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A resistive-capacitive model of pile heat exchangers with an application to thermal response tests interpretation

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Highlights

- A new finite cylindrical source model with adiabatic surface is developed.
- A new resistive-capacitive semi-analytical pile heat exchanger model is developed.
- The new model improves the assessment of pile thermal performance.
- Neglecting heat capacitance in the pile leads to an underestimation of performance.
- The new model is successfully used to interpret a thermal response test.

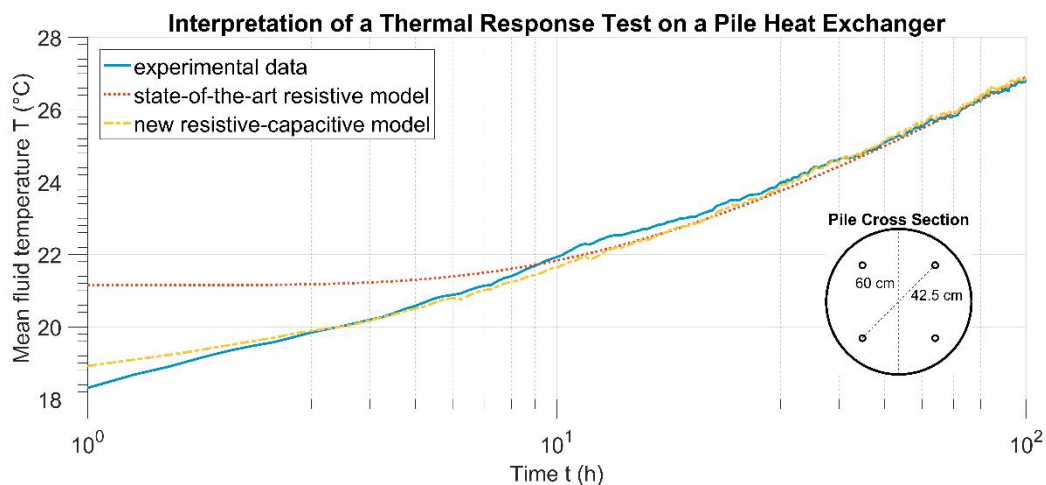
Abstract

Pile Heat Exchangers (PHE) are an attractive solution to reduce both costs and greenhouse gas emissions for new buildings. However, most state-of-the-art PHE thermal models overlook the heat capacitance of the pile concrete, which is known to be important in thermal analysis. A semi-analytical (SA) model accounting for the pile concrete inertia is developed and validated against a finite-element code. Analysis shows that accounting for PHE inertia always leads to higher performances compared to purely resistive models. Application of the model to interpretation of thermal response tests data allows estimates to be made of the minimum duration test required to obtain reliable values of ground and concrete conductivities.

Keywords

- Pile heat exchangers
- Thermal models
- Thermal response test
- Near-surface geothermal energy
- Ground source heat pumps

Graphical abstract



Nomenclature

| <i>Latin Letters</i> | | <i>Subscripts</i> | |
|----------------------|--|---------------------|------------------------------------|
| a | thermal diffusivity ($m.s^{-2}$) | 0 | undisturbed conditions |
| C | capacity of a node ($JK^{-1}.m^{-1}$) | b | borehole wall |
| e | energy per meter of pile (Jm^{-1}) | c | concrete |
| \dot{m} | flow rate ($kg.s^{-1}$) | fl | heat-carrier fluid |
| r | radius | in | inlet |
| R | thermal resistance ($K.m.W^{-1}$) | m | ground |
| p | power per meter of pile ($W.m^{-1}$) | out | outlet |
| T | temperature ($^{\circ}C$) | p | pipe |
| t | time (s) | ∞ | steady-state value |
| t^* | normalized time (Fourier number) | | |
| x,y | capacities locations in the RC circuit | | |
| | | | |
| <i>Greek letters</i> | | <i>Superscripts</i> | |
| ε | misfit (root mean square error) | | |
| λ | thermal conductivity ($W.K^{-1}.m^{-1}$) | n | time step n |
| $[\Delta]$ | Conductance matrix ($W.K^{-1}.m^{-1}$) | $*$ | normalized value |
| ρC_p | volume-specific heat capacity ($JK^{-1}.m^{-3}$) | $\phi 0$ | adiabatic condition at the surface |
| | | $T0$ | imposed temperature at the surface |
| | | | |
| <i>Acronyms</i> | | | |
| GHE | Ground Heat Exchanger | | |
| PHE | Pile Heat Exchanger | | |
| FE | Finite Elements | | |
| RC | Resistive-Capacitive | | |
| ICS | Infinite Cylindrical Source | | |
| ILS | Infinite Line Source | | |
| SA | Semi-Analytical | | |
| FLS | Finite Line Source | | |

1 Introduction

2 Ground-sourced heat pumps (GSHP) can significantly reduce CO₂ emissions associated with new
3 buildings. However high investment cost is a limitation to the deployment of this technology. As a
4 consequence in France the number of yearly installed GSHP collapsed from 15,500 to 3,200
5 between 2010 and 2014 [1] [2]. Cost-effective systems have to be found to reduce GSHP capital
6 costs. Energy geostructures such as Pile Heat Exchangers (PHE) are one solution, since they couple
7 the structural role of the geostructure with that of ground-sourced heat exchangers.

8 PHE are superficially similar to borehole heat exchangers (BHE), but although BHE sizing tools
9 are available to engineers, including pre-sizing Excel sheets (AHSRAE), bespoke software (EED)
10 and dynamics simulation tools (TRNSYS-DST, FEFLOW), there are few PHE sizing tools, with the
11 commercial software PILESIM [3] being the main example.

12 This lack of design tools is partly due to the fact that thermal modelling of PHE is more complex
13 than BHE. PHE radius can exceed 50 cm and, compared with BHE which have a typical radius of
14 <10 cm. Accurate description of the heat storage in the pile concrete is therefore needed [4].
15 Furthermore, typical PHE depths are in the range 10-30 m, where 100-200 m deep BHE are
16 typical. The aspect ratio (the ratio between the depth and the pile radius) is therefore much lower
17 for PHE than for BHE. Consequently vertical heat transfers around a PHE play a significant role
18 earlier than for BHE.

19 In addition to design, pile characterisation for determination of analysis input parameters,
20 requires realistic models of the pile capacitance. Current methodologies for the interpretation of
21 thermal response tests (TRT) overlook internal heat capacitance within the PHE. Therefore,
22 reliable PHE sizing also requires the development of relevant methodologies for the
23 implementation and interpretation of TRT.

24 This paper presents a new model of PHE. The paper starts with a brief description of the state of
25 the art concerning PHE models (Section 1). Then the construction of the new semi-analytical
26 models are discussed in Section 2. The model combines relevant step-responses (G-functions)
27 accounting for PHE aspect ratio with resistive-capacitive circuits. The model is validated against
28 a fully discretized finite-element model and its domain of validity and limitations are highlighted.
29 The model's performance compared with existing approaches is then set out (Section 4). The
30 model is finally used to analyse thermal response test (TRT) data (Section 4), and investigate the
31 reliability of interpretation based on the TRT duration. Though much work is carried out
32 considering the implication of operation of energy piles on the stresses and strains, the paper
33 focuses on thermal models does not include any thermo-mechanical assessment. The aim of the
34 article is to provide a fast and accessible algorithm for engineering practices in order to compute
35 the PHE fluid temperature evolution, avoiding the use of complex, resource-consuming and
36 expensive discretized numerical models.

37 1. Model State of the Art

38 Most PHE thermal models are either numerical, analytical or apply a combination of both these
39 techniques. Fully discretised models tend to be more accurate, but at the expense of
40 computational effort. Techniques may include finite element analysis (e.g. [5]) or finite difference
41 analysis (e.g. [6], [7], [8]). Numerical simulation is also commonly used as a research tool, for

42 example to investigation of pipe arrangements and thermal performance (e.g. [9], [10], [11], [12],
43 [13], [14], [15], [16], [17]), but is rarely practical for routine applications.

44 The state of art focuses on analytical models as they are more suitable for routine use than fully
45 discretised models. Analytical models can run over reasonable time frames, i.e. performing
46 simulations over 30 years with hourly time step, without resorting to super computing. The
47 functions produced by the analytical models are often referred to as “step-responses” or G-
48 functions (after the early work on BHEs by Eskilson [18]). Step responses describe the evolution
49 of the normalized temperature of the borehole or pile perimeter under a constant power applied
50 by unit length p ($\text{W}\cdot\text{m}^{-1}$). The evolution of the temperature change ΔT is then given by:

$$\Delta T = \frac{p}{\lambda_m} G(t^*) \quad (1)$$

51 Where $G(t^*)$ is the response function and t^* is a dimensionless time factor (Fourier number) and
52 λ_m the ground thermal conductivity ($\text{W}\cdot\text{K}^{-1}\cdot\text{m}^{-1}$). G-functions are usually configured so that the
53 temperature computed is that at the borehole (or pile) wall.

54 Common G-functions consider that heat can only be transferred by conduction. Convection, i.e.
55 heat transport by water flow, is usually overlooked. The simplest BHE G-function, the *infinite line*
56 *source (ILS)* model, represents the borehole as an infinite line emitting a constant heat flux [19].
57 Further improvements of the geometrical representation include the *finite line source (FLS)* model
58 [18] and the *hollow infinite cylindrical source (HICS)* [20] [21] and solid cylindrical heat source
59 [22].

60 The ILS, FLS and HICS G-functions are often coupled with resistive-capacitive (RC) circuits dealing
61 with the thermal transfer within the borehole itself. While early developments were purely
62 resistive, overlooking the thermal inertia of the grouting material [23] [24] [25], recent works
63 have focused on developing full resistive-capacitive circuits for single U-tube (equipped with 2
64 pipes) BHE [26] [27] [28] [29] [30] or double U-tube BHE (equipped with 4 pipes) [31] [32].
65 However, pile heat exchangers of large diameter equipped with 8 or 10 pipes are not unusual.

66 Recent research on developing G-functions dedicated to PHE have also focused on dealing with
67 the thermal inertia of the concrete, as well as accommodating a greater number of pipes and
68 reduced aspect ratio $H^* = H/r_b$. Due to the large number of parameters characterising the pile and
69 the ground, it is difficult to find a universal G-function for PHE. Loveridge and Powrie set up a
70 practical approach where they defined extreme PHE configurations, leading to lower and upper
71 bounds of numerically computed PHE G-functions. Single-pile [4] and multi-pile configurations
72 [33] are provided, along with additional step response functions to cover the pile inertia which
73 are included via superposition.

74 Bozis et al. developed an analytical method to compute the G-function for a single pile equipped
75 with multiple pipes and provided an analytical expression of G as a function of the number and
76 location of the pipes within the borehole [34]. They produced easy-to-use graphs that may be used
77 for engineering applications, though the methodology holds only if the properties of the pile
78 concrete and surrounding ground are the same.

79 Li and Lai took a different approach, developing G-functions that dealt with the pile inertia
80 explicitly [35]. They applied the infinite line-source theory in composite media accounting directly
81 for the contrast in thermal properties of the concrete and the ground. The approach is elegant but

82 requires derivation for every pipe arrangement. Hence it requires a database similar to that of
83 Eskilson for routine implementation [18]. Analytical models of spiral coils PHE have been
84 developed for homogenous ([36], [37]) or heterogeneous [38] ground conditions, some of them
85 being able to distinguish ground and concrete properties [39].

86 Zarella et al. developed a model for PHE equipped with 6 pipes, which can be generalized to any
87 number of pipe [40]. However, this model still requires a steady-state resistance, which can be
88 calculated from numerical models or from analytical formulae available in literature. While finite
89 difference or finite elements are often time-consuming, the accuracy of borehole thermal
90 resistance calculation methods is still an open question [41].

91 The final option for pile analysis is the Duct Storage model [23] which underpins the software
92 PILESIM [3]. The model superimposes three solutions, a steady state solution for within the
93 ground heat exchanger, a local ILS, and a global interaction between the underground thermal
94 store and the surrounding soil. While developed for boreholes, but later validated for piles, it is
95 limited by both use of the ILS and a steady resistance within the pile.

96 We present both a new pile G-function and importantly a resistive-capacitive model for a PHE
97 equipped with 4 pipes. This model requires material thermo-physical properties and PHE
98 geometrical properties to compute the evolution of the fluid temperature. Contrarily to the Zarella
99 model [40], no intermediate parameters must be calculated externally by the user, which results
100 in a more straight-forward workflow.

101 2. Model Development

102 In the development of the new model, the following assumptions are made:

- 103 (i) Physical properties of the materials (underground water, soil matrix, PHE heat carrier
104 fluid) do not depend upon temperature.
- 105 (ii) The initial, non-disturbed temperature T_0 is constant in the whole domain and remains
106 constant far away from the pile heat exchanger.
- 107 (iii) Both the ground and the pile concrete are regarded as homogenous and impervious
108 media.

109 The heat is transferred in the ground and in the pile by conduction. The partial derivative equation
110 for energy conservation reads:

$$(\rho C_p)_i \frac{\partial T}{\partial t} = \lambda_i \Delta T \quad (2)$$

111 λ accounts for the thermal conductivity of materials ($\text{W}\cdot\text{K}^{-1}\cdot\text{m}^{-1}$) and (ρC_p) for the volumetric heat
112 capacity ($\text{J}\cdot\text{K}^{-1}\cdot\text{m}^{-3}$). The subscripts i refers to the solid material – the ground media is subscripted
113 m and the concrete subscripted c .

