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A REMARK ON THE DIXMIER CONJECTURE

V. V. BAVULA AND V. LEVANDOVSKYY

ABSTRACT. The Dixmier Conjecture says that every endomorphism of the (first) Weyl algebra A_1 (over a field of characteristic zero) is an automorphism, i.e., if $PQ - QP = 1$ for some $P, Q \in A_1$ then $A_1 = K\langle P, Q \rangle$. The Weyl algebra A_1 is a \mathbb{Z} -graded algebra. We prove that the Dixmier Conjecture holds if the elements P and Q are sums of no more than two homogeneous elements of A_1 (there is no restriction on the total degrees of P and Q).

Key Words: the Weyl algebra, the Dixmier Conjecture, automorphism, endomorphism, a \mathbb{Z} -graded algebra.

Mathematics subject classification 2010: 16S50, 16W20, 16S32, 16W50.

1. INTRODUCTION

In the paper, K is a field of characteristic zero and $K^* := K \setminus \{0\}$. The algebra $A_1 := K\langle X, Y \mid [Y, X] = 1 \rangle$ is called the *first Weyl algebra* where $[Y, X] = YX - XY$. The n 'th tensor power of A_1 , $A_n := A_1^{\otimes n} = \underbrace{A_1 \otimes \cdots \otimes A_1}_{n \text{ times}}$, is called the *n 'th Weyl algebra*. The algebra A_n is a simple Noetherian domain of Gel'fand-Kirillov dimension

$\text{GK}(A_n) = 2n$, it is canonically isomorphic to the algebra of polynomial differential operators $K\langle X_1, \dots, X_n, \partial_1, \dots, \partial_n \rangle$ (where $\partial_i = \frac{\partial}{\partial x_i}$) via $X_i \mapsto X_i, Y_i \mapsto \partial_i$ for $i = 1, \dots, n$.

In his seminal paper [9], Dixmier (1968) found explicit generators for the group $G = \text{Aut}_K(A_1)$ of K -automorphisms of the Weyl algebra A_1 . Namely, the group G is generated by the obvious automorphisms:

$$(X, Y) \mapsto (X, Y + \lambda X^n), \quad (X, Y) \mapsto (X + \lambda Y^n, Y), \quad (X, Y) \mapsto (\mu X, \mu^{-1}Y)$$

where $\lambda \in K$, $\mu \in K^*$ and $n \in \mathbb{N}_+ := \{1, 2, \dots\}$.

In [9], Dixmier posed six problems: The first problem of Dixmier (in the list) asks if *every endomorphism of the Weyl algebra A_1 is an automorphism*, i.e., given elements P, Q of A such that $[P, Q] = 1$, do they generate the algebra A_1 ? A similar problem but for the n 'th Weyl algebra is called the *Dixmier Conjecture*. Problems 3 and 6 have been solved by Joseph [10] (1975), Problem 5 and Problem 4 (in the case of homogeneous elements) have been solved by Bavula [4] (2005).

The Dixmier Conjecture implies the *Jacobian Conjecture* (see [2]) and the inverse implication is also true (see [11] and [8]); a short proof is given in [6]; see also [1]).

In [5], it is shown that for each K -endomorphism $\phi : A_n \rightarrow A_n$ its image is very large, i.e., the left A_{2n} -module ${}^\phi A_n$ is a holonomic A_{2n} -module (where for all $a, b \in A_n$ and $c \in {}^\phi A_n$, $a \cdot c \cdot b := \phi(a)c\phi(b)$). In particular, it has finite length with simple holonomic factors over A_{2n} (see [5] for details). To prove that the Dixmier Conjecture holds for the Weyl algebra A_n it remains to show that the length is 1. Note, that the Gel'fand-Kirillov dimension of a simple A_{2n} -module can be $2n, 2n+1, \dots, 4n-1$, and the last case is the generic case.

In [7], it is shown that every algebra endomorphism of the algebra $\mathbb{I}_1 = K\langle x, \partial, f \rangle$ of polynomial integro-differential operators is an automorphism and it is conjectured that the same result holds for $\mathbb{I}_n := \mathbb{I}_1^{\otimes n} = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$.

