Article:

Berarducci, A, Kuhlmann, S, Mantova, V orcid.org/0000-0002-8454-7315 et al. (1 more author) (Accepted: 2019) Exponential fields and Conway's omega-map. Proceedings of the American Mathematical Society. ISSN 0002-9939 (In Press)

EXPONENTIAL FIELDS AND CONWAY'S OMEGA-MAP

ALESSANDRO BERARDUCCI, SALMA KUHLMANN,
VINCENZO MANTOVA, MICKAËL MATUSINSKI

Abstract. Inspired by Conway's surreal numbers, we study real closed fields whose value group is isomorphic to the additive reduct of the field. We call such fields omega-fields and we prove that any omega-field of bounded Hahn series with real coefficients admits an exponential function making it into a model of the theory of the real exponential field. We also consider relative versions with more general coefficient fields.

1. Introduction

We study real closed valued fields $\mathbb{K}$, with a convex valuation ring $\mathrm{O}(1) \subseteq \mathbb{K}$ satisfying the property that the value group $v(\mathbb{K}^\times)$ is isomorphic to the additive reduct $(\mathbb{K}, +, <)$ of the field, where v is the valuation induced by $\mathrm{O}(1)$. We call **omega-field** a field with this property. The name is motivated by the fact that any omega-field admits a map akin to Conway's omega-map $x \mapsto \omega^x$ on the field of surreal numbers $\mathbf{No}$ [3] or its fragments $\mathbf{No}(\lambda)$ studied in [6], where λ is an ϵ-number. We need to recall that any real closed field $\mathbb{K}$ admits a section of the valuation, hence it has a multiplicative subgroup $G \subseteq \mathbb{K}^{>0}$, called a group of **monomials**, given by the image of the section. Since G is a multiplicative copy of $v(\mathbb{K}^\times)$, we have that $\mathbb{K}$ is an omega-field if and only if it admits an isomorphism

$$\omega : (\mathbb{K}, +, 0, <) \cong (G, \cdot, 1, <),$$

and we shall call **omega-map** any such isomorphism. The prototypical example is Conway's omega-map on the surreal numbers, and in analogy with the surreal case, we use the exponential notation ω^x to denote the image of x under ω.

Here we explore the relation between omega-fields and exponential fields, where an **exponential field** is a real closed field $\mathbb{K}$ admitting an **exponential map**, that is an isomorphism $\exp : (\mathbb{K}, +, 0, <) \cong (\mathbb{K}^{>0}, \cdot, 1, <)$. We shall freely write e^x rather than $\exp(x)$ when convenient. Note that ω^x should not be read as $e^{\omega \log(x)}$ (the easiest way to see why is that the map $x \mapsto \omega^x$, if there is such an omega-map, is not continuous in the order topology of $\mathbb{K}$). While in general there are no containments between the class of fields admitting an omega-map and that of fields admitting an exponential map, a non-trivial inclusion of the former in the latter can be obtained by restricting the analysis to κ-bounded Hahn fields, as discussed below.

Date: Oct. 8th, 2018. Revised Jan. 17, 2019.
2010 *Mathematics Subject Classification.* Primary 03C64; Secondary 16W60.
Key words and phrases. Exponential fields, Hahn fields, Surreal Numbers.
A.B. was partially supported by the project "PRIN 2012, Logica Modelli e Insiemi". V.M. was partially supported by "Fondation Sciences Mathématiques de Paris".

In general, any real closed valued field $\mathbb{K}$ with monomials G is isomorphic to a truncation closed subfield (see Definition 2.8 (1)) of the Hahn field $\mathbf{k}((G))$ [13], where $\mathbf{k} \cong \mathrm{O}(1)/o(1)$ is the residue field and we write $o(1)$ for the maximal ideal of $\mathrm{O}(1)$. For the sake of simplicity in this introduction we focus on the typical case $\mathbf{k} = \mathbb{R}$, but our results hold more generally assuming that the residue field $\mathbf{k}$ is a model of $T_{an,\exp}$, the theory of the real exponential field $\mathbb{R}_{\exp}$ with all restricted analytic functions [5]. The full Hahn field $\mathbb{R}((G))$ is always naturally a model of the theory of restricted analytic functions T_{an} [5], but it never admits an exponential function if $G \neq 1$ [10]. However, for every regular uncountable cardinal κ, there is a group G such that the *κ-bounded* Hahn field $\mathbb{R}((G))_\kappa$ does admit an exponential function [12]. We thus restrict our analysis to fields of the form $\mathbb{K} = \mathbb{R}((G))_\kappa$ (without assuming *a priori* that they admit an exponential map). Our first result is the following. The case $G = \mathbf{No}(\kappa)$ with κ regular uncountable is in [6].

Theorem (3.8)**.** *Every omega-field of the form $\mathbb{R}((G))_\kappa$ admits an exponential function making it into a model of $T_{an,\exp}$.*

Our work was partly motivated by the desire to understand the connections between the surreal numbers, with its various subfields studied in [1, 2], and the exponential fields of the form $\mathbb{R}((G))_\kappa$ constructed by S. Kuhlmann and S. Shelah in [12]. We shall prove that the latter are not always omega-fields (Theorem 4.5), but they are omega-fields if and only if G is order-isomorphic to $G^{>1}$ (Theorem 4.1); in this case, given a chain isomorphism $\psi : G \cong G^{>1}$, there is an omega-map satisfying $\omega^g = e^{\psi(g)}$ for all $g \in G$.

Let us now discuss Theorem 3.8 in more detail. We show that given $\mathbb{K} = \mathbb{R}((G))_\kappa$ and an omega-map $\omega : \mathbb{K} \cong G$, we can construct an exponential function directly starting from ω and an auxiliary chain isomorphism

$$h : (\mathbb{K}, <) \cong (\mathbb{K}^{>0}, <),$$

where by **chain** we mean linearly ordered set. Any choice of h yields an exponential field (Theorem 3.4) and at least one choice of h will yield a model of $T_{an,\exp}$ (Theorem 3.8). Varying h we can thus produce a variety of exponential fields; some of them are models of $T_{\exp}$, while all the others are not even o-minimal (Theorem 3.10), depending on the growth properties of h (Definition 2.11).

To define the exponential function, it is more convenient to first define a **logarithm** $\log : \mathbb{K}^{>0} \to \mathbb{K}$ and let exp be the compositional inverse log. To this aim we start by putting

$$\log(\omega^{\omega^x}) = \omega^{h(x)}$$

for $x \in \mathbb{K}$ and $\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$ for $\varepsilon \in o(1)$. Note that such infinite sums make sense in the κ-bounded Hahn field $\mathbb{R}((G))_\kappa$.

The extension of log to the whole of $\mathbb{K}^{>0}$ is then carried out guided by the principle that log takes products into sums and ω takes sums into products. We simply extend this to infinite sums. More precisely, log is determined by $\log(\omega^{\sum_{i<\alpha} \omega^{\gamma_i} r_i}) = \sum_{i<\alpha} \omega^{h(\gamma_i)} r_i$ and $\log(\omega^x r(1+\varepsilon)) = \log(\omega^x) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$, where $\varepsilon \in o(1)$ and $r \in \mathbb{R}$. Another way to express the spirit of the construction is that we first define log on the representatives of the multiplicative archimedean classes ω^{ω^x}, then we extend it to the representatives of the additive archimedean classes ω^x, and finally to the whole of $\mathbb{K}$. It is not difficult to show that this construction always yields an exponential field. We now need to show that there is at least one function h such that the exponential field $\mathbb{K}$ arising from ω and h as above is a model of $T_{\exp}$. A necessary condition is that the exponential map grows faster than any polynomial, or equivalently, that its inverse log grows slower than $x^{1/n}$ for all positive $n \in \mathbb{N}$. This translates into the condition $h(x) < r \cdot \omega^x$ for every $x \in \mathbb{K}$ and $r \in \mathbb{R}^{>0}$. We shall abbreviate the above with $h(x) \prec \omega^x$.

