



Deposited via The University of Sheffield.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/145405/>

Version: Accepted Version

---

**Article:**

Zhang, R. and Smith, C.C. (2020) Upper bound limit analysis of soils with a non-linear failure criterion. *Canadian Geotechnical Journal*, 57 (3). pp. 423-432. ISSN: 0008-3674

<https://doi.org/10.1139/cgj-2018-0513>

---

© 2019 The Authors. This is an author-produced version of a paper subsequently published in *Canadian Geotechnical Journal* [<https://www.nrcresearchpress.com/journal/cgj>]. Uploaded in accordance with the publisher's self-archiving policy.

**Reuse**

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.

# Upper Bound Limit Analysis of Soils With a Non-linear Failure Criterion

Rui, Z. and Smith, C.C.

1

2           **Abstract:** Limit analysis is a widely used technique for the analysis of geotechnical collapse  
3 states and there exists a significant body of literature covering its application to soils with a  
4 linear failure criterion. However, such a failure criterion is often an idealisation of an actual  
5 non-linear response for which available analytical techniques are limited. This paper presents  
6 a new fully general solution procedure for generating upper bound multi-wedge rigid block  
7 mechanisms for a soil with a non-linear failure criterion, utilising a curved interface that  
8 obeys the non-linear yield function flow rule along its full length. This work extends the  
9 long established kinematic sliding wedge approach for linear soils and is illustrated through  
10 application to active and passive retaining wall and anchor/trapdoor problems. Through  
11 additional consideration of the lower bound solution, close bounds on the retaining wall  
12 problem to within  $\sim 1\%$  are established. The ability of the non-linear upper bound solution to  
13 predict the shear and normal stress at every point along the failure surface is discussed.

14 *Key words:* Limit analysis; Nonlinear failure; Upper Bound.

## 15 1. Introduction

16       The prediction of failure mechanisms in geotechnical engineering has many important applications  
17 in the design of such structures as retaining walls, foundations, slopes, buried pipes and culverts,  
18 ground anchors and silos. Limit analysis is a common approach applied to such problems and the  
19 theory has been extensively covered by *e.g.* Chen (1975) and Chen & Liu (1990), primarily for soils  
20 following a linear Mohr Coulomb criterion. However, the assumption of linear behaviour of soil is an  
21 idealisation and non-linear behaviour can be significant for some soils and fractured rock systems (*e.g.*  
22 Baker 2004, Hoek & Brown 1997, Mohammadi & Tavakoli 2015).

23       While recent work by *e.g.* Ukritchon & Keawsawasvong (2018) has demonstrated the modelling  
24 of non-linear behaviour in a finite element limit analysis framework, this paper is concerned with a  
25 discrete slip-line kinematic approach, which has application both for hand calculations and as part of a  
26 general purpose numerical approach *e.g.* Smith & Gilbert (2007) and Hambleton & Sloan (2013).

27       Several authors have addressed such a problem. Baker & Frydman (1983) and Chen (1975) de-  
28 scribed a method for undertaking slope stability analysis for non-linear soils using a variational ap-  
29 proach, but which required numerical integration. Soon & Drescher (2007) presented an approach  
30 based on the classic multi-wedge rigid block upper bound kinematic method, using straight slip-lines  
31 but a specific linear yield surface for each sliding interface, where this linear surface was selected as a  
32 tangent to the non-linear yield surface. The specific tangent location was chosen as part of a multivari-

---

Rui, Z. and Smith, C.C..<sup>1</sup> University of Sheffield, Department of Civil and Structural Engineering

<sup>1</sup>Corresponding author (e-mail: c.c.smith@sheffield.ac.uk).

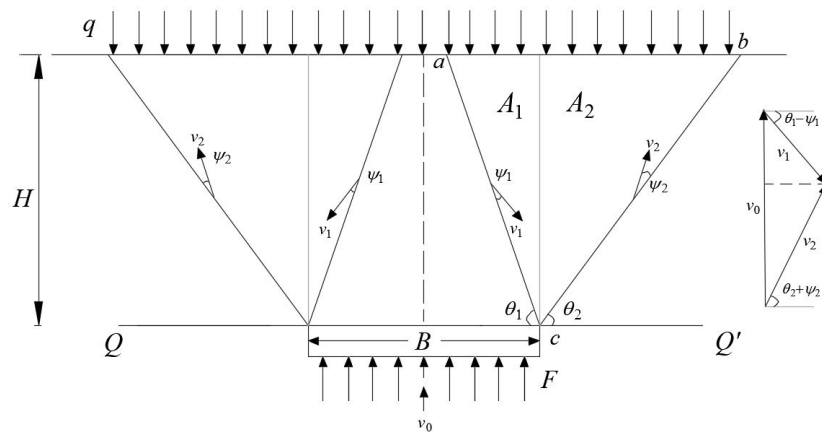
33 able optimization across all slip-lines.

34 As an alternative approach, other workers (*e.g.* Fraldi & Guarracino 2009, Yang & Huang 2011,  
 35 Yang & Long 2015 and Zhang & Yang 2018) used a variational approach to analyse the stability  
 36 of anchors, trapdoors and tunnels with a single curved slip-line, defined by a closed form equation,  
 37 that obeys the non-linear yield function flow rule along its full length. However, these analyses were  
 38 restricted to single wedges constrained to move vertically and to the authors' knowledge this approach  
 39 has not been extended to a wider range of problems. By extending the approach presented by Fraldi &  
 40 Guarracino 2009, this paper presents a new fully general form of the variational approach for analysing  
 41 translational upper bound problems using the classic multi-wedge rigid block upper bound kinematic  
 42 method, based on curved slip-lines defined by closed form equations, and following the optimisation  
 43 framework of *e.g.* Soon & Drescher (2007).

44 Examples are given for the active and passive cases of a smooth retaining wall and shown to match  
 45 to within  $\sim 1\%$  of the corresponding simple non-linear lower bounds, thus for the first time giving  
 46 almost exact plastic solutions for these cases. An example of how the approach can deal with a multi-  
 47 wedge analysis is also given for an anchor uplift problem. An intriguing aspect of the solutions are that  
 48 they give exact values of shear and normal stresses along the slip-lines, which is not normally obtained  
 49 from an upper bound analysis which can only return forces on wedge interfaces. The interpretation of  
 50 such values is discussed in the context of the examples studied.

## 51 2. Conventional upper bound mechanism analysis

52 Conventional upper bound rigid block mechanism analysis is well established in geotechnical engi-  
 53 neering and consists of postulating a failure mechanism consisting of sliding wedges. Kinematic  
 54 compatibility and an associative flow rule can be used to construct a hodograph that allows the veloci-  
 55 ties of the wedges to be determined and the relative velocities across the slip-lines computed. These  
 56 can be used to determine energy dissipation on the slip-lines, and together with external work can be  
 57 used to determine the collapse load. A simple two-wedge example for an anchor pullout problem (after  
 58 *e.g.* Murray & Geddes 1987) is shown in Fig. 1.



**Fig. 1.** Simple two-wedge anchor analysis for a linear soil showing mechanism and hodograph.

59 The essence of the method is to determine the angle of dilation and energy dissipation function  
 60 for each slip-line. For a linear Mohr-Coulomb ( $c, \phi$ ) material of unit weight  $\gamma$ , the angle of dilation  
 61  $\psi$  is equal to the angle of friction  $\phi$  for an upper bound analysis. The dissipation function for relative  
 62 slip  $s$  parallel to the slip-line is  $cls$  where  $l$  is the slip-line length and  $s = v \cos \psi$  where  $v$  is the  
 63 relative velocity jump across the slip-line. The self weight external work for each wedge is computed

64 by determining the dot product of the absolute velocity of a wedge and its self weight. This together  
65 with the hodograph leads to the following equations for the two-wedge anchor analysis.

$$[1] \quad \frac{v_0}{\sin(\theta_1 + \theta_2 - \psi_1 + \psi_2)} = \frac{v_1}{\sin(\pi/2 - \theta_2 - \psi_2)} = \frac{v_2}{\sin(\pi/2 - \theta_1 + \psi_1)}$$

$$[2] \quad v_{01} = v_1 \cdot \sin(\theta_1 - \psi_1)$$

$$[3] \quad v_{02} = v_2 \cdot \sin(\theta_2 + \psi_2)$$

with constraints:

$$[4] \quad \psi_1 \geq \theta_1 - \pi/2$$

$$[5] \quad \psi_2 \leq \pi/2 - \theta_2$$

66 The force on the anchor can be determined from the following energy balance equation.

$$[6] \quad Fv_0 = \gamma HBv_0 + qBv_0 + 2(W_{A2}v_{02} + qHv_0/\tan\theta_2 - W_{A1}v_{01} + cl_{bc}v_2 \cos\psi_2 + cl_{ac}v_1 \cos\psi_1)$$

67 For a linear soil, the optimal solution is one for which  $v_1 = 0$  and  $\theta_2 = 90 - \phi$ , giving the following  
68 equation for  $F$ :

$$[7] \quad \frac{F}{\gamma HB} = 1 + \frac{H}{B} \tan\phi + \frac{q}{\gamma H} + \frac{2q}{\gamma B} \tan\phi + \frac{2c}{\gamma B}$$

69 The aim of this paper is to demonstrate how this form of mechanism analysis for linear Mohr-  
70 Coulomb materials can be extended to a material possessing a non-linear yield surface for translational  
71 mechanisms in a fully general way.

