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The scope of Feferman's semi-intutionistic set theories and his second conjecture*

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Abstract

The paper is concerned with the scope of semi-intuitionistic set theories that relate to various foundational stances. It also provides a proof for a second conjecture of Feferman's that relates the concepts for which the law of excluded middle obtains to those that are absolute with regard to the relevant test structures, or more precisely of Δ_1 complexity. The latter is then used to show that a plethora of statements is indeterminate with respect to various semi-intuitionistic set theories.

Keywords: semi-intuitionistic set theory, law of excluded middle, indefiniteness, absoluteness, Δ_1 property,
MSC2000: 03F50; 03F25; 03E55; 03B15; 03C70

1 Introduction

Brouwer argued that limitation to constructive reasoning is required when dealing with "unfinished" totalities. As a complement to that, the predicativists such as Poincaré and Weyl (of *Das Kontinuum*) accepted the natural numbers as a "finished" or definite totality, but nothing beyond that. In addition, the "semi-intuitionistic" school of descriptive set theory (DST) of Borel et al. in the 1920s took both the natural numbers and the real numbers as definite totalities and explored what could be obtained on that basis alone. A further position (that can perhaps be associated with Zermelo) holds that the powerset operation produces "finished" totalities but the universe V of all sets is not "finished" on account of Russell's paradox. Feferman proposed to discuss these frameworks, and more broadly questions of definiteness and indeterminacy of meaning, by adopting a semi-intuitionistic point of view. From a metamathematical perspective, these and other different levels of indefiniteness/definiteness can be treated in the single framework of

*To the memory of Sol Feferman, who, with his extraordinarily broad insight and great mind was a constant source of inspiration to me, ever since my graduate days, and who, with his kindness and unfailing support has shaped my life's trajectory in crucial ways.

semi-intuitionistic theories of sets, whose basic logic is intuitionistic, but for which the law of excluded middle (LEM) is accepted for bounded formulas. This article presents semi-intuitionistic set theories and studies the scope of several systems associated with various foundational positions. As the formal counterpart of definiteness of a statement or a concept with regard to a theory T we ask whether LEM can be proved in T for it. The later parts of this paper provide a technical machinery for establishing indefiniteness of concept and statements.

We give a brief overview of the individual sections of article. Section 2 aims to present Feferman's case for semi-intuitionistic set theories as a tool in the philosophy of mathematics and relate it to Dummett's notion of indefinite extensibility in light of recent work of Øystein Linnebo (notably [47]) and others. Section 3 introduces formal systems of semi-intuitionistic set theory and connects them to various foundational schools. As the continuum hypothesis played a major role in shaping Feferman's semi-intuitionistic point of view, it will feature in section 4. Sections 5 and 6 contain new results on indeterminateness for which what I call *Feferman's second conjecture* is a crucial tool. In addition to his conjecture about the indeterminacy of CH relative to semi-intuitionistic set theory, he stated another conjecture (in [19, 20]) concerning the relationship between two types of predicates in such set theories, namely that the collection of Δ_1 predicates and the collection of predicates for which the law of excluded middle holds should coincide. First in section 5 it is shown that Δ_1 predicates satisfy LEM. Section 6 establishes the reverse implication. However, in both cases we require an extra assumption namely that the theories' axioms of choice are strengthened to global choice. But this is enough if one wants to show that a plethora of statements is indeterminate with respect to various semi-intuitionistic set theories. The last section also presents related results due to Peter Koellner and Hugh Woodin [42].

2 Semi-intuitionism in the philosophy of mathematics

Solomon Feferman, in recent years, has argued that the Continuum Hypothesis (CH) might not be a definite mathematical problem (see [19, 21, 22][1]).

> *My reason for that is that the concept of arbitrary set essential to its formulation is vague or underdetermined and there is no way to sharpen it without violating what it is supposed to be about.* ([19, p.1]).

Here CH just serves the role of most emblematic case of a statement that is seen as referring to an indefinite totality. The problem has deeper implications, though, in that it concerns the nature of logic germane to reasoning about such domains of objects. The main question is whether there are principled demarcations on the use of classical logic. In [19], Feferman proposed a logical framework for what's definite and for what's not.

> *One way of saying of a statement φ that it is definite is that it is true or false; on a deflationary account of truth that's the same as saying that the Law*

[1]The paper [19] was written for Peter Koellner's *Exploring the frontiers of incompleteness* (EFI) Project, Harvard 2011-2012.

of Excluded Middle (LEM) holds of φ *, i.e. one has* $\varphi \vee \neg\varphi$ *. Since LEM is rejected in intuitionistic logic as a basic principle, that suggests the slogan, "What's definite is the domain of classical logic, what's not is that of intuitionistic logic." [...] And in the case of set theory, where every set is conceived to be a definite totality, we would have classical logic for bounded quantification while intuitionistic logic is to be used for unbounded quantification.* ([19, p. 23])

The point of departure of his analysis are two informal notions of definiteness, namely that of a *definite domain*[2] of objects and that of a *definite concept*. Definiteness of a concept P over a domain D can be understood as saying that for every $d \in D$ either $P(d)$ is true or $P(d)$ is false. On a deflationary account of truth, more formal criteria for these distinctions can be given in logical terms:[3]

P1. A concept P is **definite** over a domain D iff the *Principle of Bivalence* (or the Law of Excluded Middle, *LEM*) holds with regard to it, i.e.,

$$\forall\vec{x} \in D\,[P(\vec{x}\,) \vee \neg P(\vec{x}\,)].$$

P2. A domain D is **definite** if and only if quantification over D is a definite logical operation, i.e., whenever $R(\vec{x}, y)$ is definite as a concept over D, so are $\forall y \in D\ R(y, \vec{x}\,)$ and $\exists y \in D\ R(y, \vec{x}\,)$.