The Weyl algebra $A_1 = \bigoplus_{i \in \mathbb{Z}} A_{1,i}$ is a \mathbb{Z} -graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where $A_{1,0} = K[H]$, $H = YX$ and, for $i \geq 1$, $A_{1,i} = K[H]X^i$ and $A_{1,-i} = K[H]Y^i$. For a nonzero element a of A_1 , the number of *nonzero homogeneous* components is called the *mass* of a , denoted by $m(a)$. For example, $m(\alpha X^i) = 1$ for all $\alpha \in K[H] \setminus \{0\}$ and $i \geq 1$. The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) \leq 2$ and $m(Q) \leq 2$. If $[P, Q] = 1$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \text{Aut}_K(A_1)$.*

2. PROOF OF THEOREM 1.1

The Weyl algebra is a generalized Weyl algebra. Let D be a ring with an automorphism σ and a central element a . The **generalized Weyl algebra** $A = D(\sigma, a)$ of degree 1, is the ring generated by D and two indeterminates X and Y subject to the relations [3]:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a \mathbb{Z} -graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), $v_0 = 1$. It follows from the defining relations that

$$v_n v_m = (n, m)v_{n+m} = v_{n+m} < n, m >$$

for some elements $(n, m) = \sigma^{-n-m} < n, m > \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$

$$n \leq m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n, m) = 1$.

Let $K[H]$ be a polynomial ring in a variable H over the field K , $\sigma : H \rightarrow H-1$ be the K -automorphism of the algebra $K[H]$ and $a = H$. The first Weyl algebra $A_1 = K\langle X, Y \mid YX - XY = 1 \rangle$ is isomorphic to the generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, H), \quad X \mapsto X, \quad Y \mapsto Y, \quad YX \mapsto H.$$

We identify both these algebras via this isomorphism, that is $A_1 = K[H](\sigma, H)$ and $H = YX$.

If $n > 0$ and $m > 0$ then

$$n \geq m : (n, -m) = (H - n) \cdots (H - n + m - 1), \quad (-n, m) = (H + n - 1) \cdots (H + n - m),$$

$$n \leq m : (n, -m) = (H - n) \cdots (H - 1), \quad (-n, m) = (H + n - 1) \cdots H,$$

in other cases $(n, m) = 1$.

The localization $B = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore subset $S = K[H] \setminus \{0\}$ of A_1 is the *skew Laurent polynomial ring* $B = K(H)[X, X^{-1}; \sigma]$ with coefficients from the field $K(H) = S^{-1}K[H]$ of rational functions where $\sigma \in \text{Aut}_K K(H)$ and $\sigma(H) = H - 1$. The map $A_1 \rightarrow B, a \mapsto a/1$ is an algebra monomorphism. We identify the algebra A_1 with its image in the algebra B via $A_1 \rightarrow B, X \mapsto X, Y \mapsto HX^{-1}$. The algebra $B = \bigoplus_{i \in \mathbb{Z}} B_i$ is a \mathbb{Z} -graded algebra where $B_i = K(H)X^i$. The algebra A_1 is a \mathbb{Z} -graded subalgebra of B .

A polynomial $f(H) = \lambda_n H^n + \lambda_{n-1} H^{n-1} + \cdots + \lambda_0 \in K[H]$ of degree n is called a *monic polynomial* if the *leading coefficient* λ_n of $f(H)$ is 1. A rational function $h \in K(H)$ is called a *monic rational function* if $h = f/g$ for some monic polynomials f, g . A homogeneous element $u = \alpha x^n$ of B is called *monic* if α is a monic rational function. We can extend the concept of degree of polynomial to the field of rational functions by the rule $\deg h = \deg f - \deg g$ where $h = f/g \in K(H)$. If $h_1, h_2 \in K(H)$ then $\deg h_1 h_2 = \deg h_1 + \deg h_2$ and $\deg(h_1 + h_2) \leq \max\{\deg h_1, \deg h_2\}$. We denote by $\text{sign}(n)$ and by $|n|$ the *sign* and the *absolute value* of $n \in \mathbb{Z}$, respectively.