Since ω^x is discontinuous (its values are the respresentatives of the archimedean classes), and h is continuous in the order topology of $\mathbb{K}$ (being a chain isomorphism from $\mathbb{K}$ to $\mathbb{K}^{>0}$), the existence of such an h is not immediate. In the case of Gonshor's h on the surreal numbers [7], the condition $h(x) \prec \omega^x$ is forced by the inductive definition of h. However, this cannot be generalized to our more general setting where similar inductive definitions make no sense, and we use instead a bootstrapping procedure (Lemma 3.6). Granted this, the final exp on $\mathbb{K}$ is easily seen to yield a model of $T_{\exp}$ using [15, 5] (Theorem 3.8).

All the logarithms considered in this paper are **analytic** (Definition 2.10): for $\varepsilon \in o(1)$, the function $x \mapsto \log(1+x)$ is given by the familiar Taylor expansion $\log(1+\varepsilon) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n}$, whereas for $g \in G$, $\log(g)$ is a *purely infinite* element of $\mathbb{R}((G))_\kappa$, and for $r \in \mathbb{R}$, $\log(r)$ is the usual real logarithm.

Theorems 3.4 and 3.8 produce analytic logarithms satisfying two additional restrictions: $\log(\omega^{\omega^x}) \in G$ for all $x \in \mathbb{K}$, and log brings "infinite products" to "infinite sums". It turns out, however, that all analytic logarithms arise in this way, up to changing the omega-map $\omega : \mathbb{K} \cong G$. More precisely, we have the following classification result.

Theorem (Corollary 4.2)**.** *Every analytic logarithm on an omega-field of the form* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *arises from some omega-map* $x \mapsto \omega^x$ *and some chain isomorphism* $h : \mathbb{K} \cong \mathbb{K}^{>0}$.

The surreal numbers fit into the above picture if we allow κ to be the proper class of all ordinals and G to be the image of Conway's omega-map $x \mapsto \omega^x$. Gonshor's exponentiation is induced by the omega-map and Gonshor's function h [7]; by the above results, any other analytic logarithm on **No** arises in this way, possibly after replacing Conway's omega-map with another isomorphism from **No** to its group of monomials and Gonshor's h with another chain isomorphism.

2. Preliminaries

2.1. **Valuations.** Let $\mathbb{K}$ be an ordered field (possibly with additional structure) and let $\mathrm{O}(1) \subseteq \mathbb{K}$ be a convex subring. Then $\mathrm{O}(1)$ is the valuation ring of a valuation v and we denote by $o(1)$ the unique maximal ideal of $\mathrm{O}(1)$. If $\mathbb{K}$ is real closed, it has a subfield $\mathbf{k} \subseteq \mathbb{K}$ isomorphic to the residue field $\mathrm{O}(1)/o(1)$ of the valuation, namely we can write $\mathrm{O}(1) = \mathbf{k} + o(1)$. We shall always assume in the sequel that $\mathbb{K}$ is real closed and $\mathrm{O}(1), o(1), \mathbf{k}$ are as above.

Definition 2.1. For $x, y \in \mathbb{K}$ we define:

- $x \preceq y$ if $|x| \leq c|y|$ for some $c \in \mathrm{O}(1)$ (domination);
- $x \asymp y$ if $x \preceq y$ and $y \preceq x$ (comparability);
- $x \prec y$ if $x \preceq y$ and $x \not\asymp y$ (strict domination);
- $x \sim y$ if $x - y \prec x$ (x is asymptotic to y).

With these notations we have $\mathrm{O}(1) = \{x : x \preceq 1\}$ and $o(1) = \{x : x \prec 1\}$.

Definition 2.2. A multiplicative subgroup G of $\mathbb{K}^{<0}$ is a group of **monomials** if it consists in a family of representatives for each $\asymp$-class. In other words a group of monomials is an embedded copy of the value group. It is well known that any real closed field admits a group of monomials.

Remark 2.3. For $x, y \in \mathbb{K}$ we have:

- $x \prec y$ if and only if $c|x| < |y|$ for all $c \in \mathrm{O}(1)$ (or equivalently for all $c \in \mathbf{k}$);
- $x \asymp y$ if and only if $x = cy(1+\varepsilon)$ for some $c \in \mathbf{k}^{\times}$ and $\varepsilon \in o(1)$;
- $x \sim y$ if and only if $x = y(1+\varepsilon)$ for some $\varepsilon \in o(1)$.
- if $x \neq 0$ there are unique $r \in \mathbf{k}^{\times}, g \in G, \varepsilon \in o(1)$ such that $x = gr(1+\varepsilon)$.

2.2. **Hahn groups.** By a **chain** we mean a linearly ordered set. We describe a well known procedure to build an ordered group starting from a chain.

Definition 2.4. Given a chain Γ and an ordered abelian group $(C, +, <)$, the **Γ-product of** C is the abelian group of all functions $f : \Gamma \to C$ with *reverse* well-ordered **support** $\{\gamma \in \Gamma : f(\gamma) \neq 0\}$ and pointwise addition, ordered by declaring $f > 0$ if $f(\gamma) > 0$, where γ is the *biggest* element in the support.[1]

If we write G in additive notation, a typical element of G can be written in the form $\sum_{\gamma\in\Gamma} \gamma r_\gamma$, representing the function sending $\gamma \in \Gamma$ to $r_\gamma \in C$, while G itself is denoted $\sum \Gamma C$. We prefer however to use a multiplicative notation and write G as $\prod t^{\Gamma C}$ and a typical element of G as $\prod_{\gamma\in\Gamma} t^{\gamma r_\gamma}$. In this notation the multiplication is given by

$$\left(\prod_{\gamma\in\Gamma} t^{\gamma r_\gamma}\right)\left(\prod_{\gamma\in\Gamma} t^{\gamma r'_\gamma}\right) = \prod_{\gamma\in\Gamma} t^{\gamma(r_\gamma + r'_\gamma)}$$

Since the supports are reverse well-ordered, we can fix a decreasing enumeration $(\gamma_i : i < \alpha)$ of the support, where α is an ordinal, and write an element of $\prod t^{\Gamma C}$ in the form

$$f = \prod_{i<\alpha} t^{\gamma_i r_i} \in \prod t^{\Gamma C}.$$

According to our conventions, $f > 1 \iff r_0 > 0$ and $t^\gamma > t^\beta \iff \gamma > \beta$.

If Γ has only one element, we may identify $\prod t^{\Gamma C}$ with a multiplicative copy t^C of $(C, +, <)$.

When $C = (\mathbb{R}, +, <)$, we obtain the **Hahn group** over Γ, which can be characterized as a maximal ordered group with a set of archimedean classes of the same order type as Γ [8]. Recall that two positive elements are in the same archimedean class if each of them is bounded, in absolute value, by an integer multiple of the other.

Notation 2.5. Let κ be a regular cardinal. If in the definition of the Γ-product we only allow supports of reverse order type $< \kappa$, we obtain the κ-bounded version

$$\left(\prod t^{\Gamma C}\right)_\kappa \subseteq \prod t^{\Gamma C}.$$

We shall also consider the case when Γ is a proper class and $\kappa = \mathbf{On}$, in which case $\left(\prod t^{\Gamma C}\right)_{\mathbf{On}}$ is the ordered group of all functions $f : \Gamma \to C$ whose support is a reverse well ordered *set* (rather than a reverse well ordered class).

[1]Other authors prefer to use well-ordered supports, but one can pass from one version to the other reversing the order of Γ.

2.3. **Hahn fields.** Given a field $\mathbf{k}$ and a multiplicative ordered abelian group G, let $\mathbf{k}((G))$ denote the Hahn field with coefficients in $\mathbf{k}$ and monomials in G. The underlying additive group of $\mathbf{k}((G))$ coincides with the G-product of $\mathbf{k}$: its elements are functions $f : G \to \mathbf{k}$ with reverse well-ordered supports, which we write either in the form $f = \sum_{g \in G} g r_g$, where $r_g = f(g)$, or in in the form

$$f = \sum_{i<\alpha} g_i r_i$$

where α is an ordinal, $(g_i)_{i<\alpha}$ is a decreasing enumeration of the support, and $r_i = f(g_i) \in \mathbf{k}^*$. Addition is defined componentwise and multiplication is given by the usual Cauchy product. We order $\mathbf{k}((G))$ according to the sign of the leading coefficient, namely $f > 0 \iff r_0 > 0$.

