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Performance study of the multiwavelet discontinuous Galerkin approach for solving the Green-Naghdi equations

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Abstract

This paper presents a multiresolution discontinuous Galerkin scheme for the adaptive solution of Boussinesq-type equations. The model combines multiwavelet-based grid adaptation with a discontinuous Galerkin (DG) solver based on the system of fully nonlinear and weakly dispersive Green-Naghdi (GN) equations. The key feature of the adaptation procedure is to conduct a multiresolution analysis using multiwavelets on a hierarchy of nested grids to improve the efficiency of the reference DG scheme on a uniform grid by computing on a locally refined adapted grid. This way the local resolution level will be determined by manipulating multiwavelet coefficients controlled by a single user-defined threshold value. The proposed adaptive multiwavelet discontinuous Galerkin solver for GN equations (MWDG-GN) is assessed using several benchmark problems related to wave propagation and transformation in nearshore areas. The numerical results demonstrate that the proposed scheme retains the accuracy of the reference scheme, while significantly reducing the computational cost.

Keywords: Multiwavelets; Discontinuous Galerkin; Boussinesq-type equations; Green-Naghdi equations; Multiresolution analysis; Nearshore wave processes

1- Introduction

The Boussinesq-type (BT) equations have been used as an alternative to the free-surface Euler equations for modelling of propagation and transformations of waves in nearshore areas. These types

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27 of applications usually correspond to the shallow water regime, where the horizontal length scale λ is
28 large compared to the water depth scale h_0 , so that the shallowness parameter is $\mu = h_0^2/\lambda^2 \ll 1$.
29 Neglecting all the terms of order $O(\mu)$ from the Euler equations leads to the so-called Nonlinear Shallow
30 Water (NSW) equations, whereas keeping them results in the simplest form of BT equations¹. While
31 this simple BT model is, in essence, weakly dispersive and only valid for long waves with $kh_0 < 0.75$
32 (k being the wavenumber), better dispersive behaviour and more accurate BT models can be achieved
33 by incorporation of more terms and related manipulations². The nonlinearity parameter is another
34 related identifier, which is defined as the ratio of the wave amplitude scale to the water depth scale, $\epsilon =$
35 a/h_0 . Most of the BT equations impose a smallness amplitude assumption as $\epsilon = O(\mu^2)$, which is too
36 restrictive for many applications in nearshore areas. Removing this assumption (i.e. let $\epsilon = O(1)$) while
37 keeping all the $O(\mu)$ terms, gives the so-called Green-Naghdi (GN) equations³⁻⁵. The GN equations
38 share the same characteristics of other BT models. However, they allow relative ease in computational
39 implementation, which makes them very favourable in coastal engineering applications^{6,7}.

40 To numerically solve various BT wave models, different approaches have been used based on
41 Finite Difference (FD), Finite Volume (FV), Finite Element (FE) and spectral element^{2,8}. The
42 Discontinuous Galerkin (DG) method is a more modern alternative for these approaches, which exploits
43 the properties of the FV and FE methods. The DG method thereby provides faster convergence rates
44 and better quality predictions on coarse meshes as compared to an equally accurate FV approach⁹⁻¹¹.
45 DG methods are becoming increasingly popular in solving BT equations^{7,11-19}. However, the runtime
46 cost of DG methods is high, given their demands for storage and evolution of local degrees of freedom
47 within each computational cell and their restrictive CFL condition when applied with explicit Runge-
48 Kutta (RK) time stepping. These costs would even be higher when modelling wave propagation and
49 transformation in coastal areas, where the multitude of spatial and temporal scales further increase the
50 wave feature and complexity.

51 Classical Adaptive Mesh Refinement (AMR) techniques were initially used in an attempt to reduce
52 fine resolution costs by adapting the mesh resolution²⁰⁻²². However, it turned out that classical AMR
53 approaches bring about new issues owing to the inherently decoupled nature between the mesh and the

54 numerical solution. In order to control grid refinement/coarsening, AMR methods usually either use
55 Richardson extrapolation²³ or heuristic criteria²⁴, which gives no information about the errors related to
56 the adaptation process, making the effectiveness of an AMR approach subject to a-posteriori error
57 estimates²⁵. Moreover, most of the available AMR developments lack a general adaptivity sensor, so
58 that they either need separate criteria for refinement/coarsening^{26,27} or problem specific criteria^{28,29} or
59 are reported to be highly dependent on the type of refinement criteria³⁰. Also, deploying a classical
60 AMR method dictates extra corrections in the numerical scheme to address the loss of well-
61 balancedness property for the case of the NSW equations^{24,31–33}.

62 Multiscale methods based on the Multiresolution Analysis (MRA) of wavelets provide an
63 alternative that can preserve the quality of numerical methods on adaptive meshes^{34–38}. Theoretical
64 analyses show that only an error threshold value is needed with this category of adaptive solvers in
65 order to bound the accumulated errors and preserve the accuracy of the reference uniform solver at the
66 finest resolution grid^{39–41}. Initially, this concept has been particularly verified with FV solvers, which
67 later appeared to give marginal computational savings and introduce unacceptably large errors for low-
68 order schemes. Therefore, the combination of DG methods with Multiwavelets⁴² (MWs) has emerged
69 recently. MWs preserve locality in line with the local and accurate structure of the DG method, which
70 enables greater compression rates alongside small computational stencil compared to wavelets.
71 Compared to the classic AMR methods, multiscale-based methods have been shown to exhibit larger
72 compression rates and more gains in CPU time^{43,44}.

73 The MW-based DG solvers have been successfully used for adaptive modelling of Euler^{41,45,46} and
74 NSW equations^{47–49}, suggesting that just by the use of a single threshold value, the adaptive MWDG
75 solver keeps the accuracy of the adaptive solution in the same order as the accuracy of the uniform
76 solution, while reducing the computational cost.

77 Among the few existing works on wavelet-based grid adaptation for solving BT models, Smith et
78 al.⁵⁰ extended the Haar Wavelet-Finite Volume (HWFV) model of Müller⁵¹ to the case of weakly
79 nonlinear, weakly dispersive model of Madsen and Sørensen⁵². Their analysis reported good

80 performance of the wavelet adaptation process, but also reported on instabilities in areas with fine
 81 resolutions, linking them to the treatment of the third spatial derivative in the BT equations.

82 This work, therefore, presents a first exploration of MW-based grid adaptation combined with DG
 83 discretization for modelling the GN equations (denoted hereafter by MWDG-GN). An existing uniform
 84 mesh DG solver for the GN equations (DG-GN)¹¹ is extended to adaptive form, following the MWDG
 85 method introduced in Kesserwani et al.⁴⁷ applied to the NSW equations (MWDG-NSW). The behaviour
 86 of the adaptive MWDG-GN solver in modelling different levels of nonlinearity and dispersion related
 87 to wave propagation is studied from both accuracy and efficiency point of view. The suitable range for
 88 the threshold parameter to reach the same quality of the solutions as the uniform DG-GN solver is also
 89 identified. The rest of the paper is organized as follows: In Section 2 we briefly recall the governing
 90 GN equations and in Section 3 the main ingredients of the uniform DG-GN solver are introduced.
 91 Section 4 describes the main ideas behind the MRA, and Section 5 explains the details of the MWDG-
 92 GN solver. Section 6 presents a series of numerical experiments that demonstrate the efficiency of the
 93 MWDG-GN solver. A summary of conclusions is presented in Section 7.