### 72 3. Non-linear yield surface

73 Various non-linear strength functions have been proposed for soils and rocks, such as bilinear  
74 functions Lefebvre (1981), trilinear functions De Mello (1977) and the Hoek-Brown failure criterion  
75 Hoek & Brown (1997). Non-linear power-type failure laws for geomaterials are increasingly being  
76 adopted for investigations of the stability of geotechnical problems (*e.g.* Baker 2004, Zhang & Chen  
77 1987, Anyaegbunam 2013).

78 In general, as shown in Fig. 2, a non-linear power-law failure criterion can be expressed as,

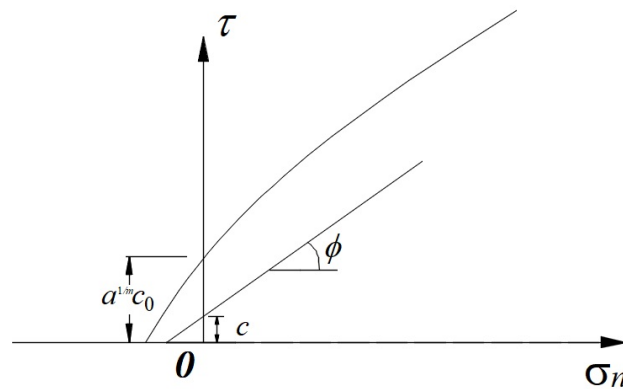
$$[8] \quad \tau/c_0 = (a + \sigma_n/\sigma_t)^{1/m}$$

79 where  $\sigma_n$  and  $\tau$  are the normal and shear stresses on the failure surface, respectively;  $c_0$  and  $\sigma_t$  are  
80 normalisation stresses; and  $a$  and  $m$  are scalar constants. When  $m = 1$ , equation (8) reduces to the  
81 well-known linear Mohr-Coulomb failure criterion:

$$[9] \quad \tau = c + \sigma_n \tan \phi$$

82 where  $a = 1$ ,  $c_o = c$  and  $\sigma_t = c / \tan \phi$ .

83 In this paper two exemplar non-linear materials will be modelled, representing (i) a dense sand  
 84 (based on the model by Bolton 1986 using a relative density index  $I_D = 1$ ) and (ii) a fractured rock  
 85 mass (approximating a Hoek-Brown material with  $\sigma_{ci} = 5 \text{ MN/m}^2$ ,  $m_i = 9.6$ ,  $GSI = 20$ ,  $m_b = 0.55$ ,  
 86 Hoek & Brown 1997 ). The properties are given in Table 1. These were obtained by generating the  
 87 relevant yield surface and carrying out a least squares best fit to equation (8). For comparison two  
 88 linear soils were also modelled using (i) a simple  $c = 1 \text{ kN/m}^2$ ,  $\phi = 30^\circ$  Mohr-Coulomb soil and (ii)  
 89 a  $c = 0$ ,  $\phi = 33^\circ$  soil corresponding to the previous Bolton model at critical state. In both these cases,  
 90 a value of  $m = 1.001$  was adopted to model a closely linear system while still capable of adopting the  
 91 non-linear solution methodology. It is shown in Appendix A that this leads to an error of  $< 0.037\%$   
 92 in modelling the linear yield surface. Since the aim of the paper is to illustrate and verify the solution  
 93 process and to contextualise it, specific engineering examples will be studied rather than undertaking  
 94 parametric studies.



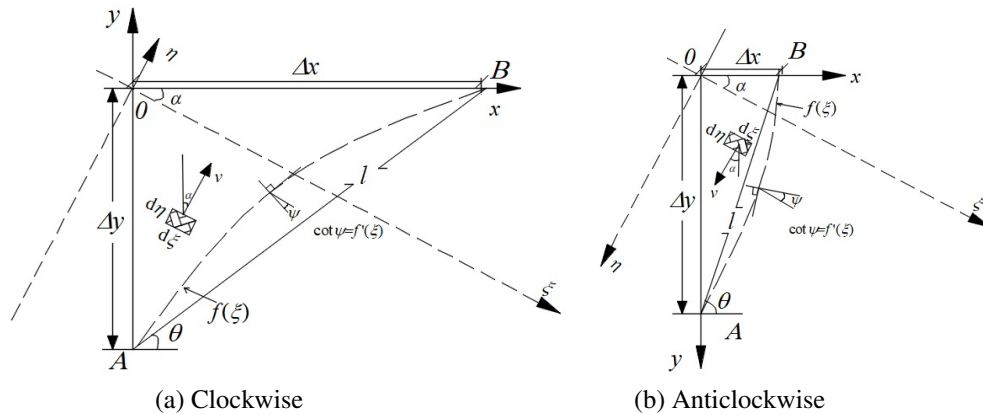
**Fig. 2.** Linear Mohr-Coulomb and non-linear power-law (equation 8) failure criteria.

Parameter	Linear		Non-linear	
	Cohesive-frictional soil (CF) $c = 1 \text{ kPa}$ , $\phi = 30^\circ$	Loose sand (LS) Bolton $I_D = 0$	Dense sand (DS) Bolton $I_D = 1$	Fractured rock (FR) Hoek-Brown
$a(-)$	1	0	0	0
$c_0(\text{kN/m}^2)$	1	1	1.697	$1.8242 \times 10^3$
$\sigma_t(\text{kN/m}^2)$	$1 / \tan 30^\circ$	$1 / \tan 33^\circ$	1	$5 \times 10^3$
$m(-)$	1.001	1.001	1.1182	1.3155
$\gamma(\text{kN/m}^3)$	15	15	15	22

**Table 1.** Exemplar linear and non-linear soil properties and unit weights.

95 **4. Non-linear upper bound failure mechanism analysis for single slip-line**  
 96 **with variational approach**

97 Consider a slip-line, which may be curved in the general case, connecting two points  $A$  and  $B$   
 98 whose secant is orientated at an angle  $\theta$  to the positive  $x$ -axis and of length  $l$ . Further consider that  
 99 there is a velocity jump of  $v$  across this slip-line orientated at a fixed angle  $\psi_s$  to its secant. This velocity  
 100 will therefore be orientated at an angle  $\alpha$  to the vertical where  $\alpha = \pi/2 - \kappa\psi_s - \theta$  and  $\kappa = \pm 1$  denotes  
 101 clockwise or anticlockwise shear respectively across the slip-line. The aim is to determine the shape  
 102 of the slip-surface joining  $A$  and  $B$  which will no longer be a secant, but will curve slightly above or  
 103 below the secant depending on the relative movement as shown in Fig. 3, such that the sum of the  
 104 local dilation  $\psi$  and slip-line gradient at any point is constant and equal to the global slip-line dilation  
 105  $\psi_s$ . This preserves the assumption of rigid body movement of adjacent wedges. To maintain the work  
 106 calculation for the general mechanism analysis similar to the linear case, a dissipation coefficient  $\hat{C}$  for  
 107 a non-linear material will be derived equivalent to the cohesion intercept term  $cl$  for a linear soil and  
 108 where the self weight of the wedges delineated by the secants may still be used, but where the slip-line  
 109 will also have its own additional self weight term  $\hat{W}$  defined by the area of soil between the curved  
 110 slip-line and the secant. This may be negative or positive depending on the direction of relative shear.



**Fig. 3.** Non-linear kinematics of a slip-line (long-dashed line between  $A$  and  $B$ ). Relative shear across slip-line: (a) clockwise; (b) anticlockwise.

111 **4.1. General analysis form for a single slip-line**

112 **4.1.1. Compatibility**

113 For a single slip-line as shown in Fig. 3, the mass of soil above the slip-line moves as a rigid block  
 114 at velocity  $v$  and at an angle  $\alpha$  to the vertical relative to the soil below. Let  $y = f(x)$  be the equation  
 115 of the velocity discontinuity surface.