114 The dimensionless time factor t^* (Fourier number) is introduced to characterize the ratio of
115 diffused heat to stored heat:

$$t^* = \frac{\lambda_i}{(\rho C_p)_i r_b^2} t \quad (3)$$

116 Note that the normalization length is r_b , the pile radius. This leads to the heat equation under its
117 normalized form:

$$\frac{\partial T^*}{\partial t^*} = \Delta^* T^* \quad (4)$$

118 2.1. Development of Hollow Finite Cylindrical Source (HFCS) G-functions

119 *Finite line source*, $G_{FLS}(t^*)$, and *hollow infinite cylindrical source*, $G_{HICS}(t^*)$, G-functions are not
 120 suitable for PHE modelling due to the short aspect ratio. However no *hollow finite cylindrical*
 121 *source*, $G_{HFCS}(t^*)$, has been developed so far. Furthermore, the “classical” FLS model assumes that
 122 a constant temperature T_0 equal to the mean temperature of the ground is imposed at the surface.
 123 However, this assumption does not seem realistic for PHE as they are located below buildings
 124 whose basement is insulated. Therefore an adiabatic condition at the surface was assumed while
 125 developing the HFCS model. The impact of the type of upper boundary condition (imposed
 126 temperature or insulation) was quantified with the FLS model. The “classical”, temperature-
 127 imposed FLS (denoted FLS^{T0}) subtracts a “mirror” term from a “source” term [42], while in the
 128 adiabatic version ($FLS^{\Phi0}$) both terms are added:

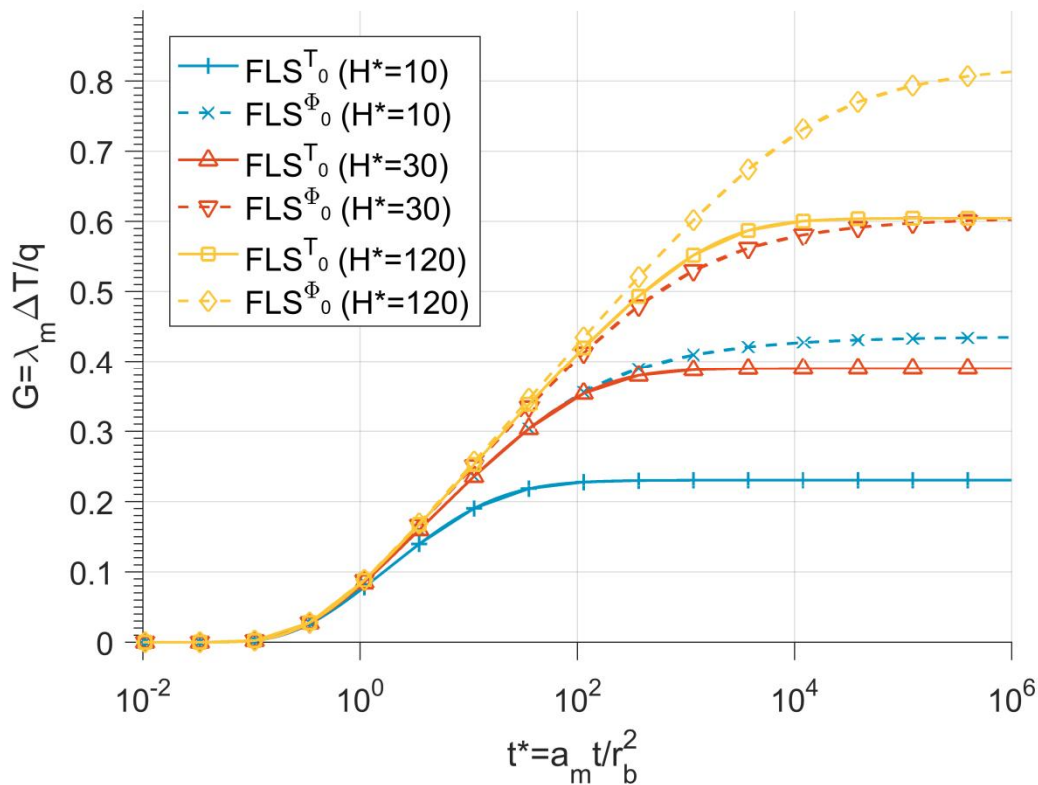
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$$\begin{aligned} G_{FLS}^{T0}(t^*) &= \frac{1}{2\pi} \left[\left(-D_A + \int_{\beta}^{\sqrt{\beta^2+1}} \frac{\operatorname{erfc}(\omega z)}{\sqrt{z^2 - \beta^2}} dz \right) - \left(D_B + \int_{\sqrt{\beta^2+1}}^{\sqrt{\beta^2+4}} \frac{\operatorname{erfc}(\omega z)}{\sqrt{z^2 - \beta^2}} dz \right) \right] \\ G_{FLS}^{\Phi0}(t^*) &= \frac{1}{2\pi} \left[\left(-D_A + \int_{\beta}^{\sqrt{\beta^2+1}} \frac{\operatorname{erfc}(\omega z)}{\sqrt{z^2 - \beta^2}} dz \right) + \left(D_B + \int_{\sqrt{\beta^2+1}}^{\sqrt{\beta^2+4}} \frac{\operatorname{erfc}(\omega z)}{\sqrt{z^2 - \beta^2}} dz \right) \right] \\ D_A &= \frac{\sqrt{\beta^2 + 1} \operatorname{erfc}(\omega \sqrt{\beta^2 + 1}) - \beta \operatorname{erfc}(\omega \beta)}{\omega \sqrt{\pi} (\exp(-\omega^2(\beta^2 + 1)) - \exp(-\omega^2 \beta^2))} \\ D_B &= \frac{\sqrt{\beta^2 + 1} \operatorname{erfc}(\omega \sqrt{\beta^2 + 1}) - \frac{1}{2} (\beta \operatorname{erfc}(\omega \beta) + \sqrt{\beta^2 + 4} \operatorname{erfc}(\omega \sqrt{\beta^2 + 4}))}{\omega \sqrt{\pi} (\exp(-\omega^2(\beta^2 + 1)) - \frac{1}{2} (\exp(-\omega^2 \beta^2) + \exp(-\omega^2(\beta^2 + 4))))} \end{aligned} \quad (5)$$

$$\begin{aligned} \omega &= \frac{H}{2\sqrt{a_m t}} = \frac{H^*}{2\sqrt{t^*}} \\ \beta &= \frac{r_b}{H} = \frac{1}{H^*} \end{aligned}$$

130

131 In eq. (5), H accounts for the ground heat exchanger depth (m), a_m the ground diffusivity ($m^2 \cdot s^{-1}$),
 132 t the time (s) and r_b the ground heat exchanger radius. Assuming an adiabatic condition leads to
 133 much higher values of the G-function than assuming a fixed temperature (Figure 1). Short
 134 boreholes lead to larger discrepancies.



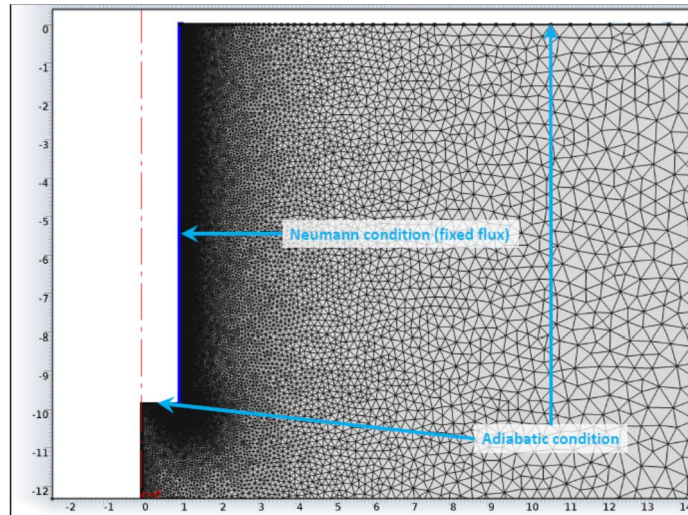
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136 *Figure 1: Comparison of the step responses G produced by the FLSmodel with 2 types of boundary conditions at the*
 137 *surface: either temperature imposed, the most common approach (denoted T_0), or adiabatic condition (denoted Φ_0)*

138

139 This highlights that for short pile heat exchangers the G-function assuming a constant
 140 temperature at the ground surface can be up to approximately 25 % lower than the G-function
 141 assuming adiabatic conditions. Some research is still needed to better understand the influence of
 142 the top boundary on the G-function.

143 In the remaining parts of the paper we will evaluate finite models assuming an adiabatic condition
 144 at the top surface (FLS $^{\Phi_0}$ and HFCS $^{\Phi_0}$). Since an analytical expression of $G_{HFCS}^{\Phi_0}(t^*)$ seems out of
 145 range, $G_{HFCS}^{\Phi_0}(t^*)$ was established from finite element (FE) simulations, achieved in COMSOL-
 146 Multiphysics software on a 2D axisymmetric model (see Figure 2). The COMSOL model solves the
 147 partial derivative equation (4), that is the normalized heat equation. An adiabatic condition was
 148 set on every domain face, except at the ground heat exchanger wall where a constant normalized
 149 power was set (Neumann condition). The size of the domain ($r \approx 2\sqrt{3 t_{max}^*}$) was large enough
 150 to ensure it did not disturb the heat transfer in the borehole vicinity. The mesh was refined in the
 151 vicinity of the borehole wall to account for sharp temperature gradient, with typical length of the
 152 triangular elements being 2 cm. Note that a coarser mesh would have been appropriate, however,
 153 as the model is 2D, a fine mesh does not compromise the execution time.



154

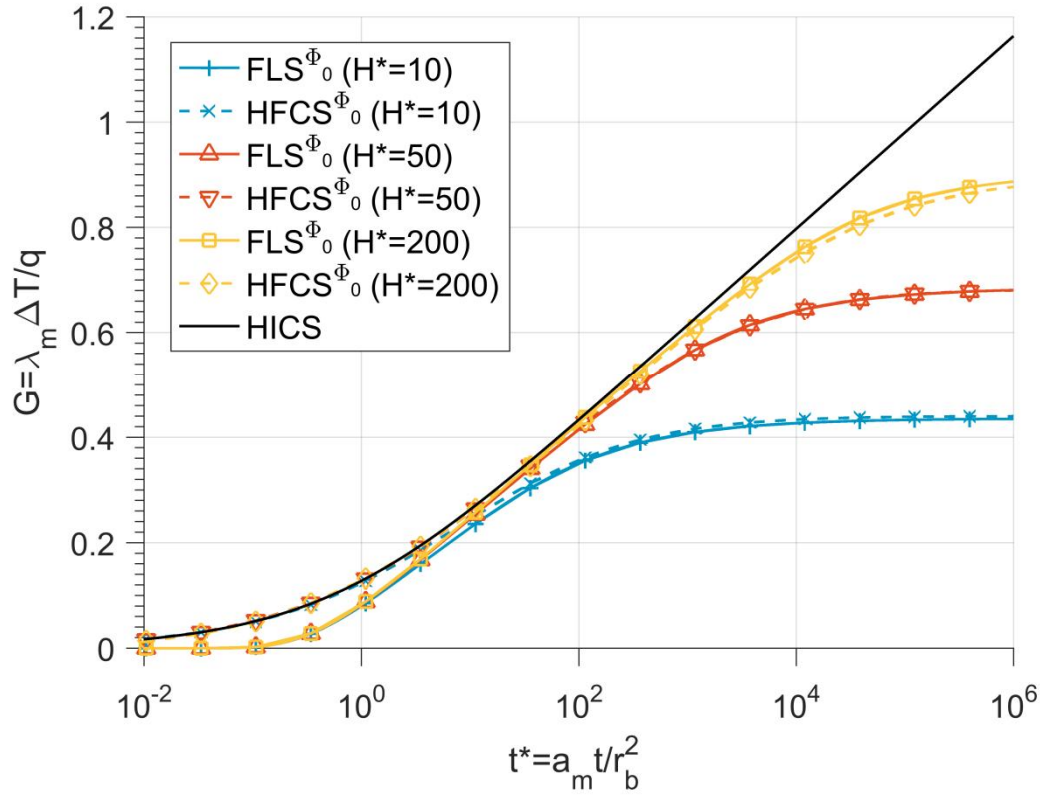
155 *Figure 2: Mesh of the Finite Element model used to compute HFCSG-function. $H^*=10$*

156

157 The FE simulations were carried out for aspect ratio ranging from $H^* = 10$ to 200. This covers
 158 radiuses up to 1 m for 10 m deep PHE, and radiuses ranging from 15 cm to 3,000 cm for a 30 m
 159 deep PHE. $G_{HFCS}^{\phi 0}(t^*)$ was computed through a parametric sweep encompassing 39 simulations
 160 ($H^* = 10, 15, \dots, 195, 200$). The response was obtained by averaging the temperature over the
 161 whole pile depth, evaluated at 109 normalized times t^* ranging from 0 to 1.2×10^6 , following a
 162 geometric progression to capture sharper changes at small time scales.

163 Figure 3 plots the results of the new $G_{HFCS}^{\phi 0}(t^*)$ in comparison with the $G_{FLS}^{\phi 0}(t^*)$. The G-function
 164 for the HICS is also included. Excellent agreement is reached between the analytical solution of
 165 the HICS and HFCS models at small times, and between FLS and HFCS for larger values of t^* . This
 166 is due to the fact that HICS model correctly describes the temperature change close to pile early
 167 in the solicitation, while the FLS model accounts for axial heat transfer which play a more
 168 significant role later [43]. The difference between the FLS and the HFCS almost vanish after $t^* =$
 169 $100 (\Delta_{FLS} < 2.5\%)$.

170



171

172 *Figure 3: Comparison of finite line source FLS^{Φ_0} and hollow infinite cylindrical source models $HICS$ to the newly*
 173 *developed hollow finite cylindrical source ($HFCS^{\Phi_0}$) step response, for three values of aspect ratio H^* . All models*
 174 *include an adiabatic ground surface condition.*

175

176 Hence, the newly developed $HFCS^{\Phi_0}$ G -function assuming adiabatic condition at the top surface is
 177 suitable for both short and long duration, accounting for both heat transfers close to the pile and
 178 vertical heat transfers. For practical applications an easy-to-use regression was established. It was
 179 based on 4251 evaluations of $G_{HFCS}(t^*)$ (39 values of $H^* \times 109$ values of t^*), for $10^{-4} < t^* < 10^6$ and
 180 $10 < H^* < 200$:

$$G_{HFCS^{\Phi_0}}(H^*, t^*) = \frac{G_{max}}{2} \left(1 + \tanh \left(\sum_{n=1}^3 A_n(H^*) (\log_{10}(t^*) - X(H^*))^n \right) \right) \quad (6)$$

181 The five coefficients of the $HFCS^{\Phi_0}$ model G_{max} , X , A_1 , A_2 , A_3 are expressed as simple functions of the
 182 aspect ratio H^* :

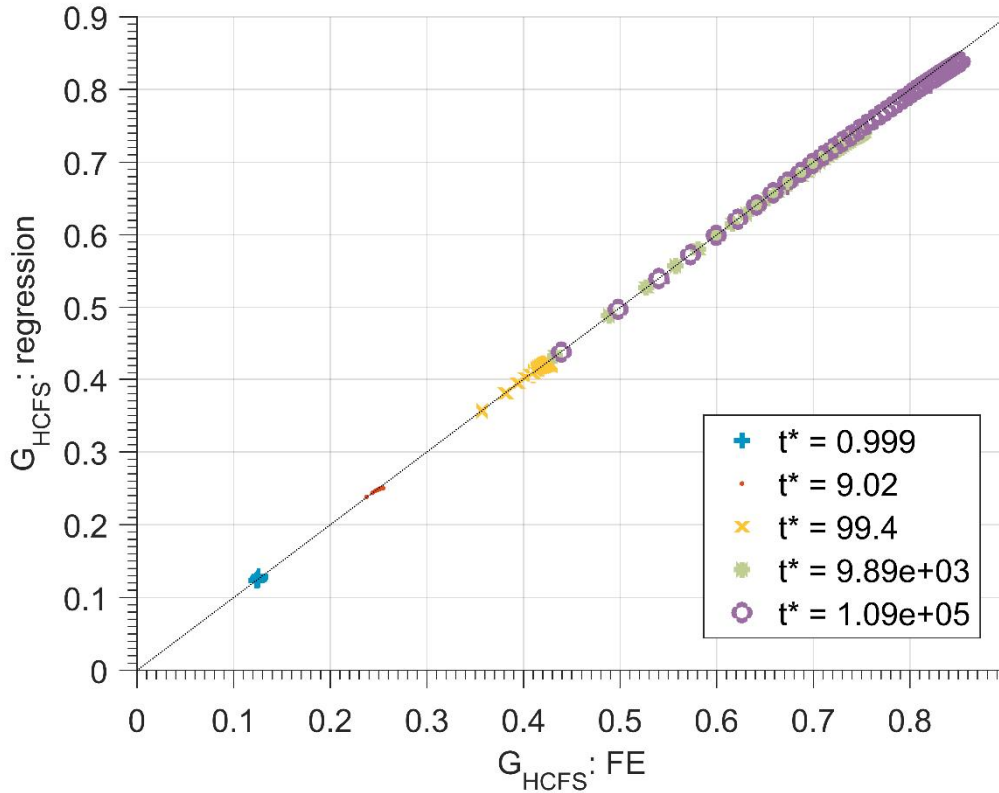
$$coefficient(H^*) = \sum_{k=1}^4 a_k (\log_{10}(H^*))^{k-1} \quad (7)$$