94 2- The Green-Naghdi (GN) equations

95 The one-dimensional (1D) GN system can be cast as the conventional NSW equations combined
 96 with source terms accounting for the dispersive effects, in the following conservative form⁵³:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, z) = \mathbf{S}_b(\mathbf{U}, z) - \mathbf{D}(\mathbf{U}, z) \quad (1)$$

$$\mathbf{U} = \begin{bmatrix} h \\ q \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}, z) = \begin{bmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}, \quad \mathbf{S}_b(\mathbf{U}, z) = \begin{bmatrix} 0 \\ -gh\partial_x z \end{bmatrix}, \quad \mathbf{D}(\mathbf{U}, z) = \begin{bmatrix} 0 \\ \mathcal{D}_c \end{bmatrix} \quad (2)$$

97 where \mathbf{U} is the vector of flow variables i.e. water depth h and discharge $q = hu$, \mathbf{F} represents the fluxes,
 98 z is the topography, g refers to the gravitational constant and \mathbf{S}_b is the topography source term. In this
 99 formulation, \mathbf{D} denotes the dispersive source term, with \mathcal{D}_c defined as

$$\mathcal{D}_c = -\frac{1}{\alpha}gh\partial_x\zeta + \quad (3)$$

$$[1 + \alpha\mathbb{T}[h_b]]^{-1} \left[\frac{1}{\alpha}gh\partial_x\zeta + h(Q_1(u) + gQ_2(\zeta)) + gQ_3 \left([1 + \alpha\mathbb{T}[h_b]]^{-1}(gh\partial_x\zeta) \right) \right]$$

101 where $u(x, t)$ is the horizontal velocity, h_b corresponds to the undisturbed state, $h(x, t) = \zeta(x, t) + h_b$
 102 is the water height, $\zeta(x, t)$ stands for the free-surface elevation and $z(x)$ is the variation of the bottom
 103 with respect to the rest state (Fig. 1), and α is an optimization parameter⁵³. The differential operators Q_1
 104 and Q_2 are expressed as follows:

$$Q_1(u) = 2h\partial_x h(\partial_x u)^2 + \frac{4}{3}h^2\partial_x u(\partial_x^2 u) + h\partial_x z(\partial_x u)^2 + uh\partial_x u(\partial_x^2 z) + u^2\partial_x \zeta(\partial_x^2 z) + \frac{h}{2}u^2(\partial_x^3 z) \quad (4)$$

$$Q_2(\zeta) = -\left(\partial_x \zeta \partial_x z + \frac{h}{2} \partial_x^2 z \right) \partial_x \zeta \quad (5)$$

105 For a given scalar function w , the second-order differential operator \mathbb{T} is defined as:

$$\mathbb{T}[h_b](w) = -\frac{h_b^3}{3} \partial_x^2 \left(\frac{w}{h_b} \right) - h_b^2 \partial_x h_b \partial_x \left(\frac{w}{h_b} \right) \quad (6)$$

106 and Q_3 admits the simplified notation:

$$Q_3(w) = \frac{1}{6} \partial_x (h^2 - h_b^2) \partial_x w + \frac{h^2 - h_b^2}{3} \partial_x^2 w - \frac{1}{6} \partial_x^2 (h^2 - h_b^2) w \quad (7)$$

107 3- The uniform DG-GN model

108 The 1D computational domain $\Omega = [x_{\min}, x_{\max}]$ is divided into N uniform and non-overlapping
 109 cells $\{I_i\}_{i=1, \dots, N}$ with cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ having size $\Delta x = x_{i+1/2} - x_{i-1/2}$ and centre $\Delta x =$
 110 $(x_{i+1/2} + x_{i-1/2})/2$. Eq. (1) is approximated with a modal DG discretization with polynomials of
 111 degree p . Let V_p be

$$V_p = \{v \in L^2(\Omega) : v|_{I_i} \in \Pi_p(I_i), i = 1, \dots, N\} \quad (8)$$

112 where $\Pi_p(I_i)$ is the space of polynomials of degree at most p on I_i . Here, Legendre polynomials will
 113 be used, define as (e.g. for $0 \leq l \leq 3$):

$$P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{1}{2}(3\xi^2 - 1), \quad P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi) \quad (\xi \in [-1,1]) \quad (9)$$

114 which are compactly-supported on $[-1,1]$, inherently discontinuous, and orthogonal for the L^2 -norm
 115 based on the following inner product:

$$\langle f, g \rangle_\Omega = \int_\Omega f(\xi)g(\xi)d\xi \quad (10)$$

116 The L^2 -orthonormal basis $\varphi_l(\xi)$ can be defined by normalizing $P_l(\xi)$ for the L^2 -norm such that
 117 $\langle \varphi_l, \varphi_{l'} \rangle_{L^2(\Omega)} = \delta_{ll'}$, where δ is the Kronecker delta. Since the reference domain spans $[-1, 1]$, the
 118 orthonormal basis is⁵⁴:

$$\varphi_l(\xi) = \sqrt{\frac{2l+1}{2}} P_l(\xi) \quad (\xi \in [-1,1]) \quad (11)$$

119 Accordingly, two sets of basis functions will be defined over I_i : the primal basis $\Phi_i =$
 120 $\{\varphi_{i,0}, \varphi_{i,1}, \dots, \varphi_{i,p}\}$, and the dual basis $\tilde{\Phi}_i = \{\tilde{\varphi}_{i,0}, \tilde{\varphi}_{i,1}, \dots, \tilde{\varphi}_{i,p}\}$ ⁴¹:

$$\varphi_{i,l}(x) = \sqrt{2}\varphi_l(\xi) \quad \text{and} \quad \tilde{\varphi}_{i,l}(x) = \frac{\varphi_{i,l}(x)}{\Delta x} \quad (12)$$

121 The primal and dual basis are chosen so that they are biorthogonal

$$\langle \varphi_{i,l}, \tilde{\varphi}_{i',l'} \rangle_{L^2(\Omega)} = \delta_{i,l} \delta_{i',l'} \quad (13)$$

122 To get an FE local weak formulation, Eq. (1) is multiplied by a test function selected as the dual basis
 123 $\tilde{\varphi}_{i,l}$, then integrated by parts over the control volume I_i to give:

$$\begin{aligned}
& \int_{I_i} \partial_t \mathbf{U}_h(x, t) \tilde{\varphi}_{i,l}(x) dx - \int_{I_i} \mathbf{F}(\mathbf{U}_h(x, t)) \partial_x \tilde{\varphi}_{i,l}(x) dx \\
& + \left[\tilde{\mathbf{F}}(\mathbf{U}_h(x_{i+1/2}, t)) \tilde{\varphi}_{i,l}(x_{i+1/2}) - \tilde{\mathbf{F}}(\mathbf{U}_h(x_{i-1/2}, t)) \tilde{\varphi}_{i,l}(x_{i-1/2}) \right] \quad (14) \\
& = \int_{I_i} \mathbf{S}_b(\mathbf{U}_h(x, t), z_h) \tilde{\varphi}_{i,l}(x) dx - \int_{I_i} \mathbf{D}_h(\mathbf{U}_h(x, t), z_h) \tilde{\varphi}_{i,l}(x) dx
\end{aligned}$$