116 This can be expressed in a rotated coordinate system as  $\eta = f(\xi)$ , where:

$$[10] \quad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

117 Use of this new coordinate system simplifies the subsequent analysis by making the velocity  $v$   
 118 parallel to the  $\eta$  axis. By assuming the plastic potential,  $\zeta$ , to be coincident with the Mohr envelope  
 119 and considering  $\tau$  is positive, the following can be defined:

$$[11] \quad \zeta = \tau - c_0(a + \sigma_n/\sigma_t)^{1/m}$$

120 Adapting the approach developed for a tunnel analysis in a Hoek-Brown material by Fraldi &  
121 Guarracino (2009), and assuming associative flow, the local angle of dilation  $\psi$  may be given by the  
122 following equation:

$$[12] \quad \tan \psi = \frac{d\tau}{d\sigma_n} = \frac{c_0(a + \sigma_n/\sigma_t)^{\frac{1-m}{m}}}{m\sigma_t}$$

123 and because relative movement of the block above the discontinuity is parallel to the  $\eta$ -axis, the fol-  
124 lowing can also be written:

$$[13] \quad \tan \psi = \frac{1}{\kappa f'(\xi)}, \quad \cos \psi = \kappa f'(\xi) [1 + f'(\xi)^2]^{-\frac{1}{2}}, \quad \sin \psi = [1 + f'(\xi)^2]^{-\frac{1}{2}}$$

125 where  $\kappa = 1$  for the clockwise relative shear case and  $\kappa = -1$  for the anti-clockwise relative shear  
126 case.

127 Combining equations (12) and (13) gives

$$[14] \quad \sigma_n = -a \cdot \sigma_t + \sigma_t \left( \frac{c_0}{m\sigma_t} \right)^{\frac{m}{m-1}} [\kappa f'(\xi)]^{\frac{m}{m-1}}$$

128 and substitution in equation (8) gives,

$$[15] \quad \tau = c_0 \left( \frac{c_0}{m\sigma_t} \right)^{\frac{1}{m-1}} [\kappa f'(\xi)]^{\frac{1}{m-1}}$$

129 Now the plastic strain rates can be written as follows:

$$[16] \quad \dot{\epsilon}_n = \lambda \frac{\partial \zeta}{\partial \sigma} = -\lambda \frac{c_0}{m\sigma_t} (a + \sigma_n/\sigma_t)^{(1-m)/m}$$

$$[17] \quad \dot{\gamma} = \lambda \frac{\partial \zeta}{\partial \tau} = \lambda$$

130 where  $\lambda$  is a scalar parameter,  $\dot{\epsilon}_n$  is the normal plastic strain rate and  $\dot{\gamma}$  is the shear plastic strain  
131 rate. Based on the kinematics occurring on the slip-line, as shown in Fig. 3, the plastic strain rate  
132 components can also be written in the form

$$[18] \quad \dot{\epsilon}_n = v_n = -\frac{v}{w} [1 + f'(\xi)^2]^{-\frac{1}{2}}$$

$$[19] \quad \dot{\gamma} = v_t = \frac{v}{w} \kappa f'(\xi) [1 + f'(\xi)^2]^{-\frac{1}{2}}$$

133 where  $\eta = f(\xi)$  is the function of velocity discontinuity surface and  $f'(\xi)$  is the first derivative of  $f(\xi)$ .  
134  $v$  is the velocity jump at the slip-line and  $w$  is the thickness of the plastic zone (assumed infinitesimal).

135 A dot denotes differentiation with respect to time and a prime with respect to  $\xi$ , i.e.  $v = \partial u / \partial t$ ,  
 136  $f'(\xi) = \partial f(\xi) / \partial \xi$

137 In order to enforce compatibility, from equation (17) and equation (19) (or, equivalently from  
 138 equation (14), equation (16) and equation (18)) it follows that:

$$[20] \quad \lambda = \frac{v}{w} \kappa f'(\xi) [1 + f'(\xi)^2]^{-\frac{1}{2}}$$

#### 139 4.1.2. Determination of internal energy dissipation and external work

140 Based on the plastic potential function equation (11), the plastic strain increment is proportional  
 141 to the gradient of the plastic potential function through the associated flow rule. The dissipation en-  
 142 ergy associated with the internal forces at any point on the surface,  $\dot{D}_i$  can therefore be obtained by  
 143 combining equation (14), equation (15), equation (18) and equation (19):

$$[21] \quad \dot{D}_i = \sigma_n v_n + \tau v_t = \frac{v}{w} [1 + f'(\xi)^2]^{-\frac{1}{2}} \left[ a \cdot \sigma_t + \sigma_t \left( \frac{c_0}{m \sigma_t} \right)^{\frac{m}{(m-1)}} (m-1) [\kappa f'(\xi)]^{\frac{m}{(m-1)}} \right]$$

144 By considering the profile of failure surface for the single wedge, as shown in Fig. 3 the energy  
 145 dissipation along the velocity discontinuity surface can be obtained by integrating  $\dot{D}_i$  over the interval  
 146  $\xi = \Delta y \sin \alpha$  to  $\Delta x \cos \alpha$  where  $\Delta x = l \cos \theta$  and  $\Delta y = l \sin \theta$ .

Hence

$$[22] \quad D = \int_{\Delta y \sin \alpha}^{\Delta x \cos \alpha} \dot{D}_i w \sqrt{1 + f'(\xi)^2} d\xi$$

$$= v \int_{\Delta y \sin \alpha}^{\Delta x \cos \alpha} \left\{ a \cdot \sigma_t + \sigma_t \left( \frac{c_0}{m \sigma_t} \right)^{\frac{m}{(m-1)}} (m-1) [\kappa f'(\xi)]^{\frac{m}{(m-1)}} \right\} d\xi$$

147 The work done ( $W_e$ ) by the external force (gravity) on the area between the curve and the secant is  
 148 given as follows (NB in the active case, the area is negative, but the integration is also negative.):

$$[23] \quad W_e = \kappa v \gamma \cos \alpha \left[ \int_{\Delta y \sin \alpha}^{\Delta x \cos \alpha} f(\xi) d\xi + \Delta y \cos \alpha (\Delta x \cos \alpha - \Delta y \sin \alpha) - 0.5 l^2 \sin(\theta + \alpha) \cos(\theta + \alpha) \right]$$

#### 149 4.1.3. Solution characterizing optimal slip-line geometry

150 In order to describe the optimal shape of the slip-line, it is necessary to obtain the explicit expression  
 151 of  $f(\xi)$  by constructing an objective function  $\Lambda$  consisting of the sum of the contribution of the slip-line  
 152 to the external work rate and the rate of the internal energy dissipation,

$$[24] \quad \Lambda = D - W_e$$

$$= v \int_{\Delta y \sin \alpha}^{\Delta x \cos \alpha} \zeta[f(\xi), f'(\xi), \xi] d\xi$$

$$- v \kappa \gamma \cos \alpha [\Delta y \cos \alpha (\Delta x \cos \alpha - \Delta y \sin \alpha) - 0.5 l^2 \sin(\theta + \alpha) \cos(\theta + \alpha)]$$

153 in which

$$[25] \quad \zeta[f(\xi), f'(\xi), \xi] = \sigma_t \left[ a + \left( \frac{c_0}{m\sigma_t} \right)^{\frac{m}{m-1}} (m-1) [\kappa f'(\xi)]^{\frac{m}{m-1}} \right] - \kappa \gamma f(\xi) \cos \alpha$$

154 In order to obtain the effective failure surface for a given slip-line of angle  $\theta$  and length  $l$ , it is  
 155 necessary to search for the extremum value of objective function  $\Lambda$  using Euler's equation through the  
 156 variational method. The expression of the variational equation of  $\Lambda$  for stationary conditions can be  
 157 written as:

$$[26] \quad \frac{\partial \Lambda}{\partial f(\xi)} - \frac{\partial}{\partial \xi} \left[ \frac{\partial \Lambda}{\partial f'(\xi)} \right] = 0$$

158 and the explicit form of the Euler's equation for the equation (25) can thus be obtained as:

$$[27] \quad \kappa \gamma \cos \alpha + \frac{m\sigma_t}{(m-1)} \left( \frac{c_0}{m\sigma_t} \right)^{\frac{m}{m-1}} [\kappa f'(\xi)]^{\frac{2-m}{m-1}} [f''(\xi)] = 0$$

159 Equation (27) is a non-linear second-order homogeneous differential equation. A first integration  
 160 yields

$$[28] \quad m\sigma_t \left( \frac{c_0}{m\sigma_t} \right)^{\frac{m}{m-1}} [\kappa f'(\xi)]^{\frac{1}{m-1}} = -\kappa \gamma \cos \alpha \cdot \xi + n_0$$

161 where  $n_0$  is integration constant coefficient. Re-arrangement of this equation gives:

$$[29] \quad f'(\xi) = \kappa m k_0 \left( \frac{n_0}{\gamma \cdot \cos \alpha} - \kappa \xi \right)^{m-1}$$

162 in which

$$[30] \quad k_0 = \frac{\sigma_t}{\gamma \cos \alpha} \left( \frac{\gamma \cdot \cos \alpha}{c_0} \right)^m = \frac{\sigma_t}{c_0} \left( \frac{\gamma \cdot \cos \alpha}{c_0} \right)^{m-1} = \frac{\sigma_t}{c_0^m} (\gamma \cos \alpha)^{m-1}$$