183 The 20 coefficients of eq. (7) (4 coefficients a_i for every 5 parameter G_{max} , X , A_1 , A_2 , A_3) were
 184 determined by minimizing the misfit (root mean square error) between the 4251 evaluations of
 185 $G_{HFCS}(t^*)$ and eq. (6) with the *fmincon* function of Matlab®. For the five $HFCS^{\Phi_0}$ parameters, the
 186 coefficients a_1 , a_2 , a_3 and a_4 are given in Table 1.

| | G_{max} | X | A_1 | A_2 | A_3 |
|-------|------------|-----------|-----------|-------------|-------------|
| a_1 | 0.14902 | 0.23592 | 0.068755 | -0.013467 | -0.33526 |
| a_2 | 1.2658 | -0.14631 | 0.027615 | 0.69358 | -0.048028 |
| a_3 | -0.070655 | 0.015128 | 0.0055503 | 0.016762 | -0.009149 |
| a_4 | 0.00082108 | -0.010045 | 0.013457 | -0.00057945 | -0.00082768 |

187 *Table 1: Regressions over the five coefficients of the HCFS model G_{max} , X , A_1 , A_2 , A_3 , established for $10^{-4} < t^* < 1.2 \times 10^6$*
188 *and $10 < H^* < 200$.*

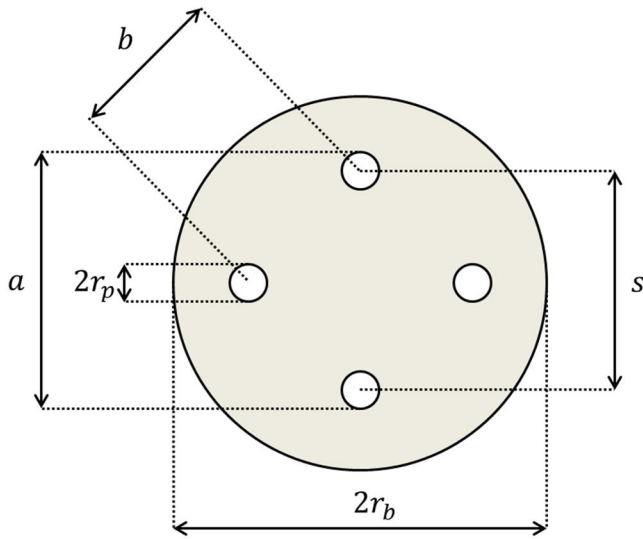
189 The regression is able to reproduce the original dataset in an excellent way (Figure 4).



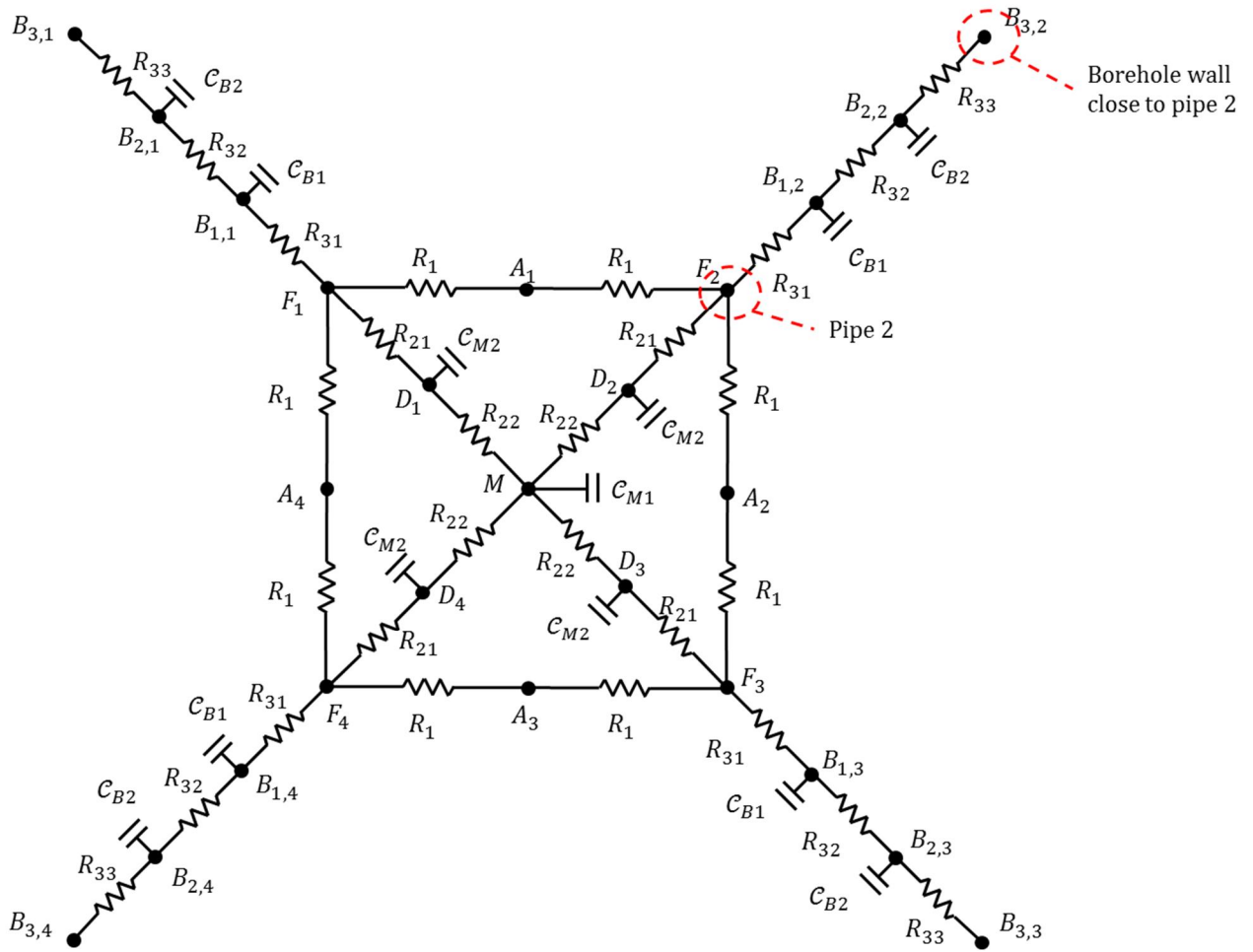
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191 *Figure 4: HCFS⁰-model: Comparison between the FE derived G-functions and the regression function (Equation 6)*
192 *with fitted parameters (Table 1): G-function at different normalized times t^* for a range of aspect ratios ($H^* = 10$ to*
193 *200)*

194
195
196

2.2. Resistive-capacitive circuit



198



199

200 Figure 5: Modelled PHE cross-section (top) and developed RC circuit (bottom).

201 The RC model was developed for configurations with 4 pipes (Figure 5). Configurations with 2
 202 pipes were not investigated as they seem to be unused in French PHE projects. It comprises six
 203 resistances ($R_1, R_{2,1}, R_{2,2}, R_{3,1}, R_{3,2}, R_{3,3}$) and four capacities ($C_{B1}, C_{B2}, C_{M1}, C_{M2}$). The number of
 204 resistances and capacities is a trade-off between accuracy and model complexity. The outer face

205 of each pipe, denoted F_i ($i=1, \dots, 4$) is connected to the adjacent portion of the heat exchanger wall
 206 $B_{3,i}$ by a serial connection of three resistances $R_{3,1}$, $R_{3,2}$, $R_{3,3}$. Two capacities C_{B1} and C_{B2} are inserted
 207 at the corresponding nodes. This outer portion of the circuit describes the temperature evolution
 208 in the outer part of the pile. Every pipe is connected to the central part of the pile by a two serial
 209 resistances $R_{2,1}$ and $R_{2,2}$ and a corresponding capacity C_{M2} . Finally, the central part of the pile is
 210 represented by its own capacity C_{M1} . Interactions between adjacent pipes (such as F_1 and F_2) are
 211 represented by two resistances R_i .

212 Normalized linear resistances R_i^* and linear capacities C_i^* are introduced:

$$\begin{aligned} R_i^* &= \lambda_c R_i \\ C_i^* &= \frac{1}{(\rho C_p)_c r_b^2} C_i \end{aligned} \quad (8)$$

213 In eq. (8) λ and (ρC_p) respectively account for the ground thermal conductivity ($\text{W.K}^{-1}.\text{m}^{-1}$) and
 214 volumetric heat capacity ($\text{J.K}^{-1}.\text{m}^{-3}$). The resistances $R_1^* = R_{2,1}^* + R_{2,2}^*$ and $R_3^* = R_{3,1}^* + R_{3,2}^* + R_{3,3}$
 215 are introduced along with three parameters x_2, y_2, y_3 to describe the location of $C_{B,1}^*, C_{B,2}^*$ and $C_{M,2}^*$:

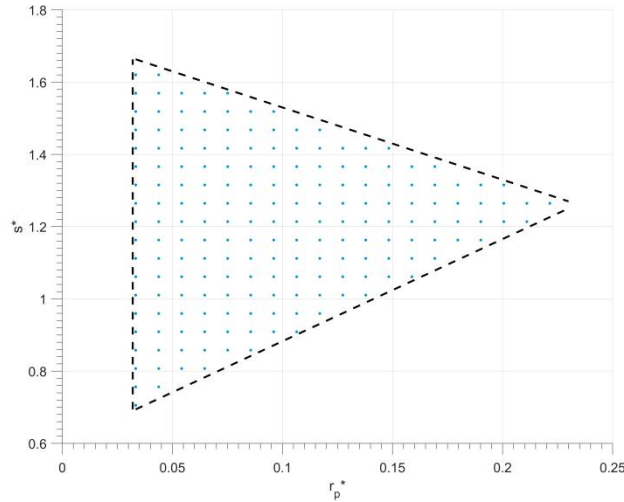
$$\begin{aligned} x_2 &= \frac{R_{21}^*}{R_2^*} \\ x_3 &= \frac{R_{31}^*}{R_3^*} \\ y_3 &= \frac{R_{32}^*}{R_3^*} \end{aligned} \quad (9)$$

216 Coefficients R_i^* and C_i^* are functions of $r_p^* = r_p / r_b$, the normalized outer pipe radius, and $s^* = s / r_b$,
 217 the normalized centre-to-centre shank spacing respectively (Figure 5). They were identified by a
 218 numerical procedure described in Annex A for 181 configurations (Figure 6). To ensure
 219 geometrical constraints, such as no overlapping between pipes, the RC parameters were
 220 estimated for (r_p^*, s^*) fulfilling the three following constraints:

$$\begin{aligned} (c_1): r_p^* &> 0.033 \\ (c_2): s^* - 2\sqrt{2}r_p^* &> 0.60 \\ (c_3): s^* + 2r_p^* &< 1.73 \end{aligned} \quad (10)$$

221

222 The first constraint (c_1) ensures a minimum normalized pipe radius r_p^* while the second and third
 223 constraints (c_2 and c_3) respectively impose minimum and maximum normalized shank spacing s^* .
 224 Let us assume a typical pipe outer radius $r_p = 1.6$ cm. For $r_b = 30.0$ cm, $r_p^* = 1.6 / 30.0 = 0.0533$ and
 225 (c_1) is checked. Since s^* must be between 0.7508 (c_2) and 1.6233 (c_3), the RC parameters can be
 226 estimated from $s = 0.7508 \times 30.0 = 22.5$ cm to $1.6233 \times 30.0 = 48.7$ cm. If $r_b = 10$ cm, then $r_p^* =$
 227 $1.6 / 10.0 = 0.160$ and s must be between 10.5 cm and 14.1 cm.



228

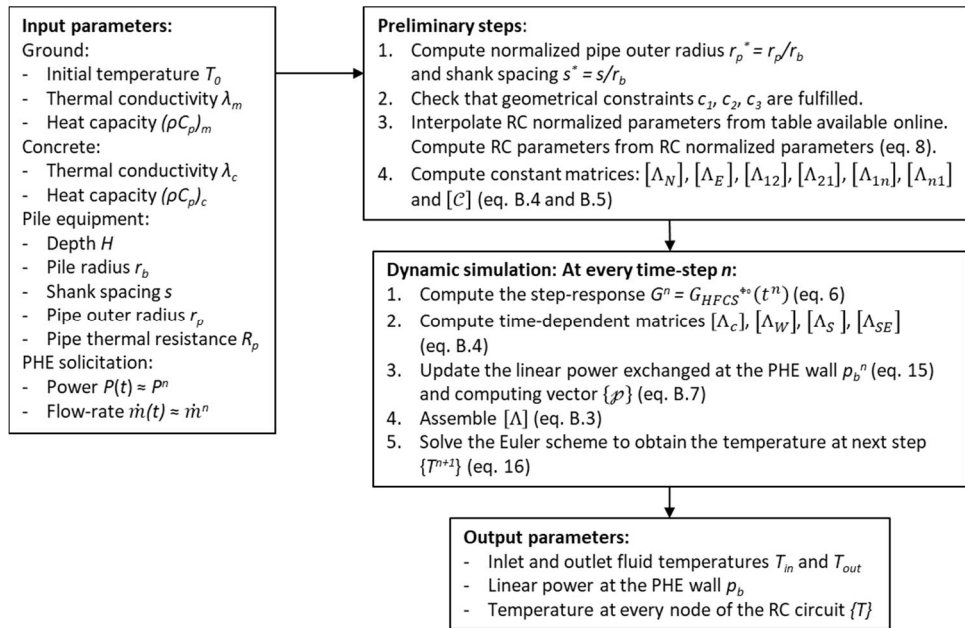
229 *Figure 6: Domain of validity for RC parameters: r_p^* and s^* shall be within the triangle. Dots indicate the 181*
 230 *configurations where the RC parameters were computed.*

231 The supporting information of the article contains a table with the resistances R_1^* , R_2^* , R_3^*
 232 capacities C_{B1}^* , C_{B2}^* , C_{M1}^* , C_{M2}^* and locations x_2 , x_3 , y_3 at the 181 configurations. Graphical
 233 representation of every parameter is also included.

234

235 2.3. Implementation of the semi-analytical PHE model

236 A model coupling the HFCS Φ^0 G-function with the RC circuit was developed to compute the
 237 temperature evolution of the PHE heat-carrier fluid (cf. Figure 7). The model is qualified as “semi-
 238 analytical” (SA) since it couples the G-functions (the analytical part) with a numerical scheme to
 239 compute the temperature in the RC circuit (the numerical part).