124 in which, \mathbf{U}_h , \mathbf{D}_h and z_h are local approximations of \mathbf{U} , \mathbf{D} and z , which are also spanned by FE
125 expansion coefficients, and $\tilde{\mathbf{F}}$ is a nonlinear numerical flux function based on an HLL approximate
126 Riemann solver⁵⁵. On I_i the local solution can be expanded using the primal basis $\varphi_{i,l}$ as:

$$\mathbf{U}_h(x, t)|_{I_i} = \sum_{l=0}^p \mathbf{U}_{i,l}(t) \varphi_{i,l}(x) \quad (x \in I_i) \quad (15)$$

$$\mathbf{D}_h(x, t)|_{I_i} = \sum_{l=0}^p \mathbf{D}_{i,l}(t) \varphi_{i,l}(x) \quad (x \in I_i) \quad (16)$$

$$z_h(x, t)|_{I_i} = \sum_{l=0}^p z_{i,l}(t) \varphi_{i,l}(x) \quad (x \in I_i) \quad (17)$$

127 where $\mathbf{U}_{i,l}$, $\mathbf{D}_{i,l}$ and $z_{i,l}$ are time-dependent expansion coefficients. These initial states are obtained by
128 projecting a given initial condition onto the dual basis. The local semi-discrete DG formulation for each
129 l -th coefficient of polynomial accuracy over a cell I_i reads:

$$\begin{aligned}
\partial_t (\mathbf{U}_{i,l}(t)) = & - \frac{\sqrt{2l+1}}{\Delta x} \left\{ \left[\tilde{\mathbf{F}}_{i+\frac{1}{2}} - (-1)^l \tilde{\mathbf{F}}_{i-\frac{1}{2}} \right] - \int_{-1}^{+1} \mathbf{F} \left(\mathbf{U}_h \left(x_i + \xi \frac{\Delta x}{2}, t \right) \right) \left(\frac{\partial [P_l(\xi)]}{\partial \xi} \right) d\xi \right. \\
& \left. - \int_{-1}^{+1} \mathbf{S}_b \left(\mathbf{U}_h \left(x_i + \xi \frac{\Delta x}{2}, t \right), z_h \right) P_l(\xi) d\xi \right\} - \mathbf{D}_{i,l}(t) \quad (18)
\end{aligned}$$

130 Here, piecewise linear polynomial basis (i.e. $l=0, 1$) are chosen, resulting in a second order DG scheme,
131 hereafter called DG2. The local integral terms are computed by the two-point Gauss-Legendre rule and
132 time integration is achieved by locally applying a 2-stage explicit RK time stepping scheme to solve
133 the ODEs in Eq. (18) with a CFL number less than 1/3 for stability. In order to consistently discretize
134 the higher order derivatives in dispersive terms, the so-called Local Discontinuous Galerkin (LDG)

135 approach⁵⁶ is used. The complete explanations regarding the DG solving procedure e.g. slope limiting,
 136 wetting/drying and solving the dispersive source terms can be found in Sharifian et al.¹¹.

137 **4- Multi-resolution analysis**

138 Considering the reference interval $[-1,1]$, a hierarchy of nested grids, $\{I_j^n\}_{j=0,1,\dots,2^n-1}$ with
 139 increasing resolution $n = 0, 1, 2, \dots$ is defined by midpoint sub-division of the reference interval, i.e.
 140 $I_j^n = [-1 + 2^{-n+1}j, -1 + 2^{-n+1}(j+1)]$. On each sub-interval I_j^n at resolution n , any continuous
 141 function is approximated as a vector space V_p^n denoting the space of piecewise polynomial functions of
 142 degree at most p . The spaces V_p^n have degrees of freedom $2^n(p+1)$ and form a nested structure of
 143 closed subspaces (Fig. 2)

$$V_p^0 \subset V_p^1 \subset \dots \subset V_p^n \subset \dots \quad (19)$$

144 For the Legendre polynomials used in the DG method (Eqs. 15-17) $\Phi = \{\varphi_0, \varphi_1, \dots, \varphi_p\}$ consisting of
 145 $p+1$ functions spanning the space V_p^0 on $[-1,1]$, it is possible to obtain the basis $\Phi_j^n =$
 146 $\{\varphi_{j,0}^n, \varphi_{j,1}^n, \dots, \varphi_{j,p}^n\}$ containing $2^n(p+1)$ functions, spanned over a sub-space V_p^n supported on I_j^n , by
 147 translation and dilation of Φ ⁴²:

$$\varphi_{jl}^n(x) = 2^{n/2} \varphi_l(2^n(x+1) - 2j - 1), \quad l = 0, \dots, p, \quad j = 0, \dots, 2^n - 1, \quad x \in I_j^n \quad (20)$$

148 in which j denotes the translation or shifting factor over sub-intervals $\{I_j^n\}_{j=0,1,\dots,2^n-1}$ and 2^n is the
 149 dilatation factor. Functions φ_l are called *scaling functions*. By considering the nested property (Eq. 19),
 150 the multiwavelet sub-space W_p^n can be defined as the orthogonal complement of V_p^n inside V_p^{n+1} , i.e.

$$V_p^n \oplus W_p^n = V_p^{n+1} \quad (21)$$

151 such that $V_p^n \perp W_p^n$ and $W_p^n \subset V_p^{n+1}$. The orthonormal basis W_p^0 comprises $p+1$ polynomials $\Psi =$
 152 $\{\psi_0, \psi_1, \dots, \psi_p\}$ defined on $[-1,1]$, also known as *multiwavelet* Legendre polynomials (Fig. 3)^{41,54,57}.
 153 Similarly, space W_p^n is spanned by functions $\Psi_j^n = \{\psi_{j,0}^n, \psi_{j,1}^n, \dots, \psi_{j,p}^n\}$, obtained by translation and
 154 dilation as

$$\psi_{jl}^n(x) = 2^{n/2} \psi_l(2^n(x+1) - 2j - 1), \quad l = 0, \dots, p, \quad j = 0, \dots, 2^n - 1, \quad x \in I_j^n \quad (22)$$

155 Using functions φ_{jl}^n , any arbitrary function $f \in L^2(-1, +1)$ can be reconstructed or decomposed across
 156 multiple scales of resolution. This is because by recursively applying Eq. (21), V_p^n can be decomposed
 157 into a single V_p^0 space along with a sequence of W_p :

$$V_p^n = V_p^0 \oplus W_p^0 \oplus W_p^1 \oplus \dots \oplus W_p^{n-1} \quad (23)$$

158 The orthogonal projection of $f(x)$ onto V_p^n takes the following form:

$$P_p^n f(x) = \sum_{j=0}^{2^n-1} \sum_{l=0}^p s_{j,l}^n \varphi_{j,l}^n(x) \quad (24)$$

159 where P_p^n is the projection operator. Eq. (24) gives the so-called *single-scale decomposition* of the
 160 approximate solution on level n . The *single-scale* coefficients, $s_{j,l}^n$, can be derived from a L^2 projection
 161 onto an orthonormal basis:

$$s_{j,l}^n = \langle f, \varphi_{j,l}^n \rangle = \int_{-1+2^{-n+1}(j)}^{-1+2^{-n+1}(j+1)} f(x) \varphi_{j,l}^n dx \quad (25)$$

162 Note that, for any $f \in V_p^n$, the following relation holds⁴²:

$$P_p^n f = f \quad (26)$$

163 It is also possible to expand f by deploying multiwavelets as

$$Q_p^n f(x) = P_p^{n+1} f(x) - P_p^n f(x) = \sum_{j=0}^{2^n-1} \sum_{l=0}^p d_{j,l}^n \psi_{j,l}^n(x) \quad (27)$$

