Journal of Mathematical Physics

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Cite as: J. Math. Phys. **55**, 102704 (2014); https://doi.org/10.1063/1.4898077
Submitted: 28 July 2014 . Accepted: 30 September 2014 . Published Online: 17 October 2014

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JOURNAL OF MATHEMATICAL PHYSICS **55**, 102704 (2014)

Duality between the trigonometric BC_n Sutherland system and a completed rational Ruijsenaars–Schneider–van Diejen system

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(Received 28 July 2014; accepted 30 September 2014; published online 17 October 2014)

We present a new case of duality between integrable many-body systems, where two systems live on the action-angle phase spaces of each other in such a way that the action variables of each system serve as the particle positions of the other one. Our investigation utilizes an idea that was exploited previously to provide group-theoretic interpretation for several dualities discovered originally by Ruijsenaars. In the group-theoretic framework, one applies Hamiltonian reduction to two Abelian Poisson algebras of invariants on a higher dimensional phase space and identifies their reductions as action and position variables of two integrable systems living on two different models of the single reduced phase space. Taking the cotangent bundle of U(2n) as the upstairs space, we demonstrate how this mechanism leads to a new dual pair involving the BC_n trigonometric Sutherland system. Thereby, we generalize earlier results pertaining to the A_n trigonometric Sutherland system as well as a recent work by Pusztai on the hyperbolic BC_n Sutherland system. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4898077]

I. INTRODUCTION

The integrable one-dimensional many-body systems of Calogero–Sutherland–Toda type and their generalizations are very important because they are ubiquitous in physical applications and have close ties to several topics of mathematics. See, for example, the reviews in Refs. 4,18,22,31,33, and 34. We here focus on their fascinating duality relations, which were first studied by Ruijsenaars.[27] We shall uncover a new case of duality between two systems of this type.

Duality between two Liouville integrable Hamiltonian systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ requires the existence of Darboux coordinates q_i, p_i on M and λ_j, ϑ_j on $\tilde{M}$ (or on dense open submanifolds of M and $\tilde{M}$) and a *global* symplectomorphism $\mathcal{R} : M \to \tilde{M}$ such that $(\lambda, \vartheta) \circ \mathcal{R}$ are action-angle variables for the Hamiltonian H and $(q, p) \circ \mathcal{R}^{-1}$ are action-angle variables for the Hamiltonian $\tilde{H}$. This means that $H \circ \mathcal{R}^{-1}$ depends only on λ and $\tilde{H} \circ \mathcal{R}$ only on q. Then one says that (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ are in action-angle duality. In addition, for the systems of our interest it also happens that when expressed in the coordinates (q, p) the Hamiltonian $H(q, p)$ admits interpretation in terms of interaction of n "particles" with position variables q_i, and $\tilde{H}(\lambda, \vartheta)$ similarly describes the interaction of n points with positions λ_i. Thus, the q_i are particle positions for H and action variables for $\tilde{H}$, and the λ_i are positions for $\tilde{H}$ and actions for H. The significance of this curious property is clear, for instance, from the fact that it persists at the quantum mechanical level as the bispectral character of the wave functions,[3,29] which are important special functions.

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Dual pairs of many-body systems were exhibited by Ruijsenaars in the course of his direct construction of action-angle variables for the many-body systems (of non-elliptic Calogero–Sutherland type and non-periodic Toda type) associated with the A_n root system.[27,28,30,31] It is natural to expect that action-angle duality also exists for many-body systems associated with other root systems. Substantial evidence to support this expectation was given in a recent paper by Pusztai,[26] where action-angle duality between the hyperbolic BC_n Sutherland[16,19] and the rational Ruijsenaars–Schneider–van Diejen (RSvD) systems[35] was established. The specific goal of the present work is to find out how this result can be generalized if one replaces the hyperbolic BC_n system with its trigonometric analogue. A similar problem has been studied previously in the A_n case, where it was found that the dual of the trigonometric Sutherland system possesses intricate global structure.[6,30] The global description of the duality necessitates a separate investigation also in the BC_n case, since it cannot be derived by naïve analytic continuation between trigonometric and hyperbolic functions. This problem turns out to be considerably more complicated than those studied in Refs. 6 and 26.

The trigonometric BC_n Sutherland system is defined by the Hamiltonian

$$H(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+\sum_{1\leq j<k\leq n}\left[\frac{\gamma}{\sin^2(q_j-q_k)}+\frac{\gamma}{\sin^2(q_j+q_k)}\right]+\sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)}+\sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)}. \tag{1.1}$$

Here, (q, p) varies in the cotangent bundle $M = T^*C_1 = C_1 \times \mathbb{R}^n$ of the domain

$$C_1=\left\{q\in\mathbb{R}^n\,\middle|\,\frac{\pi}{2}>q_1>\cdots>q_n>0\right\}, \tag{1.2}$$

and the three independent real coupling constants $\gamma, \gamma_1, \gamma_2$ are supposed to satisfy

$$\gamma>0,\quad \gamma_2>0,\quad 4\gamma_1+\gamma_2>0. \tag{1.3}$$

The inequalities in (1.3) guarantee that the n particles with coordinates q_j cannot leave the open interval $(0, \frac{\pi}{2})$ and they cannot collide. At a "semi-global" level, the dual system will be shown to have the Hamiltonian

$$\tilde{H}^0(\lambda,\vartheta)=\sum_{j=1}^{n}\cos(\vartheta_j)\left[1-\frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{4\mu^2}{(\lambda_j-\lambda_k)^2}\right]^{\frac{1}{2}}\left[1-\frac{4\mu^2}{(\lambda_j+\lambda_k)^2}\right]^{\frac{1}{2}}$$
$$-\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^{n}\left[1-\frac{4\mu^2}{\lambda_j^2}\right]+\frac{\nu\kappa}{4\mu^2}. \tag{1.4}$$

Here, $\mu > 0$, ν, κ are real constants, $\vartheta_1, \ldots, \vartheta_n$ are angular variables, and λ varies in the Weyl chamber with thick walls

$$C_2=\left\{\lambda\in\mathbb{R}^n\,\middle|\,\begin{array}{l}\lambda_a-\lambda_{a+1}>2\mu,\\(a=1,\ldots,n-1)\end{array}\quad\text{and}\quad\lambda_n>\max\{|\nu|,|\kappa|\}\right\}. \tag{1.5}$$

The inequalities defining C_2 ensure the reality and the smoothness of $\tilde{H}^0$ on the phase space $\tilde{M}^0 := C_2 \times \mathbb{T}^n$, which is equipped with the symplectic form

$$\tilde{\omega}^0=\sum_{k=1}^{n}\mathrm{d}\lambda_k\wedge\mathrm{d}\vartheta_k. \tag{1.6}$$

Duality will be established under the following relation between the coupling parameters:

$$\gamma=\mu^2,\quad \gamma_1=\frac{\nu\kappa}{2},\quad \gamma_2=\frac{(\nu-\kappa)^2}{2}, \tag{1.7}$$

where in addition to $\mu > 0$ we also adopt the condition

$$\nu>|\kappa|\geq 0. \tag{1.8}$$

This entails that Eq. (1.7) gives a one-to-one correspondence between the parameters $(\gamma, \gamma_1, \gamma_2)$ subject to (1.3) and (μ, ν, κ), and also serves to simplify our analysis. In the above, the qualification "semi-global" indicates that $\tilde{M}^0$ represents a dense open submanifold of the full dual phase space, $\tilde{M}$. The completion of $\tilde{M}^0$ into $\tilde{M}$ guarantees both the completeness of the Hamiltonian flows of the dual system and the global nature of the symplectomorphism between M and $\tilde{M}$. The structure of $\tilde{M}$ will be clarified in the paper. For example, we shall see that the action variables of the Sutherland system fill the closure of the domain $C_2 \subset \mathbb{R}^n$, with the boundary points corresponding to degenerate Liouville tori.

The integrable systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ as well as their duality relation will emerge from an appropriate Hamiltonian reduction. Specifically, we will reduce the cotangent bundle $T^*\mathrm{U}(2n)$ with respect to the symmetry group $G_+ \times G_+$, where $G_+ \cong \mathrm{U}(n) \times \mathrm{U}(n)$ is the fix-point subgroup of an involution of $\mathrm{U}(2n)$. This enlarges the range of the reduction approach to action-angle dualities,[11,12,18] which realizes[5–9] the following scenario. Pick a higher dimensional symplectic manifold (P, Ω) equipped with two Abelian Poisson algebras $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ formed by invariants under a symmetry group acting on P. Then perform Hamiltonian reduction leading to the reduced manifold $(P_{\mathrm{red}}, \Omega_{\mathrm{red}})$ carrying the reduced Abelian Poisson algebras $\mathfrak{Q}^1_{\mathrm{red}}$ and $\mathfrak{Q}^2_{\mathrm{red}}$. Under favorable circumstances, it is possible to construct two models (M, ω) and $(\tilde{M}, \tilde{\omega})$ of $(P_{\mathrm{red}}, \Omega_{\mathrm{red}})$ in such a way that when expressed in terms of (M, ω) $\mathfrak{Q}^1_{\mathrm{red}}$ and $\mathfrak{Q}^2_{\mathrm{red}}$ coincide with the Abelian Poisson algebras generated by the position and action variables of an integrable many-body Hamiltonian H, respectively, and one finds a similar picture from the dual perspective of $(\tilde{M}, \tilde{\omega}, \tilde{H})$ except that the roles of $\mathfrak{Q}^1_{\mathrm{red}}$ and $\mathfrak{Q}^2_{\mathrm{red}}$ are interchanged. In particular, the many-body Hamiltonian H on M is engendered by an element of $\mathfrak{Q}^2_{\mathrm{red}}$ and the many-body Hamiltonian $\tilde{H}$ on $\tilde{M}$ is born from an element of $\mathfrak{Q}^1_{\mathrm{red}}$. For a relatively simple and enlightening example, we recommend the reader to have a glance at the duality between the hyperbolic A_n Sutherland and rational Ruijsenaars–Schneider systems as described in Ref. 7.

The rest of the paper is organized as follows. In Sec. II, we present the necessary group-theoretic preliminaries together with the definition of the unreduced Abelian Poisson algebras $\mathfrak{Q}^1$, $\mathfrak{Q}^2$ and the symplectic reduction to be performed. Then Sec. III is devoted to the derivation of the first model (M, ω) of the reduced phase space that carries the Sutherland Hamiltonian obtained as the reduction of the free Hamiltonian governing geodesic motion on $\mathrm{U}(2n)$. The content of this section, and even its quantum analogue, is fairly standard.[10] The heart of the paper is Sec. IV, where we develop the dual model $(\tilde{M}, \tilde{\omega})$ of the reduced phase space and explain how the Hamiltonian $\tilde{H}$ arises. This section relies on a blend of ideas from Refs. 6 and 24–26, and also requires the solution of a number of rather non-trivial technical problems. Some technical details are relegated to the Appendix. Our main new results are given by Theorem 4.1 and Theorem 4.10, which yield, respectively, the "semi-global" and a fully global characterization of the reduced phase space. Finally, in Sec. V, we pull together the previous developments and discuss the duality between the two systems mentioned in the title of the paper. Here, we shall also use the action angle-duality to establish interesting properties of these Hamiltonian systems.

II. PREPARATIONS

We next describe the starting data which will lead to integrable many-body systems in duality by means of the mechanism outlined in the Introduction. We then summarize some group-theoretic facts that will be used in the demonstration of this claim.

A. Definition of the Hamiltonian reduction

Let us choose an arbitrary positive integer, n, and also introduce $N := 2n$. Our investigation requires the unitary group of degree N

$$G := \mathrm{U}(N) = \{y \in \mathrm{GL}(N, \mathbb{C}) \mid y^\dagger y = \mathbf{1}_N\}, \tag{2.1}$$

and its Lie algebra

$$\mathcal{G} := \mathfrak{u}(N) = \{Y \in \mathfrak{gl}(N, \mathbb{C}) \mid Y^\dagger + Y = \mathbf{0}_N\}, \tag{2.2}$$

where $\mathbf{1}_N$ and $\mathbf{0}_N$ denote the identity and null matrices of size N, respectively. We endow the Lie algebra $\mathcal{G}$ with the Ad-invariant bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{G} \times \mathcal{G} \to \mathbb{R}, \quad (Y_1, Y_2) \mapsto \langle Y_1, Y_2 \rangle := \mathrm{tr}(Y_1 Y_2), \tag{2.3}$$

and identify $\mathcal{G}$ with the dual space $\mathcal{G}^*$ in the usual manner. By using left-translations to trivialize the cotangent bundle T^*G, we also adopt the identification

$$T^*G \cong G \times \mathcal{G}^* \cong G \times \mathcal{G} = \{(y, Y) \mid y \in G,\ Y \in \mathcal{G}\}. \tag{2.4}$$

Then the canonical symplectic form of T^*G can be written as

$$\Omega^{T^*G} := -\mathrm{d}\langle y^{-1}\mathrm{d}y, Y\rangle. \tag{2.5}$$

It can be evaluated according to the formula

$$\Omega^{T^*G}_{(y,Y)}(\Delta y \oplus \Delta Y, \Delta' y \oplus \Delta' Y) = \langle y^{-1}\Delta y, \Delta' Y\rangle - \langle y^{-1}\Delta' y, \Delta Y\rangle + \langle [y^{-1}\Delta y, y^{-1}\Delta' y], Y\rangle, \tag{2.6}$$

where $\Delta y \oplus \Delta Y, \Delta' y \oplus \Delta' Y \in T_{(y,Y)}T^*G$ are arbitrary tangent vectors at a point $(y, Y) \in T^*G$.

