n = 0, \tag{B.6}$$

where we adopt the convention that repeated indices in a single term implies summation over all the values of the index, with $i, k, n \in \{-2, -1, 0, 1, 2\}$ and $j, m \in \{0, 1, 2\}$. The α_{ijk}'s can be calculated with a computer algebra system and are too cumbersome to reproduce here, though it is important to remember that the α_{ijk}'s are functions of p and $p_{\boldsymbol{\tau}}$; the invariants (3), (4) and (5); and the derivatives of Ψ with respect to the invariants (3) and (5). Due to the Eqs. (B.5) and (6) being symmetric, we have the symmetries $\alpha_{ijkmn} = \alpha_{njkmi} = \alpha_{nmkji}$.

Our aim here is to use Eq. (B.6) to obtain the corresponding scalar equations that hold for any choice of Ψ and the invariants (3), (4) and (5). To acheive this, we need to understand how we can vary $\boldsymbol{C}$ and $\boldsymbol{\tau}$ while keeping the invariants fixed. We can then apply these variations to Eq. (B.6), which holds true for any $\boldsymbol{C}$ and $\boldsymbol{\tau}$, to reach scalar equations in terms of the invariants and Ψ.

So first, to keep the invariants (3) and (4) fixed, we have to keep the eigenvalues of both $\boldsymbol{C}$ and $\boldsymbol{\tau}$ fixed. We can assume that the eigenvalues λ_1^2, λ_2^2 and λ_3^2 of $\boldsymbol{C}$ are all different, as this is true for most any deformation and so we do not loose generality. The same assumption can be made for τ_1, τ_2 and τ_3, the eigenvalues of $\boldsymbol{\tau}$.

The mixed invariants (5) involve the eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{\tau}$. So we write the eigen decompositions

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{V}_i \otimes \boldsymbol{V}_i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{j=1}^{3} \tau_j \boldsymbol{t}_j \otimes \boldsymbol{t}_j, \tag{B.7}$$

where the $\boldsymbol{V}_i$'s and $\boldsymbol{t}_j$'s are respectively the eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{\tau}$. Using these decompositions we see that

$$J_1 = \text{tr}(\boldsymbol{\tau C}) = \tau_j \lambda_i^2 (\boldsymbol{V}_i \cdot \boldsymbol{t}_j)^2, \; J_2 = \text{tr}(\boldsymbol{\tau C}^2) = \tau_j \lambda_i^4 (\boldsymbol{V}_i \cdot \boldsymbol{t}_j)^2, \tag{B.8}$$

$$J_3 = \text{tr}(\boldsymbol{\tau}^2 \boldsymbol{C}) = \tau_j^2 \lambda_i^2 (\boldsymbol{V}_i \cdot \boldsymbol{t}_j)^2, \; J_4 = \text{tr}(\boldsymbol{\tau}^2 \boldsymbol{C}^2) = \tau_j^2 \lambda_i^4 (\boldsymbol{V}_i \cdot \boldsymbol{t}_j)^2, \tag{B.9}$$

where summation over i and j is implied. Setting the above four terms to be constant can be seen as four independent equations for the $\boldsymbol{V}_i$'s and $\boldsymbol{t}_j$'s. Being unit eigenvectors of symmetric matrices, they must also satisfy the twelve equations

$$\boldsymbol{V}_i \cdot \boldsymbol{V}_m = \delta_{im} \quad \text{and} \quad \boldsymbol{t}_j \cdot \boldsymbol{t}_n = \delta_{jn}, \tag{B.10}$$

for $i, j, m, n = 1, 2, 3$.