163 By a further integral calculation process the equation for the velocity discontinuity surface is given  
 164 by:

$$[31] \quad f(\xi) = -k_0 \left( \frac{n_0}{\gamma \cdot \cos \alpha} - \kappa \xi \right)^m + n_1$$

165 and a further integration provides an expression required later for the weight correction term:

$$[32] \quad \int f(\xi) \cdot d\xi = \frac{\kappa k_0}{m+1} \left( \frac{n_0}{\gamma \cdot \cos \alpha} - \kappa \xi \right)^{m+1} + n_1 \xi + \text{const}$$

166 where  $n_0$  and  $n_1$  are two unknowns representing the integration constant coefficients. These can  
 167 be determined using the two boundary conditions:

$$[33] \quad f(\xi = \Delta y \sin \alpha) = -\Delta y \cos \alpha$$

$$[34] \quad f(\xi = \Delta x \cos \alpha) = \Delta x \sin \alpha$$

168 Hence:

$$[35] \quad -k_0 \left( \frac{n_0}{\gamma \cos \alpha} - \kappa \Delta x \cos \alpha \right)^m + n_1 - \Delta x \sin \alpha = 0$$

169 and

$$[36] \quad n_1 = -\Delta y \cdot \cos \alpha + k_0 \left( \frac{n_0}{\gamma \cdot \cos \alpha} - \kappa \Delta y \cdot \sin \alpha \right)^m$$

170 It is not possible to derive closed form expressions for  $n_0$  and  $n_1$ , however by substituting equation  
171 (36) into equation (35), these may be determined straightforwardly numerically using standard root  
172 finding algorithms. There is a small range of values of  $\alpha$  and hence  $\psi_s$  that will give valid solutions.

173 This solution may be expressed as a dilation (relative to the secant) of  $\psi_s = \kappa(\pi/2 - \alpha - \theta)$ ,  
174 a coefficient of dissipation  $\hat{C}(\psi_s, \theta, l)$  based on equation (22) and a correction to the wedge weight  
175 above the slip-line  $\hat{W}(\psi_s, \theta, l)$  based on equation (23).  $\hat{C}$  is equivalent to the term  $cl$  for the linear  
176 case. Thus  $\hat{C}$  is multiplied by  $s = v \cos \psi_s$  to give the full dissipation. Hence expressions for  $\hat{C}$  and  
177  $\hat{W}$  can be written in terms of the derived function  $f$ :

178 From equation (22):

[37]

$$\hat{C}(\psi_s, \theta, l) = \frac{\kappa \sigma_t}{\cos \psi_s} \left( \frac{m-1}{m+1} \right) \left( \frac{\gamma \cos \alpha}{c_0} \right)^m \left[ \left( \frac{n_0}{\gamma \cos \alpha} - \kappa \Delta y \sin \alpha \right)^{m+1} - \left( \frac{n_0}{\gamma \cos \alpha} - \kappa \Delta x \cos \alpha \right)^{m+1} \right] \\ + a \cdot \frac{\sigma_t}{\cos \psi_s} (\Delta x \cos \alpha - \Delta y \sin \alpha)$$

179 and from equation (23) and equation (32):

$$[38] \quad \frac{\hat{W}(\psi_s, \theta, l)}{\kappa \gamma} = \frac{\kappa k_0}{m+1} \left[ \left( \frac{n_0}{\gamma \cos \alpha} - \kappa \Delta x \cos \alpha \right)^{m+1} - \left( \frac{n_0}{\gamma \cos \alpha} - \kappa \Delta y \sin \alpha \right)^{m+1} \right] \\ + n_1 (\Delta x \cos \alpha - \Delta y \sin \alpha) \\ + \Delta y \cos \alpha (\Delta x \cos \alpha - \Delta y \sin \alpha) - 0.5 l^2 \sin(\theta + \alpha) \cos(\theta + \alpha)$$

180 These functions are straightforward to compute using a spreadsheet or computer program.

## 181 5. General solution procedure

182 The following outlines a typical hand solution process following the standard form of linear upper  
183 bound wedge analysis.

- 184 1. Postulate an appropriate multi-wedge failure mechanism, involving a series of nodes linking  
185 slip-lines that delineate each wedge, allocating an appropriate value of global dilation  $\psi_s$  to each  
186 slip-line.  $\psi_s$  must be chosen to generate real values of  $n_0$  and  $n_1$  for each slip-line.

- 187 2. Based on the straight lines joining each node and the values of  $\psi_s$ , draw the corresponding  
188 hodograph.
- 189 3. Determine the acting weight  $W$  of each wedge based on the area of the wedge delineated by  
190 straight lines joining each node, and adjusted according to the term  $\hat{W}$  for each slip-line edge of  
191 the wedge.
- 192 4. Determine the external work done using the velocities from the hodograph and the weight of  
193 each wedge, and the external live and dead loads.
- 194 5. Determine the internal energy dissipation  $\hat{C}v \cos \psi_s$  based on the relative velocities  $v$  across  
195 each slip-line.
- 196 6. Equate external work and internal energy dissipation to determine the live load.

## 197 6. Application to specific problems

198 Having derived a generic solution process, its application will be illustrated through a range of spe-  
199 cific geotechnical problem types: (i) active/passive smooth retaining wall, and (ii) an anchor/trapdoor.  
200 This may be done by deriving the full energy equation for the specific problem and then minimising the  
201 energy by varying the assumed values of wedge angle and dilation angle  $\psi_s$  for each slip-line, to give  
202 the optimal upper bound for the slip-line mechanism. In this note, the optimization of the upper bound  
203 solution is carried out numerically using MATLAB's built in multi-parameter minimisation functions.

### 204 6.1. Active and passive retaining wall

205 Consider a frictionless vertical wall of height  $H$  with horizontal active or passive load  $F$ , and a  
206 surface surcharge  $q$  with a single wedge at angle  $\theta$  to the horizontal and of area  $0.5H^2 / \tan \theta$  as shown  
207 in Fig. 4 together with the hodograph. The slip-line length  $l = H / \sin \theta$ .

208 If the dilation is assumed to be  $\psi_s$ , the wedge moves at a velocity  $v_0$  at an angle  $\theta + \kappa\psi_s$  to the  
209 horizontal and the full energy equation may be expressed as:

$$[39] F\kappa v_0 \cos(\theta + \kappa\psi_s) = \left[ \frac{H^2\gamma}{2 \tan \theta} - \kappa\hat{W}(\psi_s, \theta, l) + \frac{qH}{\tan \theta} \right] \kappa v_0 \sin(\theta + \kappa\psi_s) + \hat{C}(\psi_s, \theta, l)v_0 \cos \psi_s$$

210 This is identical to a conventional linear analysis with the addition of the  $\hat{W}$  term and the replace-  
211 ment of the  $cl$  term by  $\hat{C}$ . To find the optimal upper bound, it is necessary to find  $\max F(\psi_s, \theta)$  for the  
212 active case and  $\min F(\psi_s, \theta)$  for the passive case.

213 The optimization must thus be done in two parameters rather than the one ( $\theta$ ) for the linear prob-  
214 lem and is straightforward to carry out numerically using equation (39). Solutions using the example  
215 parameter sets given in Table 1, are given in Table 2. A single maximum/minimum exists in each case  
216 as shown in Fig. 5 for the fractured rock material. Note that the solutions assume that tensile stresses  
217 are sustainable on the back of the wall for the cohesive-frictional soil.

218 Each model was checked against a simple Rankine lower bound based on the non-linear yield  
219 surface (see Appendix B). Very close matches were found for the approximately linear materials  
220 as expected, and matches within  $\sim 1\%$  for the non-linear materials. The values thus bracket the true  
221 solution very closely. Comparison of the lower bound solutions to the known linear solution for the first  
222 two materials show they are close. Further checks show that they do converge as would be expected as  
223  $m$  is reduced towards 1.0.

224 Fig. 6 shows the feasible range of optimal slip-lines for the passive wall fractured rock case for  
225 different values of  $\theta$  with the optimum value of  $\psi_s$  in each case annotated on each line.