Let us introduce the $N \times N$ Hermitian, unitary matrix partitioned into four $n \times n$ blocks

$$C := \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \in G, \tag{2.7}$$

and the involutive automorphism of G defined as conjugation with C

$$\Gamma : G \to G, \quad y \mapsto \Gamma(y) := CyC^{-1}. \tag{2.8}$$

The set of fix-points of Γ forms the subgroup of G consisting of $N \times N$ unitary matrices with centro-symmetric block structure,

$$G_+ = \{y \in G \mid \Gamma(y) = y\} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in G \right\} \cong \mathrm{U}(n) \times \mathrm{U}(n). \tag{2.9}$$

We also introduce the closed submanifold G_- of G by the definition

$$G_- = \{y \in G \mid \Gamma(y) = y^{-1}\} = \left\{ \begin{bmatrix} a & b \\ c & a^\dagger \end{bmatrix} \in G \,\middle|\, b, c \in \mathrm{i}\mathfrak{u}(n) \right\}. \tag{2.10}$$

By slight abuse of notation, we let Γ stand for the induced involution of the Lie algebra $\mathcal{G}$, too. We can decompose $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-, \quad Y = Y_+ + Y_-, \tag{2.11}$$

where $\mathcal{G}_\pm$ are the eigenspaces of Γ corresponding to the eigenvalues ± 1, respectively, i.e.,

$$\begin{aligned} \mathcal{G}_+ &= \ker(\Gamma - \mathrm{id}) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \,\middle|\, A, B \in \mathfrak{u}(n) \right\}, \\ \mathcal{G}_- &= \ker(\Gamma + \mathrm{id}) = \left\{ \begin{bmatrix} A & B \\ -B & -A \end{bmatrix} \,\middle|\, A \in \mathfrak{u}(n),\ B \in \mathrm{i}\mathfrak{u}(n) \right\}. \end{aligned} \tag{2.12}$$

We are interested in a reduction of T^*G based on the symmetry group $G_+ \times G_+$. We shall use the shifting trick of symplectic reduction,[21] and thus we first prepare a coadjoint orbit of the symmetry group. To do this, we take any vector $V \in \mathbb{C}^N$ that satisfies $CV + V = 0$, and associate to it the element $\upsilon^\ell_{\mu,\nu}(V)$ of $\mathcal{G}_+$ by the definition

$$\upsilon^\ell_{\mu,\nu}(V) := \mathrm{i}\mu\big(VV^\dagger - \mathbf{1}_N\big) + \mathrm{i}(\mu - \nu)C, \tag{2.13}$$

where $\mu, \nu \in \mathbb{R}$ are real parameters. The set

$$\mathcal{O}^\ell := \big\{\upsilon^\ell \in \mathcal{G}_+ \mid \exists\, V \in \mathbb{C}^N,\ V^\dagger V = N,\ CV + V = 0,\ \upsilon^\ell = \upsilon^\ell_{\mu,\nu}(V)\big\} \tag{2.14}$$

represents a coadjoint orbit of G_+ of dimension $2(n - 1)$. We let $\mathcal{O}^r := \{\upsilon^r\}$ denote the one-point coadjoint orbit of G_+ containing the element

$$\upsilon^r := -\mathrm{i}\kappa C \quad \text{with some constant } \kappa \in \mathbb{R}, \tag{2.15}$$

and consider

$$\mathcal{O} := \mathcal{O}^\ell \oplus \mathcal{O}^r \subset \mathcal{G}_+ \oplus \mathcal{G}_+ \cong (\mathcal{G}_+ \oplus \mathcal{G}_+)^*, \tag{2.16}$$

which is a coadjoint orbit of $G_+ \times G_+$. The same coadjoint orbit was used in Ref. 26. Our starting point for symplectic reduction will be the phase space (P, Ω) with

$$P := T^*G \times \mathcal{O} \quad \text{and} \quad \Omega := \Omega^{T^*G} + \Omega^{\mathcal{O}}, \tag{2.17}$$

where $\Omega^{\mathcal{O}}$ denotes the Kirillov–Kostant–Souriau symplectic form on $\mathcal{O}$. The natural symplectic action of $G_+ \times G_+$ on P is defined by

$$\Phi_{(g_L, g_R)}(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big(g_L y g_R^{-1}, g_R Y g_R^{-1}, g_L \upsilon^\ell g_L^{-1} \oplus \upsilon^r\big). \tag{2.18}$$

The corresponding momentum map $J : P \to \mathcal{G}_+ \oplus \mathcal{G}_+$ is given by the formula

$$J(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big((yYy^{-1})_+ + \upsilon^\ell\big) \oplus \big(-Y_+ + \upsilon^r\big). \tag{2.19}$$

We shall see that the reduced phase space

$$P_{\mathrm{red}} = P_0/(G_+ \times G_+), \qquad P_0 := J^{-1}(0), \tag{2.20}$$

is a smooth symplectic manifold, which inherits two Abelian Poisson algebras from P.

Using the identification $\mathcal{G}^* \cong \mathcal{G}$, the invariant functions $C^\infty(\mathcal{G})^G$ form the center of the Lie–Poisson bracket. Denote by $C^\infty(G)^{G_+\times G_+}$ the set of smooth functions on G that are invariant under the $(G_+ \times G_+)$-action on G that appears in the first component of (2.18). Let us also introduce the maps

$$\pi_1 : P \to G, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto y, \tag{2.21}$$

and

$$\pi_2 : P \to \mathcal{G}, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto Y. \tag{2.22}$$

It is clear that

$$\mathfrak{Q}^1 := \pi_1^*(C^\infty(G)^{G_+\times G_+}) \quad \text{and} \quad \mathfrak{Q}^2 := \pi_2^*(C^\infty(\mathcal{G})^G) \tag{2.23}$$

are two Abelian subalgebras in the Poisson algebra of smooth functions on (P, Ω) and these Abelian Poisson algebras descend to the reduced phase space P_{red}.

Later we shall construct two models of P_{red} by exhibiting two global cross-sections for the action of $G_+ \times G_+$ on P_0. For this, we shall apply two different methods for solving the constraint equations that, according to (2.19), define the level surface $P_0 \subset P$

$$(yYy^{-1})_+ + \upsilon^\ell = \mathbf{0}_N \quad \text{and} \quad -Y_+ + \upsilon^r = \mathbf{0}_N, \tag{2.24}$$

where $\upsilon^\ell = \upsilon^\ell_{\mu,\lambda}(V)$ (2.13) for some vector $V \in \mathbb{C}^N$ subject to $CV + V = 0$, $V^\dagger V = N$ and $\upsilon^r = -\mathrm{i}\kappa C$. We below collect the group-theoretic results needed for our constructions.

B. Recall of group-theoretic results

To start, let us associate the diagonal $N \times N$ matrix

$$Q(q) := \mathrm{diag}(q, -q) \tag{2.25}$$

with any $q \in \mathbb{R}^n$. Notice that the set

$$\mathcal{A} := \{\mathrm{i}Q(q) \mid q \in \mathbb{R}^n\} \subset \mathcal{G}_- \tag{2.26}$$

is a maximal Abelian subalgebra in $\mathcal{G}_-$. The corresponding subgroup of G has the form

$$\exp(\mathcal{A}) = \big\{e^{\mathrm{i}Q(q)} = \mathrm{diag}\big(e^{\mathrm{i}q_1}, \ldots, e^{\mathrm{i}q_n}, e^{-\mathrm{i}q_1}, \ldots, e^{-\mathrm{i}q_n}\big) \mid q \in \mathbb{R}^n\big\}. \tag{2.27}$$

The centralizer of $\mathcal{A}$ inside G_+ (2.9) (with respect to conjugation) is the Abelian subgroup

$$Z := Z_{G_+}(\mathcal{A}) = \left\{ e^{\mathrm{i}\xi} = \mathrm{diag}\left(e^{\mathrm{i}x_1}, \dots, e^{\mathrm{i}x_n}, e^{\mathrm{i}x_1}, \dots, e^{\mathrm{i}x_n}\right) \mid x \in \mathbb{R}^n \right\} < G_+. \tag{2.28}$$

The Lie algebra of Z is

$$\mathcal{Z} = \{\mathrm{i}\xi = \mathrm{i}\,\mathrm{diag}(x, x) \mid x \in \mathbb{R}^n\} < \mathcal{G}_+. \tag{2.29}$$

The results that we now recall (see, e.g., Refs. 15, 17, and 32) will be used later. First, for any $y \in G$ there exist elements y_L, y_R from G_+ and unique $q \in \mathbb{R}^n$ satisfying

$$\frac{\pi}{2} \geq q_1 \geq \cdots \geq q_n \geq 0 \tag{2.30}$$

such that

$$y = y_L e^{\mathrm{i}Q(q)} y_R^{-1}. \tag{2.31}$$

If all components of q satisfy strict inequalities, then the pair y_L, y_R is unique precisely up to the replacements $(y_L, y_R) \to (y_L\zeta, y_R\zeta)$ with arbitrary $\zeta \in Z$. The decomposition (2.31) is referred to as the generalized Cartan decomposition corresponding to the involution Γ.

Second, every element $g \in G_-$ can be written in the form

$$g = \eta e^{2\mathrm{i}Q(q)} \eta^{-1} \tag{2.32}$$

with some $\eta \in G_+$ and uniquely determined $q \in \mathbb{R}^n$ subject to (2.30). In the case of strict inequalities for q, the freedom in η is given precisely by the replacements $\eta \to \eta\zeta, \forall\, \zeta \in Z$.

Third, every element $Y_- \in \mathcal{G}_-$ can be written in the form

$$Y_- = g_R \mathrm{i} D g_R^{-1}, \qquad D = \mathrm{diag}(d_1, \dots, d_n, -d_1, \dots, -d_n), \tag{2.33}$$

with $g_R \in G_+$ and uniquely determined real d_i satisfying

$$d_1 \geq \cdots \geq d_n \geq 0. \tag{2.34}$$

If the d_i satisfy strict inequalities, then the freedom in g_R is exhausted by the replacements $g_R \to g_R\zeta, \forall\, \zeta \in Z$.

The first and the second statements are essentially equivalent since the map

$$G \to G_-, \quad y \mapsto y^{-1} C y C \tag{2.35}$$

descends to a diffeomorphism from

$$G/G_+ = \{G_+ g \mid g \in G\} \tag{2.36}$$

onto G_-.[15]

III. THE SUTHERLAND PICTURE

We here exhibit a symplectomorphism between the reduced phase space $(P_{\mathrm{red}}, \Omega_{\mathrm{red}})$ and the Sutherland phase space

$$M = T^*C_1 = C_1 \times \mathbb{R}^n \tag{3.1}$$

equipped with its canonical symplectic form, where C_1 was defined in (1.2). As preparation, we associate with any $(q, p) \in M$ the $\mathcal{G}$-element

$$Y(q, p) := K(q, p) - \mathrm{i}\kappa C, \tag{3.2}$$

where $K(q, p)$ is the $N \times N$ matrix

$$\begin{aligned} K_{j,k} = -K_{n+j,n+k} &= \mathrm{i} p_j \delta_{j,k} - \mu(1 - \delta_{j,k})/\sin(q_j - q_k), \\ K_{j,n+k} = -K_{n+j,k} &= (\nu/\sin(2q_j) + \kappa \cot(2q_j))\delta_{j,k} + \mu(1 - \delta_{j,k})/\sin(q_j + q_k), \end{aligned} \tag{3.3}$$

with $j, k = 1, \dots, n$. We also introduce the N-component vector

$$V_{\mathbb{R}} := (\underbrace{1, \dots, 1}_{n \text{ times}}, \underbrace{-1, \dots, -1}_{n \text{ times}})^{\top}. \tag{3.4}$$

Notice from (2.12) that $K(q, p) \in \mathcal{G}_-$.

Throughout the paper we adopt the conditions (1.8) and take $\mu > 0$, although the next result requires only that the real parameters μ, ν, κ satisfy

$$\mu \neq 0 \quad \text{and} \quad |\nu| \neq |\kappa|. \tag{3.5}$$

Theorem 3.1. *Using the notations introduced in (2.13), (2.25), and (3.2), the subset S of the phase space P (2.17) given by*

$$S := \left\{ (e^{\mathrm{i}Q(q)}, Y(q,p), \upsilon^{\ell}_{\mu,\nu}(V_{\mathbb{R}}), \upsilon^r) \mid (q,p) \in M \right\}, \tag{3.6}$$

is a global cross-section for the action of $G_+ \times G_+$ on $P_0 = J^{-1}(0)$. Identifying P_{red} with S, the reduced symplectic form is equal to the Darboux form $\omega = \sum_{k=1}^n \mathrm{d}q_k \wedge \mathrm{d}p_k$. Thus, the obvious identification between S and M provides a symplectomorphism

$$(P_{\mathrm{red}}, \Omega_{\mathrm{red}}) \simeq (M, \omega). \tag{3.7}$$

Proof. We saw in Sec. II that the points of the level surface P_0 satisfy the equations

$$(yYy^{-1})_+ + \upsilon^{\ell}_{\mu,\nu}(V) = \mathbf{0}_N \quad \text{and} \quad -Y_+ - \mathrm{i}\kappa C = \mathbf{0}_N, \tag{3.8}$$

for some vector $V \in \mathbb{C}^N$ subject to $CV + V = 0$, $V^\dagger V = N$. Remember that the block-form of any Lie algebra element $Y \in \mathcal{G}$ is

$$Y = \begin{bmatrix} A & B \\ -B^\dagger & D \end{bmatrix} \quad \text{with} \quad A + A^\dagger = \mathbf{0}_n = D + D^\dagger, \quad B \in \mathbb{C}^{n\times n}. \tag{3.9}$$

Now the second constraint equation in (3.8) can be written as

$$2Y_+ = \begin{bmatrix} A + D & B - B^\dagger \\ B - B^\dagger & A + D \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n & -2\mathrm{i}\kappa\mathbf{1}_n \\ -2\mathrm{i}\kappa\mathbf{1}_n & \mathbf{0}_n \end{bmatrix} = -2\mathrm{i}\kappa C, \tag{3.10}$$

which implies that

$$D = -A \quad \text{and} \quad B^\dagger = B + 2\mathrm{i}\kappa\mathbf{1}_n. \tag{3.11}$$

Thus, every point of P_0 has $\mathcal{G}$-component Y of the form

$$Y = \begin{bmatrix} A & B \\ -B - 2\mathrm{i}\kappa\mathbf{1}_n & -A \end{bmatrix} \quad \text{with} \quad A + A^\dagger = \mathbf{0}_n, \quad B \in \mathbb{C}^{n\times n}. \tag{3.12}$$

By using the generalized Cartan decomposition (2.31) and applying a gauge transformation (the action of $G_+ \times G_+$ on P_0), we may assume that $y = e^{\mathrm{i}Q(q)}$ with some q satisfying (2.30). Then the first equation of the momentum map constraint (3.8) yields the matrix equation

$$\frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}Q(q)}Ye^{-\mathrm{i}Q(q)} + e^{-\mathrm{i}Q(q)}CYCe^{\mathrm{i}Q(q)}\right) + \mu(VV^\dagger - \mathbf{1}_N) + (\mu - \nu)C = \mathbf{0}_N. \tag{3.13}$$

If we introduce the notation $V = (u, -u)^\top$, $u \in \mathbb{C}^n$, and assume that Y has the form (3.12) then (3.13) turns into the following equations for A and B:

$$\frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}q}Ae^{-\mathrm{i}q} - e^{-\mathrm{i}q}Ae^{\mathrm{i}q}\right) + \mu(uu^\dagger - \mathbf{1}_n) = \mathbf{0}_n, \tag{3.14}$$

and

$$\frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}q}Be^{\mathrm{i}q} - e^{-\mathrm{i}q}Be^{-\mathrm{i}q}\right) - \kappa e^{-2\mathrm{i}q} - \mu uu^\dagger + (\mu - \nu)\mathbf{1}_n = \mathbf{0}_n. \tag{3.15}$$

Since $\mu \neq 0$, Eq. (3.14) implies that $|u_j|^2 = 1$ for all $j = 1, \ldots, n$. Therefore, we can apply a "residual" gauge transformation by an element $(g_L, g_R) = (e^{\mathrm{i}\xi(x)}, e^{\mathrm{i}\xi(x)})$, with suitable $e^{\mathrm{i}\xi(x)} \in Z$ (2.28) to transform $\upsilon^{\ell}_{\mu,\nu}(V)$ into $\upsilon^{\ell}_{\mu,\nu}(V_{\mathbb{R}})$. This amounts to setting $u_j = 1$ for all $j = 1, \ldots, n$. After having done this, we return to Eqs. (3.14) and (3.15). By writing out the equations entry-wise, we obtain that the diagonal components of A are arbitrary imaginary numbers (which we denote by $\mathrm{i}p_1, \ldots, \mathrm{i}p_n$) and we also obtain the following system of equations

$$\begin{aligned} A_{j,k} \sin(q_j - q_k) &= -\mu = -B_{j,k} \sin(q_j + q_k), \quad j \neq k, \\ B_{j,j} \sin(2q_j) &= \nu + \kappa \cos(2q_j) - \mathrm{i}\kappa \sin(2q_j), \quad j, k = 1, \ldots, n. \end{aligned} \tag{3.16}$$

So far we only knew that q satisfies $\pi/2 \geq q_1 \geq \ldots \geq q_n \geq 0$. By virtue of the conditions (3.5), the system (3.16) can be solved if and only if $\pi/2 > q_1 > \cdots > q_n > 0$. Substituting the unique solution for A and B back into (3.12) gives the formula $Y = Y(q, p)$ as displayed in (3.2).