So in total, the $\boldsymbol{V}_i$'s and $\boldsymbol{t}_j$'s have to satisfy 16 equations for the 18 unknown components of $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$, $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$. This leaves us with two degrees of freedom which we can carefully use to reduce Eq. (B.6). For example, let $\boldsymbol{V}_1 \cdot \boldsymbol{t}_1 = \boldsymbol{V}_2 \cdot \boldsymbol{t}_1 = 0$, which implies that $\boldsymbol{t}_1 = \pm \boldsymbol{V}_3$ and so $\boldsymbol{t}_1$ is an eigenvector of both $\boldsymbol{C}$ and $\boldsymbol{\tau}$. By taking the dot product of Eq. (B.6) with $\boldsymbol{t}_1$ on the left and right side we conclude that

$$\alpha_{ijkmn} \tau_1^{j+m} \lambda_3^{2(i+k+n)} = 0. \tag{B.11}$$

Now, in the same way, we could have used the degrees of freedom to choose $\boldsymbol{V}_1 \cdot \boldsymbol{t}_2 = \boldsymbol{V}_2 \cdot \boldsymbol{t}_2 = 0$ and concluded that

$$\alpha_{ijkmn} \tau_2^{j+m} \lambda_3^{2(i+k+n)} = 0. \tag{B.12}$$

and similarily choosing $\boldsymbol{V}_1 \cdot \boldsymbol{t}_3 = \boldsymbol{V}_2 \cdot \boldsymbol{t}_3 = 0$ we conclude that

$$\alpha_{ijkmn} \tau_3^{j+m} \lambda_3^{2(i+k+n)} = 0. \tag{B.13}$$

We can rewrite the three above equations by using the characteristic polynomial of λ_q^2 and of τ_p,

$$\lambda_3^6 = I_1 \lambda_3^4 - I_2 \lambda_3^2 + 1, \quad \tau_3^3 = I_{\tau 1} \tau_3^2 - I_{\tau 2} \tau_3 + I_{\tau 3}, \tag{B.14}$$

to replace powers of λ_3 higher than 4 and lower than 0 with λ_p^2, λ_p^4 and the invariants of $\boldsymbol{C}$, and the analogous for powers of τ_1, τ_2 and τ_3. In doing so Eqs. (B.11), (B.12) and (B.13) respectively become

$$\beta_{ij} \lambda_3^{2i} \tau_1^j = 0, \;\; \beta_{ij} \lambda_3^{2i} \tau_2^j = 0, \;\; \beta_{ij} \lambda_3^{2i} \tau_3^j = 0, \tag{B.15}$$

for $i, j = 0, 1$ and 2, and where the β_{ij}'s can be written in terms of α_{ijkmn} and the invariants of $\boldsymbol{\tau}$ and $\boldsymbol{C}$. This is best done with a computer algebra system.

As a reminder, the eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{\tau}$ in each of the above three equations are most likely different. However, the eigenvectors only appear in the form of the mixed invariants (5), which are the same for the three equations because the of Eqs. (B.8) and (B.9). Thus we may solve all the above three Eqs. (B.11), (B.12) and (B.13) simultaneously.

The three equations state that τ_1, τ_2 and τ_3 are the roots of the second-order polynomial $\beta_{ij}\lambda_3^{2i}x^j = 0$. As these eigenvalues are assumed to be all different, this is only possible if

$$\beta_{ij}\lambda_3^{2i} = 0, \quad \text{for } j = 0, 1, 2. \tag{B.16}$$

Finally, there was nothing special about $\boldsymbol{V}_3$ and λ_3, using analogous arguments we can conclude that

$$\beta_{ij}\lambda_1^{2i} = 0, \;\; \beta_{ij}\lambda_2^{2i} = 0, \;\; \beta_{ij}\lambda_3^{2i} = 0, \quad \text{for } j = 0, 1, 2. \tag{B.17}$$

Again, λ_1,λ_2 and λ_3 are assumed to be all different, so the only way that λ_1, λ_2 and λ_3 are the roots of a second-order polynomial is if the polynomials coefficients are all zero, leading to

$$\beta_{ij} = 0, \quad \text{for } i, j = 0, 1, 2. \tag{B.18}$$

This result suggest a simple way to reach expression for the β_{ij}'s: we can substitute scalars λ and τ, in place of the tensors (B.6), followed by repeatedly using the characteristic polynomials of both λ and t as explained around Eq. (B.14), and then the β_{ij}'s will be the coefficients of the multivariate polynomial in λ and τ. The results are explicity given by Eqs. (37) and (38) combined with Appendix C.