Surely, the above description provides just a template for further discussions. Crucially one needs an account of how quantification over a domain is to be understood.

Feferman's critique of using classical logic for indefinite domains is reminiscent of Michael Dummett's diagnosis of the failure of Frege's logicist project in the final chapter of [15], where he singles out the adoption of classical quantification over domains comprised of objects falling under an *indefinitely extensible concept* as the main reason for the paradoxes.[4] A concept is indefinitely extensible just in case, whenever

> *we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality characterize a larger totality all of whose members fall under it* ([16, p. 441].

His contention is that the classical view is illegitimate and that the correct logic for quantification over such a domain must be intuitionistic logic. Dummett's notion of indefinite extensibility did not please everybody. Boolos [6, p. 224] and Burgess [7, p. 205] found it hopelessly obscure and questioned its explanatory value. Recently, however, Dummett's analysis has received a careful reconstruction by Øystein Linnebo [47] that also illuminates some aspects of Feferman's position. We will turn to it in the next subsection.

[2] Feferman often uses the term *definite totality* rather than *definite domain*.

[3] Below $\forall\vec{x} \in D$ stands for $\forall x_1 \in D \ldots \forall x_n \in D$ where $\vec{x} = x_1, \ldots, x_n$.

[4] I couldn't find a single reference to Dummett in Feferman's papers on semi-intuitionistic theories. I conjecture that he was not aware of this connection. Unfortunately, I missed the opportunity to ask him about it.

2.1 Dummett on definiteness

Dummett considers two notions of definiteness.

> *... a definite totality is one quantification over which always yields a statement determinately true or false.* (Dummett, 1991, [15, p. 316])

This notion of definite totality has a lot in common with that of a definite domain in (P2). The way Dummett understands classical quantification is further illuminated by the following passage.

> *We cannot take quantification over the totality of all objects as a sentence-forming operation which will always generate a sentence with a determinate truth-value; we cannot, in other words, interpret it classically as infinitary conjunction or disjunction.* (Dummett, 1981, [14, p. 53])

Classical quantification over a domain D can be rendered as

$$\begin{aligned} \forall x \in D\, P(x) \quad &:\Leftrightarrow \quad \bigwedge_{c \in D} P(c) \\ \exists x \in D\, P(x) \quad &:\Leftrightarrow \quad \bigvee_{c \in D} P(c). \end{aligned}$$

Linnebo and Shapiro call a predicate P over the domain D *traversable* if such an interpretation is available. To handle quantification over non-traversable domains, a non-instance based conception of universal quantification is called for. Pictorially the idea behind a Dummettian definite totality seems to be that a completable search through all its element is (at least in principle) available. As a result, if for each $c \in D$ it can always be determined whether P holds true of c or not then the statements $\forall x \in P(x)$ and $\exists x \in P(x)$ have a definite truth value, too.[5] Another way of construing definite totalities is in terms of the old Aristotelian distinction between the actual and the potential. From this viewpoint definite (traversable) totalities are then equated with actual domains and only quantification over them will follow the rules of classical quantifier logic whereas reasoning over potential domains will be regimented by intuitionistic logic.

The second notion of definiteness one finds in Dummett's work applies to concepts.

> *A concept is definite provided that it has a definite criterion of application - it is determinate what has to hold good of an object for it to fall under the concept - and a definite criterion of identity - it is determinate what is to count as one and the same object.* (Dummett, 1991, [15, p. 314])

One interpretation of the above is that a definite concept P should satisfy bivalence, i.e., $P(c) \vee \neg P(c)$ for all objects c (to which P can be meaningfully applied). Thus a Dummettian definite concept would be a *intensionally definite* concept in the sense of Linnebo [47]. Looking for further clues, one finds the following passage:

[5]Of course, if we restrict the possibilities to humanly or physically possible searches we will be restricted to some form of ultra-finitism. By contrast, if we allow for supertasks to be completable then we will get a hierarchy of foundational schools.

> *We know well enough what is needed for something to be recognized as a set or as an ordinal number, and when an entity given in a certain way is the same set or ordinal number as one given in some another.* (p. 315).

Thus Dummett appears to view the concepts of set and ordinal as definite. Together with bivalence for definite concepts this entails classical quantification over sets and thus Dummett seems to endorse something very close to Feferman's preferred framework for discussing questions of definiteness, i.e., semi-intuitionistic set theory.

There is a further notion that plays a crucial role in Dummett's philosophy of mathematics. He is famous for singling out intuitionistic logic on meaning-theoretic grounds. In addition to his more familiar argument based on the learnability and intersubjective functioning of language, he put forward another one particularly germane to mathematics. It is based on the notion of *indefinite extensibility.*

> *A concept is indefinitely extensible if, for any definite characterization of it, there is a natural extension of this characterization, which yields a more inclusive concept; this extension will be made according to some general principle for generating such extensions, and, typically, the extended characterization will be formulated by reference to the previous, unextended, characterization.* (Dummett, 1963, [13, pp. 195–196])

Here Dummett is referring to Russell's famous analysis to the effect that paradoxes

> *result from the fact that [...] there are what we may call self-reproductive processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.* (Russell, 1906, [70])

Linnebo's analysis connects the Dummettian notion of extension with classical quantification.

> *A concept has an extension (that is, ED ["ED" stands for "extensionally definite".]) just in case quantification over its instances can be interpreted classically.* [47, p. 11]

On this interpretation a concept possesses an extension in Dummett's sense exactly when the collection of objects that fall under it form a definite totality in Feferman's sense. Linnebo also stresses another feature of extensions.