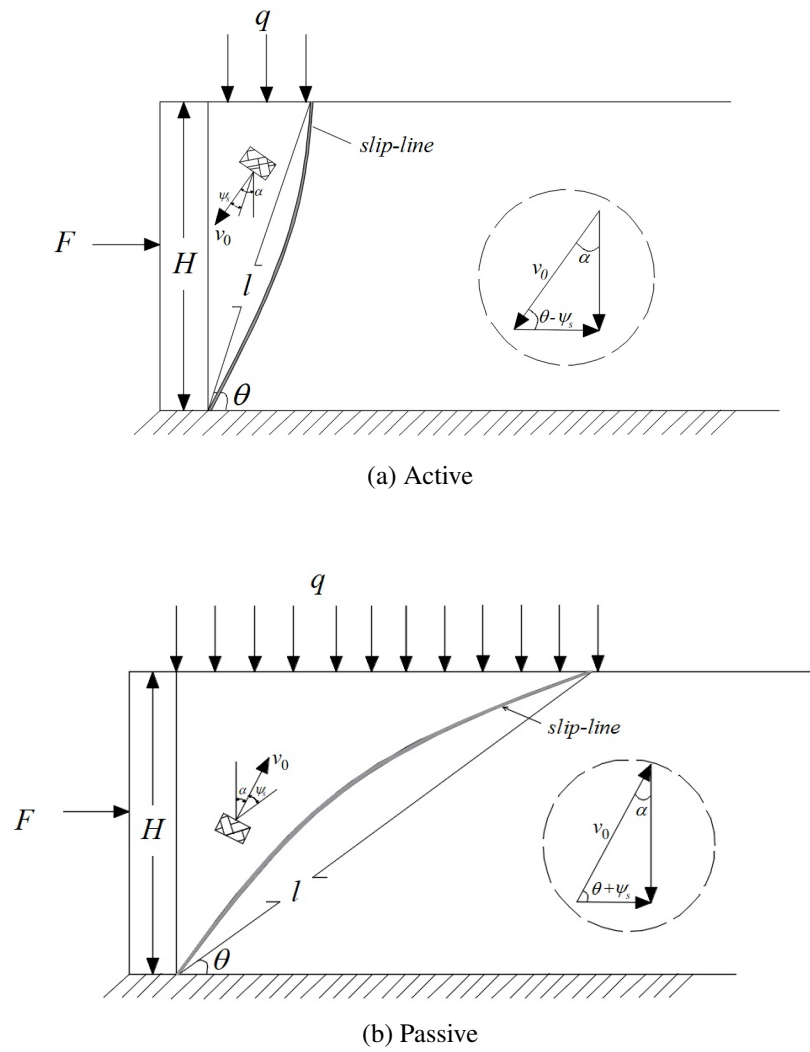
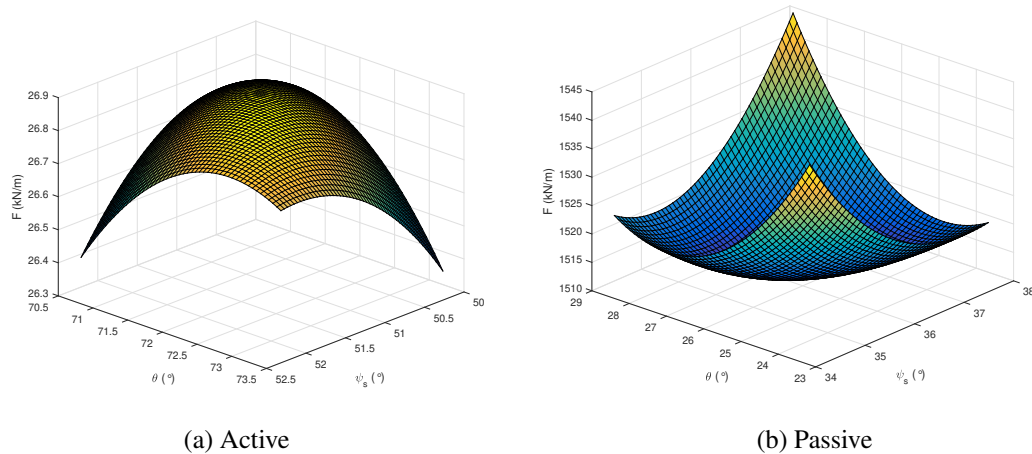


Fig. 4. Failure mechanism analysis for smooth retaining wall with surcharge load.

Failure mode	Parameter set	$\theta_{opt}$	$\psi_{s,opt}$	$n_0$	$F_{upper}$ (kN/m)	$F_{lower}$ (kN/m)	% difference	$F_{linear}$ (kN/m)
Active	CF	$59.96^\circ$	$29.92^\circ$	-8.927	65.2573	65.2573	0.000	65.05983
	LS	$61.46^\circ$	$32.91^\circ$	-7.836	62.8278	62.8278	0.000	62.6451
	DS	$70.90^\circ$	$50.78^\circ$	-1.5629	23.8231	23.8633	0.169	-
	FR	$71.99^\circ$	$51.24^\circ$	-2.791	26.8704	27.3218	1.680	-
Passive	CF	$30.05^\circ$	$29.89^\circ$	102.93	652.3262	652.3262	0.000	654.8205
	LS	$28.56^\circ$	$32.88^\circ$	111.32	717.7809	717.7809	0.000	720.8255
	DS	$22.38^\circ$	$44.39^\circ$	168.81	1349.0075	1347.7533	0.093	-
	FR	$26.18^\circ$	$36.10^\circ$	201.42	1511.5016	1506.3272	0.342	-

**Table 2.** Retaining wall solutions for the case  $q = 5 \text{ kN/m}^2$ , and  $H = 5 \text{ m}$ , using material properties from Table 1. The values of  $F_{linear}$  are computed using conventional Rankine equations for a smooth retaining wall where e.g.  $\sigma'_h = K_a \sigma'_v - K_{ac} c'$ ,  $K_a = \tan^2(\pi/4 - \phi/2)$  and  $K_{ac} = 2\sqrt{K_a}$ , and similarly for the passive case.



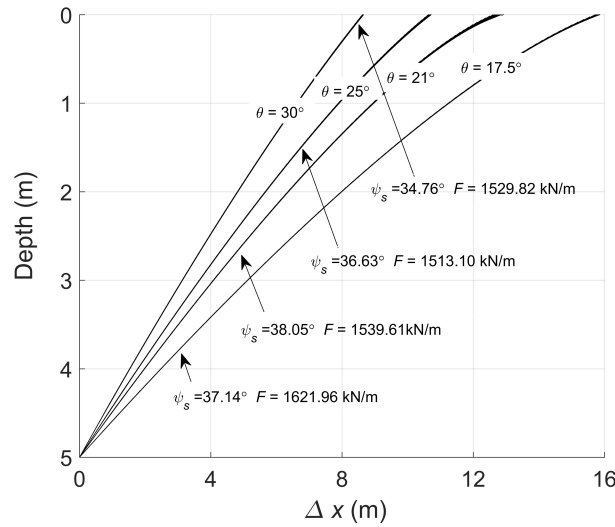
**Fig. 5.** Variation of thrust  $F$  with  $\theta$  and  $\psi_s$  for a smooth retaining wall with surcharge load for the fractured rock material (properties given in Table 1) and  $q = 5 \text{ kN/m}^2$ , and  $H = 5 \text{ m}$ .

226 To put the results into context in comparison with a conventional linear analysis, a simple conservative  
 227 analysis of the fractured rock problem could be carried out using a secant angle of friction across  
 228 a suitable stress range. The non-linear lower bound analysis in Appendix B predicts a horizontal stress  
 229 of  $\sim 450 \text{ kN/m}^2$  at the wall base for the fractured rock parameters. Thus, selecting a range of 0 to 450  
 230  $\text{kN/m}^2$ , which roughly spans the range of stresses expected in the problem, gives a secant angle of  $33^\circ$   
 231 (almost the same as the loose sand material). This gives a passive load of 1017  $\text{kN/m}$  using a simple  
 232 Rankine analysis (using the fractured rock self weight of  $22 \text{ kN/m}^3$ , and the linear angle of shearing  
 233 resistance of  $33^\circ$ ). This is about  $2/3$  of the non-linear result.

## 234 6.2. Anchor/trapdoor (two-wedge)

235 The general approach is now illustrated with a two-wedge analysis for an anchor/trapdoor following  
 236 the geometry shown in Fig. 1 and adopting an anchor width  $B = 5 \text{ m}$ .

237 There are now four variables to be optimised which are  $\theta_1$ ,  $\theta_2$ ,  $\psi_{s1}$  and  $\psi_{s2}$ . The hodograph and  
 238 corresponding equations (1) - (5) remain the same as for the linear case and equation (6) is extended to



**Fig. 6.** Sample set of possible passive slip-lines for fractured rock case for different values of  $\theta$  and with  $\psi_s$  optimized for this value of  $\theta$  (the actual feasible range of values for  $\theta$  ranges between  $0.1^\circ$  and  $84^\circ$ ).