Remark 2.6. It can be proved that if $\mathbf{k}$ is real closed and G is divisible, then $\mathbf{k}((G))$ is real closed [9]. Moreover, $\mathbf{k}((G))$ is **spherically complete**: any decreasing intersection of valuation balls has a non-empty intersection.

Notation 2.7. Inside $\mathbf{k}((G))$, we let $\mathrm{O}(1)$ be the valuation ring of all the elements x with $|x| \leq r$ for some $r \in \mathbf{k}$, and $o(1)$ be the corresponding maximal ideal. We then have $\mathrm{O}(1) = \mathbf{k} + o(1)$. With respect to this valuation ring, $\mathbf{k}$ is a copy of the residue field and G is a group of monomials. We shall use similar notations for any subfield $\mathbb{K} \subseteq \mathbf{k}((G))$ containing $\mathbf{k}$ and G.

2.4. **Restricted analytic functions.** A family $(f_i)_{i \in I}$ of elements of $\mathbf{k}((G))$ is **summable** if the union of the supports of the elements f_i is reverse well-ordered and, for each $g \in G$, there are only finitely many $i \in I$ such that g is in the support of f_i. In this case $\sum_{i \in I} f_i$ is defined as the element $f = \sum_g g r_g$ of $\mathbf{k}((G))$ whose coefficients are given by $r_g = \sum_{i \in I} r_{g,i} \in \mathbf{k}$ where $r_{g,i}$ is the coefficient of g in f_i. This makes sense since, given $g \in G$, only finitely many $r_{g,i}$ are non-zero.

By Neumann's lemma [14] for any power series $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ with coefficients in $\mathbf{k}$ and for any $\varepsilon \prec 1$ in $\mathbf{k}((G))$, the family $(a_n \varepsilon^n)_{n \in \mathbb{N}}$ is summable, so we can evaluate $P(x)$ at ε obtaining an element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n$ of $\mathbf{k}((G))$. Similar considerations apply to power series in several variables.

Definition 2.8. Let $\mathbb{K} \subseteq \mathbf{k}((G))$ be a subfield. We say that $\mathbb{K}$ is an **analytic subfield** if

1. $\mathbb{K}$ is truncation closed: if $\sum_{i<\alpha} g_i r_i$ belongs to $\mathbb{K}$, then $\sum_{i<\beta} g_i r_i$ belongs to $\mathbb{K}$ for every $\beta \leq \alpha$;
2. $\mathbb{K}$ contains $\mathbf{k}$ and G;
3. If $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ is a power series with coefficients in $\mathbf{k}$ and $\varepsilon \prec 1$ is in $\mathbb{K}$, then the element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n \in \mathbf{k}((G))$ lies in the subfield $\mathbb{K}$. Similarly for power series in several variables.

We recall that T_{an} is the theory of the real field with all analytic functions restricted to the poly-intervals $[-1,1]^n \subseteq \mathbb{K}^n$ [5]. (By rescaling, we can equivalently use any other closed poly-interval.)

Fact 2.9. *We have:*

1. *The field $\mathbb{R}((G))$ admits a natural intepretation of the analytic functions restricted to the poly-interval $[-1,1]^n \subseteq \mathbb{K}$, making $\mathbb{K}$ into a model of T_{an}.*
2. *The same holds for any analytic subfield of $\mathbb{R}((G))$, and in particular for $\mathbb{R}((G))_\kappa$ for every regular uncountable κ.*

(3) *More generally, if* $\mathbf{k}$ *is a model of* T_{an}*, then any analytic subfield* $\mathbb{K}$ *of* $\mathbf{k}((G))$ *is naturally a model of* T_{an}.

The proof of (1) is in [5] and is based on a quantifier elimination result in the language of T_{an}. The other points follow by the same argument. We interpret the restricted analytic functions in the analytic subfield $\mathbb{K} \subseteq \mathbf{k}((G))$ as follows. Given a real analytic function f converging on a neighbourhood of $[-1,1]^n \cap \mathbb{R}^n$, we need to define $f(x+\varepsilon)$ where $x \in [-1,1]^n \cap \mathbf{k}^n$ and $\varepsilon \in o(1)^n \subseteq \mathbb{K}^n$. We do this by using the Taylor expansion $f(x+\varepsilon) = \sum_i \frac{D^i f(x)}{i!} \varepsilon^i$ where i is a multi-index in $\mathbb{N}^n$. Here $D^i f(x) \in \mathbf{k}$ (using the fact that $\mathbf{k}$ is a model of T_{an}) and the infinite sum is in the sense of the Hahn field $\mathbf{k}((G))$.

2.5. **Exponential fields.** A **prelogarithm** on a real closed field $\mathbb{K}$ is a morphism from $(\mathbb{K}^{>0}, \cdot, 1, <)$ to $(\mathbb{K}, +, 0, <)$ and a **logarithm** is a surjective prelogarithm. An **exponential map** is the compositional inverse of a logarithm, that is an isomorphism from $(\mathbb{K}, +, 0, <)$ to $(\mathbb{K}^{>0}, \cdot, 1, <)$. We say that $\mathbb{K}$ is an **exponential** field if it has an exponential map. Given a logarithm log, we write exp for the corresponding exponential map and we write e^x instead of $\exp(x)$ when convenient. Now assume $\mathbf{k}$ has a logarithm and consider the Hahn field $\mathbf{k}((G))$. It turns out that if $G \neq 1$, $\mathbf{k}((G))$ never admits a logarithm extending that on $\mathbf{k}$ [10]. On the other hand if κ is a regular uncountable cardinal, then for suitable choices of G, the logarithm on $\mathbf{k}$ can be extended to $\mathbf{k}((G))_\kappa$, and when $\mathbf{k} = \mathbb{R}$ this can be done in such a way that $\mathbf{k}((G))_\kappa$ is a model of $T_{\exp}$ [12].

Definition 2.10. Let $\mathbf{k}$ be an exponential field and let $\mathbb{K}$ be an analytic subfield of $\mathbf{k}((G))$, for instance $\mathbb{K} = \mathbf{k}((G))_\kappa$ with κ regular uncountable. An **analytic logarithm** on $\mathbb{K}$ is a logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ with the following properties:

(1) $\log : \mathbb{K}^{>0} \to \mathbb{K}$ extends the given logarithm on $\mathbf{k}$.
(2) $\log(1+\varepsilon) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \varepsilon^i$ for all $\varepsilon \prec 1$ in $\mathbb{K}$ (the assumption $\varepsilon \prec 1$ ensures the summability).
(3) $\log(G) = \mathbb{K}^{\uparrow}$, where $\mathbb{K}^{\uparrow} := \mathbf{k}((G^{>1})) \cap \mathbb{K}$ is the group of **purely infinite** elements, namely the series of the form $\sum_{i<\alpha} g_i r_i$ with $g_i \in G^{>1}$ for all i.

Conditions (1) and (2) are rather natural, and ensure that the restrictions of $\log(1+x)$ to small finite intervals agree with the natural T_{an}-interpretations of Fact 2.9. A motivation for (3) is the following. The multiplicative group $\mathbb{K}^{>0}$ admits a direct sum decomposition

$$\mathbb{K}^{>0} = G\,\mathbf{k}^{>0}(1 + o(1)),$$

namely any element x of $\mathbb{K}^{>0}$ can be uniquely written in the form $x = gr(1+\varepsilon)$ where $r \in \mathbf{k}^{>0}$, $g \in G$ and $\varepsilon \in o(1)$. Applying log to both sides of the above equation, we get (by (1) and (2)) a direct sum decomposition

$$\mathbb{K} = \log(G) \oplus \mathbf{k} \oplus o(1)$$

of the additive group $(\mathbb{K}, +)$. Indeed by (1) we have $\log(\mathbf{k}^{>0}) = \mathbf{k}$ and $\log(\mathbb{K}^{>0}) = \mathbb{K}$, while (2) implies that the logarithm maps $1 + o(1)$ bijectively to $o(1)$ with inverse given by $\exp(\varepsilon) = \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{n!}$. We have thus proved that $\log(G)$ is a direct complement of $\mathrm{O}(1) = \mathbf{k} + o(1)$. Although there are several choices for such a complement, the most natural one is $\log(G) = \mathbb{K}^{\uparrow}$, as required in point (3), since it is the unique one closed under truncations.