240

241 *Figure 7: Flowchart with input/output parameters of the developed semi-analytical model. Equations (B.i) refers to*
 242 *annex B.*

243 Time is discretized in n steps $t_n = n \Delta t$. The heat-carrier fluid feeds pipes 1, 2, 3 and 4. The 4 pipes
 244 are connected in series, since this matches observed engineering practices. Note that if other
 245 arrangements, e.g. parallel, are required, the RC parameters computed in the previous section
 246 remain appropriate. However, the matrices assembling procedures described in Annex B is no
 247 longer valid and should be modified.

248 A time-varying power P^n is applied to the heat-carrier fluid:

$$P^n = \dot{m}^n C_{p,fl} (T_{in}^n - T_{out}^n) \quad (11)$$

249 With \dot{m}^n the mass flow rate in the PHE ($\text{kg}\cdot\text{s}^{-1}$), $C_{p,fl}$ the mass-specific heat capacity of the heat
 250 carrier fluid ($\text{JK}^{-1}\cdot\text{kg}^{-1}$), T_{in}^n and T_{out}^n respectively PHE inlet and outlet temperatures ($^{\circ}\text{C}$).

251 The temperature at the PHE nodes is described by a vector $\{T^n\}$. This vector encompasses all the
 252 temperature in the RC circuit along with PHE inlet outlet temperatures. The temperatures are
 253 assumed to be constant upon the whole PHE depth (one temperature per pipe), though this
 254 assumption may be inaccurate for borehole heat exchangers [24,41,44]. For instance Zeng et al.
 255 models can predict fluid temperature along the depth [24]. For a short pile ($H = 10$ m) this
 256 assumption is validated by comparison with a 3D FE model which solves the fluid temperatures
 257 along the depth in every pipe (see 2.4). This assumption should be discussed for deeper PHE.

258 Heat exchange between the fluid in pipe i and the outer face of a pipe i is accounted for by an
 259 effective thermal resistance R_p ($\text{K}\cdot\text{m}\cdot\text{W}^{-1}$) accounting for the convection within the fluid R_{conv} and
 260 the pipe thermal resistance R_{p0} :

$$R_p = R_{conv} + \underbrace{\frac{1}{2\pi\lambda_p} \ln\left(\frac{r_p}{r_{p,i}}\right)}_{=R_{p0}} \quad (12)$$

261 λ_p and $r_{p,i}$ are the pipe thermal conductivity and inner radius respectively. R_{conv} has been computed
 262 by correlations reported in [45]. Establishing a heat balance at the nodes of the RC circuit leads to
 263 a differential equation on $\{T^n\}$:

$$[C] \frac{d}{dt} \{T\} + [\Lambda] \{T\} = \{\varphi\} \quad (13)$$

264

265 $[C]$, $[\Lambda]$ and $\{\varphi\}$ are respectively a capacitance matrix ($\text{JK}^{-1}\cdot\text{m}^{-1}$), a conductance matrix
 266 ($\text{W}\cdot\text{K}^{-1}\cdot\text{m}^{-1}$) and a linear power vector ($\text{W}\cdot\text{m}^{-1}$). Their expressions are given in Annex B.

267 Here is the key point to couple the analytical model with the numerical model. The pile wall
 268 temperature T_p is computed through the superposition principle [18]:

$$T_p^n - T_0 = \frac{1}{\lambda_m} \left(p_b^1 G^n + \sum_{l=1}^{n-1} (p_b^{l+1} - p_b^l) G^{n-l} \right) \quad (14)$$

269

270 Where p_b is the linear power (or power by unit length of pile) exchanged at the PHE wall ($\text{W}\cdot\text{m}^{-1}$),
 271 which is supposed to be constant along the depth, and T_p the temperature at this node. All the
 272 nodes B_{3j} are connected to the PHE wall, leading to:

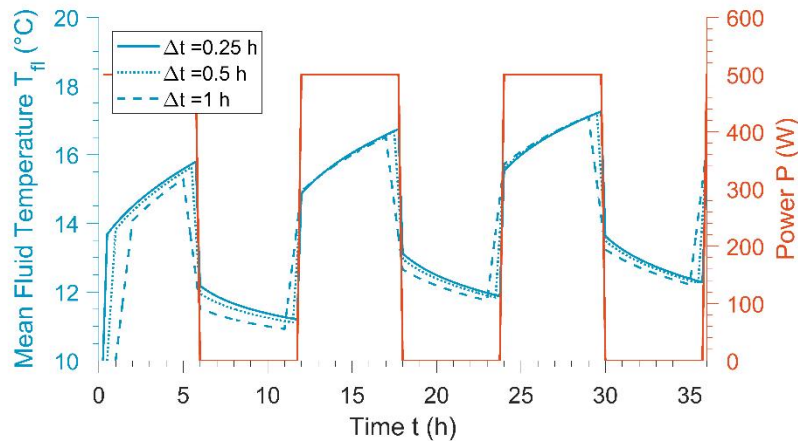
$$p_b^n = \sum_{i=1}^4 \frac{T_{B2,i}^n - T_p^n}{R_{33}} \quad (15)$$

273 $\{T^{n+1}\}$ is determined by solving equation (13) with an implicit Euler scheme implemented in
 274 Matlab®, which results in the following linear system:

$$\begin{aligned} [C] \frac{\{T^{n+1}\} - \{T^n\}}{\Delta t} + [\Lambda] \{T^n\} &= \{p\} \\ \Rightarrow \left(\frac{1}{\Delta t} [C] + [\Lambda] \right) \{T^{n+1}\} &= \frac{1}{\Delta t} [C] \{T^n\} + \{p\} \end{aligned} \quad (16)$$

275

276 The influence of time step on the temperature precision was investigated for Δt ranging from 0.25
 277 h to 1 h. The tested configuration is similar to case *a* presented in section 2.4, except that the 500
 278 W power is applied for 6 h followed by 6 h of relaxation (cf. Figure 8). The implicit Euler scheme
 279 has proven to be stable. Temperatures computed for $\Delta t = 0.25$ h and 0.5 h are almost superposed,
 280 suggesting that a 0.25 h time step is unnecessary small. $\Delta t = 1$ h can lead to an underestimation of
 281 the temperature change when the power changes sharply by c.a. 0.7 °C. Given this, $\Delta t = 0.5$ h is a
 282 good starting point to run the SA model, and the authors recommend to test the influence of Δt on
 283 the result precision.



284

285 *Figure 8: Influence of time step for a periodic solicitation.*

286 The main advantage of the semi-analytical model is that its execution is fast. For instance,
 287 evaluation of 1 000 time-step requires about 10 s on a desktop PC. Its implementation requires
 288 no advanced skills in programming and could be implemented in open software like Python, given
 289 the detailed flowchart (Figure 7). In the author's view, the most significant shortcoming of the SA
 290 model is that it considers a single PHE with an adiabatic condition at the surface, corresponding
 291 to a building of infinite extension whose floor is perfectly insulated. Recent research shows how
 292 the ground thermal regime is affected by the boundary condition at the surface [46–48]. Further
 293 development of the SA models will cope with multiple PHE with more realistic boundary
 294 conditions. Another shortcoming is that the model holds only for 4 pipes.

295 **2.4. Model validation against FE code**

296 The SA model was validated against a finite element (FE) model developed in COMSOL-
 297 Multiphysics® software for 5 cases (Table 2). For all simulations the PHE was 10 m deep and a

298 linear constant power 50 W.m^{-1} was applied. The parameters used in the simulation are
299 summarized in Table 3. The initial temperature was set to $T_0 = 0^\circ\text{C}$. The flow is turbulent; as a
300 result, the convective thermal resistance is an order of magnitude lower than the conductive one.

301

| Validation case | Pile radius r_b (cm) | Distance between two opposite pipes s (cm) | Concrete thermal conductivity ($\text{W.K}^{-1}.\text{m}^{-1}$) | Dimension of the FE model | Simulation time |
|-----------------|------------------------|--|---|---------------------------|-----------------|
| <i>a</i> | 30 | 30 | 1.2 | 2D | 200 h |
| <i>b</i> | 30 | 40 | 1.2 | 2D | 200 h |
| <i>c</i> | 30 | 30 | 1.8 | 2D | 200 h |
| <i>d</i> | 30 | 40 | 1.8 | 2D | 200 h |
| <i>e</i> | 30 | 30 | 1.8 | 3D | 5.0 y |

302 *Table 2 : Cases used for the validation of the semi-analytical model*

303

| Pipe characteristics | | | | | Solicitation | | |
|---|---------------------------------|---|--|--|---|---------------------|--|
| External pipe radius | Internal pipe radius | Pipe thermal conductivity | Pipe resistance | Effective pipe resistance | Mass flow-rate | Power | |
| $r_p = 10 \text{ mm}$ | $r_{p,i} = 2.5 \text{ mm}$ | $\lambda_p = 0.40 \text{ W.K}^{-1}.\text{m}^{-1}$ | $R_{p0} = 0.0776 \text{ K.m.W}^{-1}$ | $R_p = R_{p0} + R_{conv} = 0.089 \text{ K.m.W}^{-1}$ | $\dot{m} = 0.1 \text{ kg.s}^{-1}$ | $P = 500 \text{ W}$ | |
| Fluid characteristics and flow conditions | | | | | | | |
| Heat capacity | Dynamic viscosity ^a | Thermal conductivity ^a | Prandtl number ^a | Nusselt number | Reynolds number | Flow regime | Convective resistance |
| $C_{p,fl} = 4180 \text{ kJ.kg}^{-1}.\text{m}^{-3}$ | $\nu_{fl} = 1.31 \text{ mPa.s}$ | $\lambda_{fl} = 0.578 \text{ W.K}^{-1}.\text{m}^{-1}$ | $Pr = 9.47$ | $Nu = 48.21$ | $Re = 5925$ | Turbulent | $R_{conv} = 0.0114 \text{ K.m.W}^{-1}$ |
| Ground characteristics | | | | | Concrete characteristics | | |
| Thermal conductivity | | | Volume-specific heat capacity | | Volume-specific heat capacity | | |
| $\lambda_m = 2.3 \text{ W.K}^{-1}.\text{m}^{-1}$ | | | $(\rho C_p)_m = 2.4 \text{ MJ.K}^{-1}.\text{m}^{-3}$ | | $(\rho C_p)_c = 2.16 \text{ MJ.K}^{-1}.\text{m}^{-3}$ | | |
| ^a : Computed with CoolProps tool at a reference temperature of 10 °C (http://www.coolprop.org/) | | | | | | | |

305 Table 3 : Parameters common to all validations of the semi-analytical model

306 The fluid in every pipe is modelled in one dimension along the pipe axis s . A power balance (W)
 307 on an elementary volume V of fluid contained between s and $s+ds$ reads:

$$\int \frac{De}{Dt} dV = \oint \underline{\phi} \cdot d\underline{S} \quad (17)$$

308 In eq. (17), D/Dt is the material (lagragian) derivative. The term in the left accounts for the
 309 variations of fluid energy per volume e (Jm^{-3}):

$$\begin{aligned} \int \frac{De}{Dt} dV &= \int \frac{D(\rho_{fl} C_{p,fl} T_{fl})}{Dt} dV = \rho_{fl} C_{p,fl} \left(\frac{\partial T_{fl}}{\partial t} + v \frac{\partial T_{fl}}{\partial s} \right) \pi r_{p,i}^2 ds \\ &= \pi r_{p,i}^2 ds \rho_{fl} C_{p,fl} \frac{\partial T_{fl}}{\partial t} + \dot{m} C_{p,fl} \frac{\partial T_{fl}}{\partial s} ds \end{aligned} \quad (18)$$

310 In eq. (18), ρ_{fl} is the fluid density (kg.m^{-3}), and v the fluid velocity (m.s^{-1}), given that:

$$\dot{m} = \rho_{fl} v (\pi r_{p,i}^2) \quad (19)$$

311 The term in the right of eq. (17) accounts for the incoming flux from the outer side of the pipe at
 312 temperature T_p to the fluid through conduction in the pipe and advection:

$$\oint \underline{\phi} \cdot d\underline{S} = \frac{T_p - T_{fl}}{R_p} ds \quad (20)$$

313 Combining eq. (18) and (20), the heat balance (17) can be rewritten:

$$\pi r_{p,i}^2 \rho_{fl} C_{p,fl} \frac{\partial T_{fl}}{\partial t} + C_{p,fl} \dot{m} \frac{\partial T_{fl}}{\partial s} + \frac{T_{fl} - T_p}{R_p} = 0 \quad (21)$$

314 As the heat-carrier fluid volume is negligible compared to the concrete volume, the thermal inertia
 315 of the fluid is overlooked, which results in:

$$\dot{m}C_{p,fl} \frac{\partial T_{fl}}{\partial s} + \frac{T_{fl} - T_p}{R_p} = 0 \quad (22)$$

316 Cases *a*, *b*, *c* and *d* focus on transient thermal effects within the pile over $t = 200$ h. Since the
 317 deviation between HICS and HFCS model is low, vertical heat transfers are expected to play only
 318 a small role on the fluid temperature evolution. Therefore, the benchmark was run with a 2D
 319 horizontal FE model against the SA model with the HICS G-function. The 2D FE model considers
 320 independent fluid and pipe temperature $T_{fl,i}$ and $T_{p,i}$, as does the SA model. In the 2D FE model, eq.
 321 (22) is integrated over every pipe i from the pipe inlet to the pipe outlet:

$$\dot{m}C_{p,fl} \frac{(T_{fl,i} - T_{fl,j})}{H} + \frac{1}{R_p} \left(\frac{T_{fl,i} + T_{fl,j}}{2} - T_{p,i} \right) = 0 \quad (23)$$

$$T_{fl,j} = \begin{cases} T_{fl,4} + \frac{P}{\dot{m}C_{p,fl}} & \text{if } i = 1 \\ T_{fl,i-1} & \text{if } i = 2,3,4 \end{cases}$$

322

323 In eq. (23), R_p is the effective thermal resistance defined by eq. (12). In the given pipe i , the
 324 coupling term (second term) is evaluated at the mean fluid temperature.

