Emergence and Topological Order in Classical and Quantum Systems

Abstract

There has been growing interest in systems in condensed matter physics as a potential source of examples of both epistemic and ontological emergence. One of these case studies is the fractional quantum Hall state (FQHS). In the FQHS a system of electrons displays a type of holism due to a pattern of long-range quantum entanglement that some argue is emergent. Indeed, in general, quantum entanglement is sometimes cited as the best candidate for one form of ontological emergence. In this paper we argue that there are significant formal and physical parallels between the quantum FQHS and classical polymer systems. Both types of system cannot be explained simply by considering an aggregation of local microphysical properties alone, since important features of each are globally determined by topological features. As such, we argue that if the FQHS is a case of ontological emergence then it is not due to the quantum nature of the system and classical polymer systems are ontologically emergent as well.

1. Introduction

This paper addresses the question of which properties determine the state of a collective aggregate system. We will consider two different aggregates. Firstly, the fractional quantum Hall state: a collection of electrons confined to a 2D plane with a magnetic field applied (Lancaster and Blundell 2014, Fradkin 2013, Wen 2004). Secondly, polymer melts: a collection of polymers with different shapes (Jones 2002). A polymer is a molecule resembling a pearl necklace - a long string connecting 'beads' called monomers (which are particular atoms/molecules).

The fractional quantum Hall state/effect (FQHS/FQHE for short) is a quantum phenomenon, involving electrons in states of quantum entanglement (Lancaster and Blundell 2014, Fradkin 2013, Wen 2004). The FQHS is of great interest in condensed matter physics as it represents a 'topologically ordered' state of matter (Wen 2004), in which the collective properties are determined in part by a particular pattern of entanglement - long-range entanglement (Chen, Gu and Wen 2010). Topology is the mathematics of shape without distances (Lancaster and Blundell 2014). Two objects have the same topology if one could imagine the objects being made of infinitely malleable rubber, then reshaping one object into the other without cutting or breaking. So, a cup and a doughnut have the same topology, as they both have only one hole and one could (mathematically speaking) turn one shape into the other continuously. The study of topology doesn't care about distances (or time intervals), nor how big or small an object is, only the properties of its shape (Lancaster and Blundell 2014).

Increasingly many physicists regard the FQHS as one of the most striking examples of emergence in physics, and this has led to increasing interest in the philosophical community to see if the FQHS is emergent in philosophical terms (Lancaster and Pexton 2015). One might be forgiven for thinking that if the FQHS is emergent then its *quantum nature* plays a crucial role in that status. In this paper we shall argue to the contrary. There are striking parallels in the mathematics and modelling of the FQHS and classical polymer systems (Edwards 1965). We argue these *representational similarities are reflective of a deeper underlying set of physical similarities*. Moreover, these similarities are not in the

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microphysics of each aggregate system (they are radically different), but in the *role topological ordering plays as a counterfactual difference maker with respect to certain properties*.[1]

If the parallels between polymer systems and FQHS are scientifically and philosophically valid then we can make three claims:

Claims of this paper:

1) *The FQHS is not emergent because it is a quantum system but instead because of a specific type of emergent topological property (topological ordering) that need not be quantum in nature. Once a system possesses topological ordering it determines other properties of the system. Topological ordering is an emergent control structure that pushes the system into areas of the available parameter space at different levels (both macroscopic and microscopic).*

2) *There are classical systems which are also topologically ordered such as polymer systems. Polymer systems and the FQHS share many parallel features. These features are not merely mathematical or representational, but reflect higher-level structural similarities not found in the disjunctive microphysics.*

3) *If the FQHS is emergent because of these topological properties, then there are many classical systems which are emergent in the same way as the FQHS.*

4) *Topologically ordered systems are ontologically emergent not merely epistemically emergent.*

Before we begin with assessing the evidence for these claims it is important to make clear what we do not claim:

- We do not claim that there aren't extant accounts of emergence that apply to classical systems in the literature already. For example, we are not assessing whether accounts of emergence based on universality or phase transitions in condensed matter systems are valid or not. We simply take the view that whatever the merits of those accounts the FQHS/topological considerations play no special evidential role in them, since those states of matter are not topological.

- We do not claim that certain accounts of metaphysical emergence based on quantum entanglement are invalidated by our analysis. Rather simply that those accounts would apply to any simple two-body entangled system and again the many-body/FQHS/topological aspects of entanglement would not provide any special form of evidence for them. If simple entanglement is emergent (and we don't argue it is or isn't) then there is arguably little of additional interest about the FQHS. We start from assuming physicists' interest in the FQHS as emergent is worth looking at closely to see what is different about the FQHS from other simple or many body quantum systems, i.e. what is special about topological ordering.

- We do not claim that our definition of ontological emergence will meet all tests of strength in comparison with accounts in the literature. For example, whether topologically ordered states

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[1] This paper is not intended as a general discussion of emergence aimed at convincing reductionists that emergence is a real phenomenon in the world. We will outline how we feel the FQHS and polymer systems are emergent, but the case cannot be made (given space considerations) in a way that accounts for every metaphysical objection reductionists might make. Nor is it aimed at metaphysical system building with respect to emergence. We will not present a complete emergentist account, although our description of the type of emergence in topologically ordered systems such as the FQHS will fit more easily with some metaphysical approaches to emergence that others. Rather we are concerned with examining one type of quantum system that has been claimed to be emergent, the fractional quantum Hall state, and assessing whether there are parallels between it and some classical systems.

should be described as examples of top down causation or not is not central to our claims. This is in part because all metaphysical assessments of the nature of the causal structure of a system depend on what philosophical account of causation one starts with. We are largely agnostic as to whether one wants to think of these cases in causal terms or not. In a similar vein, we argue that these cases of emergence are ontological (in the sense they are not artefacts of epistemology alone) but, since the philosophical literature has no universally agreed upon definitions of terms such as 'strong' or 'weak' emergence, these terms carry too much baggage to be useful. So, whether our account of emergence is strong or weak simply reflects how the reader typically uses the terms. What we are clear about is our claim that topological ordering is an ontological category in the world that is ineliminable in principle from any explanans of certain phenomena of the FQHS and some polymer systems.

- Similarly, we do not claim our account of emergence is incompatible with other extant accounts of emergence, if suitable links are made. For example, polymer systems may well be examples of contextual emergence, or linked to asymptotic phenomena or accounts of emergence based on degrees of freedom. But *we do claim* that *the emergent nature of these polymer systems can be argued for outside the context of those other accounts*. Being potentially compatible with those accounts *shouldn't be interpreted as topologically ordered systems being just another example of them with no unique insights to provide*. We see it as a strength of our analysis that it is compatible with other accounts of emergence not a weakness.

2. The Fractional Quantum Hall State

In condensed matter physics we describe qualitatively different states of the same substance as phases (Anderson 1984, Jones 2002). For example, liquid water is very different from solid water in the form of ice; each is a different phase, and the transition between states is called a phase transition. Most (but not all) phase transitions can be described in terms of broken symmetry. Consider a ferromagnet for example. The metal is partially composed of electrons; each electron has a quantum property called spin. We can think of spin as an arrow pointing in a direction. When the magnet is symmetrical all the individual spins point in random directions, there is no preferred direction overall. When the symmetry is broken more arrows point in one direction than any other, so the system as a whole has a net arrow. This breaking of symmetry means that the system changes from a state of no magnetization overall to one with a non-zero magnetization. The change between each state is a phase transition (Anderson 1984).

The fractional quantum Hall effect (FQHE) and the corresponding fractional quantum Hall state (FQHS) is unusual. Firstly, this is because it is not explained by broken symmetry (though it is not unique in this) and secondly, because it is explained in terms of topological ordering (a very surprising result). It is unknown how many systems are topologically ordered, but such states of matter are a hot topic in condensed matter physics because they are both rare and specially structured.

The FQHE (Lancaster and Blundell 2014, Fradkin 2013, Wen 2004) is a collective behaviour of a system of electrons. At low temperatures the electrons behave like a liquid. We can measure a particular property of the electrons called the Hall resistivity, by applying both a magnetic field and a current. If we apply a magnetic field to the electrons and measure the Hall resistivity, we see that the it rises in proportion with the magnetic field, except for special plateaux where the resistivity stays the same even though the magnetic field does not. The existence of these plateaux is not the relevant feature, rather it is the values of the magnetic fields at which they are found.

In the integer quantum Hall effect (Lancaster and Blundell 2014, Fradkin 2013, Wen 2004), which was discovered first, the plateaux appear at values corresponding to energy levels being filled with electrons. But in the FQHE the plateaux appear at (usually odd) fractions of a filled level. This is significant because implies that that the electronic excitations of the system carry a fraction of a electron's charge. Each electron carries 1 charge, so any aggregate system made of electrons simply has a charge equal to the sum of all those charges – at least this is what a one might have thought. The existence of fractional charges implies that the electrons in the FQHE are coordinating in a very unusual way, it is as if at certain points the electrons are carrying only a fraction of an electron charge (thought impossible before the discovery of the effect). Note that despite sharing a name the integer Hall effect and the FQHE are completely different states of matter with radically different explanations.

There are different but formally equivalent ways of describing the FQHS (whether these descriptions are physically equivalent is a different matter). The three approaches are:

1) To write an effective field theory (Wen 2004). One can think of a field theory as a description of a system and an effective field theory as replacing a full description with one that captures only those properties of interest. Imagine producing a set of technical architectural drawings of a high-rise building. The full set of drawings might have, say, 50 levels to it. But perhaps we are only interested in what happens on the first 3 levels, so we instead replace our full set of drawings with a different set, this time only going to the third floor. Now, it's important to understand what physicists are doing with effective field theories. They are not just leaving parts of the description out; producing the same drawing with parts missing. Instead, they are producing a *new drawing of a different building*, one with the roof on the fourth floor and saying it is equivalent for describing the actual building as long as we are only interested in the low levels. *The effective field theory is not an approximation to the full theory, it is a formally different theory. In the FQHS the effective field theory is a topological field theory.*

2) To describe the system in terms of a new set of particles (quasiparticles called composite fermions) (Fradkin 2013). Composite fermions are electrons with flux quanta attached to them. The quasiparticles then describe the system in a more standard way (it's as if the system looks strange if we think of it as made of electrons, but if we think of it as comprising these composite fermions it is 'normal'). The important thing is that composite fermions are not local, the flux and how it is tied to each electron is determined by the whole system and the topological ordering.