> *It follows that the extension is modally rigid: it has the same members in all the circumstances in which it exists at all. It is this rigidity that enables us to compare extensions across different circumstances.* [47, p. 7-8]

There is an interesting connection to Feferman's arguments in [23], where he employs Kripke models to take into account differences as to the definiteness of concepts and domains with the result that definiteness is also explained as rigidness across contexts.[6]

In sections 5 and 6 we will throw a third notion into the mix: this is the familiar logical notion of *absoluteness* of concepts. Results from this section will show that provable definiteness and provable absoluteness are closely linked for semi-intuitionistic set theories. Interestingly, definiteness has been construed as determinateness of sense in a paper by Chris Scambler [71] and the latter notion has been linked to absoluteness between certain test structures, which he takes to be the transitive models of **ZFC**.[7]

2.2 The actual and the potential

Another way of discerning the correct logic governing collections of objects is informed by an old Aristotelian distinction. Often the line of demarcation between platonic realism on the one hand and constructivism, nominalism and fictionalism on the other hand is described as follows: Platonic realism holds that set theory (**ZFC**) has cut the nature of the mind independent mathematical world at the joints whereas constructivists and nominalists view these pretensions as an elaborately disguised game of make-believe, insisting that objects of mathematical thought exist only as mental or intersubjectively shared symbolic constructions. The boundary remains contested. An important move to potentially more neutral and fruitful grounds, that also renders the boundary more permeable, is the distinction between *actualism* and *potentialism*. It plays a central role in [48, 47] and has also been independently employed by Peter Koellner. This is a rough distinction with a long history that can be traced back to Aristotle if not further.[8] One way of formally delineating this intuitively appealing opposition is by employing intuitionistic logic for domains for which one is a potentialist whilst earmarking classical logic for domains for which one is an actualist.

To summarize, the purpose of this section was to show that semi-intuitionism in mathematics arises naturally from philosophical reflections about the existence of mathematical objects, the extension of mathematical concepts, and the meaning of mathematical statements.

[6]It should perhaps be stressed that these ideas aren't entirely new. The idea of using modality to interpret mathematical existence occurs in Parsons writings (e.g. [56]) much earlier.

[7]The referee of this paper queried whether I was gesturing at the claim: "If an informal concept is given several independent analyses and these different analyses are subsequently shown to be equivalent (as in the case of computability and predicativity) this is evidence that the concept is a robust one." I think this is a very good point, although I hesitate to articulate it in such a strong form as I think that more technical work is required to get a more thorough understanding of the connections.

[8]Recall that Aristotle, along with just about every major mathematician and philosopher before the nineteenth century, rejected the very notion of the actual infinite. They argue that the only sensible notion is that of potential infinity (at least for scientific or otherwise non-theological purposes).

3 Semi-intuitionistic set theory

The framework that Feferman proposed for the study of structural conceptions that are considered to be lacking some aspect of definiteness is semi-intuitionistic set theory. The first crucial step is to view the set-theoretic universe as an unfinished domain in which a subcollection is a definite totality if and only if it forms a set. Furthermore, the predicates of elementhood and equality are taken to be definite. As a result, classical reasoning is available for bounded set-theoretic formulas.

We will first present the system **SCS** of semi-constructive set theory. This will be followed by a discussion of the rationale for choosing these particular axioms.

Definition 3.1 **SCS** is formulated in the usual language of set theory containing $\in$ as the only non-logical symbol besides $=$. Formulas are built from atomic formulas $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Quantifiers of the forms $\forall x \in a$, $\exists x \in a$ are called *bounded. Bounded* or Δ_0*-formulas* are those wherein all quantifiers are bounded. **SCS** is based on intuitionistic logic. Basic axioms are the *restricted Law of Excluded Middle*:

$$(\Delta_0\text{-LEM}) \quad \varphi \vee \neg\varphi, \text{ for all } \Delta_0\text{-formulae } \varphi,$$

and *Extensionality, Pair,* and *Union* in their usual form. **SCS** also has an axiom asserting the existence of an infinite set, though in the specific version that there is a smallest set containing the empty set 0 which is closed under the successor operation, $x' = x \cup \{x\}$), i.e.,

Infinity Axiom

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u = 0 \vee \exists v \in x\, u = v')].$$

Further axioms are the following.

Bounded Separation

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u \in a \wedge \varphi(u))]$$

for all bounded formulas $\varphi(u)$.

Set Induction

$$\forall x\ [(\forall y \in x\, \theta(y)) \rightarrow \theta(x)] \rightarrow \forall x\, \theta(x)$$

for all formulas $\theta(x)$.

(BOS) (*Bounded Omniscience Scheme*)

$$(\text{BOS}) \quad \forall x \in a\, [\varphi(x) \vee \neg\varphi(x)] \rightarrow [\forall x \in a\, \varphi(x) \ \vee\ \exists x \in a\, \neg\varphi(x)]$$

for all formulas $\varphi(x)$.

$$(\text{AC}_{\text{Set}}) \quad \forall x \in a\, \exists y\, \psi(x, y) \rightarrow \exists f\, [\text{Fun}(f) \wedge \text{dom}(f) = a \wedge \forall x \in a\, \psi(x, f(x))]$$

for *all* formulas $\psi(x, y)$, where $\text{Fun}(f)$ expresses in the usual set-theoretic form that f is a function, and $\text{dom}(f) = a$ expresses that the domain of f is the set a.