The above arguments show that every gauge orbit in P_0 contains a point of S (3.6), and it is immediate by turning the equations backwards that every point of S belongs to P_0. By using that q satisfies strict inequalities and that all components of $V_{\mathbb{R}}$ are non-zero, it is also readily seen that no two different points of S are gauge equivalent. Moreover, the effectively acting symmetry group, which is given by

$$(G_+ \times G_+)/\mathrm{U}(1)_{\mathrm{diag}}, \tag{3.17}$$

where U(1) contains the scalar unitary matrices, acts *freely* on P_0.

It follows from the above that P_{red} is a smooth manifold diffeomorphic to M. Now the proof is finished by direct computation of the pull-back of the symplectic form Ω of P (2.17) onto the global cross-section S. □

Let us recall that the Abelian Poisson algebras $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ (2.23) consist of $(G_+ \times G_+)$-invariant functions on P, and thus descend to Abelian Poisson algebras on the reduced phase space P_{red}. In terms of the model $M \simeq S \simeq P_{\mathrm{red}}$, the Poisson algebra $\mathfrak{Q}^2_{\mathrm{red}}$ is obviously generated by the functions $(q, p) \mapsto \mathrm{tr}((-\mathrm{i}Y(q, p)))^m$ for $m = 1, \ldots, N$. It will be shown in Sec. IV that these functions vanish identically for the odd integers, and functionally independent generators of $\mathfrak{Q}^2_{\mathrm{red}}$ are provided by the functions

$$H_k(q, p) := \frac{1}{4k}\mathrm{tr}(-\mathrm{i}Y(q, p))^{2k}, \qquad k = 1, \ldots, n. \tag{3.18}$$

(In fact, we shall see that $Y(q, p)$ is conjugate to a diagonal matrix $\mathrm{i}\Lambda$ of the form in Eq. (4.7).) The first of these functions reads

$$\begin{aligned} H_1(q, p) = \frac{1}{4}\mathrm{tr}(-\mathrm{i}Y(q, p))^2 = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\mu^2}{\sin^2(q_j - q_k)} + \frac{\mu^2}{\sin^2(q_j + q_k)}\right) \\ + \frac{1}{2}\sum_{j=1}^{n}\frac{\nu\kappa}{\sin^2(q_j)} + \frac{1}{2}\sum_{j=1}^{n}\frac{(\nu-\kappa)^2}{\sin^2(2q_j)}. \end{aligned} \tag{3.19}$$

That is, upon the identification (1.7) it coincides with the Sutherland Hamiltonian (1.1). This implies the Liouville integrability of the Hamiltonian (1.1). Since its spectral invariants yield a commuting family of n independent functions in involution that include the Sutherland Hamiltonian, the Hermitian matrix function $-\mathrm{i}Y(q, p)$ (3.2) serves as a Lax matrix for the Sutherland system (M, ω, H).

As for the reduced Abelian Poisson algebra $\mathfrak{Q}^1_{\mathrm{red}}$, we notice that the cross-section S permits to identify it with the Abelian Poisson algebra of the smooth functions of the variables $q_1, \ldots, q_n$. This is so since the level set P_0 lies completely in the "regular part" of the phase space P, where the G-component y of $(y, Y, \upsilon^\ell, \upsilon^r)$ is such that $Q(q)$ in its decomposition (2.31) satisfies strict inequalities $\pi/2 > q_1 > \cdots > q_n > 0$. It is a well-known fact that in the regular part the components of q are smooth (actually real-analytic) functions of y (while globally they are only continuous functions). To see that every smooth function depending on $q \in C_1$ is contained in $\mathfrak{Q}^1_{\mathrm{red}}$, one may further use that every $(G_+ \times G_+)$-invariant smooth function on P_0 can be extended to an invariant smooth function on P. Indeed, this holds since $G_+ \times G_+$ is compact and $P_0 \subset P$ is a regular submanifold, which itself follows from the free action property established in the course of the proof of Theorem 3.1.

We can summarize the outcome of the foregoing discussion as follows. Below, the generators of Poisson algebras are understood in the functional sense, i.e., if some $f_1, \dots, f_n$ are generators then all smooth functions of them belong to the Poisson algebra.

Corollary 3.2. By using the model (M, ω) *of the reduced phase space* $(P_{\rm red}, \Omega_{\rm red})$ *provided by Theorem 3.1, the Abelian Poisson algebra* $\mathfrak{Q}^2_{\rm red}$ (2.23) *can be identified with the Poisson algebra generated by the spectral invariants* (3.18) *of the "Sutherland Lax matrix"* $-\mathrm{i}Y(q, p)$ (3.2), *which according to* (3.19) *include the many-body Hamiltonian* $H(q, p)$ (1.1), *and* $\mathfrak{Q}^1_{\rm red}$ *can be identified with the algebra generated by the corresponding position variables* q_i $(i = 1, \dots, n)$.

IV. THE DUAL PICTURE

It follows from the group-theoretic results quoted in Sec. II B that the Abelian Poisson algebra $\mathfrak{Q}^1$ is generated by the functions

$$\tilde{\mathcal{H}}_k(y, Y, \upsilon^\ell, \upsilon^r) := \frac{(-1)^k}{2k}\mathrm{tr}\big(y^{-1}CyC\big)^k, \quad k = 1, \dots, n, \tag{4.1}$$

and thus the unitary and Hermitian matrix

$$L := -y^{-1}CyC \tag{4.2}$$

serves as an "unreduced Lax matrix." It is readily seen in the Sutherland gauge (3.6) that these n functions remain functionally independent after reduction. Here, we shall prove that the evaluation of the invariant function $\tilde{\mathcal{H}}_1$ in another gauge reproduces the dual Hamiltonian (1.4). The reduction of the matrix function L will provide a Lax matrix for the corresponding integrable system. Before turning to details, we advance the group-theoretic interpretation of the dual position variable λ that features in the Hamiltonian (1.4), and sketch the plan of this section.

To begin, recall that on the constraint surface $Y = Y_- - \mathrm{i}\kappa C$, and for any $Y_- \in \mathcal{G}_-$ there is an element $g_R \in G_+$ such that

$$g_R^{-1}Y_-g_R = \mathrm{diag}(\mathrm{i}d_1, \dots, \mathrm{i}d_n, -\mathrm{i}d_1, \dots, -\mathrm{i}d_n) = \mathrm{i}D \in \mathcal{A} \quad \text{with} \quad d_1 \geq \cdots \geq d_n \geq 0. \tag{4.3}$$

Then introduce the real matrix $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ whose diagonal components are

$$\lambda_j := \sqrt{d_j^2 + \kappa^2}, \quad j \in \mathbb{N}_n. \tag{4.4}$$

In this equation, we use the notation $\mathbb{N}_n := \{1, \dots, n\}$, which will be frequently applied below together with $\mathbb{N}_N := \{1, \dots, N\}$. One can diagonalize the matrix $D - \kappa C$ by conjugation with the unitary matrix

$$h(\lambda) = \begin{bmatrix} \alpha(\boldsymbol{\lambda}) & \beta(\boldsymbol{\lambda}) \\ -\beta(\boldsymbol{\lambda}) & \alpha(\boldsymbol{\lambda}) \end{bmatrix}, \tag{4.5}$$

where the real functions $\alpha(x)$, $\beta(x)$ are defined on the interval $[|\kappa|, \infty) \subset \mathbb{R}$ by the formulae

$$\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 - \kappa^2}}}{\sqrt{2x}}, \quad \beta(x) = \kappa\frac{1}{\sqrt{2x}}\frac{1}{\sqrt{x + \sqrt{x^2 - \kappa^2}}}, \tag{4.6}$$

at least if $\kappa \neq 0$. If $\kappa = 0$, then we set $\alpha(x) = 1$ and $\beta(x) = 0$. Indeed, it is easy to check that

$$h(\lambda)\Lambda h(\lambda)^{-1} = D - \kappa C \quad \text{with} \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n). \tag{4.7}$$

Note that $h(\lambda)$ belongs to the subset G_- of G (2.10).

The above diagonalization procedure can be used to define the map

$$\mathfrak{L} : P_0 \to \mathbb{R}^n, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto \lambda. \tag{4.8}$$

This is clearly a continuous map, which descends to a continuous map $\mathfrak{L}_{\rm red} : P_{\rm red} \to \mathbb{R}^n$. One readily also sees that these maps are smooth (even real-analytic) on the open submanifolds $P_0^{\rm reg} \subset P_0$ and $P_{\rm red}^{\rm reg} \subset P_{\rm red}$, where the N eigenvalues of Y_- are pairwise different.

The image of the constraint surface P_0 under the map $\mathfrak{L}$ will turn out to be the closure of the domain

$$C_2 = \left\{ \lambda \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \lambda_a - \lambda_{a+1} > 2\mu, \\ (a = 1, \ldots, n-1) \end{array} \quad \text{and} \quad \lambda_n > \nu \right\}. \tag{4.9}$$

By solving the constrains through the diagonalization of Y, we shall construct a model of the open submanifold of P_{red} corresponding to the open submanifold $\mathfrak{L}^{-1}(C_2) \subset P_0$. This model will be symplectomorphic to the semi-global phase-space $C_2 \times \mathbb{T}^n$ of the dual Hamiltonian (1.4).

In Subsection IV A, we present the construction of the aforementioned model of $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$. The proof that also enlightens the origin of the construction is given in Subsection IV B. In Subsection IV C, we demonstrate that $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2)$ is a dense subset of P_{red} and finally, in Subsection IV D we present the global characterization of the dual model of P_{red}.

Many of the local formulae that appear in this section have analogues in Ref. 24–26, which inspired our considerations. However, the global structure is different.

A. The dual model of the open subset $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$

We first prepare some functions on $C_2 \times \mathbb{T}^n$. Denoting the elements of this domain as pairs

$$(\lambda, e^{\mathrm{i}\vartheta}) \quad \text{with} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in C_2, \quad e^{\mathrm{i}\vartheta} = (e^{\mathrm{i}\vartheta_1}, \ldots, e^{\mathrm{i}\vartheta_n}) \in \mathbb{T}^n, \tag{4.10}$$

we let

$$f_c := \left[1 - \frac{\nu}{\lambda_c}\right]^{\frac{1}{2}} \prod_{\substack{a=1 \\ (a \neq c)}}^{n} \left[1 - \frac{2\mu}{\lambda_c - \lambda_a}\right]^{\frac{1}{2}} \left[1 - \frac{2\mu}{\lambda_c + \lambda_a}\right]^{\frac{1}{2}}, \quad \forall c \in \mathbb{N}_n,$$

$$f_{n+c} := e^{\mathrm{i}\vartheta_c} \left[1 + \frac{\nu}{\lambda_c}\right]^{\frac{1}{2}} \prod_{\substack{a=1 \\ (a \neq c)}}^{n} \left[1 + \frac{2\mu}{\lambda_c - \lambda_a}\right]^{\frac{1}{2}} \left[1 + \frac{2\mu}{\lambda_c + \lambda_a}\right]^{\frac{1}{2}}. \tag{4.11}$$

For $\lambda \in C_2$ (4.9), all factors under the square roots are positive. Using the column vector $f := (f_1, \ldots, f_{2n})^\top$ together with $\Lambda_c = \lambda_c$ and $\Lambda_{c+n} = -\lambda_c$ for $c \in \mathbb{N}_n$, we define the $N \times N$ matrices $\check{A}(\lambda, \vartheta)$ and $B(\lambda, \vartheta)$ by

$$\check{A}_{j,k} = \frac{2\mu f_j \overline{(Cf)}_k - 2(\mu - \nu) C_{j,k}}{2\mu + \Lambda_k - \Lambda_j}, \quad j, k \in \mathbb{N}_N, \tag{4.12}$$

and

$$B(\lambda, \vartheta) := -\big(h(\lambda)\check{A}(\lambda, \vartheta)h(\lambda)\big)^\dagger. \tag{4.13}$$

We shall see that these are unitary matrices from $G_- \subset G$ (2.10). Then we write B in the form

$$B = \eta e^{2\mathrm{i}Q(q)} \eta^{-1} \tag{4.14}$$

with some $\eta \in G_+$ and unique $q = q(\lambda, \vartheta)$ subject to (2.30). (It turns out that $q(\lambda, \vartheta) \in C_1$ (1.2) and thus η is unique up to replacements $\eta \to \eta\zeta$ with arbitrary $\zeta \in Z$ (2.28).) Relying on (4.14), we set

$$y(\lambda, \vartheta) := \eta e^{\mathrm{i}Q(q(\lambda, \vartheta))} \eta^{-1} \tag{4.15}$$

and introduce the vector $V(\lambda, \vartheta) \in \mathbb{C}^N$ by

$$V(\lambda, \vartheta) := y(\lambda, \vartheta) h(\lambda) f(\lambda, \vartheta). \tag{4.16}$$

It will be shown that $V + CV = 0$ and $|V|^2 = N$, which ensures that $\upsilon_{\mu,\nu}^{\ell}(V) \in \mathcal{O}^{\ell}$ (2.14).

Note that $\check{A}$, y, and V given above depend on ϑ only through $e^{\mathrm{i}\vartheta}$ and are C^∞ functions on $C_2 \times \mathbb{T}^n$. It should be remarked that although the matrix element $\check{A}_{n,2n}$ (4.12) has an apparent singularity at $\lambda_n = \mu$, the zero of the denominator cancels. Thus, $\check{A}$ extends by continuity to $\lambda_n = \mu$ and remains smooth there, which then also implies the smoothness of y and V.

Theorem 4.1. *By using the above notations, consider the set*

$$\tilde{S}^0 := \{(y(\lambda, \vartheta), \mathrm{i}h(\lambda)\Lambda(\lambda)h(\lambda)^{-1}, \upsilon^\ell_{\mu,\nu}(V(\lambda, \vartheta)), \upsilon^r) \mid (\lambda, e^{\mathrm{i}\vartheta}) \in C_2 \times \mathbb{T}^n\}. \tag{4.17}$$

This set is contained in the constraint surface $P_0 = J^{-1}(0)$ and it provides a cross-section for the $G_+ \times G_+$-action restricted to $\mathfrak{L}^{-1}(C_2) \subset P_0$. In particular, $C_2 \subset \mathfrak{L}(P_0)$ and $\tilde{S}^0$ intersects every gauge orbit in $\mathfrak{L}^{-1}(C_2)$ precisely in one point. Since the elements of $\tilde{S}^0$ are parametrized by $C_2 \times \mathbb{T}^n$ in a smooth and bijective manner, we obtain the identifications

$$\mathfrak{L}^{-1}_{\mathrm{red}}(C_2) \simeq \tilde{S}^0 \simeq C_2 \times \mathbb{T}^n. \tag{4.18}$$

Letting $\tilde{\sigma}_0 : \tilde{S}^0 \to P$ denote the tautological injection, the pull-backs of the symplectic form Ω (2.17) and the function $\tilde{\mathcal{H}}_1$ (4.1) obey

$$\tilde{\sigma}_0^*(\Omega) = \sum_{c=1}^n \mathrm{d}\lambda_c \wedge \mathrm{d}\vartheta_c, \qquad (\tilde{\mathcal{H}}_1 \circ \tilde{\sigma}_0)(\lambda, \vartheta) = \frac{1}{2}\mathrm{tr}\big(h(\lambda)\check{A}(\lambda, \vartheta)h(\lambda)\big) = \tilde{H}^0(\lambda, \vartheta) \tag{4.19}$$

with the RSvD type Hamiltonian $\tilde{H}^0$ in (1.4). Consequently, the Hamiltonian reduction of the system $(P, \Omega, \tilde{\mathcal{H}}_1)$ followed by restriction to the open submanifold $\mathfrak{L}^{-1}_{\mathrm{red}}(C_2) \subset P_{\mathrm{red}}$ reproduces the system $(\tilde{M}^0, \tilde{\omega}^0, \tilde{H}^0)$ defined in the Introduction.