2.6. **Growth axiom and models of $T_{\exp}$.** Ressayre proved in [15] that an exponential field is a model of $T_{\exp}$ if and only it satisfies the elementary properties of the real exponential restricted to $[0,1]$ and satisfies the growth axiom scheme $x \geq n^2 \to \exp(x) > x^n$ for all $n \in \mathbb{N}$.

Definition 2.11. Given an analytic subfield $\mathbb{K} \subseteq \mathbf{k}((G))$, we say that an analytic logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ satisfies the **growth axiom at infinity** if $\log(x) < x^{1/n}$ for all $x > \mathbf{k}$ and all positive integers n.

Proposition 2.12. *If $\mathbf{k}$ is a model of $T_{an,exp}$ (for instance $\mathbf{k} = \mathbb{R}$) and $\mathbb{K} \subseteq \mathbf{k}((G))$ is an analytic subfield with an analytic logarithm satisfying the growth axiom at infinity, then $\mathbb{K}$ (with the natural intepretation of the symbols) is a model of $T_{an,\exp}$.*

Proof. This follows from [15, 5] but we include some details. The inverse exp of an analytic logarithm is easily seen to satisfy $e^{\varepsilon} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}$ for all $\varepsilon \in o(1)$. Since moreover exp extends the given exponential on $\mathbf{k}$, it follows that the restriction of exp to $[-1,1]$ agrees with the natural T_{an}-interpretation of Fact 2.9. This shows that $\mathbb{K}$ is at least a model of $T_{\exp\,|[-1,1]}$, as it is in fact the restriction of a model of T_{an} to a sublanguage. Since the interpretation of exp grows faster than any polynomial (by the growth axiom at infinity plus the fact that $\mathbf{k}$ is a model of $T_{\exp}$), we can conclude by the axiomatisations of [15, 5]. □

The above result rests on the quantifier elimination result for $T_{an,\exp}$. We do not know whether it suffices that $\mathbf{k}$ is a model of $T_{\exp}$ to obtain that $\mathbf{k}((G))_\kappa$ is a model of $T_{\exp}$ (or even $T_{\exp\,|[0,1]}$).

3. Omega fields

Definition 3.1. A real closed field $(\mathbb{K}, +, \cdot, <)$ with a convex valuation ring $\mathrm{O}(1)$ and corresponding group of monomials $G \subseteq \mathbb{K}^{>0}$ is an **omega-field** if $(\mathbb{K}, +, <)$ is isomorphic to $(G, \cdot, <)$ as an ordered group. Given an omega-field $\mathbb{K}$ we shall call **omega-map** any isomorphism of ordered groups

$$\omega : (\mathbb{K}, +, 0, <) \cong (G, \cdot, 1, <).$$

Since the group G of monomials is isomorphic to the value group of $\mathbb{K}$, we have that $\mathbb{K}$ is an omega-field if and only if $(\mathbb{K}, +, <)$ is isomorphic to its value group. The definition of omega-map is inspired by Conway's omega map ω^x on the surreal numbers. We recall that the surreals can be presented in the form $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$, with the image of the omega-map being the group $\omega^{\mathbf{No}}$ of monomials. The subscript **On** indicates that we only consider series whose support is a set, rather than a proper class. The surreals should thus be considered as a bounded Hahn field rather than a full Hahn field.

3.1. **Construction of omega-fields.** In the sequel let κ be a regular uncountable cardinal.

Theorem 3.2. *Given an exponential field $\mathbf{k}$, there is a group G such that the field $\mathbb{K} = \mathbf{k}((G))_\kappa$ admits an omega-map $\omega : \mathbb{K} \to G$.*

When $\mathbf{k} = \mathbb{R}$ one can take $G = \mathbf{No}(\kappa)$ as in [6]. In the general case the proof is a variant of a similar construction in [12]. Given a chain Γ and an additive ordered group C (in our application $C = (\mathbf{k}, +, <)$), let $H(\Gamma)$ denote, in the following Lemma, the ordered group $\left(\prod t^{\Gamma C}\right)_\kappa$.

Lemma 3.3. *Fix a chain* Γ_0 *and a chain embedding* $\eta_0 : \Gamma_0 \to H(\Gamma_0)$ *(for instance* $\eta_0(\gamma) = t^\gamma$*). Then there is a chain* $\Gamma \supseteq \Gamma_0$ *and a chain isomorphism* $\eta : \Gamma \cong H(\Gamma)$ *extending* η_0*.*

Proof. We consider H as a functor from chains to ordered abelian groups: if $j : \Gamma' \to \Gamma''$ is a chain embedding, we define $H(j) : H(\Gamma') \to H(\Gamma'')$ as the group embedding which sends $\prod_i t^{\gamma_i r_i}$ to $\prod_i t^{j(\gamma_i) r_i}$. We do an inductive construction in κ-many steps. At a certain stage $\beta < \kappa$ we are given

$$G_\beta = H(\Gamma_\beta)$$

and a chain embedding $\eta_\beta : \Gamma_\beta \to G_\beta$ together with embeddings $j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta$ for $\alpha < \beta$. Let $\Gamma_{\beta+1}$ be a chain isomorphic to $(G_\beta, <)$ (for instance G_β itself considered as a chain) and fix a chain isomorphism $f_\beta : G_\beta \to \Gamma_{\beta+1}$. Now let $j_\beta : \Gamma_\beta \to \Gamma_{\beta+1}$ be the composition $f_\beta \circ \eta_\beta$ and let $G_{\beta+1} = H(\Gamma_{\beta+1})_\kappa$. We can then find a commutative diagram of embeddings

$$\begin{array}{ccc} \Gamma_\beta & \xrightarrow{\eta_\beta} & H(\Gamma_\beta) \\ {\scriptstyle j_\beta}\downarrow & \swarrow{\scriptstyle f_\beta} & \downarrow{\scriptstyle H(j_\beta)} \\ \Gamma_{\beta+1} & \xrightarrow{\eta_{\beta+1}} & H(\Gamma_{\beta+1}), \end{array} \tag{1}$$

by letting $\eta_{\beta+1} = H(j_\beta) \circ f_\beta^{-1}$. We can now define $j_{\beta,\beta+1} = j_\beta$ and $j_{\alpha,\beta+1} = j_{\beta,\beta+1} \circ j_{\alpha,\beta}$ for $\alpha < \beta$.

We iterate the contruction along the ordinals. At a limit stage λ, let $\Gamma_\lambda = \varinjlim_{\beta<\lambda} \Gamma_\beta$ and let $j_{\beta,\lambda} : \Gamma_\beta \to \Gamma_\lambda$ be the natural embedding for $\beta < \lambda$.

We then define $\eta_\lambda : \Gamma_\lambda \to H(\Gamma_\lambda)$ as the composition

$$\Gamma_\lambda = \varinjlim_{\beta<\lambda} \Gamma_\beta \to \varinjlim_{\beta<\lambda} H(\Gamma_\beta) \to H(\varinjlim_{\beta<\lambda} \Gamma_\beta) = H(\Gamma_\lambda).$$

More explicitly, for each $\gamma \in \Gamma_\lambda$, pick some $\beta < \lambda$ and $\theta \in \Gamma_\beta$ such that $\gamma = j_{\beta,\lambda}(\theta)$, and define $\eta_\lambda(\gamma) \in G_\lambda$ as the image under $H(j_{\beta,\lambda}) : G_\beta \to G_\lambda$ of $\eta_\beta(\theta) \in G_\beta$. Since κ is regular, when we arrive at stage κ we have an isomorphism

$$\eta_\kappa : \Gamma_\kappa \cong G_\kappa$$

and we can define $\Gamma = \Gamma_\kappa$ and $\eta = \eta_\kappa$. □

Proof of Theorem 3.2. By Lemma 3.3, there is a chain Γ and a chain isomorphism

$$\eta : \Gamma \cong G = H(\Gamma) \tag{2}$$

Now let $\mathbb{K} = \mathbf{k}((G))_\kappa$ and define an omega-map $\omega : \mathbb{K} \to G$ by setting

$$\omega^{\sum_{i<\alpha} g_i r_i} = \prod_{i<\alpha} t^{\gamma_i r_i}. \tag{3}$$

where $g_i = \eta(\gamma_i)$. In particular $\omega^{\eta(\gamma)} = t^\gamma$ for every $\gamma \in \Gamma$. □

3.2. **The logarithm.** In the sequel let κ be a regular uncountable cardinal. Our next goal is to prove the following theorem.