Markov's Principle in the form

$$\text{(MP)} \quad \neg\neg\exists x\varphi \to \exists x\varphi$$

for all Δ_0-formulas φ.

Note that **SCS** is an extension of the intuitionistic cousin of classical Kripke-Platek set theory, **KP**. The latter is an important theory that accommodates a great deal of set theory. Its transitive models, called admissible sets, have been a major source of interplay between model theory, recursion theory and set theory (cf. [3]).

Definition 3.2 *Intuitionistic Kripke-Platek set theory*, **IKP** lacks the axioms (Δ_0-LEM), (BOS), (AC_{Set}), and (MP) from **SCS** but has *Bounded Collection*

$$\forall x \in a\, \exists y\, \psi(x,y) \;\to\; \exists z\, \forall x \in a\, \exists y \in z\, \psi(x,y)$$

for all bounded formulas $\psi(x,y)$.

Note that Bounded Collection is a consequence of (AC_{Set}) in **SCS**, so in this way **IKP** is a subtheory of **SCS**.

The study of subsystems of **ZF** formulated in intuitionistic logic with Bounded Separation goes back a long way. They were first proposed by Pozsgay [58, 59] and then investigated more systematically by Tharp [74], Friedman [26] and Wolf [78]. These systems are actually semi-intuitionistic as they contain the law of excluded middle for bounded formulas.

Remark 3.3 (i) **SCS** proves the full replacement schema of **ZF**. Moreover, **SCS** proves *Strong Collection*, i.e. all formulas

$$\forall x \in a\, \exists y\, \varphi(x,y) \to \exists z\, [\forall x \in a\, \exists y \in z\, \varphi(x,y) \;\wedge\; \forall y \in z\, \exists x \in a\, \varphi(x,y)]$$

where $\varphi(x,y)$ is an arbitrary formula.

Strong collection is an axiom schema of Constructive Zermelo-Fraenkel set theory, **CZF** (cf. [1, 2]) and also of Tharp's set theory [74].

(ii) **SCS** is a subtheory of Tharp's quasi-intuitionistic set theory of [74], for if $\forall x \in a\, \exists y\, \varphi(x,y)$ holds, then there is a set d such that $\forall x \in a\, \exists z \in d\, \exists y[z = \langle x,y\rangle \wedge \varphi(x,y)]$ and $\forall z \in d\, \exists x \in a\, \exists y[z = \langle x,y\rangle \;\wedge\; \varphi(x,y)]$ by axiom 10 of [74] (which is the same as Strong Collection), and, by axiom 6 of that system, d is the surjective image of an ordinal, i.e., there is an ordinal α and a function g with domain α and range d. Note that d is a set of ordered pairs. Now define a function f with domain a by letting $f(x)$ be the second projection of $g(\xi)$ where ξ is the least ordinal $< \alpha$ such that the first projection of $g(\xi)$ equals x.

(iv) For the sake of its justification, it is perhaps useful to view the axiom schema (AC_{Set}) as composed of two parts, namely

1. *Strong Collection* and the
2. *Axiom of Choice*, AC,

$$\forall x[x \neq \emptyset \to \exists y \in x\, x \cap y = \emptyset].$$

Strong Collection has a constructive justification via the interpretation in Martin-Löf type theory.[9] AC on the other hand is of a local nature and maybe justified on semi-constructive grounds, allowing classical reasoning with choice for fully formed (determinate) parts of the universe (a position taken for instance by Bill Tait). On the other hand, one might raise reasonable objections against adopting AC for completed totalities. In set theory, one is also looking at large cardinal notions that are incompatible with AC and the axiom of determinacy is often discussed in contexts where $\mathbb{R}$ is not well-orderable. If AC were that easily justifiable, then the existence of such cardinals as well as the axiom of determinacy could be almost ruled out by "logic".

The axioms (Δ_0-LEM) and (BOS) of **SCS** are actually redundant.

Proposition 3.4 **IKP** + ($\mathrm{AC_{Set}}$) *proves* (Δ_0-LEM) *and* (BOS).

Proof: See [65, Proposition 2.3], where Diaconescu's [12] construction is used. □

3.1 Justifying the axioms of SCS

SCS furnishes a template for discussing questions of definiteness of concepts and statements. Formally a concept $C(\vec{x})$ (expressible in the language of set theory) is said to be *definite* with respect to **SCS** if

$$\mathbf{SCS} \vdash \forall \vec{x}\, [C(\vec{x}) \;\vee\; \neg C(\vec{x})]$$

and a set-theoretic statement θ is *definite* with respect to **SCS** if

$$\mathbf{SCS} \vdash \theta \;\vee\; \neg\theta\,.$$

What seems to underly many conceptions of the set-theoretic universe that regard it as an indefinite totality is a generative approach to its ontology. Sets are generated or arise in stages by various operations applied to sets from earlier stages. This view is clearly present in Russell and Dummett but also part of the orthodox justification of the axioms of **ZFC** by means of the the cumulative hierarchy of sets. There are also tendencies to avoid the language of action by using modal terminology, stressing the potential nature of the universe. Such explanations "replace the language of time and activity with the more bloodless language of potentiality and actuality" (Parsons 1977, [56, p. 355]). Thus the asymmetrical temporal relation of creation between a set and its elements gives rise to relations of necessity and potentiality in that the existence of a set necessarily depends

[9] One of the best accounts of the constructivity of Strong Collection can be found in [54, p. 351].