164 where the detail coefficients are obtained from

$$d_{j,l}^n = \langle f, \psi_{j,l}^n \rangle = \int_{-1+2^{-n+1}(j)}^{-1+2^{-n+1}(j+1)} f(x) \psi_{j,l}^n dx \quad (28)$$

165 Recursive use of Eq. (27), leads to *multi-scale decomposition* of f on level n

$$\begin{aligned}
P_p^n f(x) &= P_p^{n-1} f(x) + Q_p^{n-1} f(x) = P_p^{n-2} f(x) + Q_p^{n-2} f(x) + Q_p^{n-1} f(x) = \dots \\
&= P_p^0 f(x) + \sum_{m=0}^{n-1} Q_p^m f(x) = \sum_{l=0}^p s_{0,l}^0 \varphi_l(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{l=0}^p d_{j,l}^m \psi_{j,l}^m(x)
\end{aligned} \tag{29}$$

166 The single scale coefficients $\{s_{0,l}^0\}_{l=0}^p$ represent the information on the coarsest level $m = 0$, while detail
167 coefficients $\{d_{j,l}^m\}$ carry multi-scale information, or fluctuations of the solution which, if added to the
168 lowest-resolution information, enrich it up to level n of resolution⁴⁸.

169 4-1- Two-scale transformation for down- and up-scaling local information

170 In order to reconstruct or decompose the local solution expansion between two successive
171 resolution levels, a two-scale transformation can be derived. Without loss of generality, the two-scale
172 transformation is considered between levels $m = 0$ and $m = 1$. The so-called Quadrature Mirror Filter
173 (QMF) coefficients will be used in decomposition and reconstruction steps, which are of two types⁴⁷:
174 low-pass filter coefficients (derived from scaling functions), and high-pass filter coefficients (derived
175 from multiwavelet functions). The low-pass filter coefficients are defined as $h_{l,r}^j = \langle \varphi_l, \varphi_{j,r}^1 \rangle$ ($j = 0, 1$;
176 $r = 0, \dots, p$). Considering $h_{l,r}^0$ we will have:

$$h_{l,r}^0 = \langle \varphi_l, \varphi_{0,r}^1 \rangle = \int_{-1}^{+1} \varphi_l(x) \varphi_{0,r}^1(x) dx = \sqrt{2} \int_{-1}^0 \varphi_l(x) \varphi_r(2x+1) dx \tag{30}$$

177 in which $x \in [-1, 0]$ comes from the fact that $\varphi_r(2x+1)$ is nonzero only if $(2x+1) \in [-1, +1]$.
178 Accordingly, by changing the variables the following holds:

$$\begin{aligned}
h_{l,r}^0 &= \langle \varphi_l, \varphi_{0,r}^1 \rangle = \sqrt{2} \int_{-1}^0 \varphi_l(x) \varphi_r(2x+1) dx \\
&= \frac{1}{\sqrt{2}} \int_{-1}^{+1} \varphi_l\left(\frac{y-1}{2}\right) \varphi_r(y) dy \approx \frac{1}{\sqrt{2}} \sum_{G=0}^p w_G \varphi_l\left(\frac{\hat{x}_G-1}{2}\right) \varphi_r(\hat{x}_G)
\end{aligned} \tag{31}$$

179 where the Gauss-Legendre quadrature rules are deployed to compute the integral. Similarly, for $h_{l,r}^1$

$$h_{l,r}^1 = \langle \varphi_l, \varphi_{0,r}^1 \rangle = \sqrt{2} \int_0^1 \varphi_l(x) \varphi_r(2x-1) dx \tag{32}$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^{+1} \varphi_l \left(\frac{y+1}{2} \right) \varphi_r(y) dy \approx \frac{1}{\sqrt{2}} \sum_{G=0}^p w_G \varphi_l \left(\frac{\hat{x}_G+1}{2} \right) \varphi_r(\hat{x}_G)$$

180 Based on multiwavelet functions $\psi_l \in W_p^0$, $l = 0, \dots, p$, the relation $W_p^0 \subset W_p^1$ leads to $\varphi_l \in W_p^1$.

181 Therefore, following the same procedure, the *high-pass filter coefficients* will be derived as:

$$g_{l,r}^0 \approx \frac{1}{\sqrt{2}} \sum_{G=0}^p w_G \psi_l \left(\frac{\hat{x}_G-1}{2} \right) \varphi_r(\hat{x}_G) \quad (33)$$

$$g_{l,r}^1 \approx \frac{1}{\sqrt{2}} \sum_{G=0}^p w_G \psi_l \left(\frac{\hat{x}_G+1}{2} \right) \varphi_r(\hat{x}_G) \quad (34)$$

182 Now, in order to define the multiwavelet decomposition, Eqs. (20) and (25) will result in

$$s_{j,l}^{n-1} = \langle f, \varphi_{j,l}^{n-1} \rangle = \sum_{r=0}^p (h_{l,r}^0 s_{2j,r}^n + h_{l,r}^1 s_{2j+1,r}^n) \quad (35)$$

183 and in the same manner,

$$d_{j,l}^{n-1} = \sum_{r=0}^p (g_{l,r}^0 s_{2j,r}^n + g_{l,r}^1 s_{2j+1,r}^n) \quad (36)$$

184 in which $l = 0, \dots, p$, $j = 0, \dots, 2^{n-1} - 1$. By forming matrices

$$\mathbf{H}_b = \{h_{i,l}^b\}, \quad \mathbf{G}_b = \{g_{i,l}^b\}, \quad b = 0, 1; \quad i, l = 0, \dots, p \quad (37)$$

185 and introducing the following vectors (for $j = 0, \dots, 2^n - 1$)

$$\begin{aligned} \mathbf{s}_j^n &= (s_{j,0}^n \quad \dots \quad s_{j,p}^n)^T \\ \mathbf{d}_j^n &= (d_{j,0}^n \quad \dots \quad d_{j,p}^n)^T \end{aligned} \quad (38)$$

186 the decomposition relations in Eqs. (35-36) can be reformulated as⁵⁴:

$$\mathbf{s}_j^{n-1} = \mathbf{H}_0 \mathbf{s}_{2j}^n + \mathbf{H}_1 \mathbf{s}_{2j+1}^n \quad (39)$$

$$\mathbf{d}_j^{n-1} = \mathbf{G}_0 \mathbf{s}_{2j}^n + \mathbf{G}_1 \mathbf{s}_{2j+1}^n \quad (40)$$

187 Now, left-multiplying Eq. (39) by \mathbf{H}_0^T and Eq. (40) by \mathbf{G}_0^T , then summing them would result in

$$\mathbf{H}_0^T \mathbf{s}_j^{n-1} + \mathbf{G}_0^T \mathbf{d}_j^{n-1} = \mathbf{s}_{2j}^n \quad (41)$$

188 and in the same way, multiplication by \mathbf{H}_1^T and \mathbf{G}_1^T leads to

$$\mathbf{H}_1^T \mathbf{s}_j^{n-1} + \mathbf{G}_1^T \mathbf{d}_j^{n-1} = \mathbf{s}_{2j+1}^n \quad (42)$$

189 In summary, Eqs. (39-40) and (41-42) define decomposition (also called multi-scale transformation)
 190 and reconstruction (also called inverse multi-scale transformation) formulas, respectively.