3) To describe the system in terms of the pattern of long-range quantum entanglement between electrons (Chen, Gu and Wen 2010). Recall that simple entanglement is a quantum state where two particles become correlated such that a measurement on one that fixes its quantum properties also fixes the properties of the other particle. Long-range entanglement is a special pattern of quantum entanglement across the whole system that uniquely describes topologically ordered systems.

The three descriptions are for the most part simply different ways of describing the same thing formally rather than competing accounts of what is physically going on. For example, they make the same experimental predictions of the fields at which Hall plateaux are observed and of the properties of the anionic excitations. However, there is a reason for privileging the long-range entanglement explanation, since it is the description that most fully captures the system and makes explicit topological features that in the other two approaches remain somewhat implicit. Whichever way of representing the system captures the physical nature of the system topological ordering is important.

So, the fractional quantum hall state (FHQS) is a topologically ordered state of matter in which the electrons in the system form a network exhibiting long-range entanglement. Such long-range entanglements cannot be removed by renormalisation group techniques to redescribe the system, since the pattern of entanglement is responsible for key features of the FQHS, such as such a system's resilience to unitary transformations.

Recently Lancaster and Pexton (2015) have argued that the FQHS represents a form of ontological emergence. Because of entanglement there is a loss of mereological supervenience. The account of emergence of Lancaster and Pexton combines elements of the accounts of emergence of Humphreys (2016), and Teller (1986).

In Humphreys' account emergence occurs in both classical and quantum systems when aggregates form and fundamental properties (basal properties) are 'fused' together to form higher-level properties. The basal properties are completely lost. So, for example, a block of marble is made of particulates with their own mass individually, but the total mass is an emergent property and in the aggregate state it is no longer possible to say any of the parts have mass, since the basal mass ceases to exist.

The account of the FQHE by Lancaster and Pexton differs from Humphreys' fusion account in that the individual basal properties of the electrons are not lost. Instead, they (following Teller's account of relational properties) suggest that in a topologically ordered state the electron properties become fused so as to become inherently relational. The electron properties are still possessed by individual electrons but are now determined with reference to the superstructure of the whole system. This can be seen in the composite fermion account of the FQHE: the representational scheme for which one can describe the system in terms of separate particles (composite fermions) is actually describing the system in terms of electrons (with basal properties) tied ineliminably to flux quanta, where the flux is a whole system property. Equally in the representation based on long-range quantum entanglement, the electrons exist but their properties can only be determined with reference to the whole system (the topological order).

This manifests macroscopically in the FQHS being robust against unitary perturbations. In a regular quantum system, entanglements could be broken by perturbations from the environment (one of the difficulties in building an effective quantum computer). In a topological state the entanglements criss-cross the whole system in a special way, meaning no local perturbation can disrupt the system. The system responds to them as a whole, not locally, and preserves its macroscopic properties (hence a hypothetical topological computer wouldn't suffer in principle from environmental decoherence). The only way to disrupt a topological state is to do damage to the whole system at once.

This point is of crucial importance. Regular quantum systems with regular entanglements are subject to local decoherence perturbations. This is a 'smoking gun' to what is special about the FQHS. It is not just another example of quantum entanglement (with the associated philosophical argument about whether such entangled systems are emergent or not). Instead it is a quantum system with a particularly specially kind of order to the pattern of entanglements – those that lead to topological order – and it is this topological ordering that makes the system behave as a unified whole in which basal properties are only operative in relation to structural whole system properties. [2]

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[2] Note, that it is not every property of the FQHS that is inherently relational. All systems possess multiple properties, rather it is the properties that require reference to the topological ordering that have become fused and are inherently relational. The epistemology of the necessity of topological representation is evidence of a shift in the ontology of the target of that representation.

The question for this paper is: 'what is doing the ontological work in the FQHS?' Is it a case of exotic quantum entanglement in which the quantum nature of the system is responsible for the emergence, or are there classical systems that also display topological ordering and are these classical systems emergent in the sense the FQHS can be argued to be?

To establish this, we will examine a purely classical case of the role topological structure plays – in polymer physics (Edwards 1965). In some senses, these polymer cases share many formal analogues with the FQHE, and we argue that they also share the emergent status of the FQHE. To anticipate, we argue that the emergent qualities are not found in the quantum nature of a topological system, but in the contextually produced higher-level relational properties that define the possibility space of the system in question. It is the topological properties, in combination with the microphysical properties that produce a set of relations that govern the behaviour of the system.

This view presents a challenge to conventional notions of causal exclusion and provides a possible avenue for synthesis between traditional emergentist and reductionist intuitions. All levels of the system are included in feeding back into each other informationally to produce its characteristic regularities. This account of emergence is applicable to both classical and quantum systems. Quantum entanglement then becomes one striking physical mechanism for producing systems with such relational top-down dynamics, but it is not the only physical mechanism.

The philosophical claims in relation to polymer rings will be as follows:

1) Topological properties are a relevant set of coarse-grained but non-reducible properties. They are not captured or expressible by the microphysics alone.

2) Topological properties play a top-down role for determining elements of the microphysical state. To properly account for a topologically ordered system one must consider information flow in multiple directions from different scales of structure.

3. The Physics of Polymers

You can think of a polymer as a molecular string of beads (monomers) (Jones 2002). The physics of a polymer system is often derived from statistical physics – a large number of molecules subject to thermal fluctuations explores the parameter space according to different statistical regimes depending upon the constraints that operate. One of these is topology. The string can have different topologies – it could be open or tied in a loop with a knot etc. This means a system made of polymers can have different properties depending upon how the strings can cross and interact; how they interact is in part determined by their topology.

Polymer physics itself was established within the larger sub-field of soft condensed matter physics in the latter half of the last century. The nature of experimental and theoretical methodologies, epistemology and ontology in soft matter means that the field furnishes a rich conceptual hunting ground of emergence, top-down causation (Ellis 2012), potential anomalies in supervenience (Batterman 2002), and multiple realisability (Aizawa and Gillett 2009) for the philosopher of emergent phenomena. This is not surprising given that one the sources of the new subfield was the wider field of condensed matter physics (simpliciter), employed by Anderson (1972) in a landmark articulation of emergence in his seminal paper *More is Different*. Soft matter provides many illustrations of

Anderson's claim that the notion of the 'fundamental' in physics should not be tied to any one scale of length or energy.

Changes to the *topological* structure of polymer molecules themselves can have radical consequences for the viscoelasticity of their fluids. Adding, for example, just one branch point to each chain – so creating a melt of 'star polymers' (see figure 1) generates exponentially longer relaxation times (the time associated with large scale changes in the polymer) and higher viscosities (Milner and McLeish 1998). In short, the topology of a polymer chain matters. This is because the pattern of (classical physical) entanglement matters. For example, we have so-called star-branched patterns and an effective field theory of 'tubes'. Each of these patterns results in very different macroscopic properties for the polymer system.

One way of thinking about polymer physics is in terms of effective field theory. Both the static configurations and the dynamical response of the ensemble of macromolecular polymer chains is dominated by an effective 'field' of entanglements with the many other chains with which they are surrounded. The strong overlap of chains in a polymer melt is one consequence of the statistical structure of the chains being close to that a Gaussian random walk. So, their mean-square end-to-end distance R satisfies $R^2 \sim N$, where N is the number of monomer units in the chain. The volume enclosed by a single chain scales therefore as $N^{3/2}$. The number of other chains within the volume spanned by a single one scales, in turn, as $R^3/N \sim N^{1/2}$, which diverges for long chains (as $N \rightarrow \infty$). A single polymer chain overlaps with a divergent number of neighbours, and its effective interactions are therefore long-range (on the scale of its molecular monomers). It is this potential for long-range interaction between polymers, combined with their geometric string-like form, that permits the further emergence of both short-range and long-range topological order.

The latter, long-range topological order is of principle salience for the purpose of this paper, but it is worth noting that the resulting short-range topological field can be thought of as an effective 'tube' surrounding each chain (see figure 1), of a diameter that follows the density of topological constraints. It is straightforward to see that the diffusional dynamics of the chains will be constrained by this field to curvilinear motions of displacement (dominating for linear chains) and
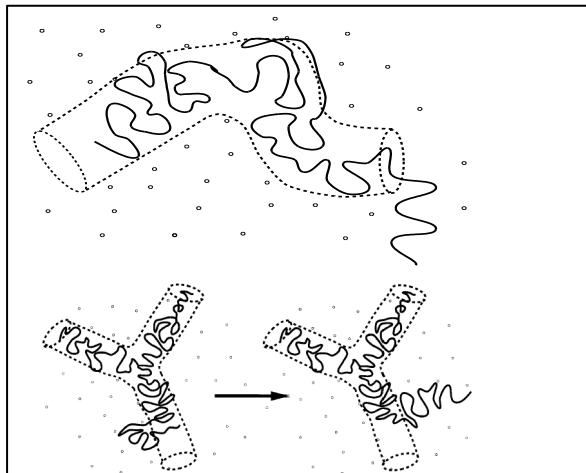

Figure 1. Entanglements have quite different effects on the dynamics of linear (above) and star-branched (below) polymers, but both find new configurations by diffusion of their free ends from the tube. For linear chains, the characteristic time for this disengagement process scales with the length L of the entanglement tube and the total chain drag, ζ, as $\tau_d \sim \zeta L^2$. Since both L and ζ scale proportionally with degree of polymerisation N, the final prediction is $\tau_d \sim N^3$.

The star polymer has to rely on much slower "retraction modes" to renew the configuration of its arms independently. These proceed *via* rare looped configurations that produce $\tau_{arm} \sim \exp(\nu N/N_e)$, where ν is a constant or order one.

arm-retraction (dominating for branched chains), providing that branch points are trapped at the meeting point of the tube-fields surrounding each joining arm. This is strongly the case for arm

numbers greater than 3 (McLeish 2002). In both cases new configurations are generated by re-exploration of the fluid by the free ends of chains.[3]

These cases are interesting, although the crucial topological effects that determine the behaviour of linear and branched chains are (i) still relatively short-ranged and (ii) temporary. It is also not possible to define the topological state of a pair of open chains in the formal mathematical sense, as this is only well-defined for closed curves.