Remark 4.2. Referring to (4.2), we have the Lax matrix

$$L(y(\lambda, \vartheta)) = h(\lambda)\check{A}(\lambda, \vartheta)h(\lambda). \tag{4.20}$$

Later we shall also prove that $\mathfrak{L}^{-1}_{\mathrm{red}}(C_2)$ is a dense subset of P_{red}, whereby the reduction of $(P, \Omega, \tilde{\mathcal{H}}_1)$ may be viewed as a completion of $(\tilde{M}^0, \tilde{\omega}^0, \tilde{H}^0)$.

B. Proof of Theorem 4.1

The proof will emerge from a series of lemmas. Our immediate aim is to construct gauge invariant functions that will be used for parametrizing the orbits of $G_+ \times G_+$ in (an open submanifold of) P_0. For introducing gauge invariants, we can restrict ourselves to the submanifold $P_1 \subset P_0$ where Y in $(y, Y, \upsilon^\ell, \upsilon^r)$ has the form

$$Y = h(\lambda)\mathrm{i}\Lambda(\lambda)h(\lambda)^{-1} \tag{4.21}$$

with some $\lambda \in \mathbb{R}^n$ for which

$$\lambda_1 \geq \cdots \geq \lambda_n \geq |\kappa|. \tag{4.22}$$

Indeed, every element of P_0 can be gauge transformed into P_1. It will be advantageous to further restrict attention to $P_1^{\mathrm{reg}} \subset P_1$ where we have

$$\lambda_1 > \cdots > \lambda_n > |\kappa|. \tag{4.23}$$

The residual gauge transformations that map P_1^{reg} to itself belong to the group $G_+ \times Z < G_+ \times G_+$ with Z defined in (2.28). Since υ^r is constant and $\upsilon^\ell = \upsilon^\ell_{\mu,\nu}(V)$, we may label the elements of P_1 by triples (y, Y, V), with the understanding that V matters up to phase. Then the gauge action of $(g_L, \zeta) \in G_+ \times Z$ operates by

$$(y, V) \mapsto (g_L y \zeta^{-1}, g_L V), \tag{4.24}$$

while Y is already invariant. Now we can factor out the residual G_+-action by introducing the G_--valued function

$$\check{A}(y, Y, V) := h(\lambda)^{-1}L(y)h(\lambda)^{-1} \tag{4.25}$$

and the $\mathbb{C}^N$-valued function

$$F(y, Y, V) := h(\lambda)^{-1} y^{-1} V. \tag{4.26}$$

Here, $\lambda = \mathfrak{L}(y, Y, V)$, which means that (4.21) holds, and we used $L(y)$ in (4.2). Like V, F is defined only up to a U(1) phase. We obtain the transformation rules

$$\check{A}(g_L y \zeta^{-1}, Y, g_L V) = \zeta \check{A}(y, Y, V)\zeta^{-1}, \tag{4.27}$$

$$F(g_L y \zeta^{-1}, Y, g_L V) = \zeta F(y, Y, V), \tag{4.28}$$

and therefore the functions

$$\mathcal{F}_k(y, Y, V) := |F_k(y, Y, V)|^2, \quad k = 1, \ldots, N \tag{4.29}$$

are well-defined, gauge invariant, smooth functions on P_1^{reg}. They represent $(G_+ \times G_+)$-invariant smooth functions on P_0^{reg}. We shall see shortly that the functions $\mathcal{F}_k$ depend only on $\lambda = \mathfrak{L}(y, Y, V)$ and shall derive explicit formulae for this dependence. Then the non-negativity of $\mathcal{F}_k$ will be used to gain information about the set $\mathfrak{L}(P_0)$ of λ values that actually occurs.

Before turning to the inspection of the functions $\mathcal{F}_k$, we present a crucial lemma.

Lemma 4.3. Fix $\lambda \in \mathbb{R}^n$ subject to (4.22) and set $\Lambda := \mathrm{diag}(\lambda, -\lambda)$ and $Y := h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}$. If $y \in G$ and $\upsilon^\ell_{\mu,\nu}(V) \in \mathcal{O}^\ell$ solve the momentum map constraint given according to the first equation in (3.8) *by*

$$yYy^{-1} + CyYy^{-1}C + 2\upsilon^\ell_{\mu,\nu}(V) = 0, \tag{4.30}$$

then $\check{A} \in G_-$ and $F \in \mathbb{C}^N$ defined by (4.25) and (4.26) solve the following equation:

$$2\mu\check{A} + \check{A}\Lambda - \Lambda\check{A} = 2\mu F(CF)^\dagger - 2(\mu - \nu)C. \tag{4.31}$$

Conversely, for any $\check{A} \in G_-$, $F \in \mathbb{C}^N$ that satisfy $|F|^2 = N$ and Eq. (4.31), pick $y \in G$ such that $L(y) = h(\lambda)\check{A}h(\lambda)$ and define $V := yh(\lambda)F$. Then $CV + V = 0$ and $(y, Y, \upsilon^\ell_{\mu,\nu}(V))$ solve the momentum map constraint (4.30).

Proof. If Eq. (4.30) holds, then we multiply it by $h(\lambda)^{-1}y^{-1}$ on the left and by $CyCh(\lambda)^{-1}$ on the right. Using (3.13), with $CV + V = 0$ and $|V|^2 = N$, and the notations (4.25) and (4.26), this immediately gives (4.31). Conversely, suppose that (4.31) holds for some $\check{A} \in G_-$ and $F \in \mathbb{C}^N$ with $|F|^2 = N$. Since $h(\lambda)\check{A}h(\lambda)$ belongs to G_-, there exists $y \in G$ such that

$$h(\lambda)\check{A}h(\lambda) = L(y). \tag{4.32}$$

Such y is unique up to left-multiplication by an arbitrary element of G_+ (whereby one may bring y into G_- if one wishes to do so). Picking y according to (4.32), and then setting

$$V := yh(\lambda)F, \tag{4.33}$$

it is an elementary matter to show that (4.31) implies the following equation:

$$yYy^{-1} + CyYy^{-1}C + 2\mathrm{i}\mu(-V(CV)^\dagger - \mathbf{1}_N) + 2\mathrm{i}(\mu - \nu)C = 0. \tag{4.34}$$

It is a consequence of this equation that

$$(V(CV)^\dagger)^\dagger = (CV)V^\dagger = V(CV)^\dagger. \tag{4.35}$$

This entails that $CV = \alpha V$ for some $\alpha \in \mathrm{U}(1)$. Then $V^\dagger = \alpha(CV)^\dagger$ also holds, and thus we must have $\alpha^2 = 1$. Hence, α is either $+1$ or -1. Taking the trace of the equality (4.34), and using that $|V|^2 = N$ on account of $|F|^2 = N$, we obtain that $\alpha = -1$, i.e., $CV + V = 0$. This means that Eq. (4.34) reproduces (4.30). □

To make progress, now we restrict our attention to the subset of $P_1^{\rm reg}$ where the eigenvalue-parameter λ of Y verifies in addition to (4.23) also the conditions

$$|\lambda_a \pm \lambda_b| \neq 2\mu \quad \text{and} \quad (\lambda_a - \nu)(\lambda_a - |2\mu - \nu|) \neq 0, \quad \forall a, b \in \mathbb{N}_n. \tag{4.36}$$

We call such λ values "strongly regular," and let $P_1^{\rm sreg} \subset P_1$ and $P_0^{\rm sreg} \subset P_0$ denote the corresponding open subsets. Later we shall prove that $P_0^{\rm sreg}$ is *dense* in P_0. The above conditions will enable us to perform calculations that will lead to a description of a dense subset of the reduced phase space. They ensure that we never divide by zero in relevant steps of our arguments. The first such step is the derivation of the following consequence of Eq. (4.31).

Lemma 4.4. The restriction of the matrix function $\check{A}$ *(4.25) to* $P_1^{\rm sreg}$ *has the form*

$$\check{A}_{j,k} = \frac{2\mu F_j \overline{(CF)}_k - 2(\mu - \nu)C_{j,k}}{2\mu + \Lambda_k - \Lambda_j}, \quad j, k \in \mathbb{N}_N, \tag{4.37}$$

where $F \in \mathbb{C}^N$ *satisfies* $|F|^2 = N$ *and* $\Lambda = \mathrm{diag}(\lambda, -\lambda)$ *varies on* $P_1^{\rm sreg}$ *according to (4.21).*

Lemma 4.5. For any strongly regular λ *and* $a \in \mathbb{N}_n$ *define*

$$\begin{aligned} w_a &:= \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \frac{(\lambda_a - \lambda_b)(\lambda_a + \lambda_b)}{(2\mu - (\lambda_a - \lambda_b))(2\mu - (\lambda_a + \lambda_b))}, \\ w_{a+n} &:= \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \frac{(\lambda_a - \lambda_b)(\lambda_a + \lambda_b)}{(2\mu + \lambda_a - \lambda_b)(2\mu + \lambda_a + \lambda_b)}, \end{aligned} \tag{4.38}$$

and set $W_k := w_k \mathcal{F}_k$ *with* $\mathcal{F}_k = |F_k|^2$*. Then the unitarity of the matrix* $\check{A}$ *as given by (4.37) implies the following system of equations for the pairs of functions* W_c *and* W_{c+n} *for any* $c \in \mathbb{N}_n$*:*

$$(\mu + \lambda_c)W_c + (\mu - \lambda_c)W_{n+c} - 2(\mu - \nu) = 0, \tag{4.39}$$

$$\lambda_c^2 W_c W_{n+c} - \mu(\mu - \nu)(W_c + W_{n+c}) + (\mu - \nu)^2 + \mu^2 - \lambda_c^2 = 0. \tag{4.40}$$

For fixed $c \in \mathbb{N}_n$ *and strongly regular* λ*, this system of equations admits two solutions, which are given by*

$$(W_c, W_{n+c}) = (W_c^+, W_{n+c}^+) = (w_c \mathcal{F}_c^+, w_{c+n}\mathcal{F}_{c+n}^+) = (1 + \frac{\nu}{\lambda_c}, 1 - \frac{\nu}{\lambda_c}), \tag{4.41}$$

and by

$$(W_c, W_{n+c}) = (W_c^-, W_{n+c}^-) = (w_c \mathcal{F}_c^-, w_{c+n}\mathcal{F}_{c+n}^-) = (-1 + \frac{2\mu - \nu}{\lambda_c}, -1 - \frac{2\mu - \nu}{\lambda_c}). \tag{4.42}$$

The functions $\mathcal{F}_k^\pm$ *satisfy the identities*

$$\sum_{k=1}^{N} \mathcal{F}_k^+(\lambda) = N \quad \text{and} \quad \sum_{k=1}^{N} \mathcal{F}_k^-(\lambda) = -N. \tag{4.43}$$

Proof. The derivation of Eqs. (4.39) and (4.40) follows a similar derivation due to Pusztai,[24] and is summarized in the Appendix. We then solve the linear equation (4.39) say for W_{c+n} and substitute it into (4.40). This gives a quadratic equation for W_c whose two solutions we can write down. We note that the derivation of Eqs. (4.39) and (4.40) presented in the Appendix utilizes the full set of the conditions (4.36).

To verify the identities (4.43), we first extend λ to vary in the open subset of $\mathbb{C}^n$ subject to the conditions $\lambda_a^2 \neq \lambda_b^2$ and $\lambda_c \neq 0$, and then consider the sums that appear in (4.43) as functions of a chosen component of λ with the other components fixed. These explicitly given sums are meromorphic functions having only first order poles, and one may check that all residues at the apparent poles vanish. Hence, the sums are constant over $\mathbb{C}^n$, and the values of the constants can be established by looking at a suitable asymptotic limit in the domain C_2 (4.9), whereby all w_k tend to 1 and the pre-factors in (4.41) and (4.42) tend to 1 and -1, respectively. □

Observe that neither any w_k nor any $\mathcal{F}_k^{\pm}$ ($k \in \mathbb{N}_N$) can vanish if λ is strongly regular. We know that the value of $\mathcal{F}_k$ (4.29) is uniquely defined at every point of P_1^{reg}. Therefore, only one of the solutions $(\mathcal{F}_c^{\pm}, \mathcal{F}_{c+n}^{\pm})$ can be acceptable at any $\lambda \in \mathfrak{L}(P_1^{\text{sreg}})$. The identities in (4.43) and analyticity arguments strongly suggest that the acceptable solutions are provided by $\mathcal{F}_k^{+}$. The first statement of the following lemma confirms that this is the case for $\lambda \in C_2$ (4.9).

Lemma 4.6. The formulae (4.41) and (4.42) can be used to define $\mathcal{F}_k^{\pm}$ as smooth real functions on the domain C_2, and none of these functions vanishes at any $\lambda \in C_2$. Then for any $\lambda \in C_2$ and $c \in \mathbb{N}_n$ at least one out of $\mathcal{F}_c^{-}$ and $\mathcal{F}_{c+n}^{-}$ is negative, while $\mathcal{F}_k^{+} > 0$ for all $k \in \mathbb{N}_N$. Hence, for $\lambda \in C_2 \cap \mathfrak{L}(P_0)$ only $\mathcal{F}_k^{+}(\lambda)$ can give the value of the function $\mathcal{F}_k$ as defined in (4.29). Taking any $\lambda \in C_2$ and any $F \in \mathbb{C}^N$ satisfying $|F_k|^2 = \mathcal{F}_k^{+}(\lambda)$, the formula (4.37) yields a unitary matrix that belongs to G_- (2.10). This matrix $\check{A}$ and vector $F \in \mathbb{C}^N$ solve Eq. (4.31).

Proof. It is easily seen that $w_k(\lambda) > 0$ for all $\lambda \in C_2$ and $k \in \mathbb{N}_N$. The statement about the negativity of either $\mathcal{F}_c^{-}$ or $\mathcal{F}_{c+n}^{-}$ thus follows from the identity $W_c^{-} + W_{n+c}^{-} = -2$. The positivity of $\mathcal{F}_k^{+}$ is easily checked. It is also readily verified that $\check{A}^{\dagger} = C\check{A}C$, which entails that $\check{A} \in G_-$ once we know that $\check{A}$ is unitary. For $\lambda \in C_2$ and $|F_k|^2 = \mathcal{F}_k^{+}(\lambda)$, the unitarity of $\check{A}$ (4.37) can be shown by almost verbatim adaptation of the arguments proving Proposition 6 in Ref. 25.