C ISS matrices

The matrices appearing in Eqs. (37) and (38) have the following expressions:

$$\boldsymbol{P}^{\{3\}}_{\{0\}} = -4 \begin{bmatrix} 2\Psi_{I_1}^2 + 2\Psi_{I_2}(p - I_2\Psi_{I_2}) \\ p^2/2 - 4\Psi_{I_1}\Psi_{I_2} - 2I_1\Psi_{I_2}^2 \\ 2I_1\Psi_{I_1}^2 + 2\Psi_{I_2}^2 - 2\Psi_{I_1}(p - 2I_2\Psi_{I_2}) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \;\; \boldsymbol{P}^{\{3\}}_{\{1\}} = 8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\Psi_{I1} \\ 2I_2\Psi_{I_1} + 2\Psi_{I_2} \\ p - 2I_1\Psi_{I_1} - 2I_2\Psi_{I_2} \end{bmatrix}, \tag{C.19}$$

$$\boldsymbol{P}^{\{3\}}_{\{2\}} = 16 \begin{bmatrix} 0 \\ 0 \\ 0 \\ p - 2I_1\Psi_{I_1} - 2I_2\Psi_{I_2} \\ 2(I_1I_2 - 1)\Psi_{I_1} + I_2(2I_2\Psi_{I_2} - p) \\ I_1p - 2(I_1^2 - I_2)\Psi_{I_1} + 2(1 - I_1I_2)\Psi_{I_2} \end{bmatrix}, \tag{C.20}$$

$$\boldsymbol{P}^{\{3\}}_{\{3\}} = -8 \begin{bmatrix} I_{\tau3}(2\Psi_{J_1} + 4I_1\Psi_{J_2} + I_{\tau1}\Psi_{J3}) \\ 4I_{\tau3}\Psi_{J_2} \\ I_{\tau3}(2I_1\Psi_{J_1} + 4(I_1^2 - I_2)\Psi_{J_2} + I_1I_{\tau1}\Psi_{J_3}) \\ (I_{\tau3} - I_{\tau1}I_{\tau2})\Psi_{J_3} - 2I_{\tau2}\Psi_{J_1} - 4I_1I_{\tau2}\Psi_{J_2} \\ 2I_2I_{\tau2}\Psi_{J_1} + 4(I_1I_2 - 1)I_{\tau2}\Psi_{J_2} + (I_{\tau1}I_{\tau2} - I_{\tau3})I_2\Psi_{J_3} \\ 4(I_2 - I_1^2)I_{\tau2}\Psi_{J_2} - 2I_1I_{\tau2}\Psi_{J_1} + (I_{\tau3} - I_{\tau1}I_{\tau2})I_1\Psi_{J_3} \end{bmatrix}, \tag{C.21}$$

$$\begin{aligned} \boldsymbol{P}^{\{3\}}_{\{4\}} = 32 &\begin{bmatrix} I_{\tau3}\left(2\left(I_2 - I_1^2\right)\Psi_{J_2} - I_1\Psi_{J_1}\right) \\ -I_{\tau3}\left(2I_1\Psi_{J_2} + \Psi_{J_1}\right) \\ I_{\tau3}\left(2\left(2I_1I_2 - I_1^3 - 1\right)\Psi_{J_2} + \left(I_2 - I_1^2\right)\Psi_{J_1}\right) \\ I_{\tau2}\left(2\left(I_1^2 - I_2\right)\Psi_{J_2} + I_1\Psi_{J_1}\right) \\ I_{\tau2}\left(2\left(I_1 + I_2^2 - I_1^2I_2\right)\Psi_{J_2} + (1 - I_1I_2)\Psi_{J_1}\right) \\ I_{\tau2}\left(2\left(I_1^3 - 2I_1I_2 + 1\right)\Psi_{J_2} + \left(I_1^2 - I_2\right)\Psi_{J_1}\right) \end{bmatrix} + \\ &32 \begin{bmatrix} -I_{\tau1}I_{\tau3}\left(\left(I_1^2 - I_2\right)\Psi_{J_4} + I_1\Psi_{J_3}\right) \\ -I_{\tau1}I_{\tau3}\left(I_1\Psi_{J_4} + \Psi_{J_3}\right) \\ -I_{\tau1}I_{\tau3}\left(\left(I_1^3 - 2I_1I_2 + 1\right)\Psi_{J_4} + \left(I_1^2 - I_2\right)\Psi_{J_3}\right) \\ (I_{\tau1}I_{\tau2} - I_{\tau3})\left(\left(I_1^2 - I_2\right)\Psi_{J_4} + I_1\Psi_{J_3}\right) \\ (I_{\tau1}I_{\tau2} - I_{\tau3})\left(\left(I_1 + I_2^2 - I_1^2I_2\right)\Psi_{J_4} + (1 - I_1I_2)\Psi_{J_3}\right) \\ (I_{\tau1}I_{\tau2} - I_{\tau3})\left(\left(I_1^3 - 2I_1I_2 + 1\right)\Psi_{J_4} + \left(I_1^2 - I_2\right)\Psi_{J_3}\right) \end{bmatrix}, \end{aligned} \tag{C.22}$$

