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Adler, I and Krause, PK (2019) A lower bound on the tree-width of graphs with irrelevant vertices. Journal of Combinatorial Theory, Series B, 137. pp. 126-136. ISSN 1096-0902

https://doi.org/10.1016/j.jctb.2018.12.008

A lower bound on the tree-width of graphs with irrelevant vertices

Isolde Adler[a], Philipp Klaus Krause[b]

[a]*School of Computing, University of Leeds*
[b]*Albert-Ludwigs-Universität Freiburg*

Abstract

For their famous algorithm for the disjoint paths problem, Robertson and Seymour proved that there is a function f such that if the tree-width of a graph G with k pairs of terminals is at least $f(k)$, then G contains a solution-irrelevant vertex (Graph Minors. XXII., JCTB 2012). We give a single-exponential lower bound on f. This bound even holds for planar graphs.

Keywords: disjoint paths problem, irrelevant vertex, vital linkage, unique linkage, planar graph, tree-width

1. Introduction

The Disjoint Paths Problem is one of the famous classical problems in the area of graph algorithms. Given a graph G, and k pairs of terminals, $(s_1, t_1), \ldots, (s_k, t_k)$, it asks whether G contains k vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i to t_i, (for $i = 1, \ldots, k$). Karp proved that the problem is NP-hard in general [4] and Lynch proved that it remains NP-hard on planar graphs [6]. Robertson and Seymour showed that it can be solved in time $g(k) \cdot |V(G)|^3$ for some computable function g, i. e. the problem is fixed-parameter tractable (and, in particular, solvable in polynomial time for fixed k). For a recursive step in their algorithm ((10.5) in [11]), they prove [13] that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if a graph G with k pairs of terminals has tree-width at least $f(k)$, then G contains a vertex that is *irrelevant* to the solution, i. e. G contains a non-terminal vertex v such that G has a solution if and only if the graph $G - v$ (with the same terminals) has a solution.

In this paper we give a lower bound on f, showing that $f(k) \geq 2^k$, even for planar graphs. For this we construct a family of planar input graphs $(G_k)_{k \geq 2}$, each with k pairs of terminals, such that the tree-width of G_k is $2^k - 1$, and every member of the family has a unique solution to the Disjoint Paths Problem,

Email addresses: `I.M.Adler@leeds.ac.uk` (Isolde Adler), `krauseph@informatik.uni-freiburg.de` (Philipp Klaus Krause)

This version has been created by modifying a preprint submitted to Elsevier; modifications are minor and were done purely to work around lack of support for xelatex on arXiv where the paths of the solution use all vertices of the graph. Hence no vertex of G_k is irrelevant. As a corollary, we obtain a lower bound of $2^k - 1$ on the tree-width of graphs having *vital linkages* (also called *unique linkages*) [12] with k components.[2] Our result contrasts the polynomial upper bound in a related topological setting [7], where two systems of curves are untangled on a sphere with holes.

For planar graphs, an upper bound of $f(k) \leq 72\sqrt{2}k^{\frac{3}{2}} \cdot 2^k$ was given in [1]. An elementary proof for a bound of $f(k) \leq (72k \cdot 2^k - 72 \cdot 2^k + 18)\lceil\sqrt{2k+1}\rceil$ was provided later [5] as well as a slightly improved bound of $f(k) \leq 26k \cdot 2^{\frac{3}{2}} \cdot 2^k$ requiring a slightly more involved proof [2]. Our lower bound shows that this is asymptotically optimal. Recently, an explicit upper bound on f on graphs of bounded genus [3] was found, then refined into one that is single exponential in k and the genus [8]. The exact order of growth of f on general graphs is still unknown.

2. Preliminaries

Let $\mathbb{N}$ denote the set of all non-negative integers. For $k \in \mathbb{N}$, we let $[k] := \{1, \dots, k\}$. For a set S we let 2^S denote the power set of S. A *graph* $G = (V, E)$ is a pair of a set of *vertices* V and a set of *edges* $E \subseteq \{e \mid e \in 2^V, |e| = 2\}$, i. e. graphs are undirected and simple. For an edge $e = \{x, y\}$, the vertices x and y are called *endpoints* of the edge e, and the edge is said to be between its endpoints. For a graph $G = (V, E)$ let $V(G) := V$ and $E(G) := E$. Let H and G be graphs. The graph H is a *subgraph* of G (denoted by $H \subseteq G$), if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $X \subseteq V(G)$, the subgraph of G *induced by* X is the graph $G[X] := (X, \{e \in E(G) \mid e \subseteq X\})$ and we let $G - v := G[V(G) \setminus \{v\}]$.