If $\lambda \in C_2$ is such that the denominators in (4.37) do not vanish, then the formula (4.37) is plainly equivalent to (4.31). Observe that only those elements $\lambda \in C_2$ for which $\lambda_n = \mu$ fail to satisfy this condition. At such λ the matrix element $\check{A}_{n,2n}$ has an apparent "first order pole," but one can check by inspection of the formula (4.12) that $\check{A}_{n,2n}$ actually remains finite and smooth even at such exceptional points, and thus solves also (4.31) because of continuity. □

Before presenting the proof of Theorem 4.1, note that at the point of $\tilde{S}^0$ labeled by $(\lambda, e^{\mathrm{i}\vartheta})$ the value of the function F (4.26) is equal to $f(\lambda, e^{\mathrm{i}\vartheta})$ given in (4.11).

Proof of Theorem 4.1. It follows from Lemma 4.3 and Lemma 4.6 that $\tilde{S}^0$ is a subset of P_1^{reg} and $\mathfrak{L}(\tilde{S}^0) = C_2$. Taking into account Theorem 3.1, this implies that $y(\lambda, \vartheta)$ (4.15) and $V(\lambda, \vartheta)$ (4.16) are well-defined smooth functions on $C_2 \times \mathbb{T}^n$. We next show that $\tilde{S}^0$ is a cross-section for the residual gauge action on $\mathfrak{L}^{-1}(C_2) \cap P_1$. To do this, pick an arbitrary element

$$(\tilde{y}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \upsilon^{\ell}_{\mu,\nu}(\tilde{V}), \upsilon^r) \in \mathfrak{L}^{-1}(C_2) \cap P_1. \tag{4.44}$$

Because $\mathcal{F}_k(\lambda) \neq 0$, we can find a unique element $e^{\mathrm{i}\vartheta} \in \mathbb{T}^n$ and an element $\zeta \in Z$ (2.28) (which is unique up to scalar multiple) such that

$$F_k(\tilde{y}\zeta^{-1}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \tilde{V}) = f_k(\lambda, e^{\mathrm{i}\vartheta}), \qquad \forall k \in \mathbb{N}_N, \tag{4.45}$$

up to a k-independent phase. We then see from (4.31) that $L(\tilde{y}\zeta^{-1}) = L(y(\lambda, \vartheta))$, which in turn implies the existence of some (unique after ζ was chosen) $\eta_+ \in G_+$ for which

$$\eta_+\tilde{y}\zeta^{-1} = y(\lambda, \vartheta). \tag{4.46}$$

Using also that $\zeta^{-1}h(\lambda)\zeta = h(\lambda)$, we conclude from the last two equations that

$$\eta_+\tilde{V} = \eta_+\tilde{y}h(\lambda)F(\tilde{y}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \tilde{V}) = y(\lambda, \vartheta)h(\lambda)f(\lambda, \vartheta) = V(\lambda, e^{\mathrm{i}\vartheta}). \tag{4.47}$$

Thus, we have shown that the element (4.44) can be gauge transformed into a point of $\tilde{S}^0$, and this point is uniquely determined since (4.45) fixes $e^{\mathrm{i}\vartheta}$ uniquely. In other words, $\tilde{S}^0$ intersects every orbit of the residual gauge action on $\mathfrak{L}^{-1}(C_2) \cap P_1$ in precisely one point.

The map from C_2 into P, given by the parametrization of $\tilde{S}^0$, is obviously smooth, and hence we obtain the identifications

$$C_2 \simeq \tilde{S}^0 \simeq (\mathfrak{L}^{-1}(C_2) \cap P_1)/(G_+ \times Z) \simeq \mathfrak{L}^{-1}(C_2)/(G_+ \times G_+) = \mathfrak{L}_{\rm red}^{-1}(C_2). \tag{4.48}$$

To establish the formula (4.19) of the reduced symplectic structure, we proceed as follows. We define $G_+ \times G_+$ invariant real functions on P by

$$\varphi_m(y, Y, V) := \frac{1}{m}\mathrm{Re}\big(\mathrm{tr}(Y^m)\big), \quad m \in \mathbb{N}, \tag{4.49}$$

and

$$\chi_k(y, Y, \upsilon) := \mathrm{Re}\big(\mathrm{tr}(Y^k y^{-1} V V^\dagger y C)\big), \quad k \in \mathbb{N} \cup \{0\}. \tag{4.50}$$

The restrictions of these functions to $\tilde{S}^0$ are the respective functions $\varphi_m^{\rm red}$ and $\chi_k^{\rm red}$

$$\varphi_m^{\rm red}(\lambda, \vartheta) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (-1)^{\frac{m}{2}} \dfrac{2}{m} \displaystyle\sum_{j=1}^n \lambda_j^m, & \text{if } m \text{ is even,} \end{cases} \tag{4.51}$$

and

$$\chi_k^{\rm red}(\lambda, \vartheta) = \begin{cases} -2(-1)^{\frac{k-1}{2}} \displaystyle\sum_{j=1}^n \lambda_j^k \left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} X_j \sin(\vartheta_j), & \text{if } k \text{ is odd,} \\ 2(-1)^{\frac{k}{2}} \displaystyle\sum_{j=1}^n \lambda_j^k \left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} X_j \cos(\vartheta_j) - \kappa \lambda_j^{k-1}\big(\mathcal{F}_j - \mathcal{F}_{n+j}\big), & \text{if } k \text{ is even,} \end{cases} \tag{4.52}$$

where $X_j = \sqrt{\mathcal{F}_j \mathcal{F}_{n+j}}$. Then we calculate the pairwise Poisson brackets of the set of functions φ_m, χ_k on P and restrict the results to $\tilde{S}^0$. This must coincide with the results of the direct calculation of the Poisson brackets of the reduced functions $\varphi_m^{\rm red}$, $\chi_k^{\rm red}$ based on the pull-back of the symplectic form Ω onto $\tilde{S}^0 \subset P$. Inspection shows that the required equalities hold if and only if we have the formula in (4.19) for the pull-back in question. This reasoning is very similar to that used in Ref. 25 to find the corresponding reduced symplectic form. Since the underlying calculations are straightforward, although rather laborious, we here omit the details. As for the formula for the restriction of $\tilde{\mathcal{H}}_1$ to $\tilde{S}^0$ displayed in (4.19), this is a matter of direct verification. □

C. Density properties

So far we dealt with the open subset $\mathfrak{L}_{\rm red}^{-1}(C_2)$ of the reduced phase space. Here, we show that Theorem 4.1 contains "almost all" information about the dual system since $\mathfrak{L}_{\rm red}^{-1}(C_2) \subset P_{\rm red}$ is a *dense* subset. This key result will be proved by combining two lemmas.

Lemma 4.7. The subset $P_0^{\rm sreg} \subset P_0$ of the constraint surface where the range of the eigenvalue map $\mathfrak{L}$ (4.8) satisfies the conditions (4.23) *and* (4.36) *is dense.*

Proof. Let us first of all note that P_0 is a connected regular analytic submanifold of P. In fact, it is a regular (embedded) analytic submanifold of the analytic manifold P since the momentum map is analytic and zero is its regular value (because the effectively acting gauge group (3.17) acts freely on P_0). The connectedness follows from Theorem 3.1, which implies that P_0 is diffeomorphic to the product of S (3.6) and the group (3.17), and both are connected.

For any $Y \in \mathcal{G}$ denote by $\{\mathrm{i}\Lambda_a\}_{a=1}^N$ the set of its eigenvalues counted with multiplicities. Then the following formulae

$$\mathcal{R}(y, Y, V) := \prod_{\substack{a,b=1 \\ (a \neq b)}}^N (\Lambda_a - \Lambda_b) \prod_{a=1}^N (\Lambda_a^2 - \kappa^2), \tag{4.53}$$

$$\mathcal{S}(y, Y, V) := \prod_{\substack{a,b=1\\(a\neq b)}}^{N} [(\Lambda_a - \Lambda_b)^2 - 4\mu^2] \prod_{a=1}^{N} \left[(\Lambda_a^2 - \mu^2)(\Lambda_a^2 - \nu^2)(\Lambda_a^2 - (2\mu - \nu)^2)\right], \tag{4.54}$$

define analytic functions on P_0. Indeed, $\mathcal{R}$ and $\mathcal{S}$ are symmetric polynomials in the eigenvalues of Y, and hence can be expressed as polynomials in the coefficients of the characteristic polynomial of Y, which are polynomials in the matrix elements of Y. The product $\mathcal{RS}$ is also an analytic function on P_0, and the subset P_0^{sreg}, which we considered in Subsection IV B, can be characterized as

$$P_0^{\text{sreg}} = \{x \in P_0 \mid \mathcal{R}(x)\mathcal{S}(x) \neq 0\}. \tag{4.55}$$

It is clear from Theorem 4.1 that $\mathcal{RS}$ does not vanish identically on P_0. Since the zero set of a non-zero analytic function on a connected analytic manifold cannot contain any open set, Eq. (4.55) implies that P_0^{sreg} is a dense subset of P_0. □

Let $\overline{C}_2$ be the closure of the domain $C_2 \subset \mathbb{R}^n$. Eventually, it will turn out that $\mathfrak{L}(P_0) = \overline{C}_2$. For now, we wish to prove the following.

Lemma 4.8. For every boundary point $\lambda^0 \in \partial\overline{C}_2$, there exists an open ball $B(\lambda^0) \subset \mathbb{R}^n$ around λ^0 that does not contain any strongly regular λ which lies outside $\overline{C}_2$ and belongs to $\mathfrak{L}(P_0)$.

Proof. We start by noticing that for any boundary point $\lambda^0 \in \partial\overline{C}_2$ there is a ball $B(\lambda^0)$ centered at λ^0 such that any strongly regular $\lambda \in B(\lambda^0) \setminus \overline{C}_2$ is subject to either of the following: (i) there is an index $a \in \{1, \dots, n-1\}$ such that

$$\lambda_a - \lambda_{a+1} < 2\mu \quad \text{and} \quad \lambda_b - \lambda_{b+1} > 2\mu \quad \forall\, b < a, \tag{4.56}$$

or (ii) we have

$$\lambda_a - \lambda_{a+1} > 2\mu, \quad a = 1, \dots, n-1 \quad \text{and} \quad \lambda_n < \nu. \tag{4.57}$$

Let us consider a strongly regular $\lambda \in B(\lambda^0)$ that falls into case (i) (4.56) and is so close to C_2 that we still have

$$\lambda_k - \lambda_{k+1} > \mu, \quad \forall\, k \in \{1, \dots, n-1\}. \tag{4.58}$$

It then follows that

$$\lambda_a - \lambda_b > 2\mu, \quad \forall\, b > a+1, \tag{4.59}$$

and

$$\lambda_a + \lambda_b > 2\mu, \quad \forall\, b \in \{1, \dots, n\}. \tag{4.60}$$

Inspection of the signs of $w_a(\lambda)$ and $w_{a+n}(\lambda)$ in (4.38) gives

$$w_a(\lambda) < 0 < w_{a+n}(\lambda). \tag{4.61}$$

Since every boundary point $\lambda^0 \in \partial\overline{C}_2$ satisfies $\lambda_a^0 > \lambda_n^0 \geq \nu$ for all $a \in \{1, \dots, n-1\}$, we may choose a small enough ball centered at λ^0 to ensure that for λ inside that ball the above inequalities as well as $\lambda_a > \nu$ hold. On account of $\lambda_a > \nu > 0$ and $\mu > 0$ we then have

$$1 - \frac{\nu}{\lambda_a} > 0 \quad \text{and} \quad -1 - \frac{2\mu - \nu}{\lambda_a} < 0. \tag{4.62}$$

By combining (4.41) and (4.42) with (4.61) and (4.62) we conclude that

$$\mathcal{F}_a^+(\lambda) < 0 \quad \text{and} \quad \mathcal{F}_{a+n}^-(\lambda) < 0. \tag{4.63}$$

By Lemma 4.5, these inequalities imply that $\mathcal{F}_a(\lambda)$ and $\mathcal{F}_{a+n}(\lambda)$ cannot be both non-negative, which contradicts the defining Eq. (4.29). This proves the claim in the case (i) (4.56).

Let us consider a strongly regular λ satisfying (ii) (4.57). In this case, we can verify that

$$1 - \frac{\nu}{\lambda_n} < 0, \quad w_n(\lambda) > 0, \quad w_{n+a}(\lambda) > 0. \tag{4.64}$$

Thus, we see from (4.41) that $\mathcal{F}_{2n}^+(\lambda) < 0$. Since the sum of the two components on the right hand side of (4.42) is negative, we also see that at least one out of $\mathcal{F}_n^-(\lambda)$ and $\mathcal{F}_{2n}^-(\lambda)$ is negative. Therefore, Eqs. (4.39) and (4.40) exclude the unitarity of $\check{A}$ (4.37) in the case (ii) (4.57) as well. □

Proposition 4.9. The λ-image of the constraint surface is contained in $\overline{C}_2$, i.e., we have

$$\mathfrak{L}(P_0) \subseteq \overline{C}_2. \tag{4.65}$$

As a consequence, $\mathfrak{L}_{\rm red}^{-1}(C_2)$ is dense in $P_{\rm red}$.

Proof. Since $P_0^{\rm sreg} \subset P_0$ is dense and $\mathfrak{L} : P_0 \to \mathbb{R}^n$ (4.8) is continuous, $\mathfrak{L}(P_0^{\rm sreg}) \subset \mathfrak{L}(P_0)$ is dense. Thus, it follows from Lemma 4.8 that for any $\lambda^0 \in \partial C_2$ there exists a ball around λ^0 that does not contain *any* element of $\mathfrak{L}(P_0)$ lying outside $\overline{C}_2$.

Suppose that (4.65) is not true, which means that there exists some $\lambda^* \in \mathfrak{L}(P_0) \setminus \overline{C}_2$. Taking any element $\hat{\lambda} \in \mathfrak{L}(P_0)$ that lies in C_2, it is must be possible to connect λ^* to $\hat{\lambda}$ by a continuous curve in $\mathfrak{L}(P_0)$, since P_0 is connected. Starting from the point λ^*, any such continuous curve must pass through some point of the boundary ∂C_2. However, this is impossible since we know that $\mathfrak{L}(P_0) \setminus \overline{C}_2$ does not contain any series that converges to a point of ∂C_2. This contradiction shows that (4.65) holds.