Let A be an algebra and $a \in A$. The subalgebra of A , $C_A(a) = \{b \in A \mid ab = ba\}$, is called the *centralizer* of the element a in A .

Proposition 2.1 ([4], Proposition 2.1). (Centralizer of a Homogeneous Element of the Algebra B)

(1) Let $u = \alpha X^n$ be a monic element of B_n with $n \neq 0$. Then the centralizer $C_B(u) = K[v, v^{-1}]$ is a Laurent polynomial ring for a unique element $v = \beta X^{\text{sign}(n)s}$ where s is the least positive divisor of n for which there exists an element $\beta = \beta_s \in K(H)$, necessarily monic and uniquely defined, such that

$$(1) \quad \beta \sigma^s(\beta) \sigma^{2s}(\beta) \cdots \sigma^{(n/s-1)s}(\beta) = \alpha, \quad \text{if } n > 0,$$

$$(2) \quad \beta \sigma^{-s}(\beta) \sigma^{-2s}(\beta) \cdots \sigma^{-(|n|/s-1)s}(\beta) = \alpha, \quad \text{if } n < 0.$$

(2) Let $u \in K(H) \setminus K$. Then $C_B(u) = K(H)$.

Let $A_{1,+} := K[H][X; \sigma]$ and $A_{1,-} := K[H][Y; \sigma^{-1}]$. The algebras $A_{1,+}$ and $A_{1,-}$ are (skew polynomial) subalgebras of A_1 .

Lemma 2.2 ([4]). If $u \in A_{1,\pm} \setminus \{0\}$ then $C_A(u) \subseteq A_{1,\pm}$.

The K -automorphism of the Weyl algebra A_1 ,

$$(3) \quad \xi : A_1 \rightarrow A_1, \quad X \mapsto Y, \quad Y \mapsto -X,$$

reverses the \mathbb{Z} -grading of the Weyl algebra A_1 , that is

$$(4) \quad \xi(A_{1,i}) = A_{1,-i} \text{ for all } z \in \mathbb{Z}.$$

By the *degree* of an element of A_1 we mean its *total degree* with respect to the canonical generators X and Y of A_1 . Let $A_{1,\leq i} := \{p \in A \mid \deg(p) \leq i\}$ for $i \in \mathbb{N}$. Then $\{A_{1,\leq i}\}_{i \in \mathbb{N}}$ is the standard filtration of the algebra A_1 associated with the generators X and Y . For all $i \in \mathbb{Z} \setminus \{0\}$ and $f \in K[H] \setminus K$,

$$(5) \quad \deg \sigma^i(f) = \deg f \text{ and } \deg(1 - \sigma^i)(f) = \deg f - 1.$$

Proof of Theorem 1.1: (i) If $P, Q \in A_{1,\leq 1}$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some $\tau \in \text{Aut}_K(A_1)$: Clearly, $P = aY + bX + \lambda$ and $Q = cY + dX + \mu$ for some $a, b, c, d, \lambda, \mu \in K$. Then $1 = [P, Q] = ad - bc$. So, the automorphism τ can be chosen of the form

$$\tau(Y) = aY + bX + \lambda \text{ and } \tau(X) = cY + dX + \mu.$$

So, till the end of the proof we assume that at least one of the polynomials P or Q does not belong to the space $A_{1,\leq 1}$. In view of the relation $1 = [P, Q] = [-Q, P]$, we can assume that $P \notin A_{1,\leq 1}$. In view of Equation (4), we can assume that the highest homogeneous part of P , say $P_p \in A_{1,p}$, satisfies the condition that $p \geq 2$. Since $m(P) \leq 2$, either $P = P_p$ (if $m(P) = 1$) or otherwise $P = P_r + P_p$ for some nonzero $P_r \in A_{1,r}$ where $r < p$.