A radically different situation arises in a dense system of overlapping polymer *rings*. In this case there are no free ends, so reconfiguration must happen by other processes. Second, just as in the theory of mathematical knots, the topological state of two rings is well-defined in terms of knot (or rather braid 'invariants') (Berrick *et al.,* 2009). Third, the structure at length-scales above the entanglement strand can be strongly non-Gaussian, as a consequence of *long-range topological* interactions between chains an entire chain-size apart. There are radical consequences of this set of true topological variables that make the melt or concentrated solution of ring polymers an extremely important case.

3.1 Melts and Solutions of Ring-Polymers – Microscopic, Emergent and Macroscopic Variables

So, ring polymer melts are a special class of polymer system. All polymer systems have topological variables (even if trivially) as background constraints. But ring polymer melt systems display topological ordering which is active in being the salient counterfactual difference maker in determining key properties of the system.

In all polymer systems explicit variables can be written down that track the microscopic degrees of freedom of the molecular chains themselves, and also the macroscopic variables of (for example) elastic stress. But in the case of ring polymers there is an intermediate set of emergent variables that is determinative on the future time-evolution of both microscopic and macroscopic variables. These represent the topological state of the polymer fluid. We will need to differentiate these variables formally, and mathematically, from the relevant microscopic co-ordinates, so in the following sections introduce the common, coarse-grained, mathematical notation used in polymer physics for these sets of variables.

Microscopic Variables

The distribution of polymer chain configurations in fluid states of polymers is dominated by entropy (Doi and Edwards 1986). When self-avoidance is cancelled or screened, the distribution of configurations is close to that of a 'Gaussian random walk' because this is simply the set of configurations with the highest entropy. The properties of this statistical object can be captured in terms of the *microscopic variables* of the system *r(n)*, by a 'two-point propagator', the probability density that a monomer *m* is at position *r'* given that *n* is at *r* is the Gaussian function:

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[3] The self-consistent dynamics of the polymer chains and their topological fields has been calculated within the mean-field limit justified by the strong overlapping of chains, and is in remarkable agreement with experiments on the macroscopic flows and stresses, and on the mesoscopic chain configurations predicted. The physics survives even in non-linear response (Bent *et al*. 2003), which is of interest to the polymer processing industry (Read *et al*. 2011).

$$P(\boldsymbol{r}' - \boldsymbol{r}; m - n) = \left(\frac{3}{2\pi b^2 |m-n|}\right)^{3/2} e^{-3(r-r')^2/2|m-n|b^2}$$

<div align="right">Eqn. 1</div>

The parameter b is the mean step length between monomers. The mean square displacement follows from averaging over this distribution (or more direct methods), so that a chain segment, or entire chain of length N has a mean square end to end distance familiar from the theory of general random walks.

$$R^2 \equiv \langle |\boldsymbol{r}(N) - \boldsymbol{r}(0)|^2\rangle = Nb^2$$

<div align="right">Eqn. 2</div>

If the chain is cyclised into a ring, again with excluded volume interactions screened, then the Gaussian form of the propagator is retained, but with the mean square displacement between two monomers calculated from the parallel addition of the two sub-chains that connect them. This leads to the expression of Eqn. 3.
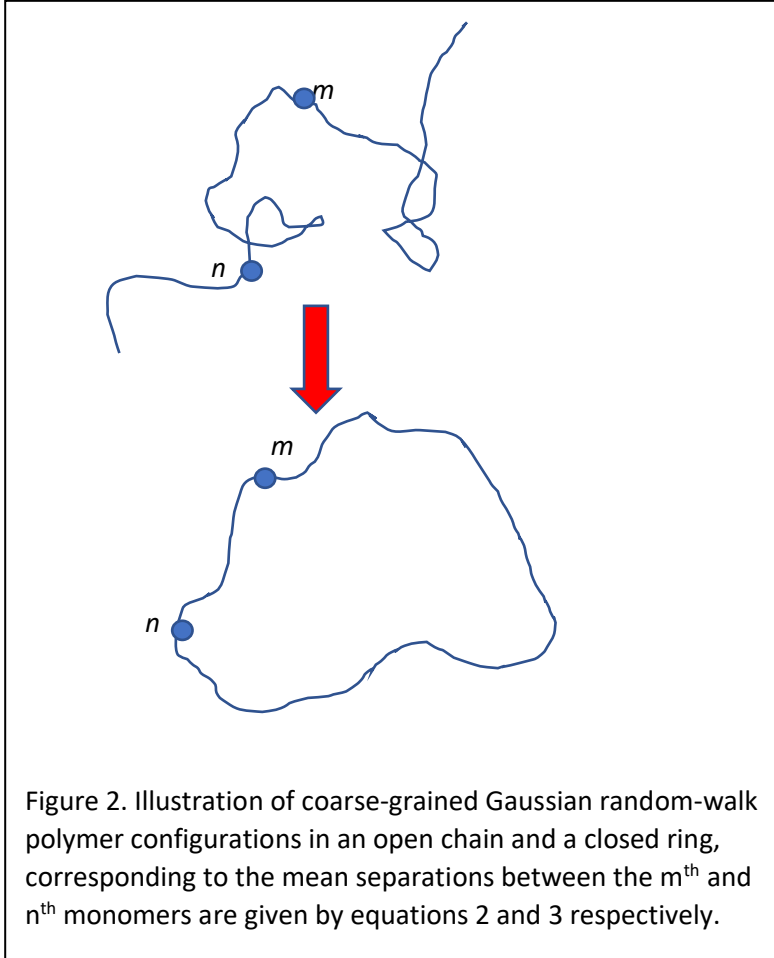
Figure 2. Illustration of coarse-grained Gaussian random-walk polymer configurations in an open chain and a closed ring, corresponding to the mean separations between the mth and n[th] monomers are given by equations 2 and 3 respectively.

$$P(\boldsymbol{r}' - \boldsymbol{r}; m - n) = \left(\frac{3}{2\pi b^2 \langle |r(m)-r(n)|^2\rangle}\right)^{\frac{3}{2}} e^{-\frac{3(r-r')^2}{2\langle |r(m)-r(n)|^2\rangle}}$$

$$\langle |\boldsymbol{r}(m) - \boldsymbol{r}(n)|^2\rangle = \frac{|m-n|(N-|m-n|)}{N}$$

<div align="right">Eqn. 3</div>

We note that the microscopic physics of the rings, as individual polymers, does not differ from that of simple linear chains, apart from details of the set of mean displacements between pairs of monomers.

Emergent Variables

As noted above, Gaussian chains embedded in 3-dimensional space are sparse, and do not fill space. A way to see this is to invert the scaling of displacement with chain length so that $N{\sim}R^2$ is read as a

mass-size scaling of an object with fractal dimension $d_f=2$. A procedure in which chains are cyclised after they are condensed into an overlapping melt therefore forms Gaussian rings which are individually overlapped by a number $n_o \sim R^3/N \sim N^{1/2}$ of other chains. This in turn will create a highly complex topological state – each of these cycles will, in the limit of large N, have with probability 1 that it loops around any other cycle with which is it overlaps in a non-trivial way. This is essentially because, although two chains are no more space-filling in a scaling sense than one chain, they do touch at a large number of points (divergent as the small-scale cut off – or step length b – tends to zero). In the neighbourhood of each of these contact points, large 'winding numbers' will typically be generated between pairs of rings (Grosberg and Frisch 2003).

A winding number is just a number recording how many times a curve loops around a specified point (on a plane) or around another curve (in 3D space). If the winding number is positive it means the loops flow counterclockwise, if it is negative it means they flow clockwise.

Winding numbers, or more-generally (and formally) the braid groups of any finite volume of ring-polymer melt (Quin and Milner 2011) are mathematical constants: if any two rings are linked (as in figures 3 and 4c), then they will remain linked at all future times. This is because chemically linked polymers cannot pass through each other. Like pieces of macroscopic strings, to un-link a pair of loops would require cutting one of them, passing the other through the cut, and re-attaching the cut ends of the first loop once more. We say that the polymers are 'uncrossable'. Because this cutting and rejoining operation can be performed at any point along one of the loops, and because this is the only operation that is able to change the winding numbers, they also qualify as *topological* properties of the system. A topological property is one that remains invariant under continuous deformation, and requires cutting and/or joining to change. An example of this set of topological variables is shown for one pair of rings in figure 3.
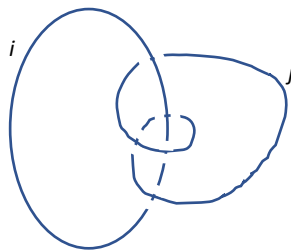

Figure 3. Schematic diagram of two rings *i* and *j* with a mutual winding number of $W_{ij}=+2$.

The set of winding numbers, W_{ij}, and their higher-order variants (there are irreducible topological invariants for triplets, quadruplets, etc. of rings that give additional information over and above their pair-wise topology), constitute our proposed set of *emergent variables*. They are reliant on the microstructural/molecular variable set for their definition, but supply additional information about the system, essential in the determination of its future time evolution.

Macroscopic Variables

The consequence of this history of cyclisation is that the system, while continuing to exhibit all the molecular diffusion over short ranges characteristic of a fluid, is actually a macroscopic elastic solid, for the same reason that a cross-linked polymer melt is a solid. The quenched degrees of freedom

(cross-links in the network, topological invariants in the ring-fluid) break translational symmetry and create solid (but non-crystalline) order.