on the existence of its elements whereas its own existence is only contingent or potential relative to them. Feferman in [23, p. 16] argues that semi-intuitionism is the logic of partially open-ended structures.[10] He uses Kripke models as a motivating idea, where the elements of its non-empty partially ordered set $(K, \leq)$ represent stages of knowledge. More precisely, a Kripke model is a quadruple $(K, \leq, D, v)$ where D is a function that assigns to each $\alpha \in K$ a non-empty set $D(\alpha)$ such that $D(\alpha) \subseteq D(\beta)$ whenever $\alpha \leq \beta$, and v is a function onto $\{0, 1\}$ at each $\alpha \in K$ such that

$$\alpha \leq \beta \wedge \vec{d} \in D(\alpha) \wedge v(\alpha, R(\vec{d})) = 1 \quad \Rightarrow \quad v(\beta, R(\vec{d})) = 1 \tag{1}$$

holds for each r-ary relation symbol R, where $\vec{d} \in D(\alpha)$ stands for $d_1, \ldots, d_r \in D(\alpha)$. Here $v(\alpha, R(\vec{d})) = 1$ means that $R(\vec{d})$ has been recognized as true at level α. The domain $D(\alpha)$ has been surveyed at stage α. Thus (1) expresses that an atomic statement, once recognized as true, stays true. The domains $D(\alpha)$ may increase forever as α increases. They may also bifurcate as $(K, \leq)$ need not be a total ordering so that one cannot speak of a final domain.[11]

Next, in light of some of the foregoing explanations of the ontology of V, we attempt to provide intuitive reasons for adopting the axioms of **SCS**. We will be brief, though. As a result of taking sets to be the definite subdomains of V and endorsing $\in$ and $=$ as definite concepts, the governing logic of Δ_0-formulas is classical logic, whence Δ_0-LEM holds true. Since Δ_0-formulas do not change their meaning according to the conception that V is a Kripke-model whose domains are equated with the sets, Bounded Separation is also justified. Pair and Union refer to basic constructions on sets while Extensionality reflects the fact that in mathematics we like to work with extensional objects, and more to the point, it is part of what we 'mean' by a set. The Infinity Axiom should perhaps be accepted on grounds that it axiomatizes the most elementary inductively generated set which is widely accepted in Martin-Löf type theory and other constructive frameworks. Clearly one has the option of eschewing this axiom. Calling this axiomatic system $\mathbf{SCS}^-$, we will investigate the scope of $\mathbf{SCS}^-$ in subsection 3.2.1. The Set Induction axiom reflects the generative nature of V as being built in stages. $\mathrm{AC}_{\mathrm{Set}}$ is perhaps the most daring axiom scheme of **SCS**. As indicated before, it can be separated into two parts, namely Strong Collection and AC. One could justify AC by saying that the generation of sets in stages puts a well-ordering on the universe V. This would actually endorse the stronger global axiom of choice, which we will consider later. One could raise the objection that this would render the universe perhaps too "L-like". We already raised this point in Remark 3.3(iv). The layering of the universe in stages is not necessarily sufficient to secure AC as is obvious in the case of the usual von Neumann hierarchy.

Turning to Strong Collection, we have to justify

$$\forall x \in a \, \exists y \, \varphi(x, y) \to \exists z \, [\forall x \in a \, \exists y \in z \, \varphi(x, y) \wedge \forall y \in z \, \exists x \in a \, \varphi(x, y)]$$

for an arbitrary formula $\varphi(x, y)$. Here a word of explanation is needed as to why we impose a restriction on the formula in Bounded Separation but not on this axiom. One

[10] Parsons [56] also uses these structures.

[11] Again, the same or similar observations were made by Parsons [56].

important difference is its hypothetical logical nature. Here one works on the assumption that $\forall x \in a\, \exists y\, \varphi(x, y)$ is true in V. The concept corresponding to $\varphi(x, y)$ may contain arbitrarily many unbounded quantifiers referring to an "unfinished" universe. An instance-based account of truth is not available for such a formula, so the recognition of its validity must have come about in an entirely different way. Its truth must be based on facts that only refer to a definite part of V in combination with other universal truth which, echoing Weyl [77, p. 54], must be of a generic form that is based on the fact that it "lies in the essence" of the concept set. As a result, the insight that the antecedent is true must stem from a procedure that produces for every $x \in a$ a set y standing in the relation $\varphi(x, y)$, however wildly complicated that relation may be.[12]

It remains to argue for Markov's principle. So assume that $\neg\neg\exists x\varphi(x)$ is true where $\varphi(x)$ is Δ_0. If $\neg\neg\exists x\varphi(x)$ is true its truth must be based on facts that only refer to a definite part B of V combined with generic knowledge about the collection of all sets. We know that the assumption of the non-existence of a set x with $\varphi(x)$ leads to a contradiction. But the only fact about V that could yield a contradiction from $\neg\exists x\varphi(x)$ would be a set u such that $\varphi(u)$ holds. If we can systematically search through V as more and more parts become actual, such a search, set by set, will eventually be successful since the predicate $\varphi(x)$ is checkable (on account of being determinately true or false for every set x) and we have a guarantee stemming from the truth of $\neg\exists x\varphi(x)$ that we will eventually hit upon a set b such that $\varphi(b)$ holds. Note that this argument in favor of (MP) seems to depend on an assumption of 'searchability' or 'surveyability' of sets and therefore appears to require a global well-ordering or at least a global choice principle. Since the global axiom of choice is less familiar and will play an important role in sections 5 and 6, we introduce it at this point.