191 **5- The Adaptive MWDG-GN model**

192 In order to combine the DG-GN solver with the MW-based grid adaptation, the multi-resolution
 193 analysis introduced in Section 4 is applied to each cell I_i of the baseline grid. The DG formulation of
 194 multi-resolution scheme follows the same procedure as the non-adaptive case (Section 3), however, in
 195 the adaptive framework the computational domain would be a heterogeneous grid comprised of
 196 selectively chosen resolution levels of the grid hierarchy (see Section 5-2), on which the time evolution
 197 is actually performed.

198 **5-1- Local multi-scale DG formulation**

199 Therefore, each cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ is recursively subdivided into 2^n sub-intervals
 200 $\{I_{i,j}^n\}_{j=0,1,\dots,2^n-1}$ in a way that each cell $I_{i,j}^n = [x_{i-1/2} + \Delta x^{(n)}j, x_{i-1/2} + \Delta x^{(n)}(j+1)]$ would have the
 201 local resolution-dependent size of $\Delta x^{(n)} = 2^{-n}\Delta x$, centred by $x_{i,j}^n = x_{i-1/2} + \Delta x^{(n)}(j+1/2)$. In this
 202 notation, the sub-index i is introduced for referring to the respective baseline cell. Also, since DG
 203 approximations and scaling function expansions are composed of the same basis functions, there is a
 204 direct relation between them. By considering the baseline cell $I_i = [a, b]$ and denoting $\Delta x^{(n)} = (b -$
 205 $a)/2^n$ as the mesh size on level n , and $x_j^n = a + (j + \frac{1}{2})\Delta x^{(n)}$ as the centre of cell I_j^n , using Eq. (20)
 206 the global DG approximation of the solution on the domain can be expanded as:

$$\begin{aligned}
\mathbf{U}_h(x, t) &= \sum_{j=0}^{2^n-1} \sum_{l=0}^p \mathbf{U}_{j,l}^{(n)}(t) \varphi_l(\xi) = \sum_{j=0}^{2^n-1} \sum_{l=0}^p \mathbf{U}_{j,l}^{(n)}(t) \varphi_l\left(\frac{2}{\Delta x^{(n)}}(x - x_j^n)\right) \\
&= \sum_{j=0}^{2^n-1} \sum_{l=0}^p \mathbf{U}_{j,l}^{(n)}(t) \varphi_l\left(\frac{2^{n+1}}{b-a}(x-a) - 2j - 1\right) \\
&= \sum_{j=0}^{2^n-1} \sum_{l=0}^p \mathbf{U}_{j,l}^{(n)}(t) \varphi_l(2^n(y+1) - 2j - 1) = 2^{-\frac{n}{2}} \sum_{j=0}^{2^n-1} \sum_{l=0}^p \mathbf{U}_{j,l}^{(n)}(t) \varphi_{j,l}^n(y)
\end{aligned} \tag{43}$$

207 in which $y = -1 + 2(x - a)/(b - a)$. On the other hand, over the reference domain $[-1,1]$, based on
208 properties of Eqs. (24) and (26), it holds that

$$\mathbf{U}_h(x, t) = P_p^n \mathbf{U}_h(x, t) = \sum_{j=0}^{2^n-1} \sum_{l=0}^p s_{j,l}^n \varphi_{j,l}^n(y) \tag{44}$$

209 Therefore, Eqs. (43) and (44) will lead to

$$2^{-\frac{n}{2}} \mathbf{U}_{j,l}^{(n)} = s_{j,l}^n \tag{45}$$

210 which gives the relation between DG and single-scale coefficients.

211 5-2- Resolution adaptivity

212 In order to select the appropriate resolution levels to form the adaptive grid, a selection process is
213 applied on the multiwavelet coefficients resulting in a set of the significant details denoted by $SD \in$
214 $\{(i, s, m), 0 \leq i \leq N, 0 \leq s \leq 2^m - 1\}$, which will be used to determine those sub-cells across the
215 different resolution levels that need to be active within the adaptive grid. In other words, the set of active
216 cells $\{I_{i,s}^m\}_{(i,s,m) \in SD}$ will merge as $\cup_{(i,s,m) \in SD} \{I_{i,s}^m\} = [x_{\min}, x_{\max}]$. To clarify this concept, if we express
217 the local polynomial approximation of the flow vector \mathbf{U}_h , over a cell $I_{i,s}$ in the form of the multi-scale
218 decomposition up to a highest resolution n (Eq. 29) with coefficients $s_{i,0,l}^0$ and $d_{i,s,l}^m$ (index i refers to
219 the baseline cell), respective detail coefficients at a resolution $m \leq n$, i.e. $d_{i,s,l}^m$, would become
220 increasingly smaller with increasing level of spatial resolutions, when the underlying function is
221 smooth. In the opposite way, if discontinuities are present, the details usually stay significant for

222 increasing refinement level^{41,46}. This property enables us to select the active cells by comparing the
 223 magnitudes of these details with a user-specified threshold value.

224 **5-2-1- Thresholding and prediction**

225 To apply the thresholding, a prescribed value ε will be defined by the user, based on which the
 226 level-dependent threshold value $\varepsilon_m = 2^{m-n}\varepsilon$ is introduced. The detail coefficients $d_{i,s,l}^m$ whose absolute
 227 values, scaled with the maximum value of the solution, fall below ε_m will be discarded from selection,
 228 i.e.

$$\hat{d}_{i,s,l}^m = \begin{cases} d_{i,s,l}^m & \text{if } \max_{\substack{l=0,\dots,p \\ i,s,m \in SD}} \left(\frac{|(d_{i,s,l}^m)_r|}{\max\left\{\max_{i,s,m \in SD} |(\mathbf{U}_{i,s,0}^m)_r|, 1\right\}} \right) > \varepsilon_m \\ 0 & \text{else} \end{cases} \quad (46)$$

229 in which $(\mathbf{U}_{i,s,0}^m)_r$ is the average coefficient of the respective conserved quantity r , in cell $I_{i,s}^m$. This
 230 procedure is called *hard thresholding*. Also, since the flow field evolves in time, after each evolution
 231 step, adaptivity is performed to update the grid at the new time level. To guarantee that no significant
 232 future of the solution is lost at the new time level, a prediction step will be further applied on a selected
 233 set of significant details, in which the following constraints are considered^{46,49}:

- 234 1- On account of the finite speed of propagation, the details in a local neighbourhood (in the same
 235 level) of a cell with significant detail may also become significant within one time step, and
 236 will be refined subsequently.
- 237 2- Formation of shocks may steepen the gradients, resulting in significant details on higher levels.
 238 Therefore, another constraint with the criterion $2^{\bar{M}+1}\varepsilon_m$ ($\bar{M} = p$ denotes the number of
 239 vanishing moments of the multiwavelets) is introduced, according to which the details at the
 240 higher level $m + 1$ will be set as significant.
- 241 3- The set of cells characterized with significant details possess a tree-like structure; i.e. if a cell
 242 in level m is detected significant, all its substructure cells on lower resolution levels are set as
 243 significant, regardless of the thresholding based on their respective details.

244 **5-2-2- Adaptive MWDG-GN algorithm**

245 In order to apply the adaptivity procedure, first the initialization is performed in the following steps:

246 1- The initial grid is formed by projection of the initial data on a fully refined grid at the finest
247 level n (Fig. 4(a)).

248 2- The multi-scale transformations (Eqs. (39-40)) are applied to determine the detail coefficients
249 on levels $m = 0, \dots, n - 1$ (Fig. 4(b)).