239 the following:

$$[40] \quad Fv_0 = \gamma HBv_0 + qBv_0 + 2qHv_0 / \tan \theta_2 + 2[W_{B2} \cdot v_{02} - \hat{W}(\psi_{s2}, \theta_2, l_2) \cdot v_2 - W_{B1} \cdot v_{01} \\ - \hat{W}(\psi_{s1}, \theta_1, l_1) \cdot v_1 + \hat{C}(\psi_{s2}, \theta_2, l_2)v_2 \cos \psi_{s2} + \hat{C}(\psi_{s1}, \theta_1, l_1)v_1 \cos \psi_{s1}]$$

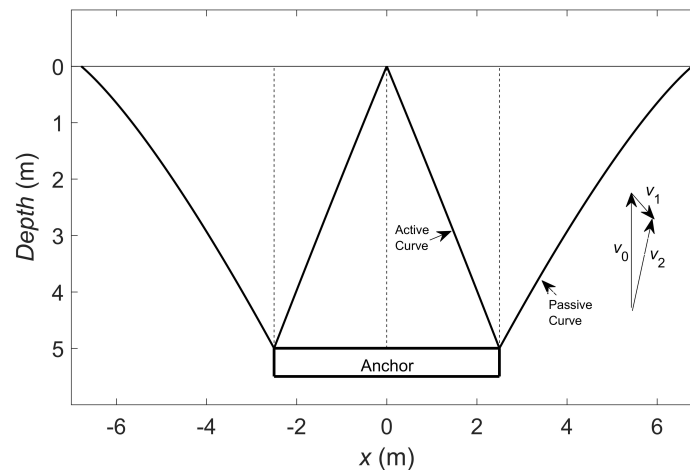
240 where  $l_1 = H / \sin \theta_1$  and  $l_2 = H / \sin \theta_2$ .

Parameter set	$\theta_1$	$\psi_{s1}$	$\theta_2$	$\psi_{s2}$	$F_{upper}$ (kN/m)	$F_{prev}$ (kN/m)
Cohesive frictional	any	any	$60.00^\circ$	$29.99^\circ$	655.28	655.29
Loose sand	any	any	$57.00^\circ$	$32.99^\circ$	675.05	675.05
Dense sand	any	any	$43.78^\circ$	$46.22^\circ$	878.51	879.45
Fractured rock	any	any	$49.45^\circ$	$40.55^\circ$	1188.40	1190.00

**Table 3.** Two-wedge anchor solutions for the case  $q = 5 \text{ kN/m}^2$ , and  $H = 5\text{m}$ , using material properties from Table 1.  $F_{prev}$  are values computed using the approach of Fraldi & Guarracino (2009), for the non-linear soils and using equation (6) for the linear soils.

241 To illustrate the general behaviour, first a solution where all four parameters  $\theta_1$ ,  $\theta_2$ ,  $\psi_{s1}$  and  $\psi_{s2}$  are  
 242 fixed is shown in Fig. 7. Solutions were then investigated where the solution was optimised for all four  
 243 parameters. However, it was found that the optimal solution was always that for which  $\theta_2 + \psi_{s2} = \pi/2$ ,  
 244 independent of the values of  $\theta_1$  and  $\psi_{s1}$  as shown in Table 3 for each of the soil types in Table 1. From  
 245 the hodograph in Fig. 1,  $v_1$  must be zero if  $\theta_2 + \psi_{s2} = \pi/2$  and  $\theta_1 - \psi_{s1}$  can take on any value. This  
 246 agrees well with the linear case (e.g. Murray & Geddes 1987). There are no known non-linear lower  
 247 bound solutions to the anchor problem. A linear lower bound solution was derived by Smith (1998)  
 248 and gives an almost identical answer as the upper bound result in equation (6).

249 An illustration of the variation of the results for fixed  $\theta_1 = 63.43^\circ$  and  $\psi_{s1} = 20.0^\circ$  with  $\theta_2$  and  
 250  $\psi_{s2}$  allowed to vary, are shown graphically in Fig. 8 for the fractured rock case and shows there is one  
 251 minimum solution. In Fig. 8 the magenta dot represents the solution for  $\theta_2 = 50^\circ$  and  $\psi_{s2} = 25^\circ$  for  
 252 fixed  $\theta_1 = 63.43^\circ$  and  $\psi_{s1} = 20.0^\circ$  corresponding to the mechanism in Fig. 7.

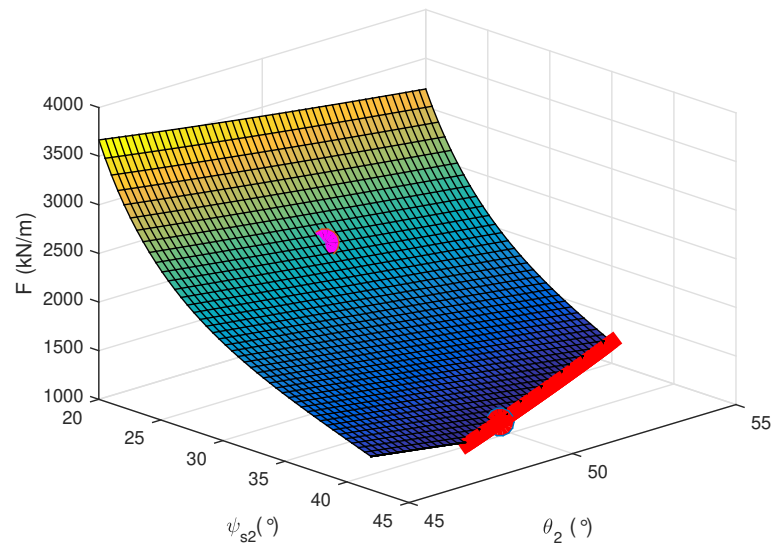


**Fig. 7.** Example results of non-linear analysis showing mechanism and hodograph for two-wedge anchor embedded in fractured rock (properties given in Table 1) with  $q = 5 \text{ kN/m}^2$  and  $H = 5 \text{ m}$ . The active curves were selected to meet at the surface on the symmetry line and use a specified dilation angle:  $\theta_1 = 63.43^\circ$ ,  $\psi_{s1} = 20^\circ$ . The passive curve used values  $\theta_2 = 50^\circ$  and  $\psi_{s2} = 25^\circ$ . The predicted upper bound load  $F = 2323.0 \text{ kN/m}^2$ .

## 253 7. Discussion

254 The above examples clearly illustrate how the method may be applied to a general multiple-wedge  
 255 rigid-block analysis, giving it a very broad applicability, and has verified it against lower bound and  
 256 other solutions in the literature. Essentially the method replicates the nature of a conventional linear  
 257 soil analysis, but doubles the number of variables to be optimised (slip-line orientation and equivalent  
 258 dilation on the slip-line), in cases where optimization is required. The examples shown which were  
 259 of a smooth retaining wall and anchor uplift display similar characteristics as for their linear coun-  
 260 terparts. The optimal smooth retaining wall single line upper bound solution is very close to the true  
 261 solution, and the optimal two-wedge anchor solution reduces to a single wedge solution.

262 The approach presented in this paper determined the optimal upper bound by full application of the  
 263 conventional energy minimisation approach. This is in contrast to some previous authors *e.g.* Fraldi &  
 264 Guarracino (2009, 2010, 2011), Yang & Huang (2011), Zhang & Yang (2018)) who adopted a partial  
 265 optimization of energy minimisation to obtain a variational form of the slip-line but then used a stress  
 266 boundary condition at the soil surface to complete the solution. This assumed that the slip-line had to  
 267 meet the (horizontal) surface at an angle consistent with a simple active or passive Rankine stress state  
 268 at the surface. Solutions invoking such a boundary condition are still valid upper bounds, but were  
 269 found to give collapse loads approx 0.3% higher than the full minimization approach as used in this  
 270 paper as shown in Table 3. While this boundary condition assumption may be valid for the smooth  
 271 retaining wall problem, it does not hold universally. In reality it is expected that the anchor/trapdoor



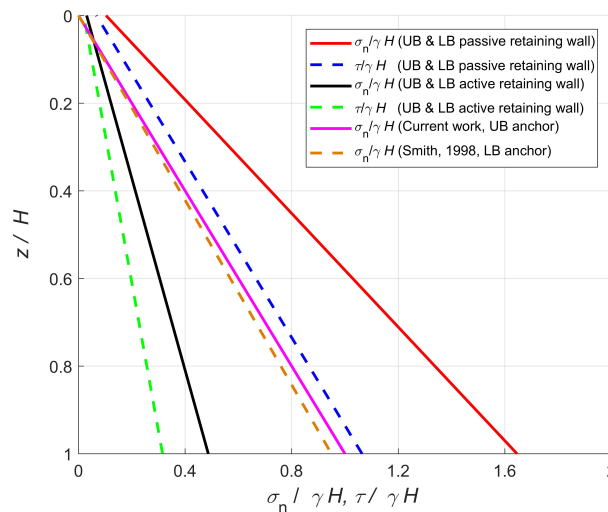
**Fig. 8.** Variation of limit load  $F$  with fixed values of  $\theta_1 = 63.43^\circ$  and  $\psi_{s1} = 20.0^\circ$  for the fractured rock case (properties given in Table 1) and  $q = 5 \text{ kN/m}^2$ ,  $H = 5\text{m}$  and  $B = 5\text{m}$ . The red line represents the kinematic limits of feasibility for the problem (equation 5). The optimal solution (red dot) lies on this line. At lower values of  $\theta_2$ , the solution is limited by feasibility of the non-linear solution. The magenta dot represents the solution depicted in Fig. 7 for  $\theta_2 = 50^\circ$  and  $\psi_{s2} = 25^\circ$ .