Theorem 3.4. *Every omega-field of the form* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *admits an analytic logarithm. More generally, if* $\mathbf{k}$ *is an exponential field, then every omega-field of the form* $\mathbb{K} = \mathbf{k}((G))_\kappa$ *admits an analytic logarithm.*

Proof. We construct a logarithm depending both on the omega-map and on an auxiliary function h. Let $h : \mathbb{K} \to \mathbb{K}^{>0}$ be a chain isomorphism (any ordered field admits such a function, for instance $h(x) = (-x+1)^{-1}$ for $x \leq 0$ and $h(x) = x+1$ for $x \geq 0$). For $x \in \mathbb{K}$, we let

$$\log(\omega^{\omega^x}) = \omega^{h(x)}.$$

This defines log on the subclass $\omega^{\omega^{\mathbb{K}}}$ of G, which we call the class of **fundamental monomials**. They can be seen as the representatives of the multiplicative archimedean classes.

Next we define $\log(g)$ for an arbitary g in G. Since $G = \omega^{\mathbb{K}}$, we can write $g = \omega^x$ for some $x \in \mathbb{K}$. We then write $x = \sum_{i<\alpha} g_i r_i = \sum_{i<\alpha} \omega^{x_i} r_i$ and set $\log(g) = \sum_{i<\alpha} \omega^{h(x_i)} r_i$. Summing up, the definition of $\log_{|G}$ takes the form

$$\log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \omega^{h(x_i)} r_i. \tag{4}$$

The idea is that $\omega^{\sum_{i<\alpha} g_i r_i}$ should be thought as an infinite product $\prod_{i<\alpha} \omega^{g_i r_i}$, and we stipulate that log maps infinite products into infinite sums.

We can now extend log to the whole of $\mathbb{K}^{>0}$ as follows. For $x \in \mathbb{K}^{>0}$ we write $x = gr(1+\varepsilon)$ with $g \in G, r \in \mathbf{k}^{>0}$ and $\varepsilon \prec 1$, and we define

$$\log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}. \tag{5}$$

The infinite sum makes sense because the terms under the summation sign are summable and the sum belongs to $\mathbf{k}((G))_\kappa$ (becacuse κ is regular and uncountable).

We must verify that with these definitions log is an analytic logarithm (Definition 2.10). It is not difficult to see that log is an increasing morphism from $(\mathbb{K}^{>0}, \cdot, 1, <)$ to $(\mathbb{K}, +, 0, <)$. To prove the surjectivity let us first observe that $\mathbf{k} = \log(\mathbf{k}^{>0}) \subseteq \log(\mathbb{K}^{>0})$. Moreover, for $\varepsilon \prec 1$ we have $\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$ with inverse given by $e^\varepsilon = \sum_n \frac{\varepsilon^n}{n!}$, and therefore $\log(1 + o(1)) = o(1)$. Since $\mathbb{K} = \mathbb{K}^{\uparrow} + \mathbf{k} + o(1)$, to finish the proof of the surjectivity it suffices to show that $\log(G) = \mathbb{K}^{\uparrow}$. So let $x = \sum_{i<\alpha} g_i r_i \in \mathbb{K}^{\uparrow}$, namely $g_i \in G^{>1}$ for all i. We must show that x is in the image of log. Since $h : \mathbb{K} \to \mathbb{K}^{>0}$ is surjective and $G = \omega^{\mathbb{K}}$, we have $G^{>1} = \omega^{(\mathbb{K}^{>0})} = \omega^{h(\mathbb{K})}$, so we can choose $x_i \in \mathbb{K}$ so that $g_i = \omega^{h(x_i)}$ for all i. Now by definition $\log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \omega^{h(x_i)} r_i = x$ concluding the proof of surjectivity. □

In the above theorem we have considered $\mathbf{k}((G))_\kappa$, rather than an arbitrary analytic subfield $\mathbb{K}$ of $\mathbf{k}((G))$, because for the proof to work we need to know that whenever $\sum_{i<\alpha} \omega^{x_i} r_i \in \mathbb{K}$, we also have $\sum_{i<\alpha} \omega^{h(x_i)} r_i \in \mathbb{K}$.

Definition 3.5. We call $\log_{\omega,h} : \mathbb{K}^{>0} \to \mathbb{K}$ the analytic logarithm induced by the omega-map $\omega : \mathbb{K} \to G$ and the chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ as given by (4)-(5) in the proof of Theorem 3.4, and we call $\exp_{\omega,h}$ its compositional inverse.

3.3. **Getting a logarithm satisfying the growth axiom.** The structures constructed so far are exponential fields, but not necessarily models of $T_{\exp}$. In this section we show how to get models of $T_{\exp}$. We need the following lemma to take care of the growth axiom at infinity.

Lemma 3.6. *Let $\mathbb{K} = \mathbf{k}((G))_\kappa$ be equipped with an omega-map $\omega : \mathbb{K} \cong G$. Then there exists a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ such that $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$.*

Proof. The idea is a bootstrapping procedure. Given an h we produce a log and an exp, and given the exp we produce a new h. We then glue together a couple of h's obtained in this way to produce the final h.

To begin with, consider the following chain isomorphism $\mathbb{K} \to \mathbb{K}^{>0}$, definable in any ordered field:

$$h_0(x) = \begin{cases} x+1 & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{for } x < 0, \end{cases} \qquad h_1(x) = \begin{cases} \frac{1}{2}x+1 & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{for } x < 0. \end{cases}$$

Definition 3.5 yields two logarithmic functions $\log_0 = \log_{\omega,h_0}$ and $\log_1 = \log_{\omega,h_1}$ on $\mathbf{k}((G))_\kappa$ associated with h_0 and h_1 (and the given omega-map). Since $h_1(x) \leq h_0(x)$, we have $\log_1(x) \leq \log_0(x)$ for all $x \in \mathbb{K}^{>1}$. The corresponding exponential functions $\exp_0, \exp_1$ satisfy the opposite inequality: $\exp_0(x) \leq \exp_1(x)$ for $x > 0$.

We claim that

$$\exp_0(x) \prec \omega^x \text{ for } x > \mathbf{k} \quad \text{and} \quad \exp_1(x) \prec \omega^x \text{ for } x \leq -\omega^3.$$

Indeed, note that $h_0(x) > x$ for all $x \in \mathbb{K}$ and $h_1(x) < x$ for $x > 2$. Taking the compositional inverse we obtain $x > h_0^{-1}(x)$ for all $x \in \mathbb{K}$ and $x < h_1^{-1}(x)$ for $x > 2$. Now let $y \in \mathbb{K}^{>\mathbf{k}}$, and let $r\omega^x$ be the leading term of y (where $r \in \mathbf{k}^{>0}$, $x \in \mathbb{K}^{>0}$). Then

$$\exp_0(y) \prec \exp_0(2r\omega^x) = \omega^{2r\omega^{h_0^{-1}(x)}} \prec \omega^{\frac{r}{2}\omega^x} \prec \omega^y,$$

since $2r\omega^x - y > \mathbf{k}$, $y - \frac{r}{2}\omega^x > \mathbf{k}$, and $\omega^{h_0^{-1}(x)} \prec \omega^x$.

Similarly, $h_1(x) < x$ for all $x \in \mathbb{K}^{>2}$. Let $y \in \mathbb{K}^{\geq \omega^3}$, and let $r\omega^x$ be the leading term of y. Then $r \in \mathbf{k}^{>0}$, $x \in \mathbb{K}^{>2}$ and

$$\exp_1(y) \succ \exp_1\left(\frac{r}{2}\omega^x\right) = \omega^{\frac{r}{2}\omega^{h_1^{-1}(x)}} \succ \omega^{2r\omega^x} \succ \omega^y.$$

Letting $z = -y \leq -\omega^3$, we obtain $\exp_1(z) = \frac{1}{\exp_1(y)} \prec \frac{1}{\omega^y} = \omega^z$, and the claim is proved.