(ii) $(m(P), m(Q)) \neq (1, 1)$: Suppose that $m(P) = m(Q) = 1$, we seek a contradiction. Then $P = \alpha X^p$ and $Q = \beta Y^p$ for some nonzero polynomials $\alpha, \beta \in K[H]$. Then

$$\begin{aligned} 1 &= [P, Q] = \alpha \sigma^p(\beta)(p, -p) - \beta \sigma^{-p}(\alpha)(-p, p) \\ &= \alpha \sigma^p(\beta)(p, -p) - \beta \sigma^{-p}(\alpha) \sigma^{-p}((p, -p)) = (1 - \sigma^{-p})(\alpha \sigma^p(\beta)(p, -p)). \end{aligned}$$

Since $p \geq 2$ (or $P \notin A_{1,\leq 1}$),

$$0 = \deg 1 = \deg(1 - \sigma^{-p})(\alpha \sigma^p(\beta)(p, -p)) = \deg \alpha + \deg \beta + \deg(p, -p) - 1$$

(by Equation (5)) $\geq 0 + 0 + p - 1 \geq 2 - 1 = 1$, a contradiction.

(iii) $(m(P), m(Q)) \neq (1, 2)$: Suppose that $m(P) = 1$ and $m(Q) = 2$. Then $P = \alpha X^p$ for some $p \geq 2$ and $Q = Q_s + Q_q$ where $Q_s \in A_{1,s}$, $Q_q \in A_{1,q}$ and $s < q$. By Lemma 2.2, the equality $[P, Q] = 1$ implies that $[P, Q_s] = 1$ and $[P, Q_q] = 0$. By the case (ii), this is not possible.

(iv) Suppose that $m(P) = 2$ and $m(Q) = 1$. Then $P = P_r + P_p$ and $Q = Q_q$. By Lemma 2.2 the equality $[P, Q] = 1$ implies that $[P_p, Q_q] = 0$ and $[P_r, Q_q] = 1$. Then, $q \geq 0$, by Lemma 2.2. The case $q = 0$ is not possible since then both $P_r, Q_q \in K[H]$ and this would contradict the equality $[P_r, Q_q] = 1$. Therefore, $q > 0$. Then $P_r = \beta Y^q$ and $Q_q = \alpha X^q$ for some nonzero elements $\beta, \alpha \in K[H]$. Then

$$-1 = [Q_q, P_r] = (1 - \sigma^{-q})(\alpha \sigma^p(\beta)(q, -q))$$

implies that

$$0 = \deg(-1) = \deg(1 - \sigma^{-q})(\alpha \sigma^p(\beta)(q, -q)) = \deg \alpha + \deg \beta + q - 1,$$

by Equation (5). Hence, $q = 1$, $\alpha, \beta \in K^*$ and $\beta = -\alpha^{-1}$. Then $P, Q \in A_{1, \leq 1}$, and, by the statement (i), the pair (P, Q) is obtained from the pair (Y, X) by applying an automorphism of A_1 .

(v) $(m(P), m(Q)) \neq (2, 2)$: Since $m(P) = m(Q) = 2$, we can write $P = P_r + P_p$ and $Q = Q_s + Q_q$ as sums of homogeneous elements where $r < p$, $P_r \in A_{1,r}$, $P_p \in A_{1,p}$ and $s < q$, $Q_s \in A_{1,s}$, $Q_q \in A_{1,q}$. The equality $[P, Q] = 1$ implies that

$$[P_r, Q_s] = 0 \quad \text{and} \quad [P_p, Q_q] = 0,$$

see Lemma 2.2. By Lemma 2.2, the elements r and s have the same sign (i.e., either $r < 0, s < 0$ or $r = s = 0$ or $r > 0, s > 0$) and also the elements p and q have the same sign. Since $p \geq 2$, we must have $q > 0$.

Suppose that $r \geq 0$, we seek a contradiction. Then $s \geq 0$ and so the elements P and Q are elements of the subring $A_{1,+} = \bigoplus_{i \geq 0} K[H]X^i$. Now,

$$K[H] \ni 1 = [P, Q] \in [A_{1,+}, A_{1,+}] \subseteq \bigoplus_{i \geq 1} K[H]X^i,$$

a contradiction. Therefore, $r < 0$ and $s < 0$.