250 3- Initial hard thresholding is applied to obtain the initial set of significant details (Fig. 4(c)).

251 After the initial significant details are determined, the main steps of the computations are performed in
252 the following steps:

253 4- The prediction step is performed based on the available set of significant details.

254 5- The inverse multi-scale transformations (Eqs. (41-42)), will be recursively applied; proceeding
255 level-wise from coarse to fine, in order to refine those cells flagged as significant from the
256 previous steps, and also determines the respective DG coefficients (Fig. 4(d)). At the end of
257 this step, the active cells which form the appropriate multi-scale adaptive grid are detected. The
258 RKDG calculations will be performed over this grid.

259 6- The RKDG evolution is performed, following the same procedure as the non-adaptive RKDG.
260 The only difference is the slope limiting. As previously stated, by choosing a suitable threshold
261 value the grid is refined up to the finest level near discontinuities and if the solution is locally
262 smooth, we expect the grid not to be refined up to the finest level in this region. This property
263 has been used as an additional indicator for the limiting process such that the limiting process
264 is only applied in cells on the finest level n .

265 7- The multi-scale transformations (Eqs. (39-40)) will be applied again for decomposition of the
266 RK-updated solutions.

267 8- The new set of significant details is computed by hard thresholding.

268 Steps 4 to 8 will be performed in the main time loop of the computations.

269 **5-2-3- Considerations regarding well-balancing**

270 To justify the well-balancing property, some considerations are applied in the adaptivity
271 process^{48,49}. Since the topography, as opposed to the flow variables, does not evolve in time, a static
272 (but not necessarily uniform) grid is considered for it. The corresponding set of significant details of
273 the topography is then added in each time step to significant details of the flow variables as an additional
274 constraint to the coarsening and the refinement procedure. Moreover, since depth is a poor indicator of
275 regularity/complexity of the solution, in all the steps involved in the adaptivity process (except for
276 RKDG evolution) the conserved variables vector must be rearranged as $\mathbf{U} = [h + z, q]^T$, so that water
277 surface elevation is analyzed by the multiresolution transformations, instead of flow depth.

278 **6- Numerical results**

279 The main idea behind the adaptive MWDG2 approach is to increase the computational efficiency
280 of the reference DG2 scheme without losing accuracy. To do this, a choice of the threshold value ε is
281 needed: a too large threshold would spoil the accuracy of the solution as a result of dominating
282 additional error, while a too small threshold leads to over-refinement and inefficiency⁴¹. With NSW
283 equations, a threshold value ranging between $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ is found to be enough for the
284 adaptive MWDG2 solver to yield an appropriate balance between accuracy and efficiency^{47,49}. For a
285 wave with dispersive behaviour, the use of BT equations generally dictates finer resolutions to resolve
286 more complex physical features of interest, and also higher computational costs compared to NSW
287 equations. To find out the effects of these characteristics, the choice for a suitable range for the threshold
288 value is here re-investigated for the GN solver for tests considering the propagation and transformation
289 of solitary waves. The solitary waves can be considered as a balance between the nonlinearity and the
290 frequency dispersion that maintains the permanent waveform. These waves have been generally used
291 to model certain behaviours of nonlinear long waves, such as the leading wave of tsunamis and storm
292 surges. Moreover, due to their locality, they are more likely to benefit from grid adaptation, compared
293 to periodic waves. The first test provides an analytical investigation of the propagation of a solitary
294 wave over a flat bottom, where convergence and thresholding criteria are analysed. The other test cases
295 show the capability of the proposed MWDG-GN model compared to experimental data with wave
296 transformation. The second test depicts the interaction of a solitary wave with a mild-slope beach and

297 the third test deals with the interaction of two solitary waves. In all the tests, the boundary conditions
 298 are imposed based on solid wall, inflow and outflow boundary and the optimization parameter of the
 299 GN equations is set to $\alpha = 1$. Regarding computational efficiency measurements, it should be noted
 300 that all the simulations are performed on a 3.6 GHz Intel i7 quad-core processor.

301 **6-1- Propagation of a solitary wave**

302 To identify and analyse the properties of the MWDG-GN solver, the propagation of a solitary wave
 303 over a flat bed is considered. The wave has a finite amplitude and permanent form resulting from the
 304 balance between nonlinear and dispersive effects, and has an exact solution given by³

$$h(x, t) = h_0 + a \operatorname{sech}^2 \left(\frac{\sqrt{3a}}{2h_0\sqrt{h_0 + a}} (x - ct) \right) \quad (47)$$

$$u(x, t) = c \left(1 - \frac{h_0}{h(x, t)} \right)$$

305 where h_0 is the initial local water depth, a the wave amplitude and $c = \sqrt{g(h_0 + a)}$ the wave speed.
 306 The solitary wave propagates in a 200 m long domain over a constant water depth of $h_0 = 1$ m, and its
 307 wave crest is initially centred at $x_0 = 50$ m. This wave is moderately nonlinear with a relative amplitude
 308 of $a/h_0 = 0.2$.

309 The adaptive MWDG-GN simulations are performed up to $t = 30$ s for a range of resolution
 310 settings, where each setting is defined by the pair $\{N_b, L\}$ with N_b and L indicating the number of cells
 311 of the baseline grid and the maximum refinement level, respectively. The settings are taken by fixing L
 312 $= 7$ and considering different size for the baseline grid $N_b = 1, 2, 3, 4$ and 5 , yielding grids with a
 313 maximum of 128, 256, 384, 512 and 640 cells, respectively. On these fine uniform grids, simulations
 314 using the DG-GN solver are also carried out to enable a relative comparison of accuracy and efficiency
 315 for adaptive MWDG-GN solver over a range of threshold values between $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-2}$.

316 **6.1.1 Choice of the threshold value with the adaptive MWDG-GN solver**

317 The accuracy of the adaptive MWDG-GN scheme is measured using the normalized L^2 errors of
 318 water height h and discharge hu based on the following formula

$$L^2_{error} = \sqrt{\frac{(U-U_T)^2 \Delta x_L}{U_T^2 \Delta x_L}} \quad (48)$$

319 where U_T is the analytical solution based on Eq. (47) and Δx_L denotes the grid size on the finest level
320 of resolution. Fig. 5 shows the water depth (Fig. 5(a)) and flow discharge (Fig. (5b)) errors computed
321 at $t = 5$ s with both MWDG-GN and DG-GN solvers for all the settings except the coarsest one with
322 $\{1,7\}$, which was not included to save space. With increasingly finer resolution of the uniform DG-GN
323 solver, the adaptive MWDG-GN solver requires increasingly smaller threshold values to keep the same
324 error magnitudes. As compared to a NSW solver for numerical modelling of non-dispersive flows, a
325 GN solver necessitates finer grid resolution to ensure capturing both dispersive and nonlinear features¹¹.
326 This implies that the adaptive MWDG-GN solver would require smaller threshold values compared to
327 an adaptive MWDG-NSW solver^{47,49} to accommodate finer resolution needs. For this test where Δx_L is
328 near 10^{-1} , settings $\{3,7\}$ - $\{5,7\}$ are identified as appropriate for the adaptive MWDG-GN solver in
329 combination with threshold values in the neighbourhood of $\varepsilon = 10^{-4}$. This seems to suggest a threshold
330 value ε that is at least 2 to 3 orders of magnitude smaller than Δx_L to meet the uniform resolution
331 accuracy required for an equivalent DG-GN solver, in line with an increase in relative wave amplitude
332 a/h_0 .