A *path* P in a graph $G = (V, E)$ is a sequence $n_0, \dots, n_k \in V$ of pairwise distinct vertices of G, such that for every $i \in \{0, \dots, k-1\}$ there is an edge $\{n_i, n_{i+1}\} \in E$. The vertices n_0 and n_k are called *endpoints* of P. The path P is called a path *from* n_0 *to* n_k (i. e. paths are *simple*). We sometimes identify the path P in G with the subgraph $(\{n_0, \dots, n_k\}, \{\{n_0, n_1\}, \dots, \{n_{k-1}, n_k\}\})$ of G. A graph G is called *connected*, if it has at least one vertex and for any two vertices $x, y \in V(G)$, there is a path from x to y in G. The inclusion-maximal connected subgraphs of a graph are called *connected components* of the graph. For $A, B \subseteq V(G)$, a set $S \subseteq V(G)$ separates A from B, if there is no path from a vertex in A to a vertex in B in the subgraph of G induced by $V(G) \setminus S$. A *tree* is a non-empty graph T, such that for any two vertices $x, y \in V(T)$ there is exactly one path from x to y in T.

[2]This result appeared in the last section of a conference paper [1]. While the main focus of the paper [1] was a single exponential upper bound on f on planar graphs, it only sketches the lower bound. Here we provide the full proof of the lower bound. A longer proof of the lower bound can also be found in the thesis [5].

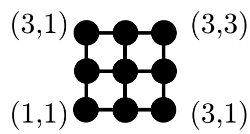

Figure 1: (3×3)-grid

Let $m, n \in \mathbb{N} \setminus \{0\}$. The $(m \times n)$-*grid* is a graph $H = (V, E)$ with $V := [m] \times [n]$ and $E := \{\{(y,x),(w,z)\} \mid (y,x) \in V, (w,z) \in V, |x-z|+|y-w| = 1\}$. In case of a square grid where $m = n$, we say that n is the *size* of the grid. An edge $\{(y,x),(w,z)\}$ in the grid is called *horizontal*, if $y = w$, and *vertical*, if $x = z$. See Figure 1 for the (3×3)-grid.

A *drawing* of a graph G is a representation of G in the Euclidean plane $\mathbb{R}^2$, where vertices are represented by distinct points of $\mathbb{R}^2$ and edges by simple curves joining the points that correspond to their endpoints, such that the interior of every curve representing an edge does not contain points representing vertices. A *planar drawing* (or *embedding*) is a drawing, where the interiors of any two curves representing distinct edges of G are disjoint. A graph G is *planar*, if G has a planar drawing (See [10] for more details on planar graphs). A *plane graph* is a planar graph G together with a fixed embedding of G in $\mathbb{R}^2$. We will identify a plane graph with its image in $\mathbb{R}^2$. Once we have fixed the embedding, we will also identify a planar graph with its image in $\mathbb{R}^2$.

Definition 1 (Disjoint Paths Problem (DPP))**.** *Given a graph G and k pairs of* terminals $(s_1,t_1) \in V(G)^2, \dots, (s_k,t_k) \in V(G)^2$*, the* Disjoint Paths Problem *is the problem of deciding whether G contains k vertex-disjoint paths $P_1, \dots, P_k$ such that P_i connects s_i to t_i (for $i \in [k]$). If such paths $P_1, \dots, P_k$ exist, we refer to them as a* solution. *We denote an* instance *of* DPP *by* $G, (s_1,t_1), (s_2,t_2), \dots, (s_k,t_k)$.

Let $G, (s_1,t_1), \dots, (s_k,t_k)$ be an instance of DPP. A non-terminal vertex $v \in V(G)$ is *irrelevant*, if $G, (s_1,t_1), \dots, (s_k,t_k)$ has a solution if and only if $G - v, (s_1,t_1), \dots, (s_k,t_k)$ has a solution.