We can now build the final chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ by taking the functions $\exp_0$, $\exp_1$ restricted to suitable convex subsets of $\mathbb{K}$, and defining h on the complement as an increasing function in such a way that globally h is increasing and bijective. A concrete choice can be the following. Let $c = \exp_1(-\omega^3) > 0$. Define

$$h(x) = \begin{cases} \exp_0(x) & \text{for } x > \mathbf{k} \\ 2c + x & \text{for } 0 < x \text{ and } x \preceq 1 \\ 2c + \frac{c}{\omega^3}x & \text{for } -\omega^3 \leq x \leq 0 \\ \exp_1(x) & \text{for } x < -\omega^3. \end{cases}$$

By construction, h is a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$: it is order preserving because $\exp_0$, $\exp_1$ are themselves chain isomorphisms, and it is surjective since $\exp_0(\mathbb{K}^{>\mathbf{k}}) = \mathbb{K}^{>\mathbf{k}}$, $\exp_1((-\infty, -\omega^3)) = (0, c)$. Moreover, $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$, as desired:

- if $x > \mathbf{k}$, then $h(x) = \exp_0(x) \prec \omega^x$;
- if $0 < x \preceq 1$, then $h(x) = 2c + x \preceq 1 \prec \omega^x$;
- if $-\omega^3 \leq x \leq 0$, then $h(x) \asymp c = \exp_1(-\omega^3) \prec \omega^{-\omega^3} \preceq \omega^x$;
- if $x < -\omega^3$, then $h(x) = \exp_1(x) \prec \omega^x$. □

We next show that an h as constructed above is sufficient to guarantee the growth axiom at infinity.

Lemma 3.7. *Let* $\log = \log_{\omega,h} : \mathbb{K}^{>0} \to \mathbb{K}$ *be as in Definition 3.5. If* h *satisfies* $h(x) \prec \omega^x$ *for every* $x \in \mathbb{K}$*, then* $\log(y) < y^r$ *for all positive* $r \in \mathbf{k}$ *and all* $y > \mathbf{k}$ *(where* y^r *is defined as* $e^{r\log(y)}$*).*

Proof. Assume $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$. This means that $h(x) < \omega^x r$ for all $r \in \mathbf{k}^{>0}$. Let $y = \omega^{\omega^x}$. Then $\log(y) = \log(\omega^{\omega^x}) = \omega^{h(x)} < \omega^{\omega^x r} = y^r$. We have thus proved that $\log(y) < y^r$ for y of the form ω^{ω^x} and $r \in \mathbf{k}^{>0}$.

We now prove the inequality for y of the form ω^x, where $x \in \mathbb{K}^{>0}$. To this aim we write the exponent x in the form $\sum_{i<\alpha} \omega^{x_i} r_i$ and observe that $r_0 > 0$ and $\log(\omega^x) = \log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \log(\omega^{\omega^{x_i}}) r_i$. By the special case we have $\log(\omega^{\omega^{x_i}}) < \omega^{\omega^{x_i} a} \leq \omega^{\omega^{x_0} a}$ for every $i < \alpha$ and $a \in \mathbf{k}^{>0}$. Letting $a = rr_0/2$ it follows that

$$\log(\omega^x) < \omega^{\omega^{x_0} a} = \left(\omega^{\omega^{x_0} r_0}\right)^{\frac{a}{r_0}} < \left(\omega^{2x}\right)^{\frac{a}{r_0}} = \omega^{xr}.$$

For a general $y > \mathbf{k}$, write y in the form $\omega^x s(1+\varepsilon)$ with $s \in \mathbf{k}^{>0}$, $x > 0$ and $\varepsilon \prec 1$, and observe that $\log(y) < \log(2s) + \log(\omega^x) < (\omega^x)^{\frac{r}{2}} < y^r$ for any $r \in \mathbf{k}^{>0}$. □

In the case when the residue field $\mathbf{k}$ is archimedian, the statement in the conclusion of Lemma 3.7 is equivalent to the growth axiom at infinity (Definition 2.11). We are now ready for the main result of this section.

Theorem 3.8. *Every omega-field of the form* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *admits an analytic logarithm making it into a model of* $T_{an,\exp}$*. More generally, if* $\mathbf{k}$ *is a model of* $T_{an,\exp}$*, then every omega-field of the form* $\mathbb{K} = \mathbf{k}((G))_\kappa$ *admits an analytic logarithm making it into a model of* $T_{an,\exp}$*.*

Proof. By Proposition 2.12 and Lemma 3.7. □

3.4. **Growth axiom and o-minimality.** We now discuss the connections between the growth axiom and o-minimality (see [4] for the development of the theory of o-minimal structures).

Lemma 3.9. *Let* $\mathbb{K}$ *be an o-minimal exponential field. Note that* exp *must be differentiable and by a linear change of variable, we can assume that* $\exp'(0) = 1$*. Then* $\exp(x) > x^n$ *for all positive* $n \in \mathbb{N}$ *and all* $x > \mathbb{N}$*.*

Proof. Given a definable differentiable unary function $f : \mathbb{K} \to \mathbb{K}$ in an o-minimal expansion of a field, its derivative f' is definable, and if f' is always positive, then f is increasing. It follows that if f, g are definable differentiable functions satisfying $f(a) \leq g(a)$ and $f'(x) < g'(x)$ for all $x \geq a$, then $f(x) < g(x)$ for every $x > a$. Starting with $0 < \exp(x)$ and integrating we then inductively obtain that for each positive $k, n \in \mathbb{N}$ there is a positive $c \in \mathbb{N}$ such that $kx^n \leq e^x$ for all $x > c$. □

By the above observation and Ressayre's axiomatization [15], an exponential field is a model of $T_{\exp}$ if and only if it satisfies the complete theory of restricted exponentiation and it is o-minimal.

Theorem 3.10. *Assume* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *has an omega-map* $\omega : \mathbb{K} \cong G$*. Fix a chain isomorphism* $h : \mathbb{K} \cong \mathbb{K}^{>0}$ *and put on* $\mathbb{K}$ *the logarithm induced by* ω *and* h *as in Definition 3.5. Then* $\mathbb{K}$ *is either a model of* $T_{\exp}$ *or it is not even o-minimal.*

Proof. We have already seen that if $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$, then $\mathbb{K}$ is a model of $T_{\exp}$ (Theorem 3.8). Now suppose that $h(x) \not\prec \omega^x$ for some x. Then there is some $n \in \mathbb{N}^{>0}$ such that $h(x) \geq \frac{1}{n}\omega^x$. Letting $y = \omega^{\frac{1}{n}\omega^x}$, we have $\log(y) = \frac{1}{n}\log(\omega^{\omega^x}) = \frac{1}{n}\omega^{h(x)} \geq \frac{1}{n}\omega^{\frac{1}{n}\omega^x} = \frac{1}{n}y$, hence $y^n \geq e^y$, contradicting o-minimality by Lemma 3.9 (since exp extends the real exponential function, we have $\exp'(0) = 1$, so the hypothesis of the lemma are satisfied). □

4. Other exponential fields of series

4.1. **Criterion for the existence of an omega-map.** In this section we try to classify all possible analytic logarithms on $\mathbf{k}((G))_\kappa$. We show that in the case of omega-fields every analytic logarithm arises from an omega-map and some h.

Theorem 4.1. *Assume that $\mathbb{K} = \mathbf{k}((G))_\kappa$ has an analytic logarithm* log. *Then:*

1. *$\mathbb{K}$ has an omega-map $\omega : \mathbb{K} \cong G$ if and only if G is isomorphic to $G^{>1}$ as a chain;*
2. *moreover, if $G \cong G^{>1}$, there is an omega-map and a chain isomorphism $h : \mathbb{K} \cong \mathbb{K}^{>0}$ such that the logarithm induced by ω and h coincides with the original logarithm.*

Proof. First note that $\mathbb{K}$, being an ordered field, is always isomorphic to $\mathbb{K}^{>0}$ as a chain. If there is an omega-map $\omega : \mathbb{K} \cong G$, we obtain an induced isomorphism from $G = \omega^{\mathbb{K}}$ to $G^{>1} = \omega^{\mathbb{K}^{>0}}$.