333 To evaluate efficiency of the adaptive MWDG-GN solver with reference to the same range of
334 threshold values, its compression rate (decrease in the number of cells due to use of wavelet adaptivity,
335 in percent) and speed up ratio (CPU time ratio of uniform to adaptive solvers) are measured after
336 completing the full 30 s simulation. Fig. 6a and Fig. 6b show speedup and compression rate against the
337 threshold values for settings $\{3,7\}$ - $\{5,7\}$, both showing an increase in speed up ratio and compression
338 rate with decreasing threshold values. In the neighbourhood of $\varepsilon = 10^{-4}$, the observed speedup ratio
339 with setting $\{5,7\}$ shows maximum efficiency (around 30 times) while setting $\{3,7\}$ shows minimum
340 efficiency (around 18 times). Nonetheless, compression rates are noted to be consistently closer, in the
341 range of 75-80%. These suggest that more costs are entailed due to wavelet adaptivity overhead with
342 decreasing size of the baseline mesh. Overall, a threshold value around $\varepsilon = 10^{-4}$ is an appropriate
343 choice in this test for the adaptive MWDG-GN solver to preserve the accuracy of an equivalent DG-

344 GN solver on the finest uniform resolution accessible to the adaptive MWDG-GN solver, while being
345 up to 30 times more efficient to run.

346 6.1.2 Mesh convergence analysis of accuracy and efficiency

347 To quantify the extent to which the adaptive MWDG-GN solver converges to the uniform DG-GN
348 solver, an error convergence analysis is performed considering both accuracy and efficiency. Accuracy
349 convergence is evaluated by plotting the L^2 errors of water height (Fig. 7(a)) and discharge (Fig. 7(b))
350 against the finest grid sizes corresponding to settings $\{N_b, L\}$ from $\{1,7\}$ to $\{5,7\}$ and computed at time
351 $t = 5$ s. The uniform DG-GN solver delivers optimal convergence rates in the order of 2.5, and the
352 adaptive MWDG-GN solver is observed to converge to the same asymptotic behaviour of the uniform
353 solver showing slightly larger errors with coarsening in grid resolution. In terms of efficiency, the same
354 L^2 errors are re-examined but with respect to the maximum number of cells entailed in the adaptive
355 MWDG-GN and the uniform DG-GN solvers at the same output time $t = 5$ s (Fig. 8). As can be seen
356 from Fig. 8, the rate of efficiency convergence of the adaptive MWDG-GN model is much faster than
357 the uniform DG-GN solver in terms of yielding errors of the same order but with considerably fewer
358 cells.

359 6.1.3 Qualitative comparisons and analysis of refinement levels

360 The predicted numerical profile of the solitary wave at different instants using the adaptive
361 MWDG-GN model with the two settings $\{3,7\}$ and $\{5,7\}$ are compared with the exact solution and the
362 predictions associated with their equivalent uniform DG-GN solvers (see Fig. 9). In order to distinguish
363 possible difference among the adaptive and uniform solver predictions, only zoom-in portions near the
364 wave crest are plotted in the sub-figures forming Fig. 9. For setting $\{5,7\}$, which allows up to a
365 maximum of 640 cells, the adaptive MWDG-GN and uniform DG-GN solver predictions are seen to
366 provide the best agreement with the exact solution throughout the 30 s simulation. For setting $\{3,7\}$,
367 the maximum number of cells within the adaptive and uniform solvers is roughly halved, which is
368 probably the main reason why these solvers consistently provided slightly poorer agreement with the
369 exact solution. In particular, by $t = 20$ s (Fig. 9c), discrepancies become clearly visible and eventually

370 intensify, by $t = 30$ s (Fig. 9d), to form small amplitude (unphysical) tails. This deficiency detected in
371 setting $\{3,7\}$ could be indicating that the finest grid resolution allowed by this setting, i.e. $\Delta x_L =$
372 0.52 m, may not be enough for the uniform DG-GN to fully capture the wave nonlinearities at a relative
373 wave amplitude close to $a/h_0 = 0.2$ ¹¹, hence for the adaptive MWDG-GN solvers too.

374 To analyse resolution prediction ability of adaptive MWDG-GN solver, Fig. 10 illustrates the
375 associated spatial refinement levels in line with the free-surface elevations over the full domain,
376 considered at the same output times as in Fig. 9. For both $\{5,7\}$ and $\{3,7\}$ settings, the adaptive MWDG-
377 GN solver is found to favourably select the finest resolution (i.e. at the maximum level $L = 7$) around
378 the crest of the solitary wave. Therein, a wider extent of fine resolution prediction (i.e. so-called over-
379 refinement) is observed with setting $\{3,7\}$ than with setting $\{5,7\}$, and this can be attributed to the
380 aforementioned discrepancies in terms of small amplitude tails at $t = 20$ s and $t = 30$ s that could have
381 exaggerated wavelet coefficients, thereby causing spurious over-refinement. In the regions of quiescent
382 flow, the adaptive MWDG-GN solver selects the coarsest resolution (i.e. levels $L = 1$ to 2) in both
383 settings.

384 To further analyse the efficiency of the MWDG-GN solver over the full 30 s simulation, its
385 instantaneous number of cells have been recorded. Fig. 11 shows the time variation of the number of
386 cells used by the adaptive solver for both settings $\{5,7\}$ and $\{3,7\}$, as well as the constant cell numbers
387 entailed in their associated DG-GN solvers. As previously shown in Fig. 6, the setting $\{3,7\}$ is found
388 less efficient than the finer setting $\{5,7\}$ despite having a compression rate of the same order. Fig. 11
389 further shows that these similar compression rates hold during the full length of the simulation, with the
390 number of cells in the adaptive grid being about 15-18% of the number of cells in the uniform
391 counterparts, for both settings $\{3,7\}$ and $\{5,7\}$. The relative decrease in efficiency with setting $\{3,7\}$ is
392 likely to have been caused by deficiencies observed previously in Fig. 10 (i.e. the trailing unphysical
393 fluctuations), which lead to over-refinements therein, in turn causing extra costs associated with wavelet
394 adaptivity overhead. In can be therefore concluded from this test, that choosing a right setting, even if
395 based on finer resolution, is central to meet both accuracy and efficiency needs within the adaptive
396 MWDG-GN solver.

397 **6-2- Run-up and run-down of a solitary wave over a sloped beach**

398 To examine the performance of the adaptive MWDG-GN model in dealing with wet/dry fronts and
399 topography, it is also tested for solitary wave run up on a sloping beach supported by the experimental
400 work of Synolakis⁵⁸. The computational domain consists of a channel with the initial free-surface
401 elevation of 1 m approaching a sloped beach (1:19.85). In order for the solitary wave to be initially
402 located in the channel, the computational domain is extended to 77 m, which is longer than the actual
403 experiments (Fig. 12). Here, a solitary wave with a non-breaking wave with weak nonlinearity is
404 selected, which has a lower relative amplitude than in test 6.1 ($a/h_0 = 0.019$). Based on the domain
405 size, the adaptive MWDG-GN model is run for $t = 25$ s using a setting $\{4,7\}$, which allows up to 512
406 cells ($\Delta x_L = 0.15$ m). The uniform DG-GN simulation is also run on the finest resolution grid to allow
407 for a relative comparison. Following the observations in test 6.1, the following three threshold values
408 are selected and tested with the adaptive MWDG-DG solver: $\varepsilon = 10^{-3}$, $\varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$
409 (i.e. being 2-3 orders of magnitude smaller than Δx_L and in the neighbourhood of 10^{-4}).