A *tree-decomposition* of a graph G is a pair (T, χ), consisting of a tree T and a mapping $\chi\colon V(T) \to 2^{V(G)}$, such that for each $v \in V(G)$ there exists $t \in V(T)$ with $v \in \chi(t)$, for each edge $e \in E(G)$ there exists a vertex $t \in V(T)$ with $e \subseteq \chi(t)$, and for each $v \in V(G)$ the set $\{t \in V(T) \mid v \in \chi(t)\}$ is connected in T. The *width* of a tree-decomposition (T, χ) is

$$\mathrm{w}(T,\chi) := \max\left\{ \big|\chi(t)\big| - 1 \;\middle|\; t \in V(T) \right\}.$$

If T is a path, (T, χ) is also called a *path-decomposition*. The *tree-width of G* is

$$\mathrm{tw}(G) := \min\left\{ \mathrm{w}(T,\chi) \;\middle|\; (T,\chi) \text{ is a tree-decomposition of } G \right\}.$$

The *path-width of G* is

$$\mathrm{pw}(G) := \min\left\{ \mathrm{w}(T,\chi) \;\middle|\; (T,\chi) \text{ is a path-decomposition of } G \right\}.$$

Obviously, every graph G satisfies $\mathrm{pw}(G) \geq \mathrm{tw}(G)$. Every tree has tree-width at most 1 and every path has path-width at most 1. It is well known that the $(n \times n)$-grid has both tree-width and path-width n. Moreover, if $H \subseteq G$, then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$ and $\mathrm{pw}(H) \leq \mathrm{pw}(G)$.

Theorem 1 (Robertson and Seymour [13])**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that if $\mathrm{tw}(G) \geq f(k)$, then $G, (s_1, t_1), \dots, (s_k, t_k)$ has an irrelevant vertex (for any choice of terminals $(s_1, t_1), \dots, (s_k, t_k)$ in G).*

A *linkage* in a graph G is a subgraph $L \subseteq G$, such that each connected component of L is a path. The *endpoints* of a linkage L are the endpoints of these paths, and the *pattern* of L is the matching on the endpoints induced by the paths, i. e. the pattern is the set

$$\bigl\{\{s,t\} \bigm| L \text{ has a connected component that is a path from } s \text{ to } t\bigr\}.$$

A linkage L in a graph G is a *vital linkage* in G, if $V(L) = V(G)$ and there is no other linkage $L' \neq L$ in G with the same pattern as L.

Theorem 2 (Robertson and Seymour [13])**.** *There are functions $g, h\colon \mathbb{N} \to \mathbb{N}$ such that if a graph G has a vital linkage with k components then $\mathrm{tw}(G) \leq g(k)$ and $\mathrm{pw}(G) \leq h(k)$.*

3. The lower bound

Our main result is the following.

Theorem 3. *Let $f, g, h\colon \mathbb{N} \to \mathbb{N}$ be as in Theorems 1 and 2. Then $f(k) \geq 2^k$, $g(k) \geq 2^k - 1$, and $h(k) \geq 2^k - 1$. Moreover, this holds even if we consider planar graphs only.*

In our proof we construct a family of graphs $G_k, k \geq 1$, of tree-width and path-width $\geq 2^k - 1$, and with a vital linkage with k components. Figure 2 shows the graph G_4.

Definition 2 (The graph G_k)**.** *Let $k, p \in \mathbb{N} \setminus \{0\}$. We inductively define an instance $G_k, (s_1, t_1), \dots, (s_k, t_k)$ of* DPP *as follows.*

The Graph $G_{1,p}$ is the path $x_1, x_2, \dots, x_p$ with p vertices, $s_1(G_{1,p}) := x_1$, $t_1(G_{1,p}) := x_p$. The bottom row *and the* top row *of $G_{1,p}$ are the graph $G_{1,p}$ itself.*

We define the graph $G_{k+1,p}$ by adding a path $y_1, y_2, \dots, y_p$ with p vertices to $G_{k,2p}$ as follows. Let $x_1, x_2, \dots, x_{2p}$ be the bottom row of $G_{k,2p}$ and let $z_1, z_2, \dots, z_{2p}$ be the top row of $G_{k,2p}$. Let

$$\begin{aligned} V(G_{k+1,p}) &:= V(G_{k,2p}) \cup \{y_1, y_2, \dots, y_p\}, \\ E(G_{k+1,p}) &:= E(G_{k,2p}) \cup \bigl\{\{y_i, y_{i+1}\} \bigm| 1 \leq i < p\bigr\} \cup \\ &\qquad \bigl\{\{y_i, x_i\}, \{y_i, x_{2p-i+1}\} \bigm| 1 \leq i \leq p\bigr\}. \end{aligned}$$