$$\boldsymbol{P}^{\{4\}}_{\{0\}} = -4 \begin{bmatrix} -4I_1\Psi_{I_2}^2 - 8\Psi_{I_1}\Psi_{I_2} + p^2 \\ -4I_1 p\Psi_{I_2} - 4p\Psi_{I_1} + 4\Psi_{I_2}^2 + I_2 p^2 \\ 4\left(\Psi_{I_1}^2 + \Psi_{I_2}\left(p - I_2\Psi_{I_2}\right)\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{C.23}$$

$$\boldsymbol{P}^{\{4\}}_{\{1\}} = 16 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\Psi_{I_2} \\ p - 2I_2\Psi_{I_2} \\ -2\Psi_{I_1} \end{bmatrix}, \quad \boldsymbol{P}^{\{4\}}_{\{2\}} = 32 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\Psi_{I_1} \\ 2\left(I_2\Psi_{I_1} + \Psi_{I_2}\right) \\ -2I_1\Psi_{I_1} - 2I_2\Psi_{I_2} + p \end{bmatrix}, \tag{C.24}$$

$$\boldsymbol{P}^{\{4\}}_{\{3\}} = -16 \begin{bmatrix} 4I_{\tau 3}\Psi_{J_2} \\ 0 \\ I_{\tau 3}\left(4I_1\Psi_{J_2} + I_{\tau 1}\Psi_{J_3} + 2\Psi_{J_1}\right) \\ -4I_{\tau 2}\Psi_{J_2} \\ 4I_2 I_{\tau 2}\Psi_{J_2} \\ \Psi_{J_3}(I_{\tau 3} - I_{\tau 1}I_{\tau 2}) - 4I_1 I_{\tau 2}\Psi_{J_2} - 2I_{\tau 2}\Psi_{J_1} \end{bmatrix}, \tag{C.25}$$

$$\boldsymbol{P}^{\{4\}}_{\{4\}} = -64 \begin{bmatrix} I_{\tau 3}\left(2I_1\Psi_{J_2} + \Psi_{J_1}\right) \\ 2I_{\tau 3}\Psi_{J_2} \\ I_{\tau 3}\left(2\left(I_1^2 - I_2\right)\Psi_{J_2} + I_1\Psi_{J_1}\right) \\ -2I_1 I_{\tau 2}\Psi_{J_2} - I_{\tau 2}\Psi_{J_1} \\ I_{\tau 2}(2(I_1 I_2 - 1)\Psi_{J_2} + I_2\Psi_{J_1}) \\ I_{\tau 2}(2\left(I_2 - I_1^2\right)\Psi_{J_2} - I_1\Psi_{J_1}) \end{bmatrix} - 64 \begin{bmatrix} I_{\tau 1}I_{\tau 3}\left(I_1\Psi_{J_4} + \Psi_{J_3}\right) \\ I_{\tau 1}I_{\tau 3}\Psi_{J_4} \\ I_{\tau 1}I_{\tau 3}\left(\left(I_1^2 - I_2\right)\Psi_{J_4} + I_1\Psi_{J_3}\right) \\ -(I_{\tau 1}I_{\tau 2} - I_{\tau 3})\left(I_1\Psi_{J_4} + \Psi_{J_3}\right) \\ (I_{\tau 1}I_{\tau 2} - I_{\tau 3})\left((I_1 I_2 - 1)\Psi_{J_4} + I_2\Psi_{J_3}\right) \\ (I_{\tau 1}I_{\tau 2} - I_{\tau 3})\left(\left(I_2 - I_1^2\right)\Psi_{J_4} - I_1\Psi_{J_3}\right) \end{bmatrix}, \tag{C.26}$$