A convenient way to encode the statement that the system is a solid is through the *macroscopic variable* of the (time-dependent) stress-tensor $\sigma(t)$. The ij^{th} component of σ_{ij} defines the ith vector component of the force transmitted across an embedded surface in the material with a vector normal in the jth direction. The stress can be measured for simple deformation histories, such as simple shear, by standard laboratory rheometric techniques. For any flexible polymeric fluid defined by the molecular variables *r(n)*, the stress tensor can be expressed in terms of a local average of them:

$$\boldsymbol{\sigma}(t) = \frac{3k_B T c}{b^2} \langle \frac{\partial r(t)}{\partial n} \frac{\partial r(t)}{\partial n} \rangle \qquad \text{Eqn. 4}$$

Here k_B is Boltzmann's constant, T the absolute temperature and c the concentration of monomers. The angular brackets denote a local spatial average over all chains in a small volume over which the macroscopic stress is defined. The statement that the fluid of linked rings is a solid can be stated in terms of the macroscopic variable $\sigma(t)$. After a finite imposed deformation, there is at least one component of the stress tensor that does not decay to zero as time tends to infinity. Note that, although there is an expression (eqn. 4) for the stress in terms of (an average over) molecular variables, the stress-tensor is macroscopic, and is exactly analogous to the pressure of a gas. Its macroscopic status is confirmed, in the same way, by observing the (extremely large) set of multiple realisations of the microscopic variables that give rise to an identical stress. It is also necessary to have a macroscopically large number of polymers (as gas molecules) before one can define the stress (as the pressure). Analogies to the gas-law equation of state have been proposed for the polymer stress (McLeish 2002), writing the macroscopic stress in terms of macroscopic properties only, but without at least approximate accounting for the topological interactions of the chains, these fail seriously, even at the level of approximation.

To track the time-evolution of the stress, for example, this expression (eqn.4) must be accompanied by the full set of dynamical equations for the polymer rings. This is where the set of emergent topological variables, generated by the polymers' long-range order, appears as determinative of the future time evolution. In complete generality (there are various candidates for the time-evolution operator $\mathcal{L}$, of a polymeric fluid, depending, for example, on whether hydrodynamics are essential or not) we must write:

$$\frac{\partial r_i(t)}{\partial t} = \mathcal{L}\big(r_i | \{W_{ij}\}\big) \qquad \text{Eqn. 5}$$

The time-evolution operator is constrained by the permanence of the values set for the emergent topological variable, denoted in equation 5 by the set of winding numbers $\{W_{ij}\}$.

A radically different outcome emerges in the case of unlinked rings. These are typically prepared in dilute solution, where the cyclisation occurs without linking one chain with another. All winding numbers are zero in this case. Removal of solvent then concentrates the ring polymers until they overlap strongly (Everaers *et al.* 2004). However, there now arises a potential paradox: the linking numbers, and higher topological invariants of the rings (of higher-order than the 2-body linking

numbers, akin to braid polynomials) are all trivially zero, and permanently so because of the uncrossability of the chains. Yet were the rings to assume their topology-free Gaussian structures, it would be impossible to sustain this trivial topology.

So, although the zero-winding number case might appear a case in which topological order plays no part this is not correct – the system still depends crucially on topology. Zero winding number is not a non-topological case, instead it is a proper subset of the topological ordering.

In consequence, the rings *collapse* on all length-scales larger than the entanglement distance. The huge 'topological pressure' on the configuration of the rings by virtue of the zero-linking criterion can be recognised from an early argument by Cates and Deutsch (1986) (which turns out actually to underestimate the true effective pressure). By assuming that one degree of freedom is lost with each ring overlapped in the Gaussian state (in reality there is at least one degree of freedom lost per entanglement) an approximate scaling form of the free energy per chain as a function of its radius of gyration *F(R)* becomes

$$\frac{F(R)}{k_B T} \approx \frac{R^3}{b^3 N} + \frac{Nb^2}{R^2}$$

Eqn. 6

The first term counts the compressive pressure of the interchain topology, the second the intra-chain free energy of confining a Gaussian walk into a region of radius $R < Nb^2$. Minimisation predicts a scaling behaviour, $R \approx N^{2/5}$ corresponding to structures intermediate between complete collapse, $R \approx N^{1/3}$ and free statistics , $R \approx N^{1/2}$.

More recently, corrections to this result (Sakaue 2012), as well as intensive simulations (Rosa and Everaers 2014) have shown that the most likely structure for sufficiently long ring-chains is one of a family of essentially completely collapsed states in which individual rings are locally dense, called the 'collapsed globule' (Halverson *et al.* 2011). This structure had previously been proposed as a candidate for the densely packed form of DNA in the cell nucleus of prokaryotes called chromatin. Though uniform in density, it contains a self-similar mapping of contour co-ordinate to spatial position, packing chain segments in a 'Russian Doll' configuration (see figure 4d).
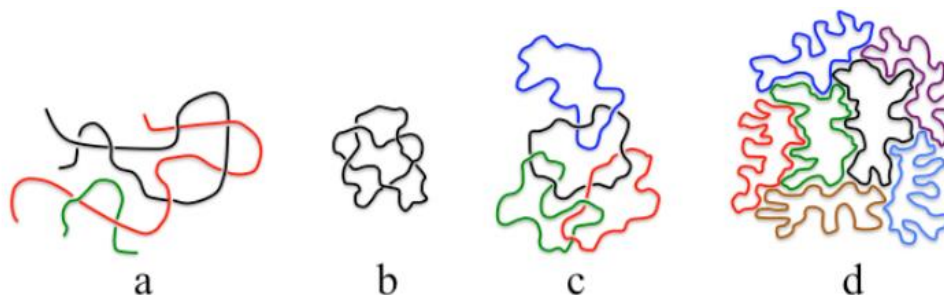
Figure 4 Possible configurations of polymer chains in different topological states in a melt of other chains; (a) linear chains in Gaussian interpenetration; (b) A single ring-chain self-knotted (the trefoil knot); (c) a melt of Gaussian ring-chains with typically complex topological invariants leading to an emergent network; (d) melt of rings in trivial topology in crumpled globule states.

Note that the use of the term 'trivial' here is in the sense used by physicists. It does not mean that topology is trivial in the ordinary language sense of irrelevant or unimportant, instead it means the topological state is of the simplest possible configuration.

The essential physical consequence of the topological difference represented by the systems of figure 4(c) and (d) could not be starker. The 'topological pressure', exerted on the ring molecules by the constraint of zero linking numbers, is enough to render them compactly collapsed, compared to an extended random walk configuration. This is as strong as the hydrophobic collapse of proteins or the chemically-attractive collapse of polymers in poor solvents. In terms of statistical mechanics it is akin to the phase change from a gas to a liquid: in the case of post-melt linking (4(c)) the chains explore the maximum translational entropy in each of their constituent parts consistent with their connectivity: a 'gas-like' entropy. In 4(d) they are self-dense, though locally mobile: a 'liquid like' state. But there are no candidates for local monomer-monomer interactions (forces) that could trigger such a collapse at all. The only difference is contained in the information at the coarse-grained level of the topological invariants, constituted non-locally at the level of entire chain configurations.

In consequence of the trivial, and permanently trivial, values of all of the emergent topological variables, the ring fluid sustains no permanent topologically connected path across a macroscopic sample. A general consequence of this is that, in this case, the stress tensor, still given by the expression in equation 4, now does decay to zero under the dynamics of equation 5. The only difference is in the set of values adopted by the emergent variables.

It is helpful to note that the 'melting' of the ring polymer system in passing from a state of high entanglement to a topologically trivial one, is not one that can be mapped onto a thermodynamic melting transition, as there is no change of symmetry. The closest phenomenon known in statistical mechanics is a percolation transition (Jones 2002), but in the case of the topological emergent variables, there does not exist a single parameter such as the probability of connection between neighbouring sites that tracks the approach to the transition in that case, which would allow the order of the transition to be well-defined.

4. Chern-Simons theory and Connections to the FQHE and Polymer System

Since the entirely classical example of the polymer ring system demonstrates a very close similarity in emergent properties to that of the Fractional Quantum Hall Effect (FQHE) of Lancaster and Pexton (2015), we might wonder if that similarity arises from structural properties in common that bear mathematical analysis, even given the very different physical regimes in which they arise. Our claim is that the FQHE and polymer ring melts are examples of extreme multiple realisability. The two cases represent very radically different microphysics, yet through the manifestation of topological order they both develop a crucial emergent structural property (the pattern of topological connections) that acts as a control variable constraining the further microscopic evolution of the system and allowing the manifestation of subsequent macroscopic bulk properties.

A remarkable connection has indeed been demonstrated between the case of classical topological long-range order exhibited by polymer rings, and the associated phenomenon in the quantum mechanical system of the FQHE. We summarise the, rather deep, mathematical mapping that this involves here, as it indicates that the cases are more closely connected, and have more consonant philosophical implications, than the usually-assumed gulf between classical and quantum physics would suggest. Although the most technical section of this paper, the details here underpin our claim that the emergent properties of the quantum and classical systems share the same structural origin.

As summarised in the introduction, the fractional quantum Hall (FQH) fluid (Lancaster and Pexton 2015, Lancaster and Blundell 2014, Fradkin 2013, Wen 2004) is a state of matter realized in materials when electrons in a metallic or semiconducting sample of solid matter are constrained to move in two dimensions, and when a sizeable magnetic field is applied. The constraint to move in two dimensions

is a quantum mechanical one, resulting from a quantization of the particles' momenta in one of the three spatial dimensions, owing to the structure of the materials that host the electrons. The applied magnetic field enhances the electron-electron interactions (that are caused by the electrostatic Coulomb interaction between the electronic charges). The dual effect of (i) the quantum mechanics of the electronic energy levels in a magnetic field; and (ii) the electron-electron interaction, lead to the formation of the FQH fluid. This state of matter is characterized by the possession of topological order, and particle-like excitations that carry charge in units of fractions of the electron charge. The ordering of the matter should not be confused with order that follows from the breaking of a symmetry: mathematically, the FQH fluid has the same symmetry as the metal /semiconductor from which the state was formed.