410 The numerical free-surface elevation profiles produced by both MWDG-GN and DG-GN solvers
411 at different (normalized) output times $t^* = t(g/h_0)^{1/2}$ are compared in Fig. 13 with reference to the
412 experimental profiles of Synolakis⁵⁸. The wave profiles computed by the adaptive MWDG-GN solver
413 closely match the profiles computed by the uniform DG-GN on the fine grid, while remaining in a good
414 agreement with the experimental data during the run-up and run-down phases. Using smaller threshold
415 values ($\varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$) only makes improvements in certain areas of the flow, e.g. the
416 wet/dry front at run-up phase at $t^* = 45$. In other areas, the adaptive MWDG-GN solver predictions
417 based on the largest threshold value ($\varepsilon = 10^{-3}$) found similar to those relative to the smallest threshold
418 ($\varepsilon = 10^{-4}$). With $\varepsilon = 10^{-3}$, the MWDG-GN solver used lower resolution levels while preserving
419 close predictive accuracy even with smaller ε , and hence is here the most efficient option. Outside the
420 vicinity of the wet/dry fronts, during the run-up and run-down phases, the adaptive MWDG-GN solver
421 predicted relatively coarse-to-moderate resolution levels, varying between $L = 2$ to 4. For this test, all
422 adaptive MWDG-GN solvers did not excessively use the finest resolution level around the wave crest

423 barely. This can be attributed to the relatively weak nonlinearity of the solitary wave in contrary to the
424 moderately nonlinear wave explored in test 6.1.

425 To analyse the efficiency of the adaptive MWDG-GN solver in relation to the choice of the
426 threshold value, the time evolution of their mesh size is plotted in Fig 14, which also contains the size
427 of the uniform grid relative to the DG-GN. This figure reinforces that $\varepsilon = 10^{-3}$ provides the most
428 efficient option with the adaptive MWDG-GN solver: it consistently activated around 8% (i.e. 45 cells)
429 of the cells accessible to it, while delivering predictions as close as the other adaptive MWDG-GN
430 solvers and the DG-GN simulation using 512 cells. With $\varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$, the adaptive
431 MWDG-GN is seen to activate higher cell percentage. However, the percentage of active cells required
432 did not exceed 18% in this test, even at $\varepsilon = 10^{-4}$. In terms of speedup, it is found between 30 to 55 in
433 this test, which is expected given the weak magnitude of wave nonlinearity and less dispersive effects
434 as compared to test 6.1.

435 6-3- Head-On Collision of Two Solitary Waves

436 A final test is introduced to study the behaviour of the adaptive MWDG-GN solver, when there are
437 more than one solitary wave propagating, each featured by a higher relative amplitude ($a/h_0 > 0.2$).
438 Therefore, the experimental test of Craig et al.⁵⁹ is selected as it involves the head-on collision of two
439 solitary waves propagating in opposite directions. This problem is characterised by the change of the
440 shape as well as a small phase-shift of the waves as a consequence of the nonlinearity and dispersion.
441 The setup consists of a 3.6 m long flume with still water depth of $h_0 = 5 \text{ cm}$. The left wave with an
442 amplitude of $a_1 = 1.063 \text{ cm}$ is initially located at $x = 0.5 \text{ m}$ while the right one is initially located at
443 $x = 3.1 \text{ m}$ with an amplitude of $a_2 = 1.217 \text{ cm}$. These values result in relative amplitudes a/h_0 equal
444 to 0.212 and 0.243, for the left and right waves respectively. Each of these solitary waves can be
445 considered to have moderate-to-high nonlinearity and are expected to cause an even higher nonlinearity
446 at the instant when they merge into a bigger solitary wave.

447 Adaptive MWDG-GN simulations are performed up to $t = 2.5 \text{ s}$ and based on setting {3,7} that
448 permits a maximum of 384 cells. As the flume experiment is 3.6 m long, adopting this setting means

449 the adaptive MWDG-GN solver can access a resolution as fine as $\Delta x_L = 0.0093 m$. As before, the
450 uniform DG-GN model is also run on the grid using the finest level of resolution. For the threshold
451 value parameter, $\varepsilon = 10^{-6}$ is selected for this test informed by the analysis of test 6.1 (see Sec.6.1.1).
452 The spatial evolution of the solitary waves simulated by the adaptive MWDG-GN solver and the
453 uniform DG-GN counterpart at different output times are shown in Fig. 15, as well as experimental
454 profiles⁵⁹. At the instant of head-on collision (around $t = 1.693 s$), the wave amplitude reaches around
455 to a level larger than the sum of the amplitudes of the two incident solitary waves (equivalent of a ratio
456 $a/h_0 = 0.5$). After the collision (around $t = 1.865 s$), two waves come out with reduced amplitudes,
457 returning to their initial form. As an outcome of this collision (during $t = 1.693$ to $1.824 s$), the two
458 waves lose momentum, which results in lower amplitudes (compared to the initial values) and a phase
459 lag.

460 In terms of free-surface elevation predictions, it can be seen from Fig. 15 that the predictions
461 produced by the adaptive and uniform solvers are almost identical, both providing a close agreement
462 with experimental profiles at all output times. Due to the high nonlinearity of the two waves and the
463 strong interactions between them, the adaptive MWDG-GN solver needed resolution levels that are
464 higher compared to the previous test cases (Sec. 6.1 and 6.2). Fine resolution levels are also observed
465 in areas away from the wave crests, with at least $L = 4$. This is also evident in Fig. 16, which shows the
466 time evolution of the number of cells in adaptive and uniform schemes. Initially, the adaptive MWDG-
467 GN solver activated 57% of cells with an increasing trend with the propagation of the two solitary
468 waves, reaching a final percentage of 85%. As a consequence of these low compression rates, the final
469 speedup is here only equal to 1.6, suggesting that the adaptive MWDG-GN solver may not be ideal for
470 the problems with poor locality such as involving multiple waveforms or periodic waves.

471 7- Conclusions

472 In this work, we applied a multiwavelet-based grid adaptation technique to Green-Naghdi (GN)
473 equations. This is achieved by extending a previously developed uniform mesh Discontinuous Galerkin
474 (DG) solver to the GN equations (DG-GN) from an adaptive Multiwavelet-based DG (MWDG) method

475 for the NSW equations (MWDG-NSW). The performance of the MWDG-GN solver is demonstrated
476 by several benchmark tests. The adaptive solver is shown to provide a robust method for driving grid
477 adaptation, with the adaptivity being controlled only by a single threshold value and with inherent error
478 control. For the threshold parameter, it has been verified that choosing a value between 2 to 3 orders of
479 magnitude smaller than the size of the grid cells on the finest level of resolution, would result in an
480 optimal combination of efficiency and accuracy to resolve small scale features of dispersive wave
481 propagations. Therefore, the same accuracy as the uniform DG-GN solver can be achieved by the
482 adaptive MWDG-GN solver, but with significantly fewer cells. For the case of single solitary waves,
483 compression rates of at least 80% and speedups around 30 are achieved. It is also found that the
484 efficiency gain of using grid adaptation depends on the amount of nonlinear and dispersive effects.
485 Accordingly, the best performance of the proposed solver MWDG-GN solver is sought to be in case of
486 moderate nonlinearity and dispersion. The 2D extension of the present MWDG scheme is the subject
487 of future work.

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