The equality $1 = [P, Q] = [P_r, Q_q] + [P_p, Q_s]$ and Lemma 2.2 imply that $r + q = 0$ and $p + s = 0$, that is $r = -q$ and $s = -p$. So,

$$P = P_{-q} + P_p \quad \text{and} \quad Q = Q_{-p} + Q_q.$$

The elements P_p and P_{-q} are homogeneous elements of the Weyl algebra A_1 . The Weyl algebra A_1 is a homogeneous subalgebra of the algebra

$$K(H)[X, X^{-1}; \sigma] = K(H)[Y, Y^{-1}; \sigma^{-1}]$$

where $K(H)$ is the field of rational functions in the variable H and the automorphism σ of $K(H)$ is given by the rule $\sigma(H) = H - 1$. By [4, Proposition 2.1(1)], the centralizer $C_B(P_p)$ of the element P_p in B is a Laurent polynomial algebra

$$K[\alpha X^n, (\alpha X^n)^{-1}]$$

for some nonzero element $\alpha \in K(H)$ and $n \geq 1$. In general, $\alpha \notin K[H]$. Similarly,

$$C_B(P_{-q}) = K[\beta Y^m, (\beta Y^m)^{-1}]$$

for some nonzero element $\beta \in K(H)$ and $m \geq 1$.

Since $[P_p, Q_q] = 0$, $Q_q \in C_B(P_p)$ and

$$P_p = \lambda(P_p)(\alpha X^n)^i = \lambda(P_p)\alpha\sigma^n(\alpha) \cdots \sigma^{n(i-1)}(\alpha)X^{ni} = \alpha_{n,i}X^p,$$

$$Q_q = \lambda(Q_q)(\alpha X^n)^j = \lambda(Q_q)\alpha\sigma^n(\alpha) \cdots \sigma^{n(j-1)}(\alpha)X^{nj} = \alpha'_{n,j}X^q,$$

for some nonzero scalars $\lambda(P_p), \lambda(Q_q) \in K^*$ and some $i \geq 1$ and $j \geq 1$ where

$$\alpha_{n,i} = \lambda(P_p)\alpha\sigma^n(\alpha) \cdots \sigma^{n(i-1)}(\alpha) \in K[H], \quad p = ni,$$

$$\alpha'_{n,j} = \lambda(Q_q)\alpha\sigma^n(\alpha) \cdots \sigma^{n(j-1)}(\alpha) \in K[H], \quad q = nj.$$

Since $[P_{-p}, Q_{-p}] = 0$, $Q_{-p} \in C_B(P_{-q})$ and

$$P_{-q} = \lambda(P_{-q})(\beta Y^m)^s = \lambda(P_{-q})\beta\sigma^{-m}(\beta) \cdots \sigma^{-m(s-1)}(\beta)Y^{ms} = \beta_{m,s}Y^p,$$

$$Q_{-p} = \lambda(Q_{-p})(\beta Y^m)^t = \lambda(Q_{-p})\beta\sigma^{-m}(\beta) \cdots \sigma^{-m(t-1)}(\beta)Y^{mt} = \beta'_{m,t}Y^q,$$

for some nonzero scalars $\lambda(P_{-q}), \lambda(Q_{-p}) \in K^*$ and some $s \geq 1$ and $t \geq 1$ where

$$\beta_{m,s} = \lambda(P_{-q})\beta\sigma^{-m}(\beta) \cdots \sigma^{-m(s-1)}(\beta) \in K[H], \quad p = ms,$$

$$\beta'_{m,t} = \lambda(Q_{-p})\beta\sigma^{-m}(\beta) \cdots \sigma^{-m(t-1)}(\beta) \in K[H], \quad q = mt.$$