For the opposite direction, assume that G is isomorphic to $G^{>1}$ as a chain and let $\psi : G \cong G^{>1}$ be a chain isomorphism. Define $\omega : \mathbb{K} \to G$ by

$$\omega^{\sum_{i<\alpha} g_i r_i} = e^{\sum_{i<\alpha} \psi(g_i) r_i}.$$

In particular we have $\omega^g = e^{\psi(g)}$. Clearly ω is a morphism from $(\mathbb{K}, +, 0, <)$ to $(G, \cdot, 1, <)$ and to prove that it is an omega-map it only remains to verify that it is surjective. To this aim recall that $\log(G) = \mathbb{K}^{\uparrow}$ (by definition of analytic logarithm), so for the corresponding exp we have $G = \exp(\mathbb{K}^{\uparrow})$. Since $e^{\sum_{i<\alpha} \psi(g_i) r_i}$ is an arbitrary element of $\exp(\mathbb{K}^{\uparrow})$, the surjectivity of ω follows. Now since $\psi : G \cong G^{>1}$ and $G = \omega^{\mathbb{K}}$, there is a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ such that

$$\psi(\omega^x) = \omega^{h(x)}.$$

Since $e^{\psi(\omega^x)} = \omega^{\omega^x}$, we obtain $\omega^{\omega^x} = e^{\omega^{h(x)}}$ and thefore $\log(\omega^{\omega^x}) = \omega^{h(x)}$. It then follows that log coincides with the analytic logarithm induced by ω and h. □

Corollary 4.2. *Every analytic logarithm on an omega-field of the form $\mathbb{K} = \mathbf{k}((G))_\kappa$ arises from some omega-map and some chain isomorphism $h : \mathbb{K} \cong \mathbb{K}^{>0}$.*

4.2. **The iota-map.** Our next goal is to show that $\mathbf{k}((G))_\kappa$ may have an analytic logarithm without being an omega-field. This will be proved in the next subsection. Here we recall the following two results from [12] with a sketch of the proofs for the reader's convenience (considering that the notations are different). We use the same notation $H(\Gamma) = \left(\prod t^{\Gamma C}\right)_\kappa$ employed in Lemma 3.3, with $C = (\mathbf{k}, +, <)$.

Fact 4.3 ([12])**.** *Let $\mathbf{k}$ be an exponential field. Let Γ be a chain and suppose there is an isomorphism of chains $\iota : \Gamma \cong H(\Gamma)^{>1}$. Let $G = H(\Gamma)$ and let $\mathbb{K} = \mathbf{k}((G))_\kappa$. Then:*

1. *there is an analytic logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ such that $\log(t^\gamma) = \iota(\gamma) \in G$.*

(2) *if* $\mathbf{k}$ *is a model of* $T_{an,\exp}$ *and* $\iota(\gamma) < t^{\gamma r}$ *for each* $r \in \mathbf{k}^{>0}$, *then* log *satisfies the growth axiom at infinity, thus making* $\mathbb{K}$ *into a model of* $T_{an,\exp}$.[2]

Proof. Define $\log = \log_\iota$ on G by

$$\log(\prod_{i<\alpha} t^{\gamma_i r_i}) = \sum_{i<\alpha} \iota(\gamma_i) r_i \in \mathbf{k}((G^{>1}))_\kappa$$

Given $x \in \mathbb{K}^{>0}$, write $x = gr(1+\varepsilon)$ for some $r \in \mathbf{k}^{>0}$, $g \in G$ and $\varepsilon \in o(1)$; now define $\log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$, where $\log(r)$ refers to the given logarithm on $\mathbf{k}$, and observe that since $\varepsilon \prec 1$ and $\kappa > \omega$ the infinite sum belongs to $\mathbb{K} = \mathbf{k}((G))_\kappa$. Clearly log is an analytic logarithm and (1) is proved. The verification of point (2) is as in Theorem 3.4. □

Fact 4.4 ([12]). *Fix a chain* Γ_0 *and a chain embedding* $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}$ *(for instance* $\iota_0(\gamma) = t^\gamma$*). Then:*

(1) *there is a chain* $\Gamma \supseteq \Gamma_0$ *and a chain isomorphism* $\iota : \Gamma \cong H(\Gamma)^{>1}$ *extending* ι_0*;*
(2) *if* $\iota_0(\gamma) < t^{\gamma r}$ *for every* $\gamma \in \Gamma_0$ *and* $r \in C^{>0}$, *then* $\iota(\gamma) < t^{\gamma r}$ *for every* $\gamma \in \Gamma$ *and* $r \in C^{>0}$.

Proof. The proof of (1) is similar to the proof of Lemma 3.3, the only difference is that we use $H(\Gamma)^{>1}$ instead of $H(\Gamma)$. Starting with the initial chain embedding $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}$ we inductively produce chain embeddings $\iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1}$ and $j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta$ for $\alpha < \beta$. The step from β to $\beta+1$ is based on the following diagram

(6)
$$\begin{array}{ccc} \Gamma_\beta & \xrightarrow{\iota_\beta} & H(\Gamma_\beta)^{>1} \\ {\scriptstyle j_\beta}\downarrow & \overset{f_\beta}{\swarrow} & \downarrow{\scriptstyle H(j_\beta)} \\ \Gamma_{\beta+1} & \xrightarrow{\iota_{\beta+1}} & H(\Gamma_{\beta+1})^{>1} \end{array}$$

where $\Gamma_{\beta+1}$ is a chain isomorphic to $H(\Gamma_\beta)^{>1}$, f_β is a chain isomorphism, and the embeddings j_β and $\iota_{\beta+1}$ are defined so that the diagram commutes. Limit stages are handled as in Lemma 3.3. Finally we set $\Gamma = \Gamma_\kappa = \varinjlim_{\beta<\kappa} \Gamma_\beta$ and $\iota = \iota_\kappa$ and observe that $\iota : \Gamma \to H(\Gamma)^{>1}$ is a chain isomorphism.

To prove (2), we show by induction on $\beta < \kappa$ that $\iota_\beta(\gamma) < t^{\gamma r}$ for every $\gamma \in \Gamma_\beta$ and $r \in C^{>0}$, provided this holds for $\beta = 0$. Since limit stages are easy, it suffices to prove the induction step from β to $\beta+1$. So let $\eta \in \Gamma_{\beta+1}$. Then $\eta = f_\beta(x)$ for some $x = \prod_i t^{\gamma_i r_i} \in \left(\prod t^{\Gamma_\beta C}\right)^{>1}_\kappa$. The embedding ι_β sends η to $\prod_i t^{j_\beta(\gamma_i) r_i}$ where $j_\beta = f_\beta \circ \iota_\beta$ is the embedding of Γ_β into $\Gamma_{\beta+1}$. We must prove that $\prod_i t^{j_\beta(\gamma_i) r_i} < t^{\eta r}$ for every $r \in C^{>0}$. This is equivalent to saying $j_\beta(\gamma_0) < \eta$, which in turn is equivalent to $\iota_\beta(\gamma_0) < \prod_i t^{\gamma_i r_i}$. The latter inequality follows from the inductive hypothesis and the proof is complete. □

4.3. **A model without an omega-map.** We can now show that there are fields of the form $\mathbb{R}((G))_\kappa$ which admit an analytic logarithm but not an omega-map.

Theorem 4.5. *Given a regular uncountable cardinal* κ, *there is* G *such that the field* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *has an analytic logarithm making it into a model of* $T_{\exp}$ *but* G *is not isomorphic to* $G^{>1}$ *as a chain (so* $\mathbb{K}$ *is not an omega-field).*

[2] In the cited paper the authors consider $\mathbf{k} = \mathbb{R}$, but the general case is the same.