The main properties of the FQH fluid are the following. In each case we identify the structural connection with the polymer case:

• *The ground state of the system is incompressible.* The FQH fluid is a macroscopic state of matter that has a gap in its excitation spectrum between its ground state and its excited states. This implies that the creation of an excited state of the system requires the input of an appreciable quantum of energy. This is distinct from the metal, where a vanishing amount of energy can excite an electron into an excited state of the system. An attempt to squeeze the FQH fluid therefore fails, owing to the absence of states for accommodating the squeezed fluid. This incompressibility is a form of *generalized rigidity* and tells us, via a classification due to Philip Anderson (Anderson 1972), that the system is ordered. As mentioned above, the order is not due to symmetry breaking; instead we call the order topological. The rigidity of the FQH fluid is the clearest property of ground state that serves as a candidate emergent property. In the ring polymer system, the analogous candidate emergent property is the liquid-like behaviour of the collapsed state.

• *The FQH system is built from electrons.* A semiclassical interpretation of the electron motions is helpful here (Wen 2004). It follows from the quantum mechanical Laughlin wavefunction (Wen 2004), and allows us to assign a correlated set of motions to the electrons in the FQHS. This is a way of visualizing the (non-symmetry breaking) order in the system as a dynamic state involving all of the electrons in the system undergoing correlated periodic motions. As described above, the polymer system is built from polymer rings, whose collective, static configuration characterizes the state of the polymer in an analogous manner.

• The excitations of the FQH system are fractionally charged identical quasiparticles with fractional statistics, known as *anyons*. That is to say, the quasiparticles carry fractions of an electronic charge and have quantum statistics that are neither those of a boson nor a fermion. If we drop an additional electron into the FQH fluid, it appears to fall apart into these quasiparticle excitations, which are the allowed expressions of the excited states of the FHQ fluid. The existence of these fractionalized quasiparticles is predicted and described by the *Chern Simons* (CS) theory of topological matter.

It might be expected that it is difficult to make a link between these particle-like states in the FQH fluid and a property of the polymer system. Remarkably, however, the path of a polymer ring in space can be reinterpreted in terms the trajectory of a quasiparticle. This involves interpreting the path in two of the three spatial dimensions as the path of the particle and the third as a time-like variable. The polymer ring, then, represents the movement of a particle in two dimensions as a function of time. This is similar to the case of the quasiparticles in the FQHS, which are constrained to two-dimensional space and undergo motion as a function of time. The result of this formal manipulation is that, just as the quasiparticles in the FQH fluid exhibit anyon statistics, the particle-like states we obtain with the

above interpretation of the polymer path also exhibit anyon statistics. This is discussed in more detail below.

From a philosophical standpoint, a remarkable feature of the FQH state is that is seems only to be explainable by considerations of the whole system. Lancaster and Pexton (2015) explain the 'whole system' nature of the state in terms of relational properties. Properties such as charge cease to be linked to particular electrons and, instead, belong to the system taken as a whole. The analogous polymer case is examined in more detail in the second half of this paper.

The most dramatic of the properties described above is the emergence of anyons in both the FQHE and polymer ring systems. We review some of the mathematical features used to describe anyons in both cases below. The first step towards finding them in the latter case is to recast the polymer problem as a topological *field theory*, a move that was first suggested by Edwards (1967). The field theories that we examine here describe topological states of matter and should be viewed as *effective theories* (Lancaster and Blundell 2014, Bain 2016), in that they aim to capture the low-energy behaviour of an interacting system without appealing to a particular mechanism for their derivation. [4]

The key distinction is that non-topological field theories usually rely on the use of the metric tensor $g_{\mu\nu}$, which may be thought of as an object that tells us how to form scalar products, and therefore determines the lengths of space and time intervals in a physical system. In contrast, *topological* field theories are blind to the details of geometry, and so do not use $g_{\mu\nu}$. Instead, these theories rely on the use of the antisymmetric symbol $\varepsilon^{\mu\nu\lambda}$ to make products, such that the theory does not possess a sense of length and time-scale. This can be seen as the reason why whole-system properties dominate the physics: many key properties in a topological theory are dominated by the shape or character of the whole manifold in which the mathematical theory is based (as this determines the number of states available to a system), while the structure encoded in the manifold by a metric is irrelevant.

Field theories are generally described by the use of a Lagrangian density (Lancaster and Blundell 2014), which is a mathematical expression that describes the energetic content of the relevant matter fields in a problem. On a technical level, a *topological* field theory will feature a *topological term* in its Lagrangian density *L*. The simplest of these is the CS term $L = -1/2\ \kappa\ \varepsilon^{\mu\nu\lambda}\ a_\mu\ d_\mu\ a_\lambda$, where a_μ is a topological gauge field, d_μ is the derivative operator and κ is a constant. The topological field a_μ is unlike many other fields in that it has no dynamics of its own (this is quite unlike the electromagnetic field, for example). Instead, the topological field provides *constraints* on other fields in the problem, which force them to exhibit whole-system, topological properties. Note that the CS term applies exclusively to the case of (2+1)-dimensional space-time (i.e. two-dimensional space and one time coordinate) and so applies to the FQH and polymer ring cases, as described above..

For the CS term to provide the necessary constraints to lead to anyons, we must couple it to a current J_μ of some field, giving us a theory like:

$L = -1/2\ \kappa\ \varepsilon^{\mu\nu\lambda}\ a_\mu\ d_\mu\ a_\lambda\ -\ J^\mu\ a_\mu,$ Eqn. 7

That is, the CS term is added to the coupling of the topological field and a current of particles of some field. The main physical consequence of this set up is that the CS term attaches the flux of the CS field to the particles of the coupled field, forming a compound variety of flux-carrying particle.

In the FQH case, the CS field a_μ is coupled to, and therefore constrains, the electromagnetic field (i.e. the field that determines the dynamics of the charged electrons) (Lancaster and Blundell 2014). We also couple the CS fields to a source of quasiparticles, which represent the excitations of the whole system. If we consider the Aharonov-Bohm phase that results from exchanging the quasiparticles, and encode in the theory that the electronic state is built from electrons or holes, whose charge is, by definition, a single unit of electronic charge, the quasiparticle excitations can be shown to exhibit fractional, anyonic statistics. [This method of assessing the characteristics of the quasiparticles has been suggested by Shech (2015) (see also Earman (2016)) to avoid possible complications derived from the dynamics of electrons not being perfectly two-dimensional. However, this is generally a small effect in real physical systems and is not a consideration in the polymer case, so we do not discuss it further.)

In the ring polymer case, the CS theory is used to encode topological constraints on the polymer rings themselves (Ferrari 2004). The CS fields can then be interpreted physically as carriers of the topological interactions that constrain the polymer system to remain in a given topological state. In order to apply the theory, the description of the rings must be cast into (2+1)-dimensional space-time as described above. The path of the ring in the three spatial dimensions of real space is interpreted in terms of a path in two spatial dimensions (the *x-y* plane), with the third dimension (the *z*-axis) playing the role of time. The polymer rings themselves then form trajectories described by the coordinate triplet ξ^μ *=(t,x,y),* corresponding to a point in the *x-y* plane moving as a function of time *t*. The CS fields are coupled to the current formed from the velocity of the ring coordinates *dξ^μ/dt*. The zeroeth component of this quantity (known as the charge) simply represents an element of the polymer chain itself. We therefore have a picture of a piece of polymer chain, coupled to the flux of the CS field as the entity that possesses fractional, anyonic statistics. If we picture the trajectories of the chains along the *z*-axis literally as paths of particles in time, then the anyon statistics follow if we regard the *x-y* coordinates of a polymer chain as analogous to quantum particle at a particular instant in time. Treating these particles as indistinguishable and then exchanging them in the *x-y* plane (just as for the FQH case) results in the wave-functions changing according to the anyon statistics that characterize a CS theory.

The mathematical structure common to the ring polymer system and the FQHE demonstrates (formally) that there are correlatives of these properties either explicit or implicit in the polymer case too. In so far as one accepts that the long-range topological order in a FQHE fluid determines the properties of the fundamental excitations, and *not* the bottom-up electronic properties of spin-1/2 electrons, or at least not entirely, then this must hold true for the classical case of entangled rings as well.

5. Ontological Emergence in Polymer Rings and FQHE

As we have seen there are many formal parallels between the physics of the FQHE and of polymer rings, the question for us now to consider is what we can philosophically learn from this.

The philosophical argument for the emergent nature of the FQHE due to Lancaster and Pexton relied on the notion that there was a failure of mereological supervenience in the FQH fluid. Supervenience has long been a topic of discussion concerning emergence (Humphreys 1997). The essential idea is that if we have two levels of a system, A and A*, then if A* supervenes on A then we cannot change

A* without changing A. Supervenience in and of itself is relatively trivial. It need not for instance imply causal relations or determination of any kind. It can also be mutual: A* could supervene on A and A supervene on A*. Most reductionist discussions of supervenience implicitly take that the particular form of supervenience at hand is delineated in certain ways that imply some form of *determination* relation not just supervenience simpliciter.

This is the basis of Kim's (Kim 2007) famous causal exclusion argument. A higher-level causal process is linked to a parallel lower-level causal process by supervenience. For Kim, causal overdetermination is undesirable, hence we exclude the high-level causal processes as being merely epistemic. The real causation is at the lower-level because changing the higher-level state demands a change to the lower-level state. The supervenience relation is used implicitly to suggest the higher-level is merely derivative of the lower level. Note the difference though. Supervenience only says that the higher-level cannot be changed without a correlate change in the lower level, but Kim's use of it implies that fixing the lower-level determines the higher-level. It is this second, stronger interpretation of supervenience, which we will call mereological supervenience, that is far from justified. Indeed, in the FQHS there are multiple directions of supervenience but there is a failure of mereological supervenience – the parts do not determine the whole, rather the parts and structural features of the whole interact to produce the particular set of properties manifested in the FQHS.

Note that one might object that mereological supervenience should be restricted to simply mean the whole supervenes on the parts. What does a breakdown of mereological supervenience mean then? For Humphreys mereological supervenience breaks down because the basal properties cease to exist. Changing the parts cannot change the whole, since there are no parts to change anymore, only the whole to directly change. We do not agree with this version of fusion. Instead, we argue that the basal properties become inherently relational. Mereological supervenience fails then because to change the parts is no longer well defined as a distinct process from changing the whole. Note that both failures of mereological supervenience imply a failure of exhaustive microphysical determination as well: in Humphreys' case because there are no microphysical properties to determine anything, in our account and Lancaster and Pexton (2015) because microphysics now includes elements of the whole and so isn't distinct from the whole.