Definition 3.5 If we add the *Axiom of Global Choice*, $\mathbf{AC}_{global}$, to a set theory T, we mean by $T + \mathbf{AC}_{global}$ an extension of T where the language contains a new binary relation symbol R and the axiom schemes of T are extended to this richer language and the following axioms pertaining to R are added:

$$(i) \qquad \forall x \forall y \forall z [\mathsf{R}(x, y) \wedge \mathsf{R}(x, z) \rightarrow y = z] \tag{2}$$

$$(ii) \qquad \forall x [x \neq \emptyset \rightarrow \exists y \in x\, \mathsf{R}(x, y)]. \tag{3}$$

Remark 3.6 (i) It is well-known that $\mathbf{ZF} + \mathbf{AC}_{global}$ is a conservative extension of **ZFC**, i.e., both prove the same theorems of the language of **ZF**.

(ii) For the set theories T of relevance to this paper, $T + \mathbf{AC}_{global}$ can obviously be viewed as a subtheory of $T + V = L$, where $V = L$ signifies the *Constructibility Axiom* (saying that every set is constructible). Under $V = L$ one has a Δ_1-definable well-ordering of the universe. It is interesting to note, though, that there are theories T such that $T + V = L$ is proof-theoretically much stronger than T while $T + \mathbf{AC}_{global}$ retains the same strength as T. For an example see Corollary 3.20 and Theorem 3.21.

[12] It's perhaps worth noting the connection of this interpretation with realizability interpretations of intuitionistic theories which we will see later.

(iii) On a personal note, let me add that I have discussed with Solomon Feferman the possibility of incorporating $\mathbf{AC}_{global}$ as a basic axiom in **SCS**. One of the reasons I put forward was that it lends more credence to (MP), or differently put that (MP) appears to presuppose $\mathbf{AC}_{global}$. He was quite sympathetic toward that view.

(iv) Adding $\mathbf{AC}_{global}$ to **SCS** does not increase the latter's proof-theoretic strength. Moreover, $\mathbf{SCS}+\mathbf{AC}_{global}$ is conservative over **SCS** for Π_2-theorems of the language of **SCS** (see Theorem 5.3(ii)).

(v) As **SCS** and its extensions are used to establish independence and indefiniteness results, a further argument in favor of $\mathbf{AC}_{global}$ is that these results still hold if one adds $\mathbf{AC}_{global}$ to the pertaining theories, thus yielding stronger consequences. One aspect of the idealization is to give the strongest *negative* results, to show that certain kinds of problems are indefinite even in the presence of $\mathbf{AC}_{global}$.

3.2 Relating SCS to various foundational schools

The introduction to this paper referred to different foundational schools that are reluctant to countenance the full use of the infinite, chiefly associated with the names of Brouwer, Poincaré, Weyl, Borel and Zermelo. Here we will attempt to associate them with formal semi-intuitionistic set theories. Historical accuracy will not be a major concern.

3.2.1 The finitistic flavor

If we view the naturals as an unfinished totality then we have to jettison the Infinity Axiom. Let $\mathbf{SCS}^-$ be **SCS** bereft of Infinity. Via the usual treatment of the natural numbers as ordinals (below ω if ω exists) we get an embedding of intuitionistic arithmetic, Heyting arithmetic **HA**, in $\mathbf{SCS}^-$.

Theorem 3.7 *(i) Via the standard translation,* **HA** *is a subsystem of* $\mathbf{SCS}^-$*.*

(ii) There are statements of arithmetic that are not provable in **HA** *but whose set-theoretic translation is provable in* $\mathbf{SCS}^-$*.*

(iii) $\mathbf{SCS}^-$ *has the same proof-theoretic strength as* **HA** *and* **PA***. The three systems prove the same* Π^0_2*-statements.*

(iv) The theory resulting from $\mathbf{SCS}^-$ *by adding classical logic possesses the same strength as* $\mathbf{SCS}^-$*. It proofs the same arithmetical theorems as* **PA***.*

Proof: (i) is straightforward. For (ii) note that **HA** does not prove Markov's principle. (iii): Note that the hereditarily finite sets form a model of $\mathbf{SCS}^-$. It is well-known that the latter sets can be modeled in **PA**. It is also known that **HA** and **PA** prove the same Π^0_2-statements of arithmetic. Thus all three theories coincide at the Π^0_2 level.

As to (iv), note that the the usual interpretation of set theory without Infinity in **PA** via the model of hereditarily finite sets validates all axioms of $\mathbf{SCS}^-$ and classical logic. □

3.2.2 The predicative flavor

SCS can be viewed as a predicative system as it only takes the infinite set of naturals for granted but does not give countenance to the Powerset Axiom. The strength of **SCS**, though, lies beyond Γ_0. In famous work of Feferman and Schütte, the latter was identified as the first ordinal that can not be reached by systems that are predicative in the sense of autonomous progressions of theories. Feferman [18, Theorem 6] shows the following.

Theorem 3.8 $\mathbf{SCS} = \mathbf{IKP} + (\Delta_0\text{−LEM}) + (\text{MP}) + (\text{BOS}) + (\text{AC}_{\text{Set}})$ *has the same proof-theoretic strength as* **KP** *(and therefore the same as* **IKP***). Thus the proof-theoretic ordinal of* **SCS** *is the famous Bachmann-Howard ordinal.*

Proof: The proof of [18, Theorem 6] uses a functional interpretation. The same result can be obtained via a realizability interpretation using codes for Σ_1 partial recursive set functions as realizers along the lines of Tharp's 1971 article [74]. □

As far as proof-theoretic strength is concerned, the foregoing Theorem shows that $(\Delta_0\text{−LEM})$ and (MP) do not contribute to it. The same holds for **AC**. Moreover, the Δ_0-Collection entailed in (AC_{Set}) suffices; and even that could be replaced by Δ_0-Replacement.