Now,

$$\begin{aligned} 1 = [P, Q] &= [P_p, Q_{-p}] + [P_{-q}, Q_q] = [\alpha_{n,i}X^p, \beta'_{m,t}Y^p] + [\beta_{m,s}Y^q, \alpha'_{n,j}X^q] \\ &= \alpha_{n,i}\sigma^p(\beta'_{m,t})(p, -p) - \beta'_{m,t}\sigma^{-p}(\alpha_{n,i})(-p, p) \\ &\quad + \beta_{m,s}\sigma^{-q}(\alpha'_{n,j})(-q, q) - \alpha'_{n,j}\sigma^q(\beta_{m,s})(q, -q). \end{aligned}$$

Using the equalities $(-p, p) = \sigma^{-p}((p, -p))$ and $(-q, q) = \sigma^{-q}((q, -q))$, the last equality above can be rewritten as follows 1=ab

$$(6) \quad 1 = (1 - \sigma^{-p})(a) + (1 - \sigma^{-q})(b)$$

where $a = \alpha_{n,i}\sigma^p(\beta'_{m,t})(p, -p) \in K[H]$ and $b = \alpha'_{n,j}\sigma^q(\beta_{m,s})(q, -q) \in K[H]$.

Recall that $P = P_{-q} + P_p$, $Q = Q_{-p} + Q_q$, 2=ab

$$(7) \quad p = mt = ni \geq 2 \text{ and } q = ms = nj \geq 1.$$

Suppose that $p = q$, and so $P = P_{-p} + P_p$, $Q = Q_{-p} + Q_p$. Then $Q = \lambda P_p$ for some $\lambda \in K^*$. Notice that

$$1 = [P, Q] = [P, Q - \lambda P], \quad m(P) = 2 \text{ and } m(Q - \lambda P) = 1.$$

By the case (iv), the pair $(P, Q - \lambda P)$ is obtained from the pair (Y, X) by applying an automorphism of the Weyl algebra A_1 .

So, either $p < q$ or $p > q$. In view of (P, Q) -symmetry $(1 = [P, Q] = [-Q, P])$, it suffices to consider, say, the first case only. Since $p < q$, the equalities (7) imply that $i < j$ and $t < s$. Then, using Equation (5) and the fact that $\deg(p, -p) = p$ for all $p \geq 1$, we see that

$$\deg a = \deg \alpha_{n,i} + \deg \beta'_{m,t} + p - 1,$$

$$\deg b = \deg \alpha'_{n,j} + \deg \beta_{m,s} + q - 1.$$

Since $i < j$ and $t < s$, $\deg \alpha_{n,i} < \deg \alpha'_{n,j}$ and $\deg \beta'_{m,t} < \deg \beta_{m,s}$. In particular, $\deg a < \deg b$. This equality contradicts Equation (6) since, by Equation (5),

$$0 = \deg 1 = \deg a - 1 - \deg b + 1 = \deg a - \deg b > 0.$$

This means that the cases $p < q$ and $p > q$ are impossible. The proof of the theorem is complete. \square

Corollary 2.3. *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) = 1$ or $m(Q) = 1$. If $[P, Q] = 1$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \text{Aut}_K(A_1)$.*

Proof: Without loss of generality we may assume $m(Q) = 1$ and $m(P) \geq 3$. That is $Q = Q_q$ and $P = \sum_{i \in I} P_i$, where $I \subset \mathbb{Z}$ is a finite set, $q \in \mathbb{Z} \setminus \{0\}$ and the elements Q_q and P_i are homogeneous in A_1 . By Equation (4), we may assume that $q > 0$. Then

$$1 = [P, Q] = \sum_i [P_i, Q_q]$$

implies that $-q \in I$, $[P_{-q}, Q_q] = 1$ and $[P_j, Q_q] = 0$ for all $j \in I$ such that $j \neq -q$. By Theorem 1.1,

$$q = 1, Q_1 = \lambda X \text{ and } P_{-1} = \lambda^{-1}Y \text{ for some } \lambda \in K^*.$$

By Lemma 2.2, $C := P - P_{-1} \in C_A(X) = K[X]$. Then $P = \tau(Y)$ and $Q = \tau(X)$ where $\tau : A_1 \rightarrow A_1$, $X \mapsto \lambda X$, $Y \mapsto \lambda^{-1}Y + C$, is an automorphism. \square

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