The FQHS is a collection of electrons where the electrons have fused their properties due to long-range quantum entanglements. In the FQHS the electrons form a composite set of quasi-particles who derive their properties from the collective state of the system. Certain properties of the electrons become 'inherently relational' in the FQHS *meaning that these properties are determined by the electrons only in the context provided by the network of entanglements, not by the electrons alone*.

When mereological supervenience holds the whole cannot change without changing the parts, in some contexts when combined with supplementary arguments such as Kim's causal exclusion, mereological supervenience is used to imply a stronger conclusion: that the whole is *determined* by the parts. In Humphreys' scheme mereological supervenience fails because the parts essentially cease to exist (or the salient basal properties cease to exist), there are no parts to change. One might be tempted to say that there is a loss of supervenience in the FQHS since the basal properties of the individual electrons are lost, but it is more accurate to say that the supervenience base is shifted away from the mereological[5]. The global properties no longer supervene on individual electron properties because those individual properties have become fused to be inherently relational collective

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[5] It is worth stressing that we are nowhere implying that the objects comprising the lower level become indistinct from the higher level. In the FQH case, for example, it is necessary to make use of the fact that the system is built from electrons in order to correctly account for the values of fractional charges and the statistics of the fractional excitations. (Lancaster and Pexton, 2015).

properties themselves, instead the individual particle properties, technically the quasi-particle 'composite electron' properties (see Lancaster and Pexton (2015) for a full discussion) are supervenient on global topological properties of the FQHS. The basal properties have not been lost in the FQHS, rather they have become inherently relational.

Is such a loss of supervenience also at work in the polymer case? Insofar as the polymer case is classical and there is no quantum entanglement and loss of basal properties, the answer is no. In the polymer case, we still have definite microphysical monomers with their own properties. But there is a corresponding tying of (some of) the properties of the parts to the (some of) properties of the whole due to topological ordering. We therefore argue that those basal properties have similarly become inherently relational with respect to the global topological ordering of the parameter space. This transition from individuated basal properties, to basal properties that must make reference to the state of the aggregate (insofar as they must make reference to the topological structure) signals a similar loss of mereological supervenience. This conclusion is additionally supported by the mapping between the two cases at the technical, structural level detailed in the last section.

However, we fundamentally believe that whether one thinks of these cases in terms of the loss of supervenience or not is not the crucial factor. Supervenience itself is trivial. The loss of mereological supervenience is simply suggestive of the important factor: *an inversion of which properties determine which other properties from bottom-up to top-down*. Even if supervenience has not been lost (although in the polymer case we believe that it is – see the technical arguments in section 6 below), we believe the case is still strong that the topological ordering constrains the lower level states in a way that requires higher-level topological structure to be taken ontologically seriously in these cases.

In the polymer case, therefore, although the loss of basal properties does not pertain, as formally expressed in terms of the direct expression of the macroscopic stress in terms of the microscopic properties (Eqn. 4), the time-evolution of those microscopic properties requires the values of the emergent topological variables for its determination.

Hence, to locate the seat of emergence in the FQHS as due to quantum entanglement is to make a mistake. What matters is that crucial properties for determining the system are inherently relational and depend on the global state of the system, not that properties necessarily become fused due to entanglement. Entanglement is one way in which such properties can become inherently relational, but it is not the only way. In the following section we describe the causal route to change inherent relations, and claim that it is adopted by the polymer ring system as much as in the FHQS.

Determination or Top-Down Causation?

Should topological order (or any topological property) be thought of as causal? We remain agnostic as to the causal status of topological order. By some accounts of causation topological properties will be explicitly causal, by other accounts they will not be. It does not matter for the question of emergence. Instead what is important is whether the topological structure/properties are necessary to determine physically what the system does.

Our intuition is to follow an explanatory indispensability argument based around a counterfactual account of explanation. That is, to explain is to provide counterfactual difference makers; to correctly account for the set of modal (counterfactual) information required to complete an explanans. These types of indispensability arguments are not infallible to be sure (see Pexton (2015)), but they are a useful guide to ontology. In such arguments we imagine any possible explanans for a given explanandum of a phenomenon. If a feature must play a role in the explanans, and it cannot be

removed in principle (not just in practice due to epistemic limitations) then that feature, or something structurally similar to it, is (to a scientific realist at least) worthy of consideration as part of the ontology of the world.

This is one way of thinking about Kim's causal exclusion argument. Consider two competing explanations, a macroscopic one in terms of A and B and a microscopic one in terms of A* and B*. A explains B. A* explains B*. If we have supervenience, such that A supervenes on A* and B on B*, then we could say any explanation in 'A and B talk' could (in principle) be translated into an explanation using 'A* and B* talk'. There are two questions then? In the polymer case can all topological order based explanations be translated into accounts that don't feature topological properties? It is our contention that they cannot. Just as in the FQHS where we had different explanans but all either explicitly (or implicitly) contained topological information so too in polymer melts any explanans for certain macroscopic properties will have to contain information about the topological ordering. If any explanans in principle must contain reference to topological ordering then we believe indispensability implies that topological ordering should be regarded as part of the ontology of the world.

6. Top-down Causation and Connections to the literature

How does our account connect to other accounts of emergence that are explicitly causal? We believe if one wants to think in causal terms then the polymer ring system is at least a plausible candidate for top-down causation.

In this section we take the characteristics of top-down causation proposed by Humphreys (1997) and argue that on these terms, the polymer case displays, like the FQH fluid, top-down causation. We anticipate and meet the objection that the emergent topological variables, at least in the classical case, are expressible in principle in terms of the microscopic ones, thus rendering them dispensable. We also point out how this account differs from the general formalism of top-down causation advanced by O'Connor and Wong (2005).

The ring-melt crumpled globule state is a *prima facie* example of a classical system responding to statistical mechanics rather than quantum mechanics, but which demonstrates the same failure of supervenience of the fine-grained variables, and instead, much like the FQHE, an example of causal flow from long-ranged topological variables onto the molecular, fine-grained ones. It satisfies all of the criteria required by Humphreys for top-down causation (Humphreys 1997):

- The *principle of equivalence*: there exist equivalence classes of fine-grained variables that correspond both the same coarse-grained, high-level variables *and* to the same macroscopic, high-level outcome. In this case the equivalence classes are all the chain configurations commensurate with the same set of topological invariants.

- *High-level changes alter low-level processes*. The change of any topological invariant will alter the range of configurations explored by at least two ring polymers; their monomers will explore local states after the change with a different distribution than before.

- In terms of Ellis' classification of top-down causation (Ellis 2008) the topological ring systems falls into the class of 'algorithmic top-down causation'; 'high-level variables have low level causal power over low level dynamics through system-structuring'.

- *Indeterminacy of low-level variables*; also referred-to as 'room at the bottom', this is a critical point to the establishment of top-down causation (rather than bottom-to-bottom causation simple *via* high-level variables). We treat this rather subtle issue in the following.

If one wishes to follow through this example in terms of a powers-based account of causation, it is only through the manifestation of topological powers that the casual structure of the system is exhausted. The monomers alone cannot exert causal power on each other to the extent required to structure the state space appropriately. It is through the expression of topological powers that the state space is structured appropriately. This can be expressed counterfactually in terms of modality. It is the top-down topological features that express the counterfactuals which would be necessary for a full casual explanation of the history of the polymer ring melt.

It might be objected that, although the highly significant topological variables are indeed determining of the local state of the fluid, they are in turn determined by the collection of fine-grained molecular variables. Specifically if the set $\{r_i(n;t_0): 1<n<N, 1<i<M\}$ of the spatial positions $r_i(n)$ of monomer n on chain i, for every one of the N monomers on M chains in the system, is specified at some time t, then in principle an objection to our claims would claim that it ought to be possible to calculate all the topological invariants that the set of rings must conform to at all future times $t>t_0$. For example, the simplest possible topological invariant is the mutual winding number Ω_2 of just two chains. In this case an ostensibly explicit formula exists in terms of the contour co-ordinates:

$$\Omega_2 = \tfrac{1}{4\pi} \oint ds_1 \oint ds_2 \, (\dot{r}(s_1) \times \dot{r}(s_2)) \frac{(r(s_1)-r(s_2))}{|r(s_1)-r(s_2)|^3}$$

Eqn. 8

There are recipes for the calculation of higher-order invariants (Jones polynomials) expressed in terms of path-crossings in two-dimensional projections of the three-dimensional knots and braids, and Edwards' early suggestion of how a subset of these might be cast into expressions of the same class as the winding number expression has recently been taken up in the application of knot-polynomials to the quantification of polymer entanglements (Quin and Milner 2011). However, this expression and others like it, but of higher order, including the chain-crossing information required to specify knots, while formally appearing to be well-determined, need some very careful qualification to be meaningful, to the extent that they support our claim that the topological variables are fundamental and not superfluous. The appearance of a microscopic, 'bottom-up' determination of the essential topological invariants is of formal status only, and expressions such as eqn. 8 require the supply of this extra information in principle (rather than simple in practice). There are several ways to see this:

F1 *Formal requirement of small-scale cut-off*. The (Wiener) Hamiltonian for the random walks whose paths are represented by the functions *r(s)* ensures continuity, but not differentiability. The walks are self-similar on all length-scales. In consequence, several properties are formally divergent – even the contour length of a polymer is one such, growing as the size of the small-scale measure of step length decreases. Another divergent quantity is the topological invariant of two interacting polymer rings. Setting the necessary cut-off scale (universally) is equivalent to setting the topological state of a dense melt of rings. The set of topological invariants is an *additional* piece of information to the existence, and the spatial locality, of the polymers themselves

F2 *Chemical history is additional information*. The way that this formal, mathematical requirement of additional, high-level, information is reflected in practice is *via* the chemical history of an experimental polymer sample. The two extremes of (i) maximal and (ii) minimal linking are given by the two time-orderings of (i) condensation then linking, and (ii) linking then condensation. Two chemically identical samples, of structurally identical rings of the same number and molecular weight distribution, are

differentiated by their chemical history. In an experimental context this information serves as a proxy for the additional information required of their topology. It is required in additional to all information about the molecular structure.