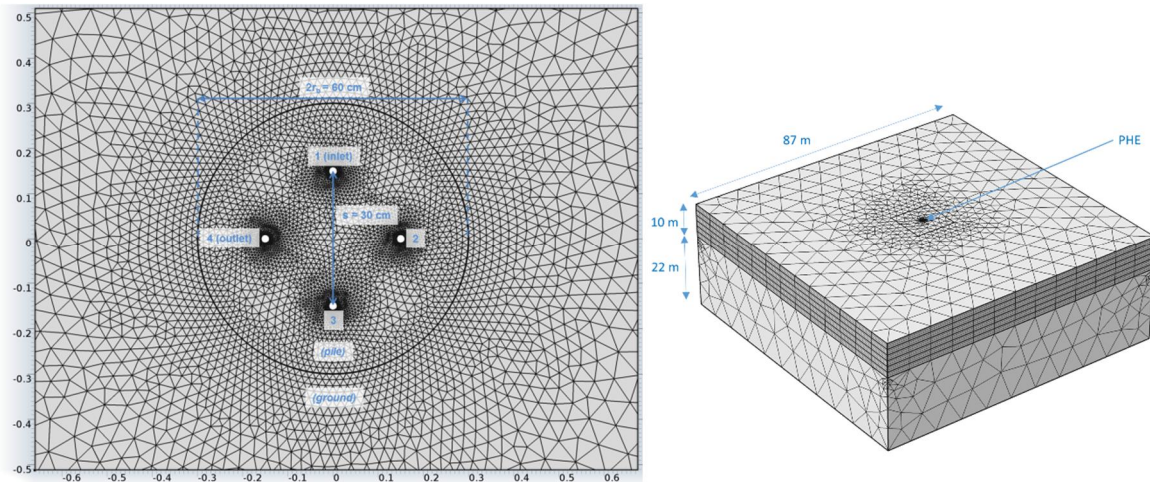
325 Case *e* tests the ability of the SA model to account both for transient thermal heat transfer within
 326 the pile and vertical heat transfer in the ground, over a long duration $t = 5$ years ($t^* = 1679$). Since
 327 the pile aspect ratio is $H^* = 10/0.3 = 33.3$, the HICS model overestimates the step response by 20%
 328 compared to HFCS model. Therefore, vertical heat transfers are expected to play a role on the fluid
 329 temperature evolution. Consequently a 3D FE model was used. The fluid temperature $T_{fl,i}(s,t)$ in
 330 pipe i is along the the pipe abscise s . The resolution of eq. (22) is implemented with linear
 331 extrusion operators in COMSOL-Multiphysics 4.2a, which allow the averaging of pipe temperature
 332 around polar coordinate θ at a given location s :

$$\dot{m}C_{p,fl} \frac{\partial T_{fl,i}}{\partial s} + \frac{1}{2\pi R_p} \int_{\theta=0}^{2\pi} (T_{fl,i} - T_{p,i}) d\theta = 0 \quad (24)$$

333

334 For both 2D and 3D models, the mesh is refined in the vicinity of the pipes (see Figure 9), and
 335 COMSOL solves the heat equation (2) in the solid parts, i.e. concrete and ground.

336



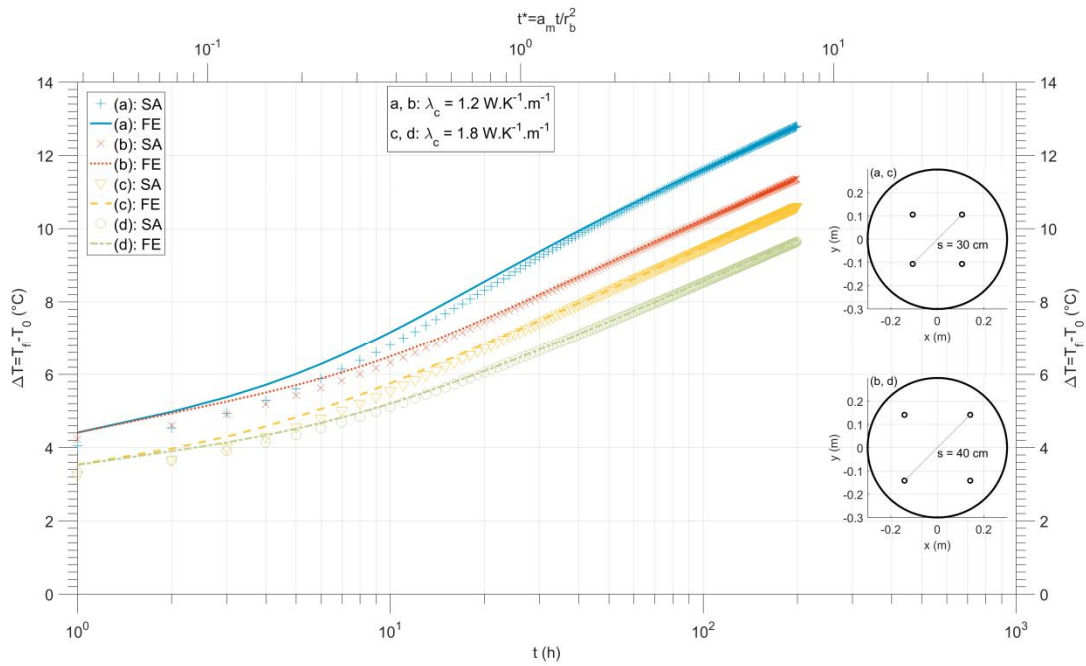
337

338 *Figure 9: Mesh of FE model used for SA model validation: 2D model (case a to d, left) and 3D model (case e, right)*

339 The SA model was run with a time step $\Delta t = 1\text{h}$ and its heat balance was checked. The sum of the
 340 internal power of the concrete and the power transferred at the borehole wall equals the linear
 341 power given by the fluid ($50\text{ W}\cdot\text{m}^{-1}$).

342 The SA and FE models are in good to excellent agreement for medium-term simulations (cases a
 343 to d) (see Figure 10). The SA model slightly underestimates the change in mean fluid temperature
 344 between approximately one hour and 25 hours. In the worst case (a), the SA model
 345 underestimates the temperature by $0.4\text{ }^{\circ}\text{C}$. The discrepancy is largest with lower concrete thermal
 346 conductivity, possibly suggesting that the R-C circuit is not 100% capturing the concrete capacity
 347 in the very short term. There may also be a small difference related to the use of the cylindrical
 348 source in the SA model which assumes application of the heat at the edge of the pile rather than
 349 within it. At longer times heat transfer to the ground becomes predominant and the SA and FE
 350 elements models give the same temperature evolution. Case c has the lowest long-term
 351 temperature evolution since the pipes are remote ($s = 40\text{ cm}$). This reduces the thermal short-
 352 circuit between the pipes. Overall, given the small discrepancies between the SA and FE solutions,
 353 this comparison validates the RC circuit along with the implementation of the semi-analytical
 354 model.

355

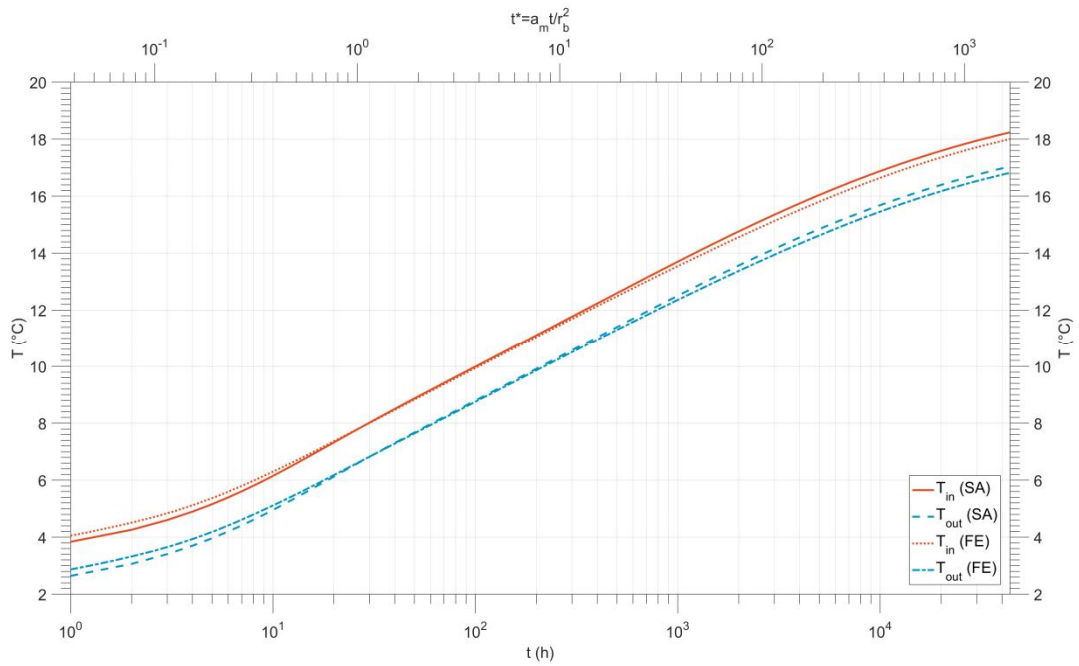


356

357 *Figure 10: Benchmark of SA model against FE model: Change in the mean fluid temperatures. Cases a, b c and d*

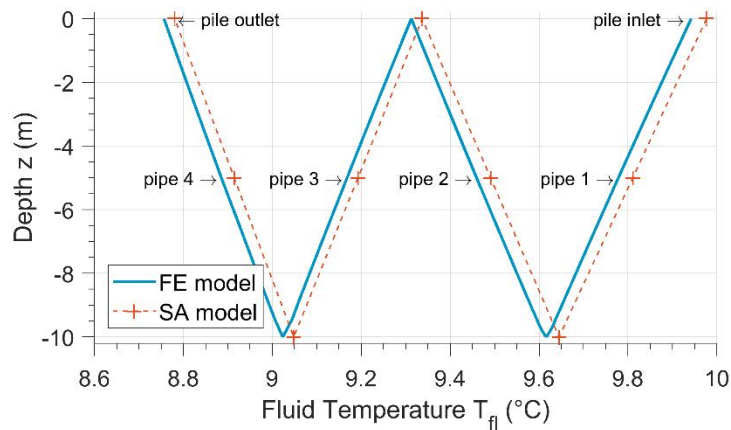
358 For the long-term simulation (case e), the agreement between the SA and the FE model is also
 359 excellent, both for inlet/ outlet temperatures and temperature distribution in the pipes (see Figure
 360 11 and Figure 12). From a numerical point of view, this validates the SA model for the short to
 361 long-term computation of the fluid temperature, from one hour to several years.

362 However the adiabatic condition imposed on the surface means that the ground and the building
 363 above the pile are assumed to exchange no heat. Further research work is needed to confirm the
 364 appropriateness of this assumption. Note that as the power exchanged was kept constant (500
 365 kW), the time step chosen by the FE model solver grown exponentially, resulting in a reasonable
 366 execution time (about 3 h for 5 years). In case of more realistic time-varying solicitation, the time
 367 step would collapse, resulting in an execution time not compatible with engineering practices.



368
369

Figure 11: Benchmark of SA model against FE model: Change in PHE inlet and outlet temperatures. Case e



370

371 Figure 12: Comparison of vertical profiles of fluid temperature at $t = 100$ h. Crosses at mid-depth refer to the mean
372 fluid temperature variables $T_{fl,1}$ to $T_{fl,4}$ (see annex 2) in the SA model. Crosses at one pipe end refer to $T_{f,ds,i}$

373 3. Comparison of the SA model to models without concrete capacity

374 Models which neglect the thermal inertia of the pile concrete (purely resistive models, or 1R) are
375 often used to predict the evolution of fluid temperature. They account for the thermal transfer in
376 the ground heat exchanger by a single resistance R_b , while the heat transfer in the ground is
377 modelled by the FLS, ILS or HICS models.

378 The influence of this assumption was investigated by comparing the new SA model with R-C
379 circuit to a state of the art model used commonly for BHE, namely the FLS with steady state
380 resistance, R_b . For consistency both the FLS and the FHCS used in the SA model were applied using
381 an adiabatic condition at the ground surface.

382 The range of conditions investigated included pile radius r_p of 15 and 30 cm, shank spacing s
383 ranging from r_b to $1.33 r_b$, low and high values of ground and concrete thermal conductivities,
384 respectively $\lambda_m = 1.3 \text{ W.K}^{-1}.\text{m}^{-1}$ and $2.3 \text{ W.K}^{-1}.\text{m}^{-1}$, $\lambda_c = 1.2 \text{ W.K}^{-1}.\text{m}^{-1}$ and $1.8 \text{ W.K}^{-1}.\text{m}^{-1}$ (Table 4). The

385 initial temperature was set to $T_0 = 10$ °C. The other parameters remained the same as for the
386 validation case (Section 2.3). The temperature evolution was simulated for 200 h to focus on the
387 effect of transient heat transfer in the pile on the temperature evolution.

388

| Input data | | | | | | Output data |
|------------|---------------|----------|--|--|---------------------------------|---|
| Case | r_b (cm) | s (cm) | λ_m (W.K ⁻¹ .m ⁻¹) | λ_c (W.K ⁻¹ .m ⁻¹) | R_b (K.m.W ⁻¹) | $T_{fl,1R-FLS}$ - $T_{fl,SA}$ at $t = 1$ h (°C) |
| 1 | 15 | 15 | 2.3 | 1.8 | 0.096 | 1.385 |
| 2 | 15 | 17.5 | 2.3 | 1.8 | 0.086 | 0.983 |
| 3 | 15 | 20 | 2.3 | 1.8 | 0.077 | 0.593 |
| 4 | 30 | 30 | 2.3 | 1.8 | 0.112 | 2.378 |
| 5 | 30 | 35 | 2.3 | 1.8 | 0.102 | 1.801 |
| 6 | 30 | 40 | 2.3 | 1.8 | 0.093 | 1.305 |
| 7 | 15 | 15 | 1.3 | 1.8 | 0.096 | 1.261 |
| 8 | 15 | 17.5 | 1.3 | 1.8 | 0.086 | 0.853 |
| 9 | 15 | 20 | 1.3 | 1.8 | 0.077 | 0.454 |
| 10 | 30 | 30 | 1.3 | 1.8 | 0.112 | 2.376 |
| 11 | 30 | 35 | 1.3 | 1.8 | 0.102 | 1.799 |
| 12 | 30 | 40 | 1.3 | 1.8 | 0.093 | 1.302 |
| 13 | 15 | 15 | 2.3 | 1.2 | 0.134 | 2.524 |
| 14 | 15 | 17.5 | 2.3 | 1.2 | 0.118 | 1.856 |
| 15 | 15 | 20 | 2.3 | 1.2 | 0.104 | 1.237 |
| 16 | 30 | 30 | 2.3 | 1.2 | 0.158 | 3.820 |
| 17 | 30 | 35 | 2.3 | 1.2 | 0.142 | 2.916 |
| 18 | 30 | 40 | 2.3 | 1.2 | 0.128 | 2.160 |
| 19 | 15 | 15 | 1.3 | 1.2 | 0.134 | 2.409 |
| 20 | 15 | 17.5 | 1.3 | 1.2 | 0.118 | 1.736 |
| 21 | 15 | 20 | 1.3 | 1.2 | 0.104 | 1.110 |
| 22 | 30 | 30 | 1.3 | 1.2 | 0.158 | 3.819 |
| 23 | 30 | 35 | 1.3 | 1.2 | 0.142 | 2.915 |
| 24 | 30 | 40 | 1.3 | 1.2 | 0.128 | 2.158 |