$$\boldsymbol{Q}^{\{1\}}_{\{1\}} = \boldsymbol{Q}^{\{1\}}_{\{2\}} = \boldsymbol{Q}^{\{2\}}_{\{1\}} = \boldsymbol{Q}^{\{2\}}_{\{2\}} = \boldsymbol{Q}^{\{n\}}_{\{0\}} = \boldsymbol{0}, \quad \text{for } n = 1, 2, 3 \text{ and } 4, \tag{C.27}$$

$$\boldsymbol{Q}^{\{1\}}_{\{3\}} = -4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{Q}^{\{1\}}_{\{4\}} = 8 \begin{bmatrix} -1 \\ I_2 \\ -I_1 \end{bmatrix}, \quad \boldsymbol{Q}^{\{2\}}_{\{3\}} = -8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{Q}^{\{2\}}_{\{4\}} = -16 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{C.28}$$

$$\boldsymbol{Q}^{\{3\}}_{\{1\}} = 8 \begin{bmatrix} -\Psi_{J_1} \\ I_2\Psi_{J_1} \\ -I_1\Psi_{J_1} \end{bmatrix}, \quad \boldsymbol{Q}^{\{3\}}_{\{2\}} = -32 \begin{bmatrix} I_1\Psi_{J_1} + \left(I_1^2 - I_2\right)\Psi_{J_2} \\ (1 - I_1 I_2)\Psi_{J_1} + \left(-I_2 I_1^2 + I_1 + I_2^2\right)\Psi_{J_2} \\ \left(I_1^2 - I_2\right)\Psi_{J_1} + \left(I_1^3 - 2I_2 I_1 + 1\right)\Psi_{J_2} \end{bmatrix}, \tag{C.29}$$

$$\boldsymbol{Q}^{\{3\}}_{\{3\}} = 8 \begin{bmatrix} -\Psi_{J_3}I_{\tau 1}^2 - 2\Psi_{J_1}I_{\tau 1} - 4I_1\Psi_{J_2}I_{\tau 1} - 2\Psi_{I_1} + I_{\tau 2}\Psi_{J_3} \\ I_2\Psi_{J_3}I_{\tau 1}^2 + 2I_2\Psi_{J_1}I_{\tau 1} + 4I_1 I_2\Psi_{J_2}I_{\tau 1} - 4\Psi_{J_2}I_{\tau 1} + 2I_2\Psi_{I_1} + 2\Psi_{I_2} - I_2 I_{\tau 2}\Psi_{J_3} \\ I_{\tau 2}\Psi_{J_3}I_1 + p - 2I_2\Psi_{I_2} + 4I_{\tau 1}I_2\Psi_{J_2} - I_{\tau 1}I_1\left(4\Psi_{J_2}I_1 + 2\Psi_{I_1} + 2\Psi_{J_1} + I_{\tau 1}\Psi_{J_3}\right) \end{bmatrix}, \tag{C.30}$$