F3 *On-lattice models require information on ordering*. A common modelling technique deployed in molecular simulation of many-body polymer simulations is to represent all chains as occupying a lattice. A single lattice-site maps onto a spatially-extended sub-chain of a real polymer. A sufficiently coarse-grained representation of real-space chains at high density therefore requires coarse-grained monomers of different chains to occupy the same site. This case makes explicit that the topological arrangement of the chains (which passes over which) constitutes an additional piece of information to the configuration (Quin and Milner 2011). Furthermore, in order to create a faithful lattice model of a system of real chains, the set of crossing orders cannot be decided locally, but must itself be set globally to honour the global invariants of the chains.

At first glance, the example of ring polymers looks like the case in which the high-level variables constitute a causal bottle-neck for processes that may ultimately be attributed to low-level variables. But F1-F3 above (each as aspect of the same underlying incompleteness of the low-level variables) indicate otherwise. Determination of the non-local, globally-defined, topological state of the ring-fluid requires the input of information not possessed by the set of low-level variables.[6] Furthermore, they determine equivalence classes of low-level variables which correspond to different, and in some cases radically different (liquid vs. solid) high-level behaviour.

How does the polymer case avoid the possibility of causal overdetermination? Consider again Kim's argument for causal exclusion. Here we have two parallel causal processes: A* causes state B* but A* supervenes on a lower level state A which causes the lower level state B, which B* supervenes on. Kim's argument is that we have two competing causal stories, overdetermination is bad, we exclude the high-level story since the lower level supervenience base fixes the higher-level state anyway.

In the polymer case, the argument would suggest that although we can define a causal chain involving topological properties, those properties supervene on local monomer properties and so we have a competing monomer causal story which ontologically excludes the topological causal story. As we have already alluded to, we believe such a causal exclusion recipe fails for the polymer case (and the FQH fluid) because topological ordering means basal properties are inherently relational and so the higher-level and lower level explanatory stories are not competing they are instead complementary accounts of the same unified relational structural determination of the available parameter space.

All causal connections require constraints. To define the causal arrow from A to B we implicitly use the constraints applying to A*-B*. In the polymer and FQH fluid cases we can see evidence for this in that there is no way of writing down a bottom level causal model of the system that hasn't already included structuring information derived from the higher-level topological features. Therefore, why do we exclude A*-B* for A-B? There is no proper causation at A-B (only retrospective epistemic mapping of causation). The causal chain described in microphysical terms alone is only possible because it implicitly relies on the context provided by the higher-level structural features. If we follow Kim we make a mistake by metaphysically prioritising a supervenience base, instead we should

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[6] We might ask how it is possible for a change in the higher-level variables not to lead to a lower-level change. This possibility arises formally owing to the existence of 'boundary states' of microscopic variables, corresponding to high-energy states of crossing polymer chains, which could belong to either of two sets of well-defined topological states. Only once the topological variables are defined do the polymers thenceforth explore configurations only within the corresponding subspace of phase space. These subspaces include other boundary states within their closed sets, but their open sets are exclusive of other topological states.

metaphysically prioritise the level at which the constraints obtain that allow the manifestation of a reliable casual relation. Therefore, we describe the polymer case and FQH fluid as examples where mereological supervenience fails, not cases where the most trivial and thinnest notion of supervenience fails. It is of no matter if there is a basic supervenience relation, what matters is if we cannot account for the manifestation of a counterfactual causal relation (or equivalently a set of causal powers) by invoking only the lower level state of the system. In the neither the FHQ nor the polymer case can we do this.

We may also discuss the polymer ring example in terms of the general scheme proposed by O'Connor and Wong (2005), who add to Kim's two-level treatment of variables A,B and A*,B* a third set of emergent variables E that rely on the existence of the A,B variables, but are not determined by them. They are, however, necessary in the determination of future evolution of both A,B and A*,B*. In their case E may also create the basis for higher levels of emergent variables E_n. In this account the E-variables also demonstrate time-evolution. The comparison may be diagrammed using the same notation as O'Connor and Wong (2005); such a representation is given in figure 5.

Dynamical evolution of system S over time

Dynamical constraint of polymer ring system for all time

⟶ upward causation of baseline emergent state E
- - -➤ upward causation of super-emergent states E_n
⟶◆ maintenance causation of emergence-sustaining configuration P*
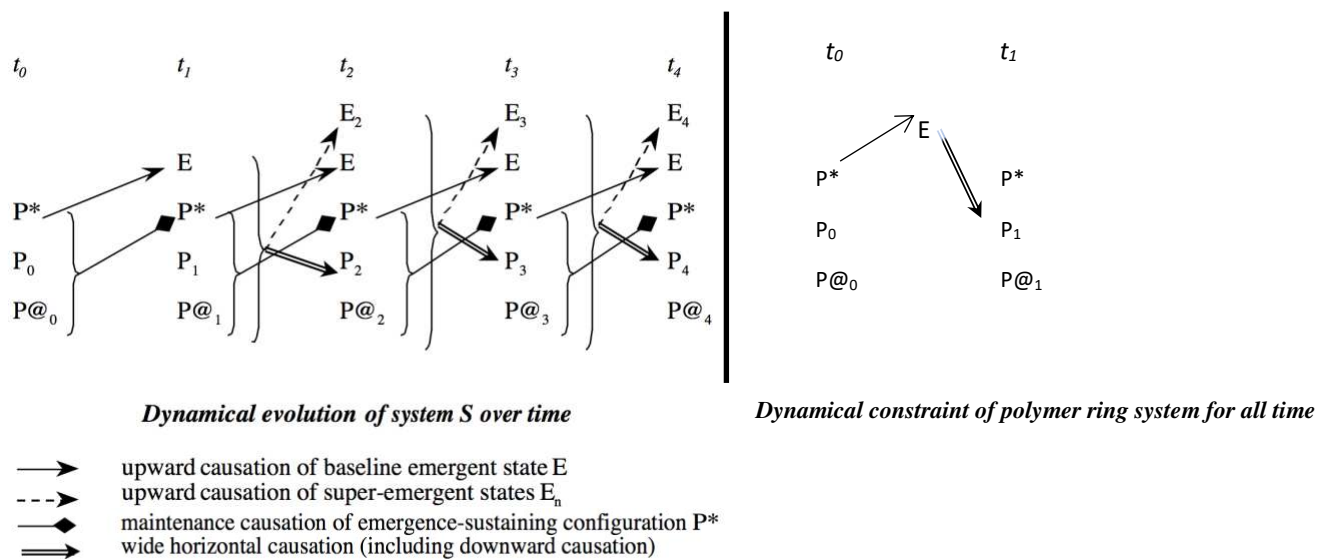⟹ wide horizontal causation (including downward causation)

Figure 5. Comparison of the emergent variable scheme of O'Connor and Wong (2005) (left) with the description, in the same notation, of the polymer ring system example (right). P* denotes a complex physical configuration of a system S at time t_0. P_0 comprises 'the remaining aspect of S's physical state at time t_0'. $P@_0$ are the environmental factors outside S that bear upon its evolution. In the polymer system a single set of emergent (topological) variables arises immediately from the configuration of the system, whose values co-determine all future microscopic configurations P_1 of the fluid.

7. Objections

We now anticipate some objections to our argument as presented:

1) **What is special about these cases; aren't there lots of topological variables in many systems?**

It is true that trivially speaking all systems have topological properties. But topological ordering/long-range quantum entanglement isn't simply a topological property, it is a special type of sensitivity of the system to a particular aspect of topological structure. Because of that sensitivity topological structure plays a crucial part in the explanans of some of the phenomena manifested in these systems. Anything that is a crucial difference maker in an explanans is a good candidate for something that is explanatory, especially if it is the salient difference maker with respect to certain counterfactuals. In the case of the FQHS long-range entanglement is the crucial difference maker with respect to the system displaying anyon statistics and a robustness to unitary transformations. The key difference between the FQHS and the integer quantum Hall state is the structuring of the system according to topological order. Topology is not just another property.

Indeed, in general properties in and of themselves aren't particularly interesting for counterfactuals. Every system that exists has a whole host of properties. What matters epistemically for us to pick out counterfactually salient information is which properties make a crucial difference. This requires the specification of a contrast class: state A versus state not-A. Many systems will have topological properties that are *irrelevant to these counterfactual contrasts*, but not the FQHS. The difference maker in a counterfactual such as whether a thrown rock broke a window is not the topology of spacetime. The topology of spacetime might be a necessary background condition but it is not the difference maker for that counterfactual. In the FQHS it is the emergent topological order that is the difference maker for counterfactuals salient to the macroscopic properties of the system. Similarly, we argue that some polymer systems also possess topological order (through a different microphysical mechanism than quantum entanglement) and the salient different maker in the explanans of some macroscopic properties is the precise structure of topological order as defined by winding numbers.

2) **Topological order itself isn't that special.**

One might reasonably object that topological order isn't as special as we suggest. Indeed, one understandably might contend that a system to be topologically ordered just when it admits a low-energy effective field theory that takes the form of a topological quantum field theory. We are resistant to such a stipulative definition since it is not clear how well-defined "low energy" is in all cases, nor how to reliably to extract low-energy properties.

While effective field theories are one way of characterising topological order it is not the only way. In the FQHE case one could use Laughlin wavefunctions or composite fermions to characterize the state. Like conventional long-range order, there are a number of equivalent ways of describing order and physicists tend to choose the most useful. But there is still widespread agreement that there is a distinction between topological properties such as topological excitations (e.g. excitations like walls and vortices) and topological order. The key point is topological order is a non-trivial form of topological structure, and it is not a background constraint, it is produced endogenously in the aggregation of microphysical parts.

Moreover, the point about 'order' generally defined is that the variables of an ordered system are restricted in the values that they assume over a disordered system (so at lattice points rather than any spatial coordinate). Topological order by contrast is represented in the

specification of a subset of possible topological characteristics taken from the full set available to a topologically disordered system.

3) Winding numbers are the wrong way of characterising topological emergence.