389 Table 4 : Cases used for the comparison between the 1R and SA models

390 As indicated, the purely resistive (1R) model was built by connecting the mean fluid
391 temperature $T_{fl,1R}$ to the ground heat exchanger wall at temperature T_p via a steady-state
392 thermal resistance R_b . The linear power transferred to the ground p reads:

$$p = \frac{T_{fl,1R} - T_p}{R_b} \quad (25)$$

393 R_b was estimated assuming a homogenous fluid temperature in the PHE. Consequently no heat is
394 transferred between pipes and resistances R_1 , $R_{2,1}$ and $R_{2,2}$ don't play any role. 4 resistances
395 ($R_{3,1}+R_{3,2}+R_{3,3}+R_p$) connect the fluid ($T_{fl,1R}$) to the borehole wall (T_p). R_b reads:

$$R_b = \frac{R_{3,1} + R_{3,2} + R_{3,3} + R_p}{4} \quad (26)$$

396 Combining (25) with the FLS Gfunction leads to:

$$T_{fl,1R} = T_0 + pR_b + \frac{p}{\lambda_m} G(t) \quad (27)$$

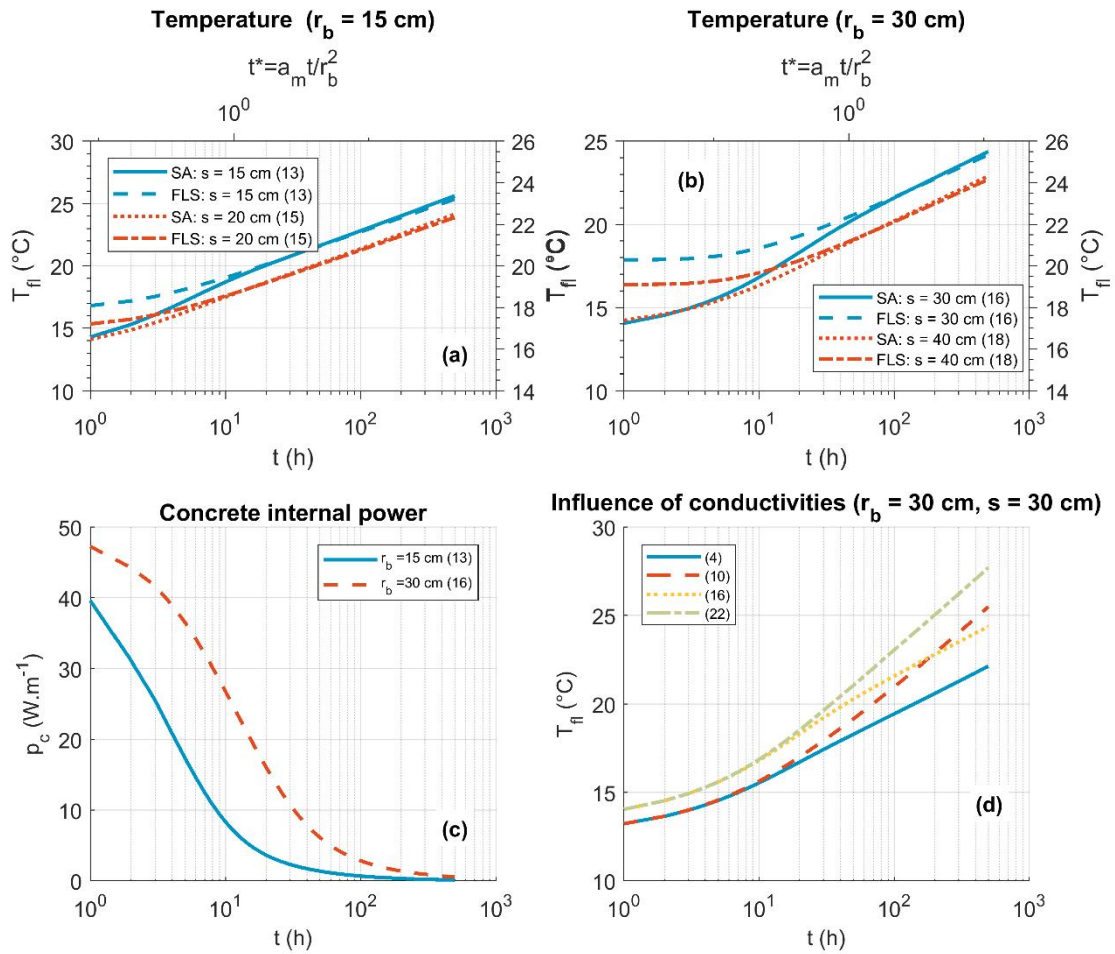
397

398 3.1. Comparison Results

399 After 1 hour of operation, the temperature discrepancy between 1R-FLS and SA models $\Delta T =$
400 $T_{fl,1R,FLS} - T_{fl,SA}$ ranges between 0.46 °C (case 9) and 3.82 °C (case 16) as shown in Table 4.
401 Approximating the PHE by a single resistance always leads to overestimations of the temperature
402 changes of the heat-carrier fluid. In other words the performances of the PHE are always
403 underestimated; no matter the pile radius, pipe spacing, concrete and ground thermal
404 conductivities

405 The main results of the comparison are summarised below:

- 406 - Both SA and 1R-FLS models converge to the same function. It is expected since both
407 models assume the same boundary condition, a perfect insulation, at the surface (see
408 Figure 13 *a* and *b*).
409
- 410 - The temperature evolution for configurations with remoter pipes ($s = 1.25 r_b$) is always
411 below the temperature for configurations with closer pipes ($s = 4/3 r_b$) (see Figure 13 *a*
412 and *b*). The discrepancy between the 1R-FLS and the SA models are larger when the pipes
413 are closer. The extreme case is for $r_b = 30$ cm and $s = 30$ cm: the 1R-FLS model then
414 overestimates the temperature change after 1 h per 3.82 °C (see Figure 13 *a* and *b*).
415
- 416 - The long-term trend is reached at shorter times for smaller PHE than for larger PHE (see
417 Figure 13 *a* and *b*). For the small radius (i.e. $r_b = 15$ cm), it takes c.a. 10 h ($s = 20$ cm) to 20
418 h ($s = 15$ cm) for both models to converge, while it requires approximately c.a. 50 h ($s =$
419 40 cm) to 100 h ($s = 30$ cm) for the large radius ($r_b = 30$ cm). The importance of the heat
420 transfer within the concrete at the beginning of the analysis illustrates this observation. 2
421 hours are necessary for the concrete of smaller PHE to be half-loaded, i.e. to reach $p_c = 25$
422 $W.m^{-1}$, while it takes 10 hours for larger PHE to do so (see Figure 13 *c*).
423
- 424 - For the large PHE, the ground thermal conductivity λ_m has no effect on the temperature
425 evolution up to 10 h (see Figure 13 *d*). Note that for narrower piles, the ground
426 conductivity will play a role much earlier. Once the pile concrete is loaded, the slope of the
427 curve is determined by λ_m , lower values of λ_m yielding to larger temperature changes.
428



430

431 *Figure 13: Main results of the comparison of the SA model to a model without concrete capacity. Temperature*
 432 *evolution refers to the evolution of the averaged inlet/ outlet temperature.*

433

434 These observations highlight that oversimplifying thermal transfers within the PHE always lead
 435 to overestimation of the temperature changes of the heat-carrier fluid and hence also
 436 underestimation of the capability of the pipes to transfer heat to the PHE and the ground. Models
 437 accounting for the pile thermal inertia offer the possibility to optimize the PHE performances and
 438 operations.

439

440 **4. Application to TRT data**

441 The SA model was applied to the analysis of a long thermal response test (TRT) carried out on a
 442 PHE with a radius of 30 cm and four heat exchange pipes installed, located in London clay, (cf.
 443 Table 5, Table 6 and Figure 14). The TRT lasted for 353 hours (14.7 days, see Table 6). Given the
 444 short duration of the test ($t^* < 10$), the boundary condition (adiabatic or isotherm) plays a
 445 negligible role over the TRT duration on the G-function. An adiabatic condition at the surface was
 446 assumed, and $G_{HFCs\phi_0}$ used (eq. (6)) in the SA model.

447 No laboratory measurement of ground and concrete thermal properties was performed, and the
 448 determination of distance between pipes s was based on standard construction details rather than
 449 as built records. The initial ground temperature T_0 was estimated by measuring the temperature
 450 of the heat-carrier fluid (water) circulating in the PHE before the TRT heater is switched on.

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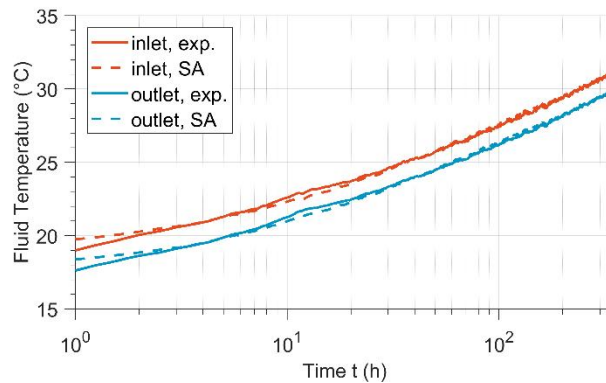
| Ground properties | | PHE properties | | | | | |
|-------------------|--------------------------------|-------------------|------------------------|---------------------------------|------------------------------|---------------------------|--|
| Lithology | Initial temperature T_0 (°C) | PHE depth H (m) | Pile radius r_b (cm) | Distance between pipes s (cm) | Pipe outer radius r_p (mm) | Pipe thickness e_p (mm) | Thermal conductivity of the pipe λ_p (W.K ⁻¹ .m ⁻¹) |
| London Clay | 14.2 | 31 | 30.0 | 42.5 | 12.5 | 2.2 | 0.4 |

452 *Table 5 : TRT: Characteristics of the ground and PHE*

453

| Fluid | Power P (kW) | Flow-rate \dot{m} (kg.s ⁻¹) | Temperature difference inlet/ outlet ΔT (°C) |
|-------|----------------|---|--|
| Water | 1.69 | 0.32 | 1.25 |

454 *Table 6 : TRT: Characteristics of the solicitation*



455

456 *Figure 14: Evolution of PHE inlet and outlet temperatures monitored during the TRT, and SA model results*
 457 *(parameters fitted with $t_{min} = 1$ h, $t_{max} = 350$ h).*

458 The ground volume-specific heat capacity was estimated based on SIA-384/ 6 guidelines [49]
 459 which indicates $(\rho C_p)_m$ in the range 2.0-2.8 MJ.K⁻¹.m⁻³ for wet clay. A value $(\rho C_p)_m = 2.4$ MJ.K⁻¹.m⁻³
 460 was used. For the concrete $(\rho C_p)_c = 2.2$ MJ.K⁻¹.m⁻³ was assumed.

461 The ability of the semi-analytical model to predict the fluid temperature evolution once the model
 462 parameters have been fitted was tested. Along with the ground thermal conductivity λ_m , the

463 concrete thermal conductivity λ_c was chosen as an effective parameter accounting for internal
 464 thermal transfers within the pile.

465 Therefore, the TRT was interpreted by minimizing the root mean square error (RMSE) ($^{\circ}\text{C}$)
 466 between measured and computed outlet temperatures was minimized [50] [51]:

$$RMSE(\lambda_m, \lambda_c) = \sqrt{\frac{\int_{t_{min}}^{t_{max}} (T_{out,exp}(t) - T_{out,SA}(\lambda_m, \lambda_c, t))^2 dt}{t_{max} - t_{min}}} \quad (28)$$

467

468 The RMSE was minimized with the local optimization algorithm *active-set* developed in
 469 MATLAB®. The SA model was run with a time step $\Delta t = 15$ min, which ensured its evaluation
 470 within a few seconds.

471 The choice of the TRT duration should result from a compromise between the limitations of costs,
 472 leading to shorter TRT, and the reliability of the results, leading to longer TRT. Therefore, the
 473 influence of the value of the lower and upper bounds, respectively t_{min} and t_{max} upon λ_m and λ_c was
 474 investigated. The following values were used: $t_{min} = 1, 5, 10, 20, 40$ h and $t_{max} = 100, 150, 200, 250,$
 475 300 and 350 , making over all 30 simulations.

476 Note that the intrinsic ground and concrete thermal conductivities are constant values
 477 independent upon the investigation duration. However, the value of effective thermal
 478 conductivities derived from the TRT analysis will depends on how much of the test data is
 479 included in the analysis. Therefore these effective thermal conductivities appears to vary with the
 480 amount of time elapsed in the test. When this variation stops and the value of thermal conductivity
 481 converges on an asymptote, then one can be confident you that appropriate values have been
 482 fitted.

483 The estimated ground and concrete conductivities λ_m and λ_c tend to converge to values of 1.48
 484 $\text{W.K}^{-1}.\text{m}^{-1}$ and $0.94 \text{ W.K}^{-1}.\text{m}^{-1}$ when t_{max} increases. Meanwhile, the dependence upon t_{min} tends to
 485 decrease (Figure 15). For instance, for $t_{max} = 100$ h, λ_m ranges between $1.63 \text{ W.K}^{-1}.\text{m}^{-1}$ ($t_{min} = 40$ h)
 486 and $1.86 \text{ W.K}^{-1}.\text{m}^{-1}$ ($t_{min} = 1$ h), leading to a difference of $0.23 \text{ W.K}^{-1}.\text{m}^{-1}$. However, when $t_{max} = 350$
 487 h, λ_m ranges between $1.44 \text{ W.K}^{-1}.\text{m}^{-1}$ ($t_{min} = 40$ h) and $1.51 \text{ W.K}^{-1}.\text{m}^{-1}$ ($t_{min} = 1$ h), with a difference
 488 of only $0.07 \text{ W.K}^{-1}.\text{m}^{-1}$. As time increases, the estimated thermal ground thermal conductivity tends
 489 to be independent upon the lower bound of integration t_{min} . Furthermore, negligible change in λ_m
 490 and λ_c ($\approx 1\%$) is observed between $t_{max} = 250$ h and $t_{max} = 350$ h.

491 The prediction of the SA model was compared to the prediction of a “classical” model for the
 492 interpretation of TRT on BHE (an approximation of the ILSG-function), which reads [52]:

$$T_{fl,cl} = T_0 + p \left[R_b + \frac{1}{4\pi\lambda_m} \left(\ln \left(\frac{4\lambda_m}{(\rho C_p)_m r_b^2} \right) - \gamma \right) \right] + \frac{p}{4\pi\lambda_m} \ln(t) \quad (29)$$

493

494 This model is valid for normalized time (Fourier number) $t^* > t_{min}^*$. The common criteria $t_{min}^* = 5$
 495 was used [52]. The ground thermal conductivity λ_m determined from the classical interpretation
 496 is in the range 1.35 to $1.45 \text{ W.K}^{-1}.\text{m}^{-1}$ while the thermal resistance R_b is between 0.128 and 0.134
 497 $\text{K.m}^{-1}.\text{W}^{-1}$ (see Table 7). Note that both λ_m and R_b increases when t_{max} increases from 250 to 350 h.

498 Reasons may be that the concrete is not fully loaded. The interpretation with the SA model yields
 499 $\lambda_m = 1.45$ to $1.50 \text{ W.K}^{-1}.\text{m}^{-1}$. For the longest integration time ($t_{\max} = 350 \text{ h}$), the SA gives a higher
 500 value of λ_m by approximately 3% to 6%. The larger range of values obtained for the classical ILS
 501 model suggest that a larger duration of data is required to use this approach. However, there is a
 502 trade off since use of the ILSbased interpretation over longer timescales will lead to errors due to
 503 neglecting the importance of axial effects with short aspect ratio piles.