Let's see what happens to **SCS** if one adds classical logic to it.

Proposition 3.9 *The strength of* **SCS** *augmented by classical logic,* $\mathbf{SCS}^c$*, is the same as that of formal second order arithmetic with the axiom of choice.*

Proof: Note that in the presence of classical logic, (AC_{Set}) implies that for every formula $\phi(x)$, the class $\{x \in \mathbb{N} \mid \phi(x)\}$ is a set just in the same way as Replacement entails full separation in classical set theory. Thus second order arithmetic plus **AC** can be seen to be contained in $\mathbf{SCS}^c$. Conversely, in second order arithmetic countable sets can be modeled via well-founded trees. The latter interpretation validates all the axioms of $\mathbf{SCS}^c$ when working in second order arithmetic with **AC**. It is also known that the latter theory is Π^1_4 conservative over second order arithmetic (without **AC**). □

3.2.3 The semi-intuitionistic descriptive set theory flavor

According to some classical *Descriptive Set Theorists*, the set $\mathbb{R}$ of real numbers is a definite totality but not the supposed totality of arbitrary subsets of $\mathbb{R}$. Here one takes the real numbers to form a definite totality and explores what can be obtained on that basis alone. A formal framework in which such a viewpoint can be explored is

$$\mathbf{SCS}^+ := \mathbf{SCS} + \mathbb{R} \text{ is a set.}$$

Since $\mathbf{SCS}^+$ has classical logic for Δ_0-formulas it is not necessary to pay much attention to how the reals are actually formalized as is so often the case in intuitionistic contexts. Thus, any of the following equivalent statements could be used to formalize the existence of $\mathbb{R}$ as a set:

- The collection of all functions from $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, is a set.

- The collection of all subsets of $\mathbb{N}$ is a set.

Proposition 3.10 *(i) The proof-theoretic strength of* $\mathbf{SCS}^+$ *resides strictly between full classical second order arithmetic and Zermelo set theory. More precisely, the proof-theoretic strength of* $\mathbf{SCS}^+$ *is greater than that of second order arithmetic but weaker than that of third order arithmetic augmented by* AC*.*

(ii) The strength of the classical version of $\mathbf{SCS}^+$ *is that of third order arithmetic plus* AC*.*

(iii) All theorems of classical second order arithmetic with the axiom of choice are theorems of $\mathbf{SCS}^+$*.*

Proof: See [65]. □

Definiteness with regard to $\mathbf{SCS}^+$ can be extended a bit above the level of second order arithmetic. An important example is that of *projective determinacy*, PD, asserting that all projective games are determined (see e.g. [39]).

Theorem 3.11 $\mathbf{SCS}^+ \vdash \mathrm{PD} \vee \neg\mathrm{PD}$.

Proof: [65, Theorem 7.2]. □

An important test case for definiteness is also provided by the continuum hypothesis, CH. CH is the statement that every infinite set of reals is either in one-one correspondence with $\mathbb{N}$ or with $\mathbb{R}$. More formally, this can be expressed as follows:

$$\forall x \subseteq \mathbb{R}\, [x \neq \emptyset \rightarrow (\exists f\, f : \omega \twoheadrightarrow x \vee \exists f\, f : x \twoheadrightarrow \mathbb{R})]$$

where $f : y \twoheadrightarrow z$ signifies that f is a surjective function with domain y and co-domain z.

3.2.4 The Zermelo or height potentialism flavor

This version approves of the powerset operation as a definite operation. The class $\mathcal{P}(x)$ of all subsets of a set x is thus considered to be a definite totality, i.e., $\mathcal{P}(x)$ is a set. However, V is still not a definite totality as the line of ordinals stretches into the potential, although each chunk V_α of the von Neumann hierarchy is actual for an actual ordinal α as the powerset operation can be iterated along α. From this point of view of height potentialism, not all of the axioms of **ZFC** seem to be justifiable. It is, e.g., unclear that Replacement applied to formulas with arbitrary alternations of quantifiers make sense on the height potentialist conception, hence the closer affinity of this conception to Zermelo set theory. One could perhaps call this *Power Predicative Set Theory.* Formally we take this to be the theory

$$\mathbf{SCS}(\mathcal{P}) := \mathbf{SCS} + \textit{Powerset Axiom}.$$

This theory lends itself to a snappy description in terms of the actualism versus potentialism view. In it the powerset $\mathcal{P}(x)$ springs from x as a fully-formed set. The canonical view of V is that it arises by iterating the powerset operation along the ordinal spine and the step from V_α to $V_{\alpha+1}$ is maximal and allows for no further width extension in the sense that no new subsets of V_α arise at stages $\beta > \alpha + 1$. Thus one could summarize this by saying that adherents of $\mathbf{SCS}(\mathcal{P})$ are width actualists but height potentialists.

It is easy to see that the classical version of $\mathbf{SCS}(\mathcal{P})$ is a familiar theory.

Proposition 3.12 $\mathbf{SCS}(\mathcal{P})$ *plus classical logic is the same as* **ZFC**.