$$\begin{aligned} \boldsymbol{Q}^{\{3\}}_{\{4\}} = {} & 32\left(I_{\tau 1}^2 - I_{\tau 2}\right)\Psi_{J_4} \begin{bmatrix} I_2 - I_1^2 \\ I_2 I_1^2 - I_1 - I_2^2 \\ 2I_2 I_1 - 1 - I_1^3 \end{bmatrix} + 32\left(I_{\tau 1}^2 - I_{\tau 2}\right)\Psi_{\mathrm{J3}} \begin{bmatrix} -I_1 \\ I_1 I_2 - 1 \\ I_2 - I_1^2 \end{bmatrix} + \\ & 16 \begin{bmatrix} p - 2I_2\Psi_{I_2} - 2I_1\left(\Psi_{I_1} + I_{\tau 1}\Psi_{J_1}\right) + 4\left(I_2 - I_1^2\right)I_{\tau 1}\Psi_{\mathrm{J2}} \\ 2(I_1 I_2 - 1)\Psi_{I_1} - I_2 p - 2I_{\tau 1}\Psi_{J_1} + 2I_2\left(I_2\Psi_{I_2} + I_1 I_{\tau 1}\Psi_{J_1}\right) - 4\left(-I_2 I_1^2 + I_1 + I_2^2\right)I_{\tau 1}\Psi_{\mathrm{J2}} \\ 2\left(I_2 - I_1^2\right)\Psi_{I_1} + I_1 p + 2(1 - I_1 I_2)\Psi_{I_2} + 2\left(I_2 - I_1^2\right)I_{\tau 1}\Psi_{J_1} - 4\left(I_1^3 - 2I_2 I_1 + 1\right)I_{\tau 1}\Psi_{\mathrm{J2}} \end{bmatrix}, \end{aligned} \tag{C.31}$$

$$\boldsymbol{Q}^{\{4\}}_{\{1\}} = -16 \begin{bmatrix} 0 \\ 0 \\ \Psi_{J_1} \end{bmatrix}, \ \boldsymbol{Q}^{\{4\}}_{\{2\}} = 64 \begin{bmatrix} -\Psi_{J_1} - I_1\Psi_{J_2} \\ I_2\Psi_{J_1} + (I_1 I_2 - 1)\Psi_{J_2} \\ \left(I_2 - I_1^2\right)\Psi_{J_2} - I_1\Psi_{J_1} \end{bmatrix}, \tag{C.32}$$

$$\boldsymbol{Q}^{\{4\}}_{\{3\}} = 16 \begin{bmatrix} 2\left(\Psi_{I_2} - 2I_{\tau 1}\Psi_{J_2}\right) \\ p - 2I_2\Psi_{I_2} + 4I_2 I_{\tau 1}\Psi_{J_2} \\ -\Psi_{J_3}I_{\tau 1}^2 - 2\Psi_{J_1}I_{\tau 1} - 4I_1\Psi_{J_2}I_{\tau 1} - 2\Psi_{I_1} + I_{\tau 2}\Psi_{J_3} \end{bmatrix}, \tag{C.33}$$

$$\begin{aligned} \boldsymbol{Q}^{\{4\}}_{\{4\}} = {} & 64\left(I_{\tau 1}^2 - I_{\tau 2}\right)\Psi_{J_3} \begin{bmatrix} -1 \\ I_2 \\ -I_1 \end{bmatrix} + 64\left(I_{\tau 1}^2 - I_{\tau 2}\right)\Psi_{\mathrm{J4}} \begin{bmatrix} -I_1 \\ I_1 I_2 - 1 \\ I_2 - I_1^2 \end{bmatrix} + \\ & 32 \begin{bmatrix} 2\left(I_2\Psi_{I_1} + \Psi_{I_2} + I_2 I_{\tau 1}\Psi_{J_1} + 2(I_1 I_2 - 1)I_{\tau 1}\Psi_{\mathrm{J2}}\right) \\ -2\left(\Psi_{I_1} + I_{\tau 1}\left(\Psi_{J_1} + 2I_1\Psi_{\mathrm{J2}}\right)\right) \\ p - 2I_2\Psi_{I_2} - 2I_1\left(\Psi_{I_1} + I_{\tau 1}\Psi_{J_1}\right) + 4\left(I_2 - I_1^2\right)I_{\tau 1}\Psi_{\mathrm{J2}} \end{bmatrix}, \end{aligned} \tag{C.34}$$