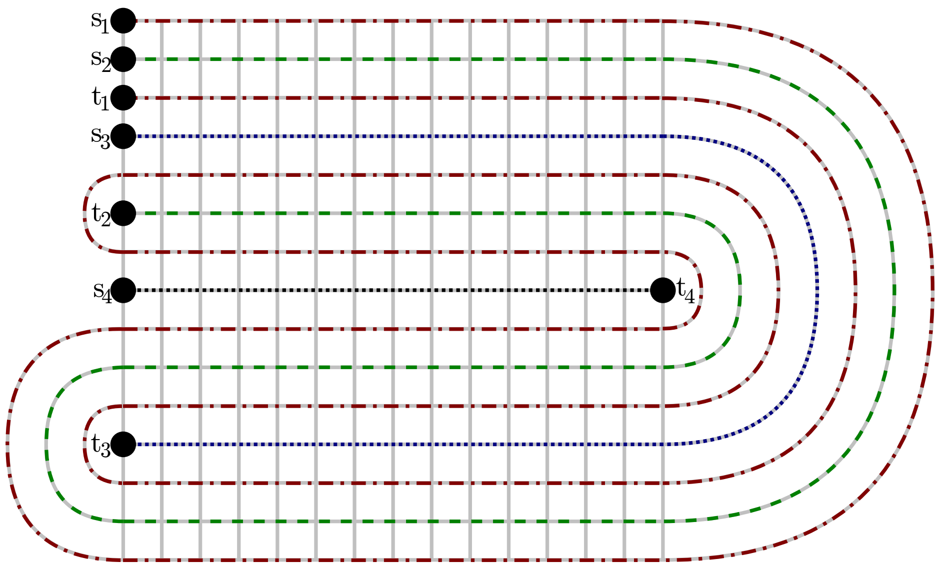

Figure 2: $G_4, (s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4)$ with solution.

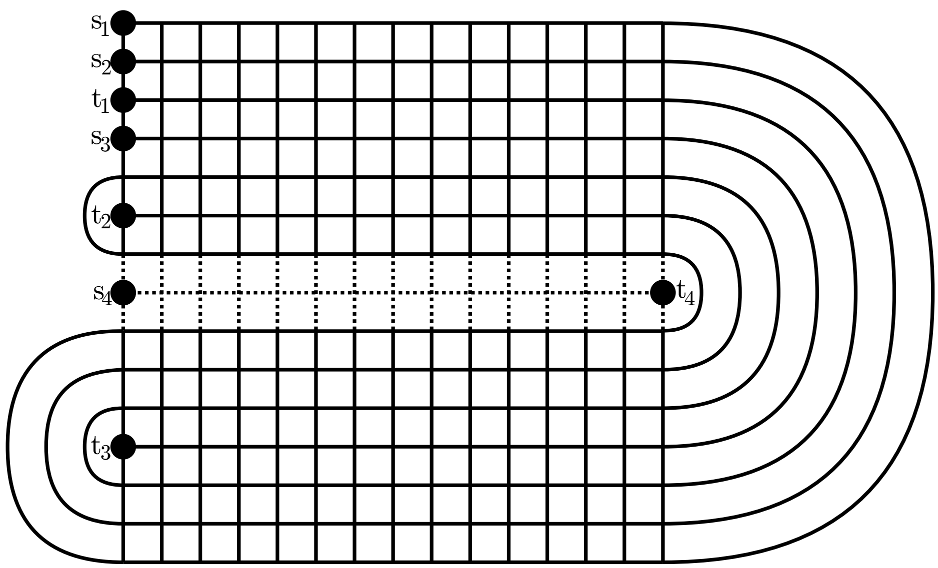

Figure 3: The construction of $G_4 = G_{4,15}$ from $G_{3,30}$.