504 The SA model fits the concrete thermal conductivity rather than the pile thermal resistance.
 505 However, the latter can be calculated from the former, e.g. by the method of shape factors [53].
 506 The values also rise with time as the thermal load on the concrete increases. As with λ_m , the values
 507 are slightly higher than that obtained from the classical ILS interpretation. The R_b values are
 508 slightly higher than might have been expected, but reflect the low thermal conductivity of the pile
 509 concrete and the relatively small number of heat exchange pipes installed.

| "Classical" interpretation (eq. (29)) | | | | | Interpretation with SA model | | | | |
|---------------------------------------|-------------------|-------------------|--|--|------------------------------|-------------------|--|--|--|
| t_{\min}^* | t_{\min} (h) | t_{\max} (h) | λ_m ($\text{W.K}^{-1}.\text{m}^{-1}$) | R_b ($\text{K.m}^{-1}.\text{W}^{-1}$) | t_{\min} (h) | t_{\max} (h) | λ_m ($\text{W.K}^{-1}.\text{m}^{-1}$) | λ_c ($\text{W.K}^{-1}.\text{m}^{-1}$) | R_b ($\text{K.m}^{-1}.\text{W}^{-1}$) |
| 5 | 200 | 250 | 1.35 | 0.128 | 40 | 250 | 1.45 | 0.96 | 0.137 |
| 5 | 200 | 300 | 1.38 | 0.129 | 40 | 300 | 1.43 | 0.97 | 0.138 |
| 5 | 200 | 350 | 1.42 | 0.132 | 40 | 350 | 1.43 | 0.97 | 0.138 |
| 7 | 260 | 350 | 1.45 | 0.134 | 1 | 350 | 1.50 | 0.93 | 0.141 |

510 *Table 7: Comparison of the methods of TRT interpretation. In the interpretation with SA model, R_b is calculated*
 511 *based on λ_c and method of shape factors described in [53].*

512 For PHE design it is also important to have accurate predictions of the outlet temperature since
 513 this effects the heat pump efficiency. To investigate this, the actual and simulated outlet
 514 temperatures are plotted in Figure 16 when $t_{\max}=250$ hours with the SA model and the 1R-FLS
 515 model. For the latter model the ground value $\lambda_m = 1.45 \text{ W.K}^{-1}.\text{m}^{-1}$ and resistance $R_b = 0.134$
 516 $\text{K.m}^{-1}.\text{W}^{-1}$ are used. After the parameters have been fitted to the earlier test data, the predictions
 517 over the later test data ($250 \text{ h} < t < 350 \text{ h}$) are shown. The temperatures computed by the classical
 518 model and the SA model with $t_{\min} = 40 \text{ h}$ are almost superposed. The SA model with parameters
 519 fitted on $t_{\min} = 1 \text{ h}$ slightly underestimate the temperature by $\approx 0.2 \text{ }^\circ\text{C}$. From a practical point of
 520 view, the classical model with parameters fitted $t_{\min}^* = 5$ predicts the overall temperature
 521 evolution well. However, the usefulness of the SA model lies in shorter times: while the 1R-FLS
 522 model overestimates the fluid temperature by $2.9 \text{ }^\circ\text{C}$ at $t = 1 \text{ h}$, the overestimation by the SA model
 523 is only $0.4 \text{ }^\circ\text{C}$, and rapidly reduces. The SA model also reproduces the fluctuations in temperature
 524 that occur due to power input variations in a way a constant resistance model never can. This
 525 means the SA model is more suitable for use in routine operation when the supplied power varies
 526 over short timescales.

527 These results suggest that for this 30 cm wide PHE the SA model can be inverted to obtain reliable
 528 values of λ_m and λ_c if the minimum TRT duration is 250 h ($t^* \sim 6.25$). The SA model is then capable
 529 of reproducing the whole sequence of temperature, from short times ($t \approx 1 \text{ h}$) to longer times (250
 530 $\text{ h} < t < 350 \text{ h}$), and could consequently be used for the dynamic simulations of PHE coupled to heat
 531 pumps. Larger errors would be expected at both short and long timescales based on the classical
 532 1R-ILS model.

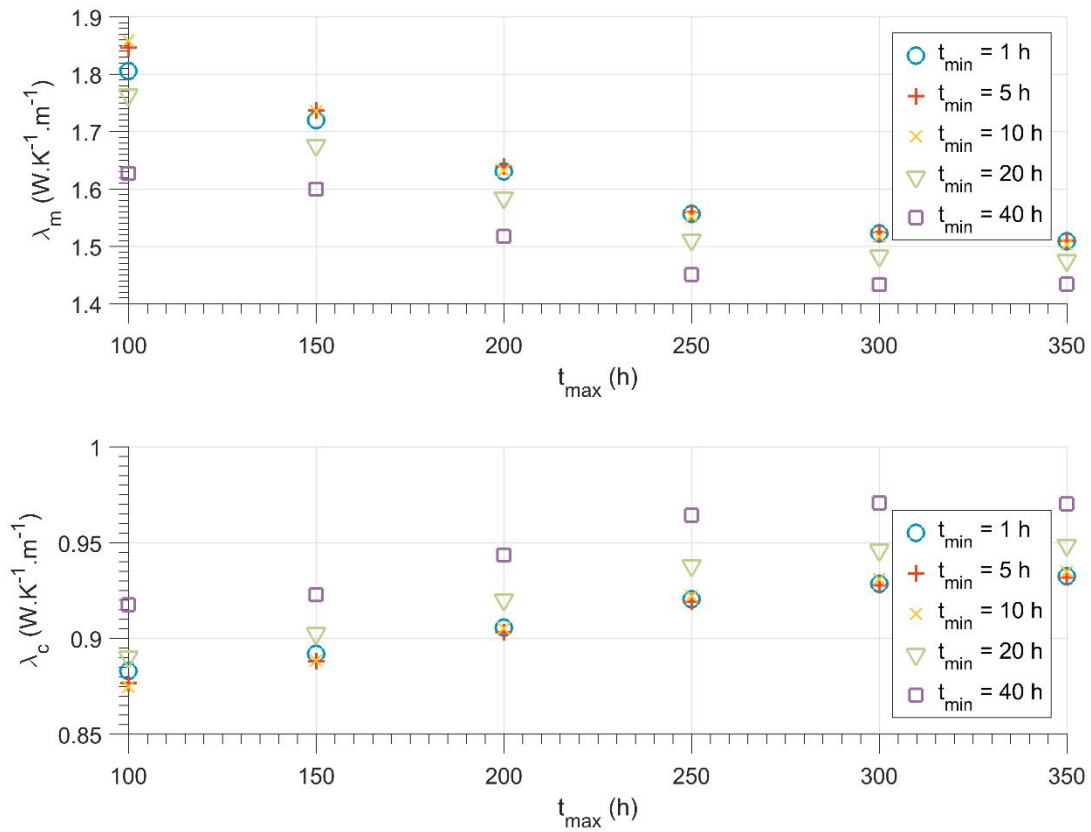


Figure 15: Interpretation with the SA model: λ_m and λ_c as a function of the integration times t_{min} and t_{max}

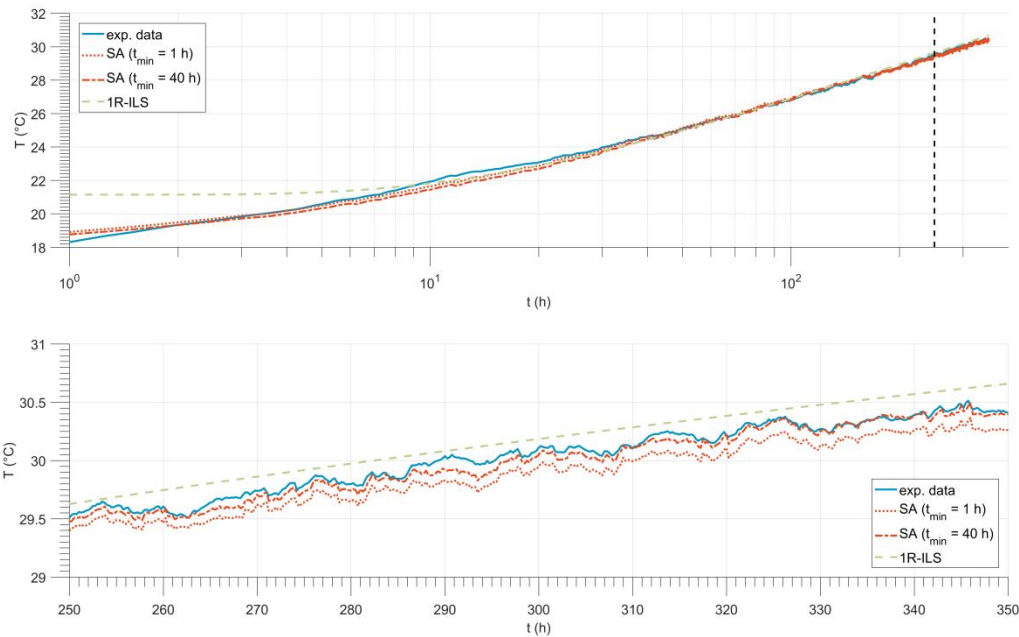


Figure 16: Evolution of the mean fluid temperature: Experimental data, SA model with $t_{min} = 1$ h, $t_{min} = 40$ h, classical model. The black-dotted vertical line accounts for $t_{max} = 250$ h, which has been used as the upper bound of integration for the three models.

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5. Conclusion

A semi-analytical (SA) model to compute the temperature evolution in PHE was developed. It relies on relevant resistive-capacitive circuits accounting for PHE internal thermal inertia and hollow semi-infinite cylindrical source step-response to account for long-term vertical heat transfer around the pile. The SA model has been checked against a finite element code. Both models are in excellent agreement at a range of timescales. However further research effort is needed to better understand the thermal interactions between the pile and the above building, and how this can be dealt with in analytical G-function.

The SA model was compared to a purely resistive (1R) model that neglects thermal inertia in the PHE. The results suggest that the 1R model always overestimate the PHE outlet temperature, no matter the pipe radius and positions, or the ground and concrete thermal conductivities. In other words, the 1R model always underestimates the PHE performances. Taking into consideration thermal transfers within the PHE in dynamic simulation tools would improve the assessment of PHE performances and their potential of development.

Purely resistive models developed for Borehole Heat Exchangers (BHE) are barely suitable for the interpretation of thermal response tests (TRT) performed on PHE. Therefore, the SA model was used to analyse a TRT performed on a PHE of radius 30 cm. The main result is that for this type of large PHE, the TRT duration should be of 250 h, so that reliable values of ground and concrete thermal conductivities are determined.

The SA model has been developed for a PHE equipped with 4 pipes and for impervious ground conditions. Further developments will focus on extending the SA model to configurations with a larger number of pipes and integrating step-responses accounting for underground water flow and a group of piles.

6. Acknowledgement

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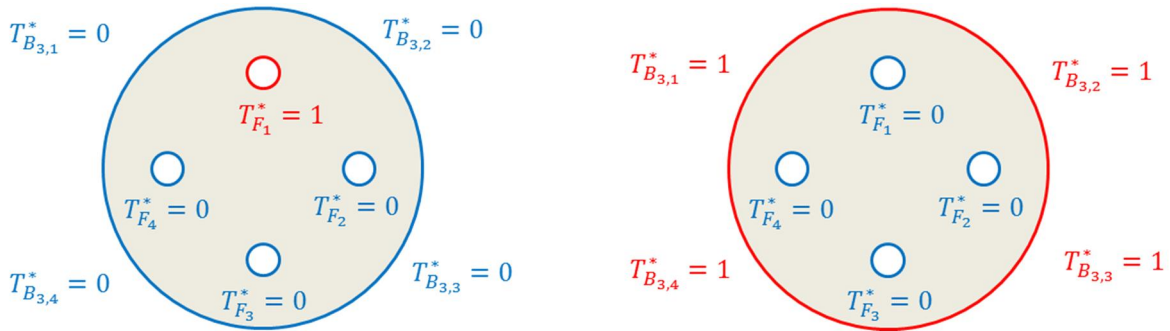
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1 Annex A: Determination of the RC parameters

2 The normalized RC parameters as defined in (8) were fitted to minimize an objective function ε
 3 (see Figure A-1). ε is defined as a combination of root mean square error between the normalized
 4 energies e^* computed by the RC model and e^* by a FE code. ε was weighted by the asymptotic
 5 values of energy in two simulations:

$$\varepsilon = \frac{e_{sim 1, \infty}^* \varepsilon_{sim 1} + e_{sim 2, \infty}^* \varepsilon_{sim 2}}{e_{sim 1, \infty}^* + e_{sim 2, \infty}^*} \quad (A.1)$$

6 For both simulations 1 and 2 the initial temperature is zero ($T^* = 0$). In simulation 1, $T^* = 1$ is set
 7 on one pipe while the borehole wall and all the other pipes are maintained to the initial
 8 temperature ($T^* = 0$). In simulation 2, $T^* = 1$ is set on all the pipes while the borehole wall is kept
 9 at the initial temperature ($T^* = 0$) (cf. Figure A.1).



10

11 *Figure A.1: Boundary condition for simulation #1 (left) and simulation #2 (right)*

12 Solving simulation 1 in steady state leads to:

$$\begin{cases} q_{F_1}^* = -\frac{1}{R_1^*} - \left(1 - \frac{1}{4}\right) \frac{1}{R_2^*} - \frac{1}{R_3^*} \\ q_{F_2}^* = \frac{1}{4 R_2^*} + \frac{1}{2 R_1^*} \\ q = \frac{1}{4 R_2^*} \end{cases} \Rightarrow \begin{cases} R_1^* = \frac{1}{2(q_{F_2}^* - q_{F_3}^*)} \\ R_2^* = \frac{1}{4 q_{F_3}^*} \\ R_3^* = -\frac{1}{q_{F_1}^* + 2q_{F_2}^* + q_{F_3}^*} \end{cases} \quad (A.2)$$

13 Where $p_{F_1}^*$, $p_{F_2}^*$ and $p_{F_3}^*$ refer to the power exchanged at pipes F_1 , F_2 and F_3 respectively; $R_1^* = R_{2,1}^*$
 14 $+ R_{2,2}^*$, $R_2^* = R_{3,1}^* + R_{3,2}^* + R_{3,3}^*$.