Proof. Start with the chain $\Gamma_0 = \omega_1 \times \mathbb{Z}$ ordered lexicographically and the initial embedding $\iota_0 : \Gamma_0 \to \left(\prod_\kappa t^{\Gamma_0 \mathbf{k}}\right)^{>1} = H(\Gamma_0)^{>1}$ given by $\iota_0((\alpha, n)) = t^{(\alpha, n-1)}$. Define $\Gamma = \varinjlim_{\beta<\kappa} \Gamma_\beta$ and $\iota : \Gamma \cong H(\Gamma)^{>1}$ as in Fact 4.4 and note that $\iota(\gamma) < t^{\gamma r}$ for every $\gamma \in \Gamma$ and $r \in \mathbf{k}^{>0}$ (since this holds for ι_0 and is preserved at later stages). Now take $G = H(\Gamma)$ and put on the field $\mathbb{K} = \mathbf{k}((G))_\kappa$ the log induced by ι as in Fact 4.3. By the above inequalities the log satisfies the growth axiom at infinity, so $\mathbb{K}$ is a model of $T_{\exp}$. It remains to show that $G \ncong G^{>1}$ as a chain. Note that the image of $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1} = \Gamma_1$ is cofinal and coinitial in $H(\Gamma_0)^{>1}$. It follows that for each $\beta \le \kappa$, the image of $\iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1}$ is cofinal and coinitial in $H(\Gamma_\beta)^{>1} = \Gamma_{\beta+1}$. Likewise, by an easy induction, for each $\beta \geq 0$ the image of Γ_0 in Γ_β is initial and cofinal. In particular the image of Γ_0 in the final chain $\Gamma_\kappa = \Gamma \cong H(\Gamma)^{>1}$ is coinitial and cofinal. Since Γ_0 has cofinality ω_1 and coinitiality ω, it follows that Γ and $H(\Gamma)^{>1}$ have cofinality ω_1 and coinitiality ω. Now observe that $1/x$ is an order-reversing bijection from $H(\Gamma)^{<1}$ to $H(\Gamma)^{>1}$, and therefore $H(\Gamma) = H(\Gamma)^{<1} \cup 1 \cup H(\Gamma)^{>1}$ has cofinality and coinitiality both equal to ω_1. We conclude that $G = H(\Gamma)$ cannot be chain isomorphic to $G^{>1}$, because they have different coinitiality. □

5. Omega-groups

A group isomorphic to the value group of an omega-field will be called **omega-group**. It would be interesting to give a characterization of the omega-groups. As a partial result, we characterise those groups G such that $\mathbf{k}((G))_\kappa$ is an omega-field. We also clarify the relation between having an omega-map and having an analytic logarithm.

Proposition 5.1. *Let $\mathbb{K}$ be a field of the form $\mathbf{k}((G))_\kappa$. Then:*

(1) *if $\mathbb{K}$ is an omega-field, then G is isomorphic to $\left(\prod t^{\Gamma \mathbf{k}}\right)_\kappa$, where the chain Γ is order-isomorphic to (the underlying chain of) G itself;*
(2) *if $\mathbb{K}$ has an analytic logarithm, then G is isomorphic to $\left(\prod t^{\Gamma \mathbf{k}}\right)_\kappa$, where Γ is order-isomorphic to $G^{>1}$.*

Proof. (1) The elements of $\mathbb{K}$ can be written in the form $\sum_{i<\alpha} g_i r_i$. So the elements of G are of the form $\omega^{\sum_{i<\alpha} g_i r_i}$. This corresponds to the element $\prod_{i<\alpha} t^{g_i r_i} \in \left(\prod t^{G \mathbf{k}}\right)_\kappa$ via an isomorphism.

(2) Since $\log(G) = \mathbb{K}^\uparrow$, we have $G = \exp(\mathbb{K}^\uparrow)$, and therefore an element g of G can be written in the form $\exp(\sum_{i<\alpha} g_i r_i)$ with $g_i \in G^{>1}$ and $r_i \in \mathbf{k}$. This corresponds to $\prod_{i<\alpha} t^{g_i r_i} \in \left(\prod t^{G^{>1} \mathbf{k}}\right)_\kappa$ via an isomorphism. □

In the following corollary we abstract some of the properties of the groups considered above. We refer to [11] for the definition of the value-set.

Corollary 5.2. *Let $\mathbb{K}$ be a field of the form $\mathbf{k}((G))_\kappa$.*

(1) *If $\mathbb{K}$ has an analytic logarithm, then G is a $\mathbf{k}$-module, the value set Γ of G is order isomorphic to $G^{>1}$, and all the $\mathbf{k}$-archimedean components of G are isomorphic to the additive group of $\mathbf{k}$.*
(2) *If $\mathbb{K}$ is an omega-field, the same properties hold (as in particular $\mathbb{K}$ has an analytic logarithm) and in addition G is isomorphic to $G^{>1}$ as a chain.*

Acknowledgements. All the authors want to thank the organizers of the trimester "Model Theory, Combinatorics and Valued fields in model theory" at the Institute Henri Poincaré, January 8 - April 6, 2018 where part of this research was conducted. We also thank the anonymous referee for useful comments.

References

[1] Alessandro Berarducci and Vincenzo Mantova. Surreal numbers, derivations and transseries. *Journal of the European Mathematical Society*, 20:339–390, 2018.

[2] Alessandro Berarducci and Vincenzo Mantova. Transseries as germs of surreal functions. *Transactions of the American Mathematical Society*, 371(5):3549–3592, 2019.

[3] John H. Conway. *On number and games*, volume 6 of *London Mathematical Society Monographs*. Academic Press, London, 1976.

[4] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248. LMS Lecture Notes Series, Cambridge Univ. Press, 1998.

[5] Lou van den Dries, Angus Macintyre, and David Marker. The elementary theory of restricted analytic fields with exponentiation. *Annals of Mathematics*, 140(1):183–205, 1994.

[6] Lou van den Dries and Philip Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167(2):173–188, 2001, and Erratum, *Fundamenta Mathematicae*, 168(3), 2001, 295–297.

[7] Harry Gonshor. *An introduction to the theory of surreal numbers*. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1986.

[8] Hans Hahn. Über die nichtarchimedischen Grössenszsteme. *Sitz. der K. Akad der Wiss., Math Nat. KL.*, 116(IIa):601–655, 1907.

[9] Irving Kaplansky. Maximal fields with valuations. *Duke Mathematical Journal*, 9:303–321, 1942.

[10] Franz-Viktor Kuhlmann, Salma Kuhlmann, and Saharon Shelah. Exponentiation in power series fields. *Proceedings of the American Mathematical Society*, 125(11):3177–3183, 1997.

[11] Salma Kuhlmann. *Ordered Exponential Fields*, volume 12 of *Fields Institute Monographs*. American Mathematical Society, Providence, Rhode Island, 2000.

[12] Salma Kuhlmann and Saharon Shelah. κ-bounded exponential-logarithmic power series fields. *Annals of Pure and Applied Logic*, 136(3):284–296, nov 2005.

[13] Marie-Hélène Mourgues and Jean-Pierre Ressayre. Every Real Closed Field has an Integer Part. *The Journal of Symbolic Logic*, 58(2):641–647, 1993.

[14] Bernhard Hermann Neumann. On ordered division rings. *Trans. Amer. Math. Soc*, 66(1):202–252, 1949.

[15] Jean-Pierre Ressayre. Integer parts of real closed exponential fields (extended abstract). In Peter Clote and J. Krajíček, editors, *Arithmetic, Proof Theory, and Computational Complexity (Prague, 1991)*, volume 23 of *Oxford Logic Guides*, pages 278–288. Oxford University Press, New York, 1993.

Alessandro Berarducci: Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy

E-mail address: `alessandro.berarducci@unipi.it`

Salma Kuhlmann: Fachbereich Mathematik und Statistik, Universität Konstanz, Universitätsstrasse 10, 78457 Konstanz, Germany

E-mail address: `salma.kuhlmann@uni-konstanz.de`

Mickaël Matusinski: Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux, 351 cours de la Libération, 33405 Talence cedex, France

E-mail address: `mickael.matusinski@math.u-bordeaux.fr`

Vincenzo Mantova: School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

E-mail address: `v.l.mantova@leeds.ac.uk`