We set $s_{k+1}(G_{k+1,p}) := y_1$, $t_{k+1}(G_{k+1,p}) := y_p$ *and* $s_i(G_{k+1,p}) := s_i(G_{k,p})$, $t_i(G_{k+1,p}) := t_i(G_{k,p})$ *for* $1 \leq i \leq k$. *The* top row *of* $G_{k+1,p}$ *is* $z_1, \ldots, z_p$ *and the* bottom row *of* $G_{k+1,p}$ *is* $z_{2p}, \ldots, z_{p+1}$.

Let $G_k := G_{k,2^k-1}$. *We define the* DPP *instance* $G_k, (s_1, t_1), \ldots, (s_k, t_k)$ *as* $G_k, (s_1(G_k), t_1(G_k)), \ldots, (s_k(G_k), t_k(G_k))$.

Figure 3 shows the construction of $G_4 = G_{4,15}$ from $G_{3,30}$.

Remark 1. *By construction, the graph* G_k *contains a* $((2^k - 1) \times (2^k - 1))$*-grid as a subgraph. The tree-width and path-width of* G_k *are thus at least* $2^k - 1$.

Remark 2. *By construction, the graph* G_k *contains a linkage (because in each step we add a path linking a new terminal pair).*

We will now show that this linkage is vital by considering a topological version.

Definition 3 (Topological DPP)**.** *Given a subset* X *of the plane and* k *pairs of terminals* $(s_1, t_1) \in X^2, \ldots, (s_k, t_k) \in X^2$ *the* TOPOLOGICAL DISJOINT PATHS PROBLEM *is the problem of deciding whether there are* k *pairwise disjoint curves in* X*, such that each curve* P_i *is homeomorphic to* $[0, 1]$ *and its ends are* s_i *and* t_i. *If such curves* $P_1, \ldots, P_k$ *exist, we refer to them as a* solution. *We denote an* instance *of the topological Disjoint Paths Problem by* $X, (s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

A *disc-with-edges* is a subset X of the plane containing a closed disc D such that the connected components of $X \setminus D$, called *edges*, are homeomorphic to open intervals $(0, 1)$. We now define a family $(X_k)_{k \in \mathbb{N} \setminus \{0\}}$ of discs-with-edges together with terminals. These will be used as instances of the topological DPP. Figure 4 illustrates the construction.

Definition 4 (X_k)**.** *Let* D *be a closed disc in the plane and* $k \in \mathbb{N} \setminus \{0\}$. *We start by inductively defining points* s_k, t_k *on the boundary* ∂D *of* D. *(These will be used as terminals and to confine the way the edges are added to* D*.) Let* s_1, t_1 *be two distinct points on* ∂D*, and let* $C_1 := \partial D \setminus \{s_1, t_1\}$. *Hence* C_1 *is the union of two curves, each homeomorphic to the open interval* $(0, 1)$. *Call one of the curves* S_1 *and the other* T_1. *Assume that* s_k, t_k, C_k, S_k, *and* T_k *are already defined, and assume that* T_k *is a curve adjacent to* t_k *and* s_1. *Place a new point* s_{k+1} *on* S_k *and a new point* t_{k+1} *on* T_k*, let* $C_{k+1} := C_k \setminus \{s_{k+1}, t_{k+1}\}$*, let* T_{k+1} *be the component of* C_{k+1} *adjacent to* t_{k+1} *and* s_1*, and let* S_{k+1} *be the component of* C_{k+1} *adjacent to* t_{k+1} *and* t_k.

Now let $X_1 := D$ *and* $E_1 := \varnothing$. *Assume the space* X_k *and the set* E_k *are already defined. We define* X_{k+1} *by adding a planar matching of* $2^k - 1$ *edges to* X_k. *We call the set of these edges* E_{k+1}. *The edges are pairwise disjoint and disjoint from* X_k. *They are added such that each end is adjacent to a point on* ∂D *and no two edges are adjacent to the same point on* ∂D. *Each edge has one end adjacent to a point on the component of* C_{k+2} *between* t_k *and* s_{k+1}*, and the other end adjacent to a point on the component of* C_{k+2} *between* t_k *and* s_{k+2}. *Finally, let* $X_{k+1} := X_k \cup E_{k+1}$.