272 stress field would involve a singularity at the point where the slip-line meets the surface with rotations  
 273 of the principle stress directions around this point as demonstrated by Smith (1998) for the linear soil  
 274 case.

275 The solution also assumed that the shape of the non-linear slip-line could be described by a function  
 276  $y = f(x)$ . This assumption gives a relatively simple solution. There may be scope to achieve higher  
 277 degrees of freedom in the solutions by adopting a parametric curve  $f_p(x, y) = 0$ , however this is  
 278 beyond the scope of the present work.

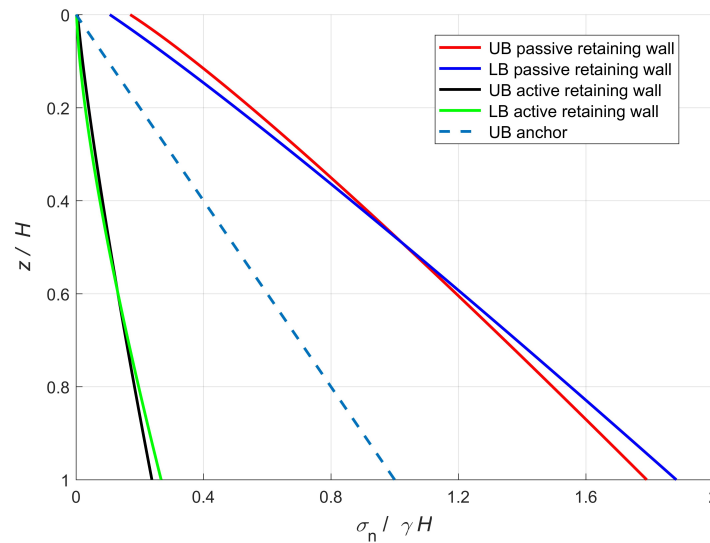
279 One intriguing aspect of the analysis as pointed out by Baker & Frydman (1983) and Chen (1975)  
 280 is that the upper bound solution not only identifies the slip-line geometry, but also part of the stress  
 281 state at every point along the line, using equation (14) and equation (8), since each point has a unique  
 282 gradient. It is thus possible to plot the shear stress and/or normal stress on the line with depth as  
 283 shown in Fig. 9 and Fig. 10. For the active and passive walls, these values match reasonably closely  
 284 to the values predicted by the lower bound approach (as would be expected). Note that the plotted  
 285 lower bound values are those corresponding to the yield condition predicted by the lower bound at the  
 286 relevant depth.

287 For the anchor, the normal stress follows a value  $\sigma_n \approx 1.0\gamma z$ . This is consistent with the order of  
 288 magnitude of values found in the stress rotation model of Smith (1998) for an anchor in a linear soil.  
 289 While the optimal mechanism is expected to involve multiple slip-lines, the single slip-line solution is  
 290 expected to be close to optimal, in a similar way to the linear soil case, and the corresponding stress  
 291 state is expected to be close to the true solution result, but not exact. This example clearly shows that  
 292 the predicted stresses are of the order expected and may be valuable in identifying the nature of lower  
 293 bound solutions, or stresses acting on structures. Further work, however, is required in this area.



**Fig. 9.** Predicted upper bound (UB) and lower bound (LB) normalised normal and shear stresses for anchor ( $q = 0\text{kN/m}^2$  and  $H = 5\text{m}$ ) and active and passive retaining wall cases ( $q = 5\text{kN/m}^2$  and  $H = 5\text{m}$ ): loose sand case. Wall UB and LB solutions are coincident.

294 Finally while the work here has been presented in the context of a classical hand calculation with  
 295 simple optimization of a few variables, it should be possible to incorporate the approach into the much  
 296 more general computational rigid block analysis approach Discontinuity Layout Optimization (Smith  
 297 & Gilbert 2007) to produce solutions of high accuracy and to also extend the approach to cover rota-  
 298 tional mechanisms in addition to translational mechanisms.



**Fig. 10.** Predicted upper bound (UB) and lower bound (LB) normalised normal stresses for anchor ( $q = 0\text{kN/m}^2$  and  $H = 5\text{m}$ ) and active and passive retaining wall cases ( $q = 5\text{kN/m}^2$  and  $H = 5\text{m}$ ): fractured rock case (NB no lower bound anchor solution is available for this case).

## 299 8. Conclusions

- 300 1. A fully general variational approach for the upper bound analysis of geotechnical collapse mech-  
301 anisms in non-linear soils has been presented. The analysis follows the form of the classic upper  
302 bound multi-wedge analysis utilised for linear soils. It is based on the use of closed form equa-  
303 tions and only requires the numerical solution of a single implicit equation in one variable.
- 304 2. The approach presented has significantly extended a methodology developed previously for the  
305 special case of deep tunnels and the anchor/trapdoor problem, and used full energy optimisation  
306 of the solution, rather than adopting a special boundary condition.
- 307 3. Application of the method to the analysis of active and passive earth pressures acting on a smooth  
308 retaining wall, demonstrated that the single wedge solutions obtained gave results very close to  
309 a simple lower bound analysis and thus established a close bracket to the true plastic solution for  
310 this case.
- 311 4. A further example addressing the anchor uplift problem demonstrated the solution process for  
312 multi-wedges and showed that the solution behaviour follows a similar pattern to that for linear  
313 soils. More accurate solutions for this problem were obtained compared to previous work in the  
314 literature.
- 315 5. Due to the non-linearity of the yield surface, for the simple types of solution utilised here, it  
316 is possible to determine the normal and shear stresses at any point on the slip-line. This is not  
317 normally available for upper bound problems. The validity of these stresses has been investigated  
318 and show strong consistency with related lower bound solutions, but further work is required in  
319 this area to establish the validity of the values generated.

320 **References**

- 321 Anyaegbunam, A. J. (2013), 'Nonlinear power-type failure laws for geomaterials: Synthesis from  
322 triaxial data, properties, and applications', *International Journal of Geomechanics* **15**(1), 04014036.
- 323 Baker, R. (2004), 'Nonlinear mohr envelopes based on triaxial data', *Journal of Geotechnical and  
324 Geoenvironmental Engineering* **130**(5), 498–506.
- 325 Baker, R. & Frydman, S. (1983), 'Upper bound limit analysis of soil with non-linear failure criterion',  
326 *Soils and Foundations* **23**(4), 34–42.
- 327 Bolton, M. (1986), 'The strength and dilatancy of sands', *Geotechnique* **36**(1), 65–78.
- 328 Chen, W.-F. (1975), *Limit Analysis and Soil Plasticity*, Vol. 7 of *Developments in Geotechnical Engi-  
329 neering*, Elsevier Scientific Publishing Company.
- 330 Chen, W. & Liu, X. (1990), *Limit Analysis in Soil Mechanics*, Vol. 41 of *Developments in Geotechnical  
331 Engineering*, Elsevier.
- 332 De Mello, V. F. (1977), 'Reflections on design decisions of practical significance to embankment  
333 dams'.
- 334 Fraldi, M. & Guarracino, F. (2009), 'Limit analysis of collapse mechanisms in cavities and tunnels  
335 according to the hoek–brown failure criterion', *International Journal of Rock Mechanics and Mining  
336 Sciences* **46**(4), 665–673.
- 337 Fraldi, M. & Guarracino, F. (2010), 'Analytical solutions for collapse mechanisms in tunnels with  
338 arbitrary cross sections', *International Journal of Solids and Structures* **47**(2), 216–223.
- 339 Fraldi, M. & Guarracino, F. (2011), 'Evaluation of impending collapse in circular tunnels by analytical  
340 and numerical approaches', *Tunnelling and Underground Space Technology* **26**(4), 507–516.
- 341 Hambleton, J. & Sloan, S. (2013), 'A perturbation method for optimization of rigid block mechanisms  
342 in the kinematic method of limit analysis', *Computers and Geotechnics* **48**, 260–271.
- 343 Hoek, E. & Brown, E. T. (1997), 'Practical estimates of rock mass strength', *International Journal of  
344 Rock Mechanics and Mining Sciences* **34**(8), 1165–1186.
- 345 Lefebvre, G. (1981), 'Fourth canadian geotechnical colloquium: Strength and slope stability in cana-  
346 dian soft clay deposits', *Canadian Geotechnical Journal* **18**(3), 420–442.
- 347 Mohammadi, M. & Tavakoli, H. (2015), 'Comparing the generalized hoek-brown and mohr-coulomb  
348 failure criteria for stress analysis on the rocks failure plane', *Geomechanics and Engineering  
349* **9**(1), 115–124.
- 350 Murray, E. & Geddes, J. D. (1987), 'Uplift of anchor plates in sand', *Journal of Geotechnical Engi-  
351 neering* **113**(3), 202–215.
- 352 Smith, C. C. (1998), 'Limit loads for an anchor/trapdoor embedded in an associative coulomb soil',  
353 *International Journal for Numerical and Analytical Methods in Geomechanics* **22**(11), 855–865.
- 354 Smith, C. & Gilbert, M. (2007), 'Application of discontinuity layout optimization to plane plasticity  
355 problems', *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering  
356 Sciences* **463**(2086), 2461–2484.