By (4.65) we have $P_0^{\rm sreg} \subset \mathfrak{L}^{-1}(C_2)$, and we know from Lemma 4.7 that $P_0^{\rm sreg} \subset P_0$ is dense. These together entail that $\mathfrak{L}_{\rm red}^{-1}(C_2) \subset P_{\rm red}$ is dense. □

D. Global characterization of the dual system

We have seen that

$$P_0^{\rm sreg} \subset \mathfrak{L}^{-1}(C_2) \subset P_0 \tag{4.66}$$

is a chain of dense open submanifolds. These project onto dense open submanifolds of $P_{\rm red}$ and their images under the map $\mathfrak{L}$ (4.8) are dense subsets of $\mathfrak{L}(P_0) = \mathfrak{L}_{\rm red}(P_{\rm red})$

$$\mathfrak{L}(P_0^{\rm sreg}) \subset C_2 \subset \mathfrak{L}(P_0). \tag{4.67}$$

Now introduce the set

$$\mathbb{C}^n_{\neq} := \{z \in \mathbb{C}^n \mid \prod_{k=1}^n z_k \neq 0\}. \tag{4.68}$$

The parametrization

$$z_j = \sqrt{\lambda_j - \lambda_{j+1} - 2\mu} \prod_{a=1}^j e^{\mathrm{i}\vartheta_a}, \quad j = 1, \dots, n-1, \qquad z_n = \sqrt{\lambda_n - \nu} \prod_{a=1}^n e^{\mathrm{i}\vartheta_a} \tag{4.69}$$

provides a diffeomorphism between $C_2 \times \mathbb{T}^n$ and $\mathbb{C}^n_{\neq}$. Thus, we can view $z \in \mathbb{C}^n_{\neq}$ as a variable parametrizing $C_2 \times \mathbb{T}^n$ that corresponds to the semi-global cross-section $\tilde{S}^0$ by Theorem 4.1. Below, we shall exhibit a *global cross-section* in P_0, which will be diffeomorphic to $\mathbb{C}^n$. In other words, the "semi-global" model of the dual systems will be completed into a global model by allowing the zero value for the complex variables z_k. This completion results from the symplectic reduction automatically.

First of all, let us note that the inverse of the parametrization (4.69) gives

$$\lambda_k(z) = \nu + 2(n-k)\mu + \sum_{j=k}^n z_j \bar{z}_j, \quad k = 1, \dots, n, \tag{4.70}$$

which extend to smooth functions over $\mathbb{C}^n$. The range of the extended map $z \mapsto (\lambda_1, \dots, \lambda_n)$ is the closure $\overline{C}_2$ of the polyhedron C_2. The variables $e^{\mathrm{i}\vartheta_k}$ are well-defined only over $\mathbb{C}^n_{\neq}$, where the parametrization (4.69) entails the equality

$$\sum_{k=1}^{n} \mathrm{d}\lambda_k \wedge \mathrm{d}\vartheta_k = \mathrm{i} \sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k. \tag{4.71}$$

An easy inspection of the formulae (4.11) shows that the functions f_a can be recast as

$$f_k(\lambda, e^{\mathrm{i}\vartheta}) = |z_k| g_k(z), \quad f_{n+k}(\lambda, e^{\mathrm{i}\vartheta}) = e^{\mathrm{i}\vartheta_k} |z_{k-1}| g_{n+k}(z), \qquad k = 1, \dots, n, \ \ z_0 := 1, \tag{4.72}$$

with uniquely defined functions $g_1(z), \dots, g_{2n}(z)$ that extend to smooth (actually real-analytic) positive functions on $\mathbb{C}^n$. Note that these functions depend on z only through $\lambda(z)$, i.e., one has

$$g_a(z) = \eta_a(\lambda(z)), \qquad a = 1, \dots, N, \tag{4.73}$$

with suitable functions η_a that one could display explicitly. The absolute values $|z_k|$ that appear in (4.72) are not smooth at $z_k = 0$, and the phases $e^{\mathrm{i}\vartheta_k}$ are not well-defined there. The crux is that both of these "troublesome features" can be removed by applying suitable gauge transformations to the elements of the cross-section $\tilde{S}^0$ (4.17). To demonstrate this, we define $m = m(e^{\mathrm{i}\vartheta}) \in Z_{G_+}(\mathcal{A})$ by

$$m_k(e^{\mathrm{i}\vartheta}) := \prod_{j=1}^{k} e^{-\mathrm{i}\vartheta_j}, \qquad k = 1, \dots, n. \tag{4.74}$$

Conforming with (2.28), we also set $m_{k+n} = m_k$. Then the gauge transformation by $(m, m) \in G_+ \times G_+$ operates on the $\mathbb{C}^N$-valued vector $f(\lambda, e^{\mathrm{i}\vartheta})$ and on the matrix $\check{A}(\lambda, e^{\mathrm{i}\vartheta})$ according to

$$f(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) f(\lambda, e^{\mathrm{i}\vartheta}) \equiv \phi(z), \qquad \check{A}(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) \check{A}(\lambda, e^{\mathrm{i}\vartheta}) m(e^{\mathrm{i}\vartheta})^{-1} \equiv \tilde{A}(z), \tag{4.75}$$

which defines the functions $\phi(z)$ and $\tilde{A}(z)$ over $\mathbb{C}^n_{\neq}$. The resulting functions have the form

$$\phi_k(z) = \bar{z}_k g_k(z), \quad \phi_{n+k}(z) = \bar{z}_{k-1} g_{n+k}(z), \quad k = 1, \dots, n, \tag{4.76}$$

and

$$\tilde{A}_{a,b}(z) = -\frac{2\mu \bar{z}_a z_{b-1} g_a(z) g_{n+b}(z)}{\lambda_a(z) - \lambda_b(z) - 2\mu}, \quad 1 \le a, b \le n, \tag{4.77}$$

$$\tilde{A}_{a,n+b}(z) = -\frac{2\mu \bar{z}_a z_b g_a(z) g_b(z)}{\lambda_a(z) + \lambda_b(z) - 2\mu} + \delta_{a,b} \frac{\mu - \nu}{\lambda_a(z) - \mu}, \tag{4.78}$$

$$\tilde{A}_{n+a,b}(z) = \frac{2\mu \bar{z}_{a-1} z_{b-1} g_{n+a}(z) g_{n+b}(z)}{\lambda_a(z) + \lambda_b(z) + 2\mu} - \delta_{a,b} \frac{\mu - \nu}{\lambda_a(z) + \mu}, \tag{4.79}$$

$$\tilde{A}_{n+a,n+b}(z) = \frac{2\mu \bar{z}_{a-1} z_b g_{n+a}(z) g_b(z)}{\lambda_a(z) - \lambda_b(z) + 2\mu}. \tag{4.80}$$

Now the important point is that, as is easily verified, the apparent singularities coming from vanishing denominators in $\tilde{A}$ all cancel, and both $\phi(z)$ and $\tilde{A}(z)$ extend to smooth (actually real-analytic) functions on the whole of $\mathbb{C}^n$. In particular, note the relation

$$\tilde{A}_{k,k+1}(z) = \tilde{A}_{k+n+1,k+n}(z) = -2\mu g_k(z) g_{k+n+1}(z), \qquad k = 1, \dots, n-1. \tag{4.81}$$

Corresponding to (4.13), we also have the matrix $\tilde{B}(z) \equiv -(h(\lambda(z))\tilde{A}(z)h(\lambda(z)))^{\dagger}$. This is smooth over $\mathbb{C}^n$ since both $\tilde{A}(z)$ and $h(\lambda(z))$ (4.5) are smooth. It follows from their defining equations that the induced gauge transformations of $y(\lambda, e^{\mathrm{i}\vartheta})$ (4.15) and $V(\lambda, e^{\mathrm{i}\vartheta})$ (4.16) are given by

$$y(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) y(\lambda, e^{\mathrm{i}\vartheta}) m(e^{\mathrm{i}\vartheta})^{-1} \equiv \tilde{y}(z), \tag{4.82}$$

and

$$V(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) V(\lambda, e^{\mathrm{i}\vartheta}) = \tilde{y}(z) h(\lambda(z)) \phi(z) \equiv \tilde{V}(z). \tag{4.83}$$

Since $\tilde{y}(z)$ is a uniquely defined smooth function of $\tilde{B}(z)$, both $\tilde{y}(z)$ and $\tilde{V}(z)$ are smooth functions on the whole of $\mathbb{C}^n$.

After these preparations, we are ready to state the main result of this paper.

Theorem 4.10. *By using the above notations, consider the set*

$$\tilde{S} := \{(\tilde{y}(z), \mathrm{i}h(\lambda(z))\Lambda(\lambda(z))h(\lambda(z))^{-1}, \upsilon^{\ell}_{\mu,\nu}(\tilde{V}(z)), \upsilon^{r}) \mid z \in \mathbb{C}^n\}. \tag{4.84}$$

This set defines a global cross-section for the $G_+ \times G_+$-action on the constraint surface P_0. The parametrization of the elements of $\tilde{S}$ by $z \in \mathbb{C}^n$ gives rise to a symplectic diffeomorphism between $(P_{\rm red}, \Omega_{\rm red})$ and $\mathbb{C}^n$ equipped with the Darboux form $\mathrm{i}\sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$. The spectral invariants of the "global RSvD Lax matrix"

$$\tilde{L}(z) \equiv h(\lambda(z))\tilde{A}(z)h(\lambda(z)) \tag{4.85}$$

yield commuting Hamiltonians on $\mathbb{C}^n$ that represent the reductions of the Hamiltonians spanning the Abelian Poisson algebra $\mathfrak{Q}^1$ (2.23).

Proof. Let us denote by

$$z \mapsto \tilde{\sigma}(z) \tag{4.86}$$

the assignment of the element of $\tilde{S}$ to $z \in \mathbb{C}^n$ as given in (4.84). The map $\tilde{\sigma} : \mathbb{C}^n \to P$ (2.17) is smooth (even real-analytic) and we have to verify that it possesses the following properties. First, $\tilde{\sigma}$ takes values in the constraint surface P_0. Second, with Ω in (2.17),

$$\tilde{\sigma}^*(\Omega) = \mathrm{i}\sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k. \tag{4.87}$$

Third, $\tilde{\sigma}$ is injective. Fourth, the image $\tilde{S}$ of $\tilde{\sigma}$ intersects every orbit of $G_+ \times G_+$ in P_0 in precisely one point.

Let us start by recalling from Theorem 4.1 the map $(\lambda, \theta) \mapsto \tilde{\sigma}_0(\lambda, \theta)$ that denotes the assignment of the general element of $\tilde{S}^0$ (4.17) to $(\lambda, \theta) \in C_2 \times \mathbb{T}^n$, where now we defined

$$\theta := e^{\mathrm{i}\vartheta}. \tag{4.88}$$

Then the first and second properties of $\tilde{\sigma}$ follow since we have

$$\tilde{\sigma}(z(\lambda, \theta)) = \Phi_{(m(\theta),m(\theta))}(\tilde{\sigma}_0(\lambda, \theta)), \quad \text{for all} \quad (\lambda, \theta) \in C_2 \times \mathbb{T}^n. \tag{4.89}$$

We know that $\tilde{\sigma}_0(\lambda, \theta) \in P_0$ for all $(\lambda, \theta) \in C_2 \times \mathbb{T}^n$, which implies the first property since $\tilde{\sigma}$ is continuous and P_0 is a closed subset of P. The restriction of the pull-back (4.87) to $\mathbb{C}^n_{\neq}$ is easily calculated using the parametrization $(\lambda, \theta) \mapsto z(\lambda, \theta)$ and using that by Theorem 4.1 $\tilde{\sigma}_0^*(\Omega) = \sum_{k=1}^n \mathrm{d}\lambda_k \wedge \mathrm{d}\vartheta_k$. Indeed, this translates into (4.87) restricted to $\mathbb{C}^n_{\neq}$, which implies the claimed equality because $\tilde{\sigma}^*(\Omega)$ is smooth on $\mathbb{C}^n$.

Before continuing, we remark that the map $(\lambda, \theta) \mapsto z(\lambda, \theta)$ naturally extends to a continuous map on the closed domain $\overline{C}_2 \times \mathbb{T}^n$ and its "partial inverse" $z \mapsto \lambda(z)$ extends to a smooth map $\mathbb{C}^n \to \overline{C}_2$. We will use these extended maps without further notice in what follows. (The extended map $(\lambda, \theta) \mapsto z(\lambda, \theta)$ is not differentiable at the points for which $\lambda \in \partial C_2$.)

In order to show that $\tilde{\sigma}$ is injective, consider the equality

$$\tilde{\sigma}(z) = \tilde{\sigma}(\zeta) \quad \text{for some} \quad z, \zeta \in \mathbb{C}^n. \tag{4.90}$$

Looking at the "second component" of this equality according to (4.84) we see that $\lambda(z) = \lambda(\zeta)$. Then the first component of the equality implies $\tilde{A}(z) = \tilde{A}(\zeta)$. The special case $\tilde{A}_{a,1}(z) = \tilde{A}_{a,1}(\zeta)$ of this equality gives

$$\frac{\bar{z}_a \eta_a(\lambda(z))\eta_{n+1}(\lambda(z))}{\lambda_a(z) - \lambda_1(z) - 2\mu} = \frac{\bar{\zeta}_a \eta_a(\lambda(\zeta))\eta_{n+1}(\lambda(\zeta))}{\lambda_a(\zeta) - \lambda_1(\zeta) - 2\mu}, \qquad 1 \le a \le n. \tag{4.91}$$

We know that the factors multiplying $\bar{z}_a$ and $\bar{\zeta}_a$ are equal and non-zero (actually negative). Thus, $z = \zeta$ follows, establishing the claimed injectivity.

Next we prove that no two different element of $\tilde{S}$ are gauge equivalent to each other, i.e., $\tilde{S}$ can intersect any orbit of $G_+ \times G_+$ at most in one point. Suppose that

$$\Phi_{(g_L,g_R)}(\tilde{\sigma}(z)) = \tilde{\sigma}(\zeta) \tag{4.92}$$

for some $(g_L, g_R) \in G_+ \times G_+$ and $z, \zeta \in \mathbb{C}^n$. We conclude from the second component of this equality that $\lambda(z) = \lambda(\zeta)$. Because $\lambda(z) \in \overline{C}_2$ holds, $\lambda(z)$ is regular in the sense that it satisfies (4.23). Thus, we can also conclude from the second component of the equality (4.92) that g_R belongs to the Abelian subgroup Z of G_+ given in (2.28). Then we infer from the first component

$$g_L \tilde{y}(z) g_R^{-1} = \tilde{y}(\zeta) \tag{4.93}$$

of the equality (4.92) that $g_L = g_R$. We here used that $\tilde{A}(\zeta)$ can be represented in the form (2.32) with strict inequalities in (2.30), which holds since S (3.6) is a global cross-section. Now denote $g_L = g_R = e^{\mathrm{i}\xi} \in Z$ referring to (2.28). Then we have $e^{\mathrm{i}\xi}\tilde{A}(z)e^{-\mathrm{i}\xi} = \tilde{A}(\zeta)$, and in particular

$$e^{\mathrm{i}x_a}\tilde{A}_{a,a+1}(z)e^{-\mathrm{i}x_{a+1}} = \tilde{A}_{a,a+1}(\zeta), \quad \forall a = 1, \dots, n-1. \tag{4.94}$$

By using (4.72) and (4.81)

$$\tilde{A}_{a,a+1}(z) = -2\mu\eta_a(\lambda(z))\eta_{n+a+1}(\lambda(z)) \neq 0, \tag{4.95}$$

and thus we obtain from $\lambda(z) = \lambda(\zeta)$ that $e^{\mathrm{i}\xi}$ must be equal to a multiple of the identity element of G_+. Hence, we have established that $\tilde{\sigma}(z) = \tilde{\sigma}(\zeta)$ is implied by (4.92).