In this way we obtain a family X_k, $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ *of instances to the topological DPP.*

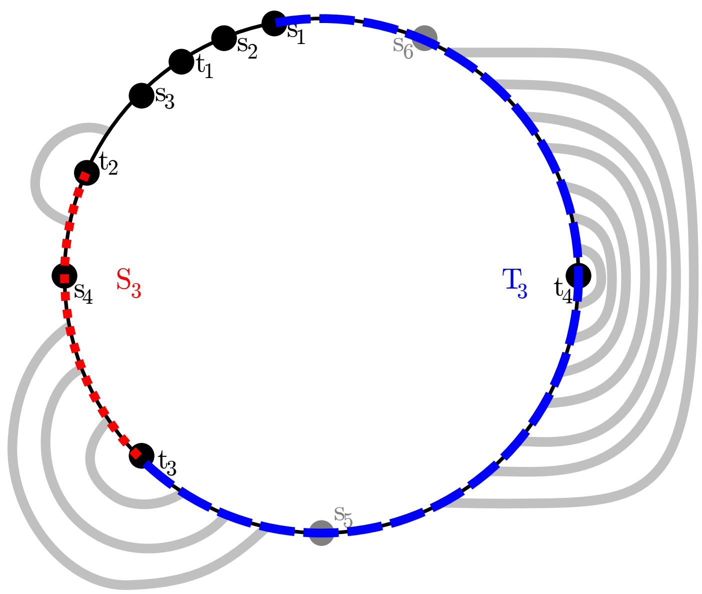

Figure 4: The construction of $X_4, (s_1, t_1), \dots, (s_4, t_4)$, for the topological DPP. Note that s_5 and s_6 are only used to place E_4 correctly.

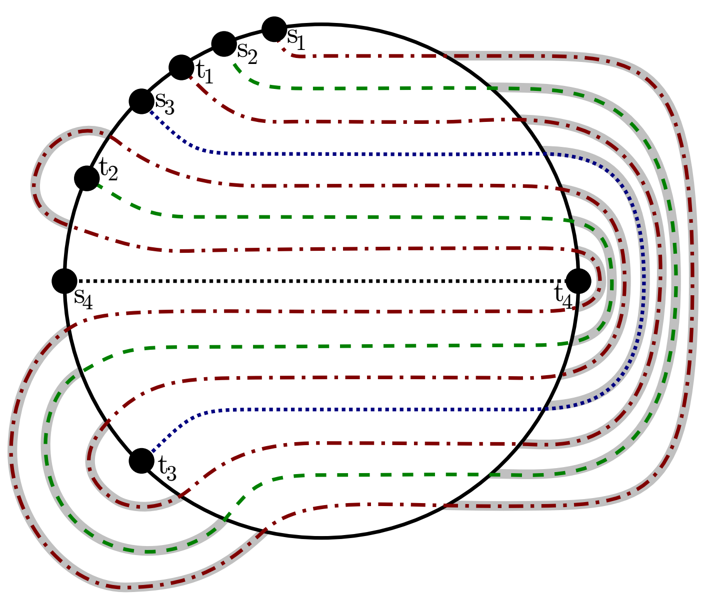

Figure 5: A solution of the topological DPP from Figure 4.

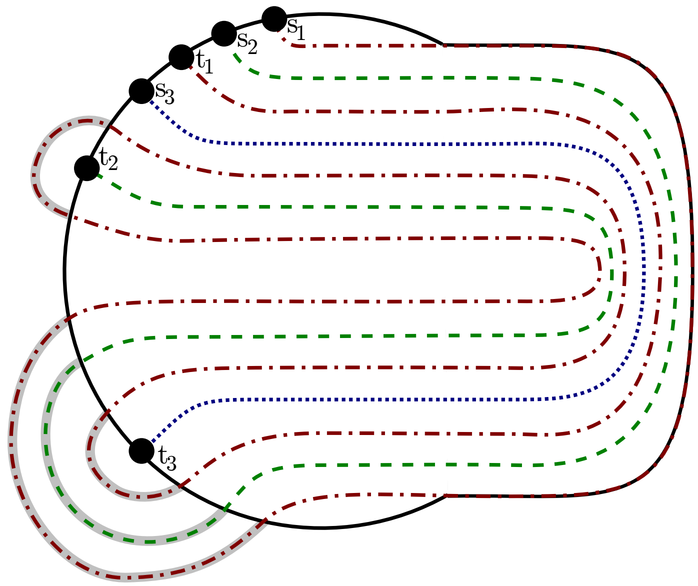

Figure 6: The solution on $X_3', (s_1, t_1), \ldots, (s_3, t_3)$ induced by the solution on $X_4, (s_1, t_1), \ldots, (s_4, t_4)$.