- 357 Soon, S.-C. & Drescher, A. (2007), 'Nonlinear failure criterion and passive thrust on retaining walls',  
358 *International Journal of Geomechanics* **7**(4), 318–322.
- 359 Ukritchon, B. & Keawsawasvong, S. (2018), 'Three-dimensional lower bound finite element limit anal-  
360 ysis of hoek-brown material using semidefinite programming', *Computers and Geotechnics* **104**, 248  
361 – 270.
- 362 Yang, X. & Huang, F. (2011), 'Collapse mechanism of shallow tunnel based on nonlinear hoek–brown  
363 failure criterion', *Tunnelling and Underground Space Technology* **26**(6), 686–691.
- 364 Yang, X. & Long, Z. (2015), 'Roof collapse of shallow tunnels with limit analysis method', *Journal of*  
365 *Central South University* **22**, 1929–1936.
- 366 Zhang, R. & Yang, X. (2018), 'Limit analysis of active and passive mechanisms of shallow tun-  
367 nels in nonassociative soil with changing water table', *International Journal of Geomechanics*  
368 **18**(7), 04018063.
- 369 Zhang, X. & Chen, W. (1987), 'Stability analysis of slopes with general nonlinear failure criterion',  
370 *International Journal for Numerical and Analytical Methods in Geomechanics* **11**(1), 33–50.

### 371 **Appendix A: Non-linear approximation of a linear yield function**

372 For validation purposes and also in order to access the stress values along a slip-line, it is useful to  
 373 use the non-linear model described to represent a linear system by adopting a value of  $m$  very close to  
 374 1.0. The accuracy of this approximation can be calculated as follows:

375 Taking  $a = 0$  for simplicity, the following can be written

$$[41] \quad \tau = c_0(\sigma_n/\sigma_t)^{1/m}$$

376 Let this equation and the linear form  $\tau = c_{0l}(\sigma_n/\sigma_t)$  intersect at the origin and when  $\sigma_n = \sigma_{n1}$ .  
 377 This defines the range of the approximation. Thus

$$[42] \quad c_{0l} = c_0(\sigma_{n1}/\sigma_t)^{(1-m)/m}$$

378 The difference between the linear and non-linear curves at any value of  $\sigma_n$ , as a proportion of the  
 379 intersection value at  $\sigma_{n1}$  is given by:

$$[43] \quad \frac{c_0(\sigma_n/\sigma_t)^{1/m} - c_{0l}(\sigma_n/\sigma_t)}{c_0(\sigma_{n1}/\sigma_t)^{1/m}}$$

380 A plot of this function shows that this has a maximum at around  $0.4\bar{\sigma}$  and is approximately equal  
 381 to  $0.37 \times (m - 1)$  for small  $m - 1$ .

### 382 **Appendix B: Lower bound solution for a smooth retaining wall**

383 For a smooth wall with vertical soil/wall interface, the same simple stress state configuration may  
 384 be used for a non-linear soil as for a linear soil, namely the assumption that principal stresses are  
 385 horizontal and vertical. The vertical stresses may thus be predicted by the following simple equation:

$$[44] \quad \sigma_v = q + \gamma z$$

386 where  $q$  is the surface surcharge and  $z$  is the depth below the surface. Hence drawing the largest or  
 387 smallest Mohr's circle through this point that touches the non-linear yield surface will determine the  
 388 passive and active lateral earth pressures respectively. Depending on the nature of the yield surface, the  
 389 circle may be limited by a tangent to the main curve, or by the apex of the yield surface when  $\tau = 0$ .  
 390 Given that:

$$[45] \quad \tau = c_0(a + \sigma_n/\sigma_t)^{1/m}$$

391 the lowest value of  $\sigma_n$  is when  $\tau = 0$ :

$$[46] \quad \sigma_{n,min} = -a\sigma_t$$

392 At any point the gradient is given by:

$$[47] \quad \tan \psi_t = \frac{d\tau}{d\sigma_n} = \frac{c_0}{m\sigma_t} (a + \sigma_n/\sigma_t)^{(1-m)/m}$$

393 Referring to Fig. 11,

$$[48] \quad s - \sigma_n = \tau \tan \psi_t = \frac{c_0^2}{m\sigma_t} (a + \sigma_n/\sigma_t)^{(2-m)/m}$$

$$[49] \quad s = \sigma_n + \frac{c_0^2}{m\sigma_t} (a + \sigma_n/\sigma_t)^{(2-m)/m}$$

$$[50] \quad \frac{ds}{d\sigma_n} = 1 + \frac{(2-m)c_0^2}{m^2\sigma_t^2} (a + \sigma_n/\sigma_t)^{(2-2m)/m}$$

394 This is always positive if  $m < 2$  and indicates a switch of sign at:

$$[51] \quad (a + \sigma_n/\sigma_t) = \left( \frac{(m-2)c_0^2}{m^2\sigma_t^2} \right)^{\frac{m}{2(m-1)}}$$

$$[52] \quad \sigma_{n0} = \sigma_t \left( -a + \left( \frac{(m-2)c_0^2}{m^2\sigma_t^2} \right)^{\frac{m}{2(m-1)}} \right)$$

395 Which corresponds to a value of  $s$  at:

$$[53] \quad s_0 = \sigma_{n0} + \frac{c_0^2}{m\sigma_t} \left( \frac{(m-2)c_0^2}{m^2\sigma_t^2} \right)^{\frac{2-m}{2(m-1)}}$$

396 It is thus necessary to work with Mohr's circles from  $s = 0$  to  $s_0$  that touch  $\sigma_{n,min}$ . Above  $s_0$  the  
397 circles are tangential to the yield surface (as shown in Fig. 11) and of radius  $t$  given as follows:

$$[54] \quad t = \sqrt{\tau^2 + (s - \sigma)^2} = \tau \sqrt{1 + \tan^2 \phi} = \tau \sec^2 \phi$$

398 For active conditions

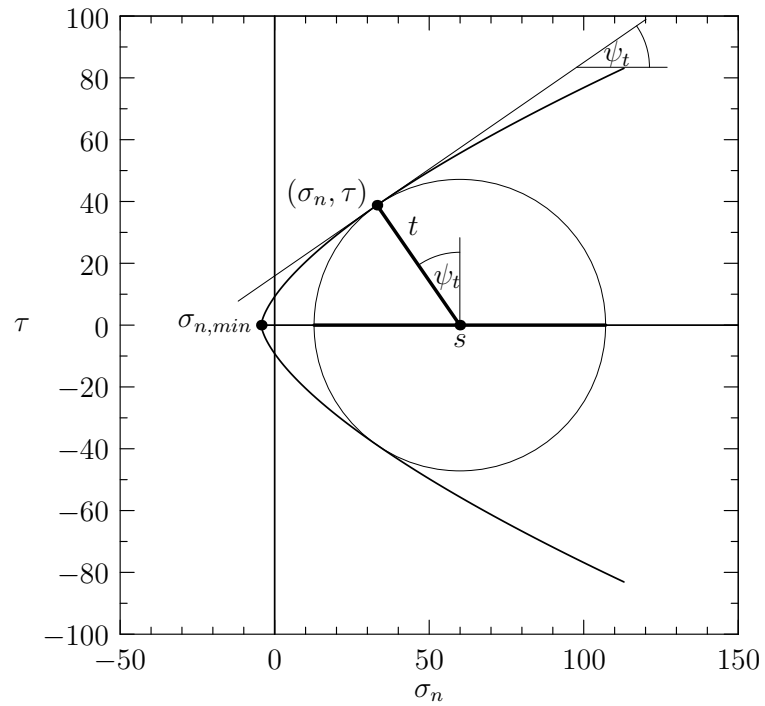
$$[55] \quad \sigma_v = s + t$$

$$[56] \quad \sigma_h = s - t$$

399 For passive conditions

$$[57] \quad \sigma_v = s - t$$

$$[58] \quad \sigma_h = s + t$$



**Fig. 11.** Mohr's circle for non-linear yield surface,  $c_0 = 5$ ,  $a = 2.5$ ,  $\sigma_t = 1.0/\tan(30)$ ,  $m = 1.5$ .