It remains to demonstrate that $\tilde{S}$ intersects every gauge orbit in P_0. We have seen previously that $\mathfrak{L}^{-1}(C_2)$ is dense in P_0 and $\tilde{S}^0$ (4.17) is a cross-section for the gauge action in $\mathfrak{L}^{-1}(C_2)$. These facts imply that for any element $x \in P_0$ there exists a series $x(k) \in \mathfrak{L}^{-1}(C_2)$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty}(x(k)) = x, \tag{4.96}$$

and there also exist series $(g_L(k), g_R(k)) \in G_+ \times G_+$ and $(\lambda(k), \theta(k)) \in C_2 \times \mathbb{T}^n$ such that

$$x(k) = \Phi_{(g_L(k),g_R(k))}\left(\tilde{\sigma}_0(\lambda(k), \theta(k))\right). \tag{4.97}$$

Since $\mathfrak{L} : P_0 \to \mathbb{R}^n$ is continuous, we have

$$\mathfrak{L}(x) = \lim_{k\to\infty}\mathfrak{L}(x(k)) = \lim_{k\to\infty}\lambda(k). \tag{4.98}$$

This limit belongs to $\overline{C}_2$ and we denote it by λ^∞. The non-trivial case to consider is when λ^∞ belongs to the boundary ∂C_2. Now, since $G_+ \times G_+ \times \mathbb{T}^n$ is compact, there exists a convergent subseries

$$(g_L(k_i), g_R(k_i), \theta(k_i)), \quad i \in \mathbb{N}, \tag{4.99}$$

of the series $(g_L(k), g_R(k), \theta(k))$. We pick such a convergent subseries and denote its limit as

$$(g_L^\infty, g_R^\infty, \theta^\infty) := \lim_{i\to\infty}(g_L(k_i), g_R(k_i), \theta(k_i)). \tag{4.100}$$

Then we define $z^\infty \in \mathbb{C}^n$ by

$$z^\infty := \lim_{i\to\infty} z(\lambda(k_i), \theta(k_i)) = z(\lambda^\infty, \theta^\infty). \tag{4.101}$$

Since $z \mapsto \tilde{\sigma}(z)$ is continuous, we can write

$$\tilde{\sigma}(z^\infty) = \lim_{i\to\infty}\tilde{\sigma}(z(\lambda(k_i), \theta(k_i))) = \lim_{i\to\infty}\Phi_{(m(\theta(k_i)),m(\theta(k_i)))}\left(\tilde{\sigma}_0(\lambda(k_i), \theta(k_i))\right), \tag{4.102}$$

where $m(\theta)$ is defined by (4.74), with $\theta = e^{\mathrm{i}\vartheta}$. By combining these formulae, we finally obtain

$$\begin{aligned} x &= \lim_{i\to\infty}\Phi_{(g_L(k_i),g_R(k_i))}\left(\tilde{\sigma}_0(\lambda(k_i), \theta(k_i))\right) \\ &= \lim_{i\to\infty}\Phi_{(g_L(k_i)m(\theta(k_i))^{-1},g_R(k_i)m(\theta(k_i))^{-1})}\left(\tilde{\sigma}(z(\lambda(k_i), \theta(k_i)))\right) \\ &= \Phi_{(g_L^\infty m(\theta^\infty)^{-1},g_R^\infty m(\theta^\infty)^{-1})}(\tilde{\sigma}(z^\infty)). \end{aligned} \tag{4.103}$$

Therefore, $\tilde{S}$ is a global cross-section in P_0.

The final statement of Theorem 4.10 about the global RSvD Lax matrix (4.85) follows since $\tilde{L}$ is just the restriction of the "unreduced Lax matrix" L of (4.2) to the global cross-section $\tilde{S}$, which represents a model of the full reduced phase space P_{red}. □

V. DISCUSSION

In this paper, we characterized a symplectic reduction of the phase space (P, Ω) (2.17) by exhibiting two models of the reduced phase space P_{red} (2.20). These are provided by the global cross-sections S and $\tilde{S}$ described in Theorem 3.1 and in Theorem 4.10. The two cross-sections naturally give rise to symplectomorphisms

$$(M, \omega) \simeq (P_{\mathrm{red}}, \Omega_{\mathrm{red}}) \simeq (\tilde{M}, \tilde{\omega}), \tag{5.1}$$

where $M = T^*C_1$ (1.2) with the canonical symplectic form $\omega = \sum_{k=1}^n \mathrm{d}q_k \wedge \mathrm{d}p_k$ and $\tilde{M} = \mathbb{C}^n$ with $\tilde{\omega} = \mathrm{i} \sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$. The Abelian Poisson algebras $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ on P (2.23) descend to reduced Abelian Poisson algebras $\mathfrak{Q}^1_{\mathrm{red}}$ and $\mathfrak{Q}^2_{\mathrm{red}}$ on P_{red}. The construction guarantees that any element of the reduced Abelian Poisson algebras possesses complete Hamiltonian flow. These flows can be analyzed by means of the standard projection algorithm as well as by utilization of the symplectomorphism (5.1).

To further discuss the interpretation of our results, consider the gauge invariant functions

$$\mathcal{H}_k(y, Y, V) = \frac{1}{4k}(-\mathrm{i}Y)^{2k} \quad \text{and} \quad \tilde{\mathcal{H}}_k(y, Y, V) = \frac{(-1)^k}{2k}\mathrm{tr}(y^{-1}CyC)^k, \quad k = 1, \ldots, n. \tag{5.2}$$

The restrictions of the functions $\mathcal{H}_k$ to the global cross-sections S and $\tilde{S}$ take the form

$$\mathcal{H}_k|_S = \frac{1}{4k}(-\mathrm{i}Y(q, p))^{2k} = H_k(q, p) \quad \text{and} \quad \mathcal{H}_k|_{\tilde{S}} = \frac{1}{2k}\sum_{j=1}^n \lambda_j(z)^{2k}. \tag{5.3}$$

According to (3.19), the H_k yield the commuting Hamiltonians of the Sutherland system, while the λ_j as functions on $\tilde{S} \simeq \mathbb{C}^n$ are given by (4.70). Since any smooth function on a global cross-section encodes a smooth function on P_{red}, we conclude that the Sutherland Hamiltonians H_k and the "eigenvalue-functions" λ_j define two alternative sets of generators for $\mathfrak{Q}^2_{\mathrm{red}}$.

The restrictions of the functions $\tilde{\mathcal{H}}_k$ read

$$\tilde{\mathcal{H}}_k|_S = \frac{(-1)^k}{k}\sum_{j=1}^n \cos(2kq_j) \quad \text{and} \quad \tilde{\mathcal{H}}_k|_{\tilde{S}} = \frac{1}{2k}\mathrm{tr}(\tilde{L}(z)^k) \tag{5.4}$$

with $\tilde{L}(z)$ is defined in (4.85). On the semi-global cross-section $\tilde{S}^0$ of Theorem 4.1, which parametrizes the dense open submanifold $\mathfrak{L}^{-1}_{\mathrm{red}}(C_2) \subset P_{\mathrm{red}}$, we have

$$\tilde{\mathcal{H}}_1|_{\tilde{S}^0} = \tilde{H}^0, \tag{5.5}$$

where $\tilde{H}^0$ is the RSvD Hamiltonian displayed in (1.4). We see from (5.4) that the functions $q_j \in C^\infty(S)$ and the commuting Hamiltonians $\tilde{\mathcal{H}}_k|_{\tilde{S}}$ engender two alternative generating sets for $\mathfrak{Q}^1_{\mathrm{red}}$. On account of the relations

$$\tilde{M}^0 \simeq \tilde{S}^0 \simeq C_2 \times \mathbb{T}^n \simeq \mathbb{C}^n_{\neq} \subset \mathbb{C}^n \simeq \tilde{S} \simeq \tilde{M}, \tag{5.6}$$

$\tilde{\mathcal{H}}_1|_{\tilde{S}}$ yields a globally smooth extension of the many-body Hamiltonian $\tilde{H}^0$.

It is immediate from our results that both $\mathfrak{Q}^1_{\mathrm{red}}$ and $\mathfrak{Q}^2_{\mathrm{red}}$ define Liouville integrable systems on P_{red}, since both have n functionally independent generators. The interpretations of these Abelian Poisson algebras that stem from the models S and $\tilde{S}$ underlie the action-angle duality between the Sutherland and RSvD systems as follows. First, the generators q_k of $\mathfrak{Q}^1_{\mathrm{red}}$ can be viewed alternatively as particle positions for the Sutherland system or as action variables for the RSvD system. Their canonical conjugates p_k are of non-compact type. Second, the generators λ_k of $\mathfrak{Q}^2_{\mathrm{red}}$ can be viewed alternatively as action variables for the Sutherland systems or as globally well-defined "particle positions" for the completed RSvD system. In conclusion, the symplectomorphism $\mathcal{R} : M \to \tilde{M}$

naturally induced by (5.1) satisfies all properties required by the notion of action-angle duality outlined in the Introduction.

We finish by pointing out some further consequences. First of all, we note that the dimension of the Liouville tori of the Sutherland system drops on the locus where the action variables encoded by λ belong to the boundary of the polyhedron $\overline{C}_2$. This is a consequence of the next statement, which can be proved by direct calculation.

Proposition 5.1. Consider the Sutherland Hamiltonians $H_k(z) = \frac{1}{2k}\sum_{j=1}^n \lambda_j(z)^{2k}$ and for any $z \in \mathbb{C}^n$ define $\mathcal{D}(z) := \#\{z_k \neq 0 \mid k = 1, \ldots, n\}$. Here, $H_k(z)$ denotes the reduction of the Hamiltonian $\mathcal{H}_k$ expressed in terms of the model $\tilde{M}$, cf. (5.3). Then one has the equality

$$\dim(\mathrm{span}\{\mathrm{d}\lambda_k(z) \mid k = 1, \ldots, n\}) = \dim(\mathrm{span}\{\mathrm{d}H_k(z) \mid k = 1, \ldots, n\}) = \mathcal{D}(z). \tag{5.7}$$

It follows from (5.7) that the dense open submanifold $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$ corresponds to the part of the Sutherland phase space where the Liouville tori have full dimension n. It is also worth noting that the special point for which $z = 0$, or equivalently

$$\lambda_j = \nu + 2(n-j)\mu, \quad \forall j = 1, \ldots, n, \tag{5.8}$$

gives the unique global minimum of the function $H_1(z)$. Equation (5.3) implies that actually each function H_k $(k = 1, \ldots, n)$ possesses a global minimum at $z = 0$. An interesting characterization of this equilibrium point in terms of the (q, p) variables can be found in Ref. 2.

Being in control of the action-angle variables for our dual pair of integrable systems, the following result is readily obtained.

Proposition 5.2. Any "Sutherland Hamiltonian" $H_k \in C^\infty(M)$ $(k = 1, \ldots, n)$ given by (3.18) defines a non-degenerate Liouville integrable system, i.e., the commutant of H_k in the Poisson algebra $C^\infty(M)$ is the Abelian algebra generated by the action variables $\lambda_1, \ldots, \lambda_n$. Any "RSvD Hamiltonian" $\tilde{H}_k \in C^\infty(\tilde{M})$, $k = 1, \ldots, n$, which by definition coincides with $\tilde{\mathcal{H}}_k|_{\tilde{S}}$ in (5.4) upon the identification $\tilde{M} = \tilde{S}$, is maximally degenerate ("superintegrable") since its commutant in the Poisson algebra $C^\infty(\tilde{M})$ is generated by $(2n-1)$ elements.

Proof. The subsequent argument relies on the "action-angle symplectomorphisms" between (M, ω) and $(\tilde{M}, \tilde{\omega})$ corresponding to (5.1).

Let us first restrict the Sutherland Hamiltonian H_k to the submanifold parametrized by the action-angle variables varying in $C_2 \times \mathbb{T}^n$. For generic λ, we see from (5.3) that the flow of H_k is dense on the torus $\mathbb{T}^n$. Therefore, any smooth function f that Poisson commutes with H_k must be constant on the non-degenerate Liouville tori of the Sutherland system. By smoothness, this implies that f Poisson commutes with all the action variables λ_j on the full phase space. Consequently, it can be expressed as a function of those variables.

Next, by a slight abuse of notation, let us write $\tilde{H}_k(q, p) = \tilde{h}_k(q)$ for the "RSvD Hamiltonian" expressed in terms of the associated "dual action-angle phase space" $M = C_1 \times \mathbb{R}^n$. By (5.4), $\tilde{h}_k(q) = \frac{(-1)^k}{k}\sum_{j=1}^n \cos(2kq_j)$ and one can verify that the matrix

$$X_{i,j}(q) := \frac{\partial \tilde{h}_i(q)}{\partial q_j} \tag{5.9}$$

is non-degenerate for all $q \in C_1$. As argued in Ref. 1, this implies that $\tilde{H}_k$ is maximally superintegrable. In fact, the commutant of $\tilde{H}_k$ is generated by the "dual actions" $q_1, \ldots, q_n$ together with the functions

$$f_i(q, p) := \sum_{j=1}^n p_j (X(q))^{-1}_{j,i}, \quad i \in \mathbb{N}_n \setminus \{k\}. \tag{5.10}$$

This concludes the proof. □

In the end, we remark that the matrix functions $-\mathrm{i}Y(q,p)$ and $\tilde{L}(z)$, which naturally arose from the Hamiltonian reduction, serve as Lax matrices for the pertinent dual pair of integrable systems. We also notice that the z_j can be viewed as "oscillator variables" for the Sutherland system since the actions λ_k are linear combinations in $|z_j|^2$ ($j = 1, \ldots, n$) and the form $\tilde{\omega}$ coincides with the symplectic form of n independent harmonic oscillators. It could be worthwhile to inspect the quantization of the Sutherland system based on these oscillator variables and to compare the result to the standard quantization.[13,14,20] We plan to return to this issue in the future.

ACKNOWLEDGMENTS

This work was supported in part by the Hungarian Scientific Research Fund (OTKA) under Grant No. K-111697 and by the project TÁMOP-4.2.2.A-11/1/KONV-2012-0060 financed by the European Union (EU) and co-financed by the European Social Fund. T.F.G.'s research was also supported by the EU and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP-4.2.4.A/2-11/1-2012-0001 "National Excellence Program."

APPENDIX: SOME TECHNICAL DETAILS

In this appendix, we complete the proof of Lemma 4.5 by a calculation based on Jacobi's theorem on complementary minors (e.g., Ref. 23), which will be recalled shortly. Our reasoning below is adapted from Pusztai.[24] A significant difference is that in our case we need the strong regularity conditions (4.23) and (4.36) to avoid dividing by zero during the calculation. In fact, this appendix is presented mainly to explain the origin of the strong regularity conditions.

For an $m \times m$ matrix M let $M\left(\begin{smallmatrix} r_1 & \cdots & r_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ denote the determinant formed from the entries lying on the intersection of the rows $r_1, \ldots, r_k$ with the columns $c_1, \ldots, c_k$ of M ($k \leq m$),

$$M\begin{pmatrix} r_1 & \cdots & r_k \\ c_1 & \cdots & c_k \end{pmatrix} = \det(M_{r_i, c_j})_{i,j=1}^{k}.$$

Theorem A.1 (Jacobi). *Let A be an invertible $N \times N$ matrix with $\det(A) = 1$ and $B := (A^{-1})^{\top}$. For a fixed permutation $\left(\begin{smallmatrix} j_1 & \cdots & j_N \\ k_1 & \cdots & k_N \end{smallmatrix}\right)$ of the pairwise distinct indices $j_1, \ldots, j_N \in \{1, \ldots, N\}$ and any $1 \leq p < N$*

$$B\begin{pmatrix} j_1 & \cdots & j_p \\ k_1 & \cdots & k_p \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} j_1 & \cdots & j_N \\ k_1 & \cdots & k_N \end{pmatrix} A\begin{pmatrix} j_{p+1} & \cdots & j_N \\ k_{p+1} & \cdots & k_N \end{pmatrix}. \tag{A1}$$

Applying Jacobi's theorem to $\check{A}$ (4.37) we now derive the two equations (4.39) and (4.40) for the pair of functions (W_a, W_{n+a}) for each $a = 1, \ldots, n$, which are defined by $W_k = w_k \mathcal{F}_k$ with $\mathcal{F}_k = |F_k|^2$ (4.29) and w_k in (4.38).