Remark 3. *The embedding of G_k (as shown in Figure 2 for G_4) corresponds to the space X_k. Thus by Remark 2 the topological* DPP *on X_k, $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ has a solution.*

For an instance of the topological DPP on X_4, this solution can be seen in Figure 5.

Lemma 1. *For $k \in \mathbb{N} \setminus \{0\}$ the topological* DPP *instance $X_k, (s_1, t_1), \ldots, (s_k, t_k)$ has a unique solution $P_1, \ldots, P_k$ (up to homeomorphism). The solution uses all edges $\bigcup_{1 \le i \le k} E_i$.*

Proof. For $k = 1$ this is true because $E_1 = \varnothing$. Inductively assume that the lemma holds for k. Let $P_1, \ldots, P_{k+1}$ be any solution to $X_{k+1}, (s_1, t_1), \ldots, (s_{k+1}, t_{k+1})$. This solution induces a solution of the topological DPP $X_k, (s_1, t_1), \ldots, (s_k, t_k)$ as follows. Every edge $e \in E_{k+1}$ together with the segment of ∂D that connects the ends of e and contains t_{k+1} bounds a disc D_e. The space $X_k' := X_{k+1} \cup \bigcup_{e \in E_{k+1}} D_e$ is homeomorphic to X_k and the paths $P_1, \ldots, P_k$ form a solution of X_k', (s_1, t_1), $\ldots$, (s_k, t_k). Figure 6 illustrates this for $k = 3$. By induction, this solution is unique up to homeomorphism and the paths $P_1, \ldots, P_k$ use all edges in $\bigcup_{1 \le i \le k} E_i$. Let $Q_1, \ldots, Q_k$ be the solution obtained by embedding the graph G_k (cf. Remark 3). By uniqueness, for each $i \in [k]$, the edges of $\bigcup_{1 \le i \le k} E_i$ used by P_i are the same as for Q_i, and the order of their appearance on P_i when walking from s_i to t_i is also the same as on Q_i. Hence the solution $P_1, \ldots, P_k$ on X'_k restricted to the closed disc D of X'_k is a planar matching of curves (the curves in $\bigcup_{1\leq i\leq k} P_i \setminus \bigcup_{1\leq i\leq k} E_i$) between pairs of points on ∂D (and the same pairs of points are obtained by restricting $Q_1, \ldots, Q_k$ to D). These pairs of points also have to be matched in X_{k+1}.

We now claim that in the solution $P_1, \ldots, P_{k+1}$ on X_{k+1}, each curve in $\bigcup_{1\leq i\leq k} P_i \setminus \bigcup_{1\leq i\leq k} E_i$ uses an edge of E_{k+1}. If not, then there is a curve

$$p \in \bigcup_{1\leq i\leq k} P_i \setminus \bigcup_{1\leq i\leq k} E_i$$

that avoids all edges in E_{k+1}. Since the edges of $\bigcup_{1\leq i\leq k} E_i$ are already used, p is routed within D. By construction of X_{k+1} and the fact that all edges of $\bigcup_{1\leq i\leq k} E_i$ are already used, this means that p separates s_{k+1} from both t_{k+1} and the endpoints of the edges in E_{k+1}, a contradiction to P_{k+1} being a path in the solution. Hence p uses an edge of E_{k+1}.

Since the sets $\bigcup_{1\leq i\leq k} P_i \setminus \bigcup_{1\leq i\leq k} E_i$ and E_{k+1} have equal size, it follows that each curve of the matching

$$\bigcup_{1\leq i\leq k} P_i \setminus \bigcup_{1\leq i\leq k} E_i$$

uses precisely one edge of E_{k+1}. Since the endpoints of the matching are fixed, they induce an order on the matching curves which determines precisely which edge of E_{k+1} is used by which curve.

Altogether, this shows that the solution to $X_{k+1}, (s_1, t_1), \ldots, (s_{k+1}, t_{k+1})$ is unique up to homeomorphism and uses all edges $\bigcup_{1\leq i\leq k} E_i$.

q. e. d.