One might object that there are states with trivial winding numbers, or indeed many other systems with winding numbers, if topological order is simply a consequence of winding numbers then aren't these systems respectively non-emergent/emergent?

It is important to stress that we do not suggest the presence or absence of a winding number is itself indicative of emergence. Instead winding numbers are simply a good way to distinguish states with different topological structures. Topological order is a particular set of structures that goes beyond a topological description in terms of winding numbers.

Also note that we claim that the trivial winding number globule state is emergent. If we define topological order as a restriction of a system to a more or less narrowly defined set of topological characteristics then the trivial winding number state can be seen as a rather well-ordered state in this sense and heavily restricted. The crumpled-globule liquid state is a special form of the ensemble with unique properties, so we argue that it is emergent in the same sense that the solid states are from non-trivial windings.

4) Why aren't topological properties just part of microphysical background conditions?

The reader might be prepared to concede that topological order is a special class of topological properties. They might even concede that for some systems topological order is explanatorily ineliminable. But does this imply emergence? Could the topology not just be part of the microphysics itself, a background condition/constraint, like the topology of spacetime for example? We do not think so. One can have a debate about the appropriate level to ascribe global topological properties (and whether they microphysical or not) but in the case of topological order *these are not pre-existing background topological constraints*.

Instead, the topological state is resultant from certain microphysical interactions, and once established that topological order acts to constrain the microphysical state space that produced it. *We do not think that a form of order that is produced in a system should be thought of as a background exogenous constraint.* Nor is it just another part of the microphysics unless one stipulates that any physical process is 'microphysical' – the topological connections clearly structure the state space beyond the influence of the microphysical variables alone, i.e. at a level above the microphysical (if one wishes to think in terms of levels). This is reflected in the requirement in principle, not just practice, for a representational model that includes terms not contained in the microphysical (see the defences F1-F3 above).

Could one object that the topological ordering is produced from microphysical interactions so is merely resultant not emergent? We think this question is ill posed unless one has a robust distinction between emergent and resultant phenomena. In our view nature can only order the system through using the mechanism of topological ordering; the system doesn't come for free once the microphysics is set, it requires the global topological properties to manifest to structure the microphysical state. This might not count as emergence if one's view is emergent phenomena are completely separate top-down processes from microphysics, but

we take the view that this is a strength of our account. We have an account of where the topological properties that matter come from and how they influence the system. This is not emergence as a type of top-down magic, nor is it a trivial exhaustive determination of the state of the system from the microphysics alone. Instead it is microphysics interacting with background constraints (dimensionality etc.) to manifest global properties which then 'take the driver's seat' in terms of determining some of the properties of the system. If one wishes to call this merely resultant rather than emergent so be it, we aren't interested in stipulative definitions, but either way the topological properties must be part of the ontology of the world to have such an effect on it. They are not merely epistemic, and they are not straightforwardly microphysical.

But we can go further. We think it may be possible to ascribe, depending on the choice of representational model, topological information to microscopic or macroscopic descriptions. But, this we believe does not undermine the emergent nature of the phenomenon. As in the FQHS one can tie quanta of flux to electrons and produce a purely 'microscopic' description but yet the nature of those microscopic variables has changed. We no longer have an aggregation of separate microscopic variables, instead we have microscopic variables which are only well defined relationally. We believe we have demonstrated that this is the case for polymer ring melts as well.

In other words, there may be epistemic representational flexibility as to whether we stipulate the topological information as microscopic or macroscopic, but what isn't epistemic is that in either scenario the state of the system is determined by variables that are now defined inherently in relation to the whole of the system. There is a physical change in the system when it becomes topologically ordered.

We therefore propose an account of emergence which is not top-down in the sense of spooky new properties come into existence from nowhere that determine the system. But nor are the counterfactually important properties purely microscopic in the resultant sense a reductionist picture might suggest.

Instead, we have a hybrid account, conceptually. The microscopic variables combine in certain ways to manifest the structural property of topological order. That structure then binds the microscopic variables (fuses them) in a way that means they are no longer divisible variables whose properties simply add up. These microscopic properties are inherently relational defined by their relation to each other and the structure.

5) Are the links between the FQHS and the polymer case just mathematical/representational?

We hope we have demonstrated that there are at least interesting formal representational/mathematical similarities between the quantum FQHS and the classical polymer case. Indeed, these kinds of surprising formal similarities between areas of physics are often found. Condensed matter physics and areas of fundamental physics often have representational formalities which lead to interesting research. In our specific cases we do not believe the similarities are merely representational.

We contend that these systems display a kind of radical multiple realisability with respect to topological order, such that in each case the microphysical means of manifesting such order are completely different, but that doesn't matter because the topological order itself is (for some properties) the crucial difference maker. As with cases of universality then, the details

of how you get to a point aren't important if once you reach that point an entirely new set of structural properties take over and direct the system. The FQHS and the polymer case are not just mathematically or representationally similar, for the very reason that we argue: it is the topological order itself which is the salient counterfactual variable with respect to certain properties of the system. The emergence is not merely epistemic since any feature of the world that is ineliminable from an explanation in principle (not just practice) is a good candidate for being part of the ontology of the world.[7]

Is the topological ordering really ineliminable in principle? Such questions are difficult to answer, after all they depend on hypothetical future science. But we believe we have argued plausibly that there is no reasonable chance of topological ordering being removed from descriptions and explanations of these systems. Indeed, the multiple descriptive strategies used by physicists for the FQHS all either explicitly or implicitly include global topological ordering. So, any particular formalism might be replaced, but the sense that global structural features are determining the system will not be. Since much of the same representational structure must be used in the case of polymer ring melts, we believe this indicates a similar ineliminablity. In short, we believe science should be used as a starting point for metaphysics, and if the claim is topology can be done away with then the burden of proof is on the reductionist to state how that plausibly might happen.

Recall as well that our first aim in this paper is not to argue for emergence as such, more it is to argue that *if the FQHS is emergent because of topological ordering then classical systems should be regarded as emergent also*. Neither case may be emergent by some definitions, but we believe it is difficult to argue that the quantum system is emergent in virtue of its topological properties if the classical case isn't. One might argue that the FQHS is emergent because of purely quantum considerations, such as entanglement, but then there is nothing philosophically interesting about the FQHS since quantum entanglements are found in many different non-topologically ordered systems, and it is left an open question to account for why physicists 'smell' something particularly 'strong emergence like' in the FQHS. (It could be that physicists' notion of emergence is so distinct from philosophers' that what physicists consider to be interesting to the emergence question is completely irrelevant for philosophers, we do not believe this should be a claim taken for granted though. While the terms may differ in usage, we believe there is enough semantic overlap to pay attention to case studies physicists think display a particularly strong type of emergence).

6) **The analogy between polymer rings and FQHS breaks down. The Chern-Simons field attaches flux quanta to electrons in the FQHS but in the polymer case there is no equivalent. The parallel would require interpreting the 3-dimensional spatially extended system as a 2-dimsnional and 1 time dimensional point particle.**

It is true that Chern-Simons theories are designed to work in 2+1 dimensional spacetime (i.e. 2 spatial dimensions and time). They won't work on 3+1 dimensional space. In the FQH case we couple:

a) The electromagnetic field to the current of the CS field;

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[7] Assuming one is a scientific realist – if one is an anti-realist the reduction/emergence debate has no weight to begin with since one would never privilege unobservable microphysical entities as the only ontologically serious parts of the world if one doesn't believe you can claim they exist in the first place!

b) the CS field to the current of quasiparticles. This quasiparticle current is a vector, whose 0th (time-like) component is the quasiparticle's CS charge.

We don't literally couple electrons to the CS field. Rather the electrons give rise to the CS field and we use this fact in constraining the charges electronic charges allowed in the system. In the polymer case we take the polymer ring as a trajectory (the z axis as time, x y as position). We couple the velocity of this trajectory, which is analogous to a current, to the CS field. This is a vector, whose 0th (time-like) component is literally an element of polymer.

This literal failure to map the microphysics is actually part of our argument though. Recall it is our case that the FQHS and the polymer ring melt represent radically different microphysics that produce the same form of structural properties. We have radical multiple realisability with extremely disjunctive microphysical bases (one quantum, one classical, each with different dimensions). The formal mathematical analogies speak to the crucial physical point of contact which is topological order itself. Once we have topological order this control variable structures the state space for both systems.

7) Why are topological properties causal?

The substantive part of our argument does not require topological properties to be causal. As we stated, causation is largely in the eye of the beholder. It fundamentally depends on having a definition of what it means to be causal in the first place, By some accounts topology is causal, by others it is non-causal. We discuss causal theories of emergence as a point of contact to suggest that some accounts of causation would seem to match with our account of topological ordering. But the causal nature of topology is irrelevant. What does matter is that the topological ordering is structurally important in determining the state of the system. It orders and constrains the state space. That structuring is a crucial difference maker in counterfactuals regarding the properties of the system. The topological ordering isn't a background condition, it doesn't pre-exist (as say the topology of spacetime does), rather it is specifically produced by the aggregation and once manifested counterfactually constrains the properties of the aggregation. We believe that is sufficient for an ontological claim that topological ordering is not simply another part of the aggregates themselves or an artefact of representation.

8. Conclusion

We have noted the similarity in consequence of the topological ring melt, and the fractional quantum Hall state. Both contain emergent high-level variables which are topological, global and determine equivalence classes of the low-level variable and the macroscopic properties. The correspondence may be extended in more formal directions, exploiting the field-theoretic description of polymers. Ferrari (2004) has shown, for example, that in the case of entangled rings up to 4-plats, the polymer states correspond to anyons, within a Chern-Simons field theory.

By looking in detail at the properties created by long range topological constraints in the case of the polymer ring melt, and its mapping onto the analogous phenomenon in the fractional quantum Hall effect, we have shown that the development of inherently relational basal properties in the latter case does not arise through the non-classical physics of quantum entanglement. The classical case of entangled ring polymers exhibits the same pattern from constraints that result in the observed causal

structures. In both cases the relevant emergent variables arise from, and constitute, long-range topological relationships.

The polymer ring system can be mapped onto general frameworks proposed in the philosophical literature by Kim, O'Connor and Wong, Humphreys and others.

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