Lemma A.2. Fix any strongly regular λ, i.e., $\lambda \in \mathbb{R}^n$ for which (4.23) and (4.36) hold, and use the above notations for (W_a, W_{n+a}). If $\check{A}$ given by (4.37) is a unitary matrix, then (W_a, W_{n+a}) satisfies the two equations (4.39) and (4.40) *for each $a = 1, \ldots, n$.*

Proof. Let $\check{B} := (\check{A}^{-1})^{\top}$, i.e., $\check{B}_{j,k} := \overline{\check{A}_{j,k}}, j, k \in \{1, \ldots, N\}$ and $a \in \{1, \ldots, n\}$ be a fixed index. Since $\det(\check{A}) = 1$, by Jacobi's theorem with $j_b = b$, ($b \in \mathbb{N}_N$) and $k_c = c$, ($c \in \mathbb{N}_N \setminus \{a, n+a\}$), $k_a = n + a$, $k_{n+a} = a$ and $p = n$ we have

$$\check{B}\begin{pmatrix} 1 & \cdots & a & \cdots & n \\ 1 & \cdots & n+a & \cdots & n \end{pmatrix} = -\check{A}\begin{pmatrix} n+1 & \cdots & n+a & \cdots & N \\ n+1 & \cdots & a & \cdots & N \end{pmatrix}. \tag{A2}$$

Denote the corresponding $n \times n$ submatrices of $\check{B}$ and $\check{A}$ by ξ and η, respectively. One can check that

$$\xi = \Psi - \frac{\mu - \nu}{\mu - \lambda_a} E_{a,a}, \quad \eta = \Xi - \frac{\mu - \nu}{\mu + \lambda_a} E_{a,a}, \tag{A3}$$

where $E_{j,k}$ stands for the $n \times n$ elementary matrix $(E_{j,k})_{j',k'} = \delta_{j,j'}\delta_{k,k'}$ and Ψ and Ξ are the Cauchy-like matrices

$$\Psi_{j,k} := \begin{cases} \dfrac{2\mu \overline{F}_j F_{n+k}}{2\mu - \lambda_j + \lambda_k}, & \text{if } k \neq a, \\ \dfrac{2\mu \overline{F}_j F_a}{2\mu - \lambda_j - \lambda_a}, & \text{if } k = a, \end{cases} \quad \text{and} \quad \Xi_{j,k} := \begin{cases} \dfrac{2\mu F_{n+j} \overline{F}_k}{2\mu + \lambda_j - \lambda_k}, & \text{if } k \neq a, \\ \dfrac{2\mu F_{n+j} \overline{F}_{n+a}}{2\mu + \lambda_j + \lambda_a}, & \text{if } k = a, \end{cases} \tag{A4}$$

$j, k \in \{1, \ldots, n\}$. Expanding $\det(\xi)$ and $\det(\eta)$ along the ath column we obtain the formulae

$$\det(\xi) = \det(\Psi) - \frac{\mu - \nu}{\mu - \lambda_a}\mathcal{C}_{a,a}, \quad \det(\eta) = \det(\Xi) - \frac{\mu - \nu}{\mu + \lambda_a}\mathcal{C}_{a,a}, \tag{A5}$$

where $\mathcal{C}_{a,a}$ is the cofactor of Ψ associated with entry $\Psi_{a,a}$. Since Ψ and Ξ are both Cauchy-like matrices we have

$$\det(\Psi) = \frac{1}{\mu - \lambda_a} D_a W_a, \quad \det(\Xi) = \frac{1}{\mu + \lambda_a} D_a W_{n+a}, \tag{A6}$$

where

$$D_a := \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \overline{F}_b F_{n+b} \prod_{\substack{c,d=1 \\ (a \neq c \neq d \neq a)}}^{n} \frac{\lambda_c - \lambda_d}{2\mu + \lambda_c - \lambda_d}. \tag{A7}$$

It can be easily seen that $\mathcal{C}_{a,a} = D_a$, therefore formulae (A2), (A5), (A6) lead to the equation

$$(\mu + \lambda_a) W_a + (\mu - \lambda_a) W_{n+a} - 2(\mu - \nu) = 0. \tag{A8}$$

It should be noticed that in the last step we divided by D_a, which is legitimate since D_a is non-vanishing due to the strong-regularity condition given by (4.23) and (4.36). To see this, assume momentarily that $F_i = 0$ for some $i = 1, \ldots, n$ at some strongly regular λ. The denominator in (4.37) does not vanish, and the unitarity of $\check{A}$ implies that we must have $\check{A}_{i,i+n} = 1$ or $\check{A}_{i,i+n} = -1$. These in turn are equivalent to

$$\lambda_i = 2\mu - \nu \quad \text{or} \quad \lambda_i = \nu, \tag{A9}$$

which are excluded by (4.36). One can similarly check that the vanishing of F_{n+i} would require

$$\lambda_i = \nu - 2\mu \quad \text{or} \quad \lambda_i = -\nu, \tag{A10}$$

which are also excluded. These remarks pinpoint the origin of the second half of the conditions imposed in (4.36).

Next, we apply Jacobi's theorem by setting $j_b = k_b = b$, $(b \in \mathbb{N}_n)$, $j_{n+1} = k_{n+1} = n + a$, $j_{n+c} = k_{n+c} = n + c - 1$, $(c \in \mathbb{N}_{n-1})$, and $p = n + 1$. Thus,

$$\check{B}\begin{pmatrix} 1 & \cdots & n & n+a \\ 1 & \cdots & n & n+a \end{pmatrix} = \check{A}\begin{pmatrix} n+1 & \cdots & \widehat{n+a} & \cdots & N \\ n+1 & \cdots & \widehat{n+a} & \cdots & N \end{pmatrix}, \tag{A11}$$

where $\widehat{n+a}$ indicates that the $(n + a)$th row and column are omitted. Now denote the submatrices of size $(n + 1)$ and $(n - 1)$ corresponding to the determinants in (A11) by X and Y, respectively. From (A11) and (4.37) it follows that $\det(X) = \det(Y) = D_a$ (A7). The submatrix X can be written in the form

$$X = \Phi - \frac{\mu - \nu}{\mu - \lambda_a} E_{a,n+1} - \frac{\mu - \nu}{\mu + \lambda_a} E_{n+1,a}, \tag{A12}$$

i.e., X is a rank two perturbation of the Cauchy-like matrix Φ having the entries

$$\Phi_{j,k} := \frac{2\mu \overline{F}_j F_{n+k}}{2\mu - \lambda_j + \lambda_k}, \quad \Phi_{j,n+1} := \frac{2\mu \overline{F}_j F_a}{2\mu - \lambda_j - \lambda_a},$$

$$\Phi_{n+1,k} := \frac{2\mu \overline{F}_{n+a} F_{n+k}}{2\mu + \lambda_a + \lambda_k}, \quad \Phi_{n+1,n+1} := \overline{F}_{n+a} F_a, \tag{A13}$$

where $j, k \in \{1, \ldots, n\}$. The determinant of Φ is

$$\det(\Phi) = -\frac{\lambda_a^2}{\mu^2 - \lambda_a^2} D_a W_a W_{n+a}, \tag{A14}$$

which cannot vanish because λ is strongly regular. Since X is a rank two perturbation of Φ we obtain

$$\det(X) = \det(\Phi) - (\mu - \nu)\left(\frac{\mathcal{C}_{a,n+1}}{\mu - \lambda_a} + \frac{\mathcal{C}_{n+1,a}}{\mu + \lambda_a}\right) + (\mu - \nu)^2 \frac{\mathcal{C}_{a,n+1}\mathcal{C}_{n+1,a} - \mathcal{C}_{a,a}\mathcal{C}_{n+1,n+1}}{(\mu - \lambda_a)(\mu + \lambda_a)\det(\Phi)}, \tag{A15}$$

where $\mathcal{C}$ now is used to denote the cofactors of Φ. By calculating the necessary cofactors we derive

$$\begin{gathered} \mathcal{C}_{a,a}\mathcal{C}_{n+1,n+1} = D_a^2 W_a W_{n+a}, \\ \mathcal{C}_{a,n+1} = -\frac{1}{\mu + \lambda_a} D_a W_{n+a}, \quad \mathcal{C}_{n+1,a} = -\frac{1}{\mu - \lambda_a} D_a W_a. \end{gathered} \tag{A16}$$

Equations (A14)–(A16) together with $\det(X) = D_a$ imply

$$\lambda_a^2 (W_a W_{n+a} - 1) - \mu(\mu - \nu)(W_a + W_{n+a} - 2) + \nu^2 = 0. \tag{A17}$$

Equations (A8) and (A17) coincide with (4.39) and (4.40), respectively. □

[1] Ayadi, V., Fehér, L., and Görbe, T. F., "Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals," J. Geom. Symm. Phys. **27**, 27–44 (2012); e-print arXiv:1209.1314 [math-ph].

[2] Corrigan, E. and Sasaki, R., "Quantum versus classical integrability in Calogero-Moser systems," J. Phys. A **35**, 7017–7062 (2002); e-print arXiv:hep-th/0204039.

[3] Duistermaat, J. J. and Grünbaum, F. A., "Differential equations in the spectral parameter," Commun. Math. Phys. **103**, 177–240 (1986).

[4] Etingof, P. I., *Calogero-Moser Systems and Representation Theory* (European Mathematical Society, Zürich, 2007).

[5] Fehér, L., "Action-angle map and duality for the open Toda lattice in the perspective of Hamiltonian reduction," Phys. Lett. A **377**, 2917–2921 (2013); e-print arXiv:1312.0404 [math-ph].

[6] Fehér, L. and Ayadi, V., "Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction," J. Math. Phys. **51**, 103511 (2010); e-print arXiv:1005.4531 [math-ph].

[7] Fehér, L. and Klimčík, C., "On the duality between the hyperbolic Sutherland and the rational Ruijsenaars-Schneider models," J. Phys. A **42**, 185202 (2009); e-print arXiv:0901.1983 [math-ph].

[8] Fehér, L. and Klimčík, C., "Poisson-Lie interpretation of trigonometric Ruijsenaars duality," Commun. Math. Phys. **301**, 55–104 (2011); e-print arXiv:0906.4198 [math-ph].

[9] Fehér, L. and Klimčík, C., "Self-duality of the compactified Ruijsenaars-Schneider system from quasi-Hamiltonian reduction," Nucl. Phys. B **860**, 464–515 (2012); arXiv:1101.1759 [math-ph].

[10] Fehér, L. and Pusztai, B. G., "Derivations of the trigonometric BC(n) Sutherland model by quantum Hamiltonian reduction," Rev. Math. Phys. **22**, 699–732 (2010); e-print arXiv:0909.5208 [math-ph].

[11] Fock, V., Gorsky, A., Nekrasov, N., and Rubtsov, V., "Duality in integrable systems and gauge theories," JHEP **07**, 028 (2000); e-print arXiv:hep-th/9906235.

[12] Gorsky, A., "Integrable many-body systems in the gauge theories," Theor. Math. Phys. **103**, 681–700 (1995),

[13] Heckman, G. J., "Root systems and hypergeometric functions II," Compos. Math. **64**, 353–373 (1987).

[14] Heckman, G. J. and Opdam, E. M., "Root systems and hypergeometric functions I," Compos. Math. **64**, 329–352 (1987).

[15] Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic Press, New York, 1978).

[16] Inozemtsev, V. I. and Meshcheryakov, D. V., "Extension of the class of integrable dynamical systems connected with semisimple Lie algebras," Lett. Math. Phys. **9**, 13–18 (1985).

[17] Matsuki, T., "Double coset decomposition of reductive Lie groups arising from two involutions," J. Algebra **197**, 49–91 (1997).

[18] Nekrasov, N., "Infinite-dimensional algebras, many-body systems and gauge theories," *Moscow Seminar in Mathematical Physics*, edited by A. Yu. Morozov and M. A. Olshanetsky (American Mathematical Society, Providence, 1999), pp. 263–299.

[19] Olshanetsky, M. A. and Perelomov, A. M., "Completely integrable Hamiltonian systems connected with semisimple Lie algebras," Invent. Math. **37**, 93–108 (1976).

[20] Opdam, E. M., "Root systems and hypergeometric functions IV," Compos. Math. **67**, 191–209 (1988).

[21] Ortega, J.-P. and Ratiu, T. S., *Momentum Maps and Hamiltonian Reduction*, Progress in Mathematics Vol. 222 (Birkhäuser, Basel, 2004).

[22] Perelomov, A. M., *Integrable Systems of Classical Mechanics and Lie Algebras* (Birkhäuser, Basel, 1990).

[23] Prasolov, V. V., *Problems and Theorems in Linear Algebra* (American Mathematical Society, Providence, 1994).

[24] Pusztai, B. G., "On the scattering theory of the classical hyperbolic C(n) Sutherland model," J. Phys. A **44**, 155306 (2011); e-print arXiv:1010.4663 [math-ph].

[25] Pusztai, B. G., "Action-angle duality between the C(n)-type hyperbolic Sutherland and the rational Ruijsenaars-Schneider-van Diejen models," Nucl. Phys. B **853**, 139–173 (2011); e-print arXiv:1106.2943 [math-ph].

[26] Pusztai, B. G., "The hyperbolic BC(n) Sutherland and the rational BC(n) Ruijsenaars-Schneider-van Diejen models: Lax matrices and duality," Nucl. Phys. B **856**, 528–551 (2012); e-print arXiv:1109.0446 [math-ph].
[27] Ruijsenaars, S. N. M., "Action-angle maps and scattering theory for some finite-dimensional integrable systems. I. The pure soliton case," Commun. Math. Phys. **115**, 127–165 (1988).
[28] Ruijsenaars, S. N. M., "Relativistic Toda systems," Commun. Math. Phys. **133**, 217 (1990).
[29] Ruijsenaars, S. N. M., "Finite-dimensional soliton systems," *Integrable and Superintegrable Systems*, edited by B. Kupershmidt (World Scientific, Singapore, 1990), pp. 165–206.
[30] Ruijsenaars, S. N. M., "Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals," Publ. RIMS **31**, 247–353 (1995).
[31] Ruijsenaars, S. N. M., "Systems of Calogero-Moser type," in *Proceedings of the 1994 CRM-Banff Summer School Particles and Fields* (Springer, Berlin, 1999), pp. 251–352.
[32] Schlichtkrull, H., "Harmonic analysis on semisimple symmetric spaces," in *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Mathematics Vol. 16, edited by G. Heckman and H. Schlichtkrull (Academic Press, New York, 1994), pp. 91–225.
[33] Sutherland, B., *Beautiful Models* (World Scientific, Singapore, 2004).
[34] Toda, M., *Theory of Nonlinear Lattices*, 2nd enlarged ed. (Springer, Berlin, 1989).
[35] van Diejen, J. F., "Deformations of Calogero-Moser systems and finite Toda chains," Theor. Math. Phys. **99**, 549–554 (1994).