Remark 4. *In a topological* DPP *instance, the number of edges around the terminals is crucial. Even just relaxing the conditions on X_k by having 2 edges instead of 1 edge around terminal t_2 allows a quite different solution to the topological* DPP*. This solution uses no edge around t_k, one edge around each of $t_3, t_3, \ldots, t_{k-1}$, and the two edges around t_2 (Figure 7 shows this for $k = 4$).*

Theorem 4. *Let $k \in \mathbb{N} \setminus \{0\}$. The graph G_k contains a vital linkage.*

Proof. Let $P_1, \ldots, P_k$ be the linkage from Remark 2. We argue that it is vital. For $k = 1$ and $k = 2$, one can easily verify that G_k has a unique embedding. For $k \geq 2$, contracting an edge at s_1 suffices to make G_k 3-connected. Since 3-connected planar graphs have unique embeddings [14], the graph G_k also has a unique embedding, and it suffices to consider our previous embedding of G_k (cf. Figure 2). Let D be the minimal disc containing the grid in G_k. The disc D together with $E(G_k)$ is the space X_k. The paths $P_1, \ldots, P_k$ thus give a solution to the topological DPP instance $X_k, (s_1, t_1), \ldots, (s_k, t_k)$, which by Lemma 1 is unique and uses all edges in E_k. Thus any linkage $P'_1, \ldots, P'_k$ with the same pattern as $P_1, \ldots, P_k$ can differ from $P_1, \ldots, P_k$ only inside the grid. Thus for

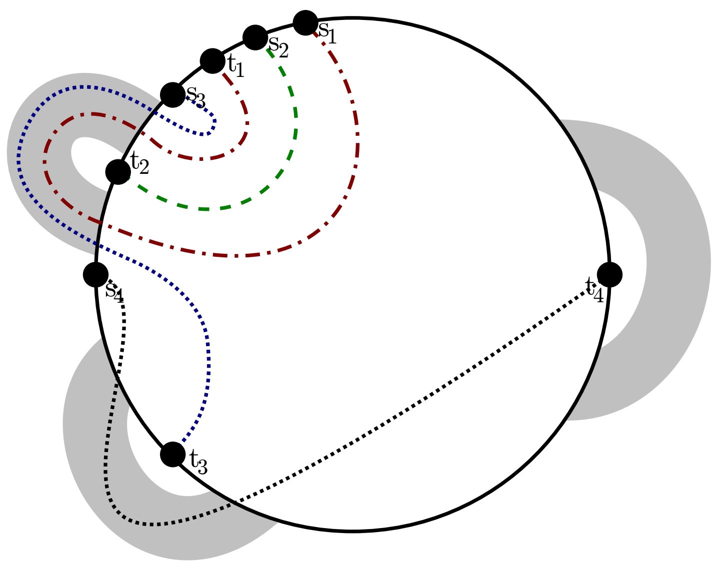

Figure 7: The number of edges around the terminals is crucial (cf. Remark 4).

each $y \in [2^k - 1]$ there is a subpath Q'_y of some path of the solution $P'_1, \ldots, P'_k$, such that the endpoints of Q'_y are $(y', 1)$ and $(y, 2^k - 1)$ for some $y' \in [2^k - 1]$. Hence the family $(Q'_y)_{y \in [2^k - 1]}$ is a linkage between the first column and the last column of the grid.

Suppose that $P'_1, \ldots, P'_k$ indeed differs from $P_1, \ldots, P_k$. Then at least one path Q'_y contains a vertical edge e in the grid. Hence the column of e contains at most $2^k - 3$ vertices that are not used by Q'_y and, by Menger's Theorem [9], the remaining $2^k - 2$ paths of the family cannot be routed, a contradiction.

q. e. d.

Proof of Theorem 3 Theorem 3 immediately follows from Theorem 4 and Remark 1. q. e. d.

Acknowledgements

This research was partially supported by the Deutsche Forschungsgemeinschaft, project Graphstrukturtheorie und algorithmische Anwendungen, AD 411/1-1 and project Graphstrukturtheorie im Übersetzerbau, KR 4970/1-1. We thank Frédéric Mazoit for valuable discussions, especially for inspiring Remark 4. We also thank an anonymous reviewer for suggesting very elegant shortenings of our construction and proof.

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