15 Three parameters x_2 , y_2 , y_3 are introduced to describe the location of $C_{B,1}$, $C_{B,2}$ and $C_{M,2}$:

$$\begin{aligned} x_2 &= \frac{R_{21}}{R_2} \\ x_3 &= \frac{R_{31}}{R_3} \\ y_3 &= \frac{R_{32}}{R_3} \end{aligned} \quad (A.3)$$

16 Simulation 2 focuses on testing the outer part of the RC circuit (i.e. from node F_1 to B_3) while
 17 simulation 1 tests both this outer part and the heart of the RC circuit. In simulation 2 only $C_{B,1}$, $C_{B,2}$
 18 and x_2 play a role. The heat balance on nodes B_1^* and B_2^* gives:

$$\begin{bmatrix} C_{B1}^* & 0 \\ 0 & C_{B2}^* \end{bmatrix} \frac{d}{dt^*} \begin{Bmatrix} T_{B1}^* \\ T_{B2}^* \end{Bmatrix} + \begin{bmatrix} \frac{1}{R_{32}^*} + \frac{1}{R_{31}^*} & -\frac{1}{R_{32}^*} \\ -\frac{1}{R_{32}^*} & \frac{1}{R_{32}^*} + \frac{1}{R_{33}^*} \end{bmatrix} \begin{Bmatrix} T_{B1}^* \\ T_{B2}^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ R_{33}^* \end{Bmatrix} \quad (A.4)$$

19 The energy in the pile section reads:

$$e_{sim 2}^*(t^*) = 4 (C_{B1}^* T_{B1}^*(t^*) + C_{B2}^* T_{B2}^*(t^*)) \quad (A.5)$$

20 Noticing that in steady state the temperature at the nodes B_1 and B_2 are respectively equal to x_3
21 and $x_3 + y_3$, the energy in steady state reads:

$$e_{sim 2, \infty}^* = 4 [C_{B1}^* x_3 + C_{B2}^* (x_3 + y_3)] \quad (A.6)$$

22 For simulation 1, a heat balance leads to:

$$\begin{bmatrix} C_{B1}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{B2}^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{M2}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{M2}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{M1}^* \end{bmatrix} \frac{d}{dt^*} \begin{Bmatrix} T_{B1}^* \\ T_{B2}^* \\ T_{A1}^* \\ T_{C1}^* \\ T_{C2}^* \\ T_M^* \end{Bmatrix} + \begin{bmatrix} \frac{1}{R_{31}^*} + \frac{1}{R_{32}^*} & -\frac{1}{R_{32}^*} & 0 & 0 & 0 & 0 \\ -\frac{1}{R_{32}^*} & \frac{1}{R_{32}^*} + \frac{1}{R_{33}^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{R_1^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_{21}^*} + \frac{1}{R_{22}^*} & 0 & -\frac{1}{R_{22}^*} \\ 0 & 0 & 0 & 0 & \frac{1}{R_{21}^*} + \frac{1}{R_{22}^*} & -\frac{1}{R_{22}^*} \\ 0 & 0 & 0 & -\frac{1}{R_{22}^*} & -\frac{1}{R_{22}^*} & \frac{1}{R_{22}^*} \end{bmatrix} \begin{Bmatrix} T_{B1}^* \\ T_{B2}^* \\ T_{A1}^* \\ T_{C1}^* \\ T_{C2}^* \\ T_M^* \end{Bmatrix} = \begin{Bmatrix} 1 \\ R_{31}^* \\ 0 \\ 1 \\ R_1^* \\ 1 \\ R_{21}^* \\ 0 \\ 0 \end{Bmatrix} \quad (A.7)$$

25

26 The energy in the pile section is:

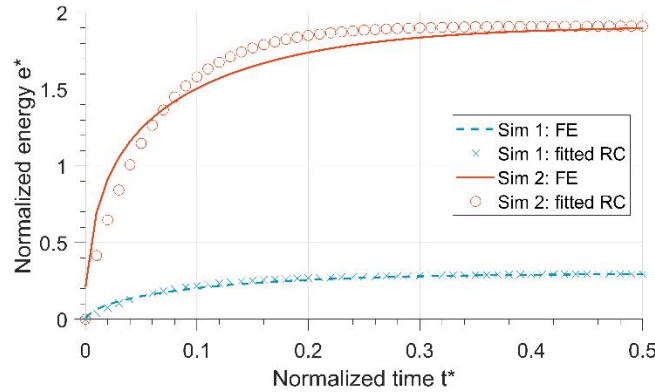
$$e_{sim 1}^*(t) = C_{B2}^* T_{B2}^*(t) + C_{B1}^* T_{B1}^*(t) + C_{M2}^* T_{C1}^*(t) + 3 C_{M2}^* T_{C2}^*(t) + C_{M1}^* T_M^*(t) \quad (A.8)$$

27 The additivity of thermal capacities leads to the following constraint:

$$4 (C_{B1}^* + C_{B2}^* + C_{M2}^*) + C_{M1}^* = \pi (1 - 4 r_p^2) \quad (A.9)$$

28 Equations A.4 and A.7 were solved with the *ode45* function for ordinary derivative equations in
29 MATLAB® Software. The internal time step used by the *ode45* was left up to *ode45*, with the
30 output with being exported at every normalized time step $\Delta t^* = 10^{-2}$ up to $t^* = 5$. $t^* = 5$ was used
31 as it ensured the steady-state to be reached. Similarly, the inner time step used by COMSOL was
32 left to the software, with output being exported on the same period. The reader is referred to the
33 supporting information for further details on this numerical procedure.

34 The objective function ε (eq. A.1) was minimized when fulfilling equality constraints and with a
 35 Particule Swarm Optimization (PSO) algorithm [54] (cf. Figure A-1 for one configuration).

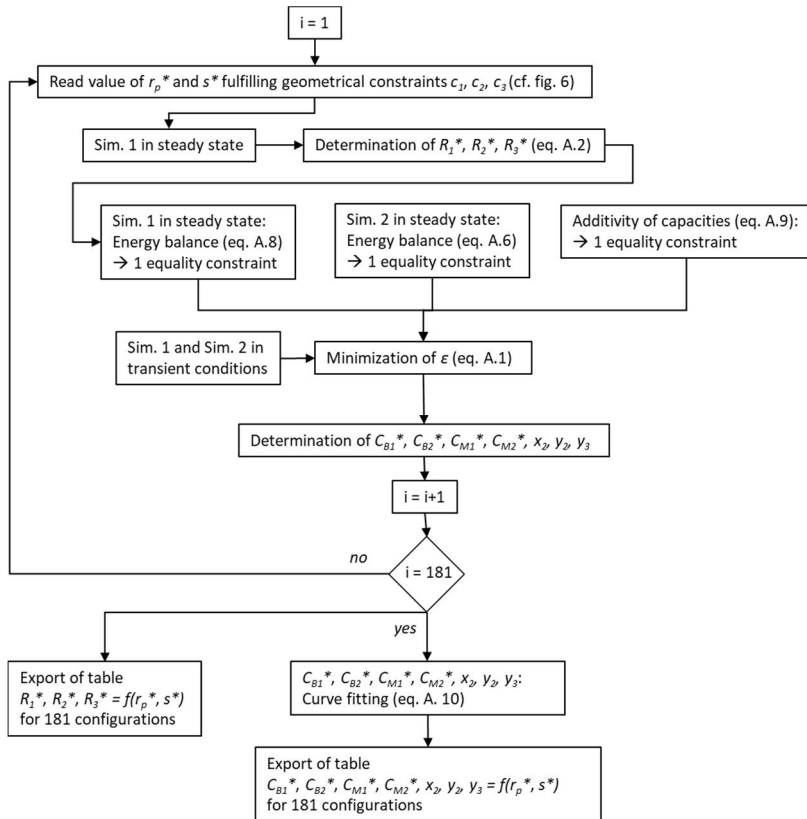


36
 37 *Figure A-1: Evolution of normalized energy for Sim 1 and Sim 2 from FE model and RC model with fitted parameters.*
 38 *Configuration defined by $r_p^* = 0.0533$ and $s^* = 1.417$.*

39 The procedure is iterated over 181 configurations, each configuration being characterized by a
 40 value of r_p^* and s^* satisfying geometric constrains c_1, c_2, c_3 as represented in Figure 6. Capacities
 41 and capacity locations exhibit some rough behaviour, and are smoothed through a quadratic form
 42 was determined for every parameter p (capacity or location):

$$p(r_p^*, s^*) = a_1 + a_2 r_p^* + a_3 s^* + a_4 r_p^* s^* + a_5 r_p^{*2} + a_6 s^{*2} \quad (A.10)$$

43
 44 Finally, values for the 10 RC model parameters for the 181 configurations are exported in a table,
 45 available in the supporting information of the paper.



47 *Figure A-2: Overall process for determination of the RC parameters*

48

49 **Annex B: Assembling matrices**

50 Let us consider a pile with 4 pipes i ($i=1,..,4$) connected in serial. $\{T\}$ contains the inlet
51 temperature T_{in} , the PHE temperature and the borehole wall temperature T_p :

$$\{T\} = \begin{Bmatrix} T_{in} \\ \{T_1\}_{8 \times 1} \\ \{T_2\}_{8 \times 1} \\ \{T_3\}_{8 \times 1} \\ \{T_4\}_{8 \times 1} \\ T_M \\ T_p \end{Bmatrix} \quad (B.1)$$

52 $\{T_i\}_{8 \times 1}$ contains the temperature in a pile section around a pile. The temperatures are assumed to
53 be independent upon the depth:

$$\{T_i\}_{8 \times 1} = \begin{Bmatrix} T_{A,i} \\ T_{C,i} \\ T_{fl,i} \\ T_{F,i} \\ T_{B1,i} \\ T_{B2,i} \\ p_i/\lambda_0 \\ T_{fds,i} \end{Bmatrix} \quad (B.2)$$

54 The conductance matrix $[\Lambda]$ is given by assembling submatrices:

$$[\Lambda] = \begin{bmatrix} [0]_{1 \times 1} & [\Lambda_N]_{1 \times 8} & [\Lambda_N]_{1 \times 8} & [\Lambda_N]_{1 \times 8} & [\Lambda_N]_{1 \times 8} & [0]_{1 \times 2} \\ [\Lambda_W]_{8 \times 1} & [\Lambda_c]_{8 \times 8} & [\Lambda_{12}]_{8 \times 8} & [0]_{8 \times 8} & [\Lambda_{1n}]_{8 \times 8} & [\Lambda_E]_{8 \times 2} \\ [0]_{8 \times 1} & [\Lambda_{21}]_{8 \times 8} & [\Lambda_c]_{8 \times 8} & [\Lambda_{12}]_{8 \times 8} & [0]_{8 \times 8} & [\Lambda_E]_{8 \times 2} \\ [0]_{8 \times 1} & [0]_{8 \times 8} & [\Lambda_{21}]_{8 \times 8} & [\Lambda_c]_{8 \times 8} & [\Lambda_{12}]_{8 \times 8} & [\Lambda_E]_{8 \times 2} \\ [0]_{8 \times 1} & [\Lambda_{n1}]_{8 \times 8} & [0]_{8 \times 8} & [\Lambda_{21}]_{8 \times 8} & [\Lambda_c]_{8 \times 8} & [\Lambda_E]_{8 \times 2} \\ [0]_{2 \times 1} & [\Lambda_S]_{2 \times 8} & [\Lambda_S]_{2 \times 8} & [\Lambda_S]_{2 \times 8} & [\Lambda_S]_{2 \times 8} & [\Lambda_{SE}]_{2 \times 2} \end{bmatrix}$$

56 (B.3)

57 The submatrices $[\Lambda_c]$, $[\Lambda_W]$, $[\Lambda_S]$, $[\Lambda_{SE}]$, $[\Lambda_E]$, $[\Lambda_N]$, $[\Lambda_{12}]$, $[\Lambda_{21}]$, $[\Lambda_{1n}]$, $[\Lambda_{n1}]$ are given by:

$$[\Lambda_c] = \begin{bmatrix} \frac{2}{R_1} & 0 & 0 & -\frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_{21}} + \frac{2}{R_1} & 0 & -\frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_p} & -\frac{1}{R_p} & 0 & 0 & -\lambda_0 & 0 & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_{21}} & -\frac{1}{R_p} & \frac{1}{R_p} + \frac{1}{R_{21}} + \frac{2}{R_1} + \frac{1}{R_{31}} & -\frac{1}{R_{31}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_{31}} & \frac{1}{R_{31}} + \frac{1}{R_{32}} & -\frac{1}{R_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{R_{32}} & \frac{1}{R_{32}} + -\frac{1}{R_{33}} & 0 & 0 & 0 \\ 0 & 0 & -2\lambda_0 & 0 & 0 & 0 & 0 & 0 & \lambda_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_0 & \lambda_0 \frac{p^{n+1}}{H\Delta T} \end{bmatrix}$$

$$[\Lambda_N] = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_0 \ 0]$$

$$[\Lambda_W] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \lambda_0 \\ -\frac{p^{n+1}}{H\Delta T} \lambda_0 \end{bmatrix}$$

$$[\Lambda_S] = \begin{bmatrix} 0 & -\frac{1}{R_{22}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4G^n}{R_{33}} & 0 \end{bmatrix}$$

$$[\Lambda_E] = \begin{bmatrix} -\frac{1}{R_{22}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{R_{33}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[\Lambda_{SE}] = \begin{bmatrix} \frac{4}{R_{22}} & 0 \\ 0 & \lambda_m + \frac{4G^n}{R_{33}} \end{bmatrix}$$

$$\begin{aligned}
[\Lambda_{12}] &= \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
[\Lambda_{21}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{P^{n+1}}{H\Delta T} \end{bmatrix} \\
[\Lambda_{1n}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
[\Lambda_{n1}] &= \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{R_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

(B.4)

58 In eq. (B.4) λ_0 is a reference thermal conductivity of the same order of magnitude as λ_m (e.g. 1 W.K-
59 $^{-1}\cdot m^{-1}$) introduced for unit consistency. R_p accounts for the effective thermal resistance of the pipe,
60 including both advection within the fluid and heat conduction in the pipe material. Note that the
61 power P is evaluated at the next time step P^{n+1} . The capacitance matrix $[C]$ reads:

$$[C] = \begin{bmatrix} [0]_{1 \times 1} & [0]_{1 \times 8} & [0]_{1 \times 8} & [0]_{1 \times 8} & [0]_{1 \times 8} & [0]_{1 \times 2} \\ [0]_{8 \times 1} & [C_0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 2} \\ [0]_{8 \times 1} & [0]_{8 \times 8} & [C_0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 2} \\ [0]_{8 \times 1} & [0]_{8 \times 8} & [0]_{8 \times 8} & [C_0]_{8 \times 8} & [0]_{8 \times 8} & [0]_{8 \times 2} \\ [0]_{8 \times 1} & [0]_{8 \times 8} & [0]_{2 \times 8} & [0]_{8 \times 8} & [C_0]_{8 \times 8} & [0]_{8 \times 2} \\ [0]_{2 \times 1} & [0]_{2 \times 8} & [0]_{2 \times 8} & [0]_{2 \times 8} & [0]_{2 \times 8} & [0]_{2 \times 2} \end{bmatrix} \quad (B.5)$$

62 With:

$$[C_0]_{8 \times 8} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{M2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{B1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{B2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (B.6)$$

63 And the right member $\{\rho\}$ reads:

$$\{\rho\} = \left\{ \begin{array}{l} \frac{p^{n+1}}{H} \\ [0]_{(35) \times 1} \\ 0 \text{ if } n = 1 \\ \lambda_m T_0 + (p_b^{-1}(G^2 - G^1)) \text{ if } n = 2 \\ \lambda_m T_0 + \left(p_b^{-1} G^n + \sum_{l=1}^{n-2} (p_b^{l+1} - p_b^l) G^{n-l} - p_b^{n-1} G^1 \right) \text{ if } n > 2 \end{array} \right\} \quad (B.7)$$

64