Proof: This is clearly the case since classically replacement yields full separation and this theory has the powerset axiom. ☐

In order to find a classical theory of the same strength as $\mathbf{SCS}(\mathcal{P})$ we have to look at less familiar places. *Power Kripke-Platek set theory* is obtained from **KP** by also viewing the creation of the powerset of any set as a basic operation performed on sets. In the classical context, subsystems of **ZF** with Bounded Separation and Power Set have been studied by Thiele [75], Friedman [27] and more recently in great depth by Mathias [51]. They also occur naturally in power recursion theory, investigated by Moschovakis [52] and Moss [53], where one studies a notion of computability on the universe of sets which regards the power set operation as an initial function. Semi-intuitionistic set theories with Bounded Separation but containing the Power Set axiom were proposed by Pozsgay [58, 59] and then studied more systematically by Tharp [74], Friedman [26] and Wolf [78]. Such theories are naturally related to systems derived from topos-theoretic notions and to type theories (e.g., see [66]). Mac Lane has singled out and championed a particular fragment of **ZF**, especially in his book *Form and Function* [49]. *Mac Lane Set Theory*, christened **MAC** in [51], comprises the axioms of Extensionality, Null Set, Pairing, Union, Infinity, Power Set, Bounded Separation, Foundation, and Choice.

To state the axioms of $\mathbf{KP}(\mathcal{P})$ it is convenient to introduce another type of bounded quantifiers.

Definition 3.13 We use subset bounded quantifiers $\exists x \subseteq y \; \ldots$ and $\forall x \subseteq y \; \ldots$ as abbreviations for $\exists x(x \subseteq y \wedge \ldots)$ and $\forall x(x \subseteq y \rightarrow \ldots)$, respectively.

The $\Delta_0^{\mathcal{P}}$-formulas are the smallest class of formulas containing the atomic formulas closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a, \; \exists x \in a, \; \forall x \subseteq a, \; \exists x \subseteq a.$$

A formula is in $\Sigma^{\mathcal{P}}$ if belongs to the smallest collection of formulae which contains the $\Delta_0^{\mathcal{P}}$-formulas and is closed under $\wedge, \vee$ and the quantifiers $\forall x \in a$, $\exists x \in a$, $\forall x \subseteq a$ and $\exists x$. A formula is $\Pi^{\mathcal{P}}$ if belongs to the smallest collection of formulas which contains the $\Delta_0^{\mathcal{P}}$-formulas and is closed under $\wedge, \vee$, the quantifiers $\forall x \in a$, $\exists x \in a$, $\forall x \subseteq a$ and $\forall x$.

Definition 3.14 $\mathbf{KP}(\mathcal{P})$ has the same language as **ZF**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation, $\Delta_0^{\mathcal{P}}$-Collection and Set Induction (or Class Foundation).[13]

The transitive models of $\mathbf{KP}(\mathcal{P})$ have been termed **power admissible** sets in [27].

[13] The system $\mathbf{KP}(\mathcal{P})$ in the present paper is not quite the same as the theory $\mathbf{KP}^{\mathcal{P}}$ in Mathias' paper [51, 6.10]. The difference between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}^{\mathcal{P}}$ is that in the latter system set induction only holds

Remark 3.15 Alternatively, $\mathbf{KP}(\mathcal{P})$ can be obtained from $\mathbf{KP}$ by adding a function symbol $\mathcal{PC}$ for the powerset function as a primitive symbol to the language and the axiom

$$\forall y\,[y \in \mathcal{PC}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0 Separation and Collection to the Δ_0-formulas of this new language.

Lemma 3.16 *$\mathbf{KP}(\mathcal{P})$ is* **not** *the same theory as $\mathbf{KP} + \mathbf{Pow}$. Indeed, $\mathbf{KP} + \mathbf{Pow}$ is a much weaker theory than $\mathbf{KP}(\mathcal{P})$ in which one cannot prove the existence of $V_{\omega+\omega}$.*

Proof: See [67, Lemma 2.4]. □

[67] featured an ordinal analysis of $\mathbf{KP}(\mathcal{P})$. As it turns out the technique can be augmented to also yield an ordinal analysis of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$. A refinement of [67, Theorem 8.1] then yields partial conservativity of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ over $\mathbf{KP}(\mathcal{P}) + \mathrm{AC}$.

Theorem 3.17 *Let A be a $\Sigma^{\mathcal{P}}$ sentence of the language of set theory without R. If $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash A$ then $\mathbf{KP}(\mathcal{P}) + \mathrm{AC} \vdash A$.*

Proof: [68, Theorem 3.3]. □

The latter result has been improved in [10].

Theorem 3.18 *Let B be $\Pi_2^{\mathcal{P}}$-sentence of the language without the predicate R. If $\mathbf{KP}(\mathcal{P})+ \mathbf{AC}_{global} \vdash B$, then $\mathbf{KP}(\mathcal{P}) + \mathrm{AC} \vdash B$.*

Proof: [10, Theorem 9.1]. □

Finally we would like to calibrate the proof-theoretic strength of $\mathbf{KP}(\mathcal{P})$ in ordinal terms.

Theorem 3.19 *If $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash \theta$, where θ is a $\Sigma^{\mathcal{P}}$-sentence not containing R, then one can explicitly find an ordinal (notation) τ less than the Bachmann-Howard ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ such that*

$$\mathbf{KP} + \mathrm{AC} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha\le\tau} \texttt{ exists} \vdash \theta.$$

Proof: This is a consequence of [68, Theorem 3.3] and Theorem 3.17. □

Corollary 3.20 *$\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ and $\mathbf{KP}(\mathcal{P})$ have the same proof-theoretic strength. They prove the same Π^1_4-sentences of second order arithmetic.*

An important fact to be mentioned is that $\mathbf{AC}_{global}$ behaves very differently from the hypothesis $V = L$ in the context of $\mathbf{KP}(\mathcal{P})$. Whereas the former does not add any proof-theoretic strength the latter raises it considerably.

for $\Sigma_1^{\mathcal{P}}$-formulas, or what amounts to the same, $\Pi_1^{\mathcal{P}}$ foundation ($A \neq \emptyset \to \exists x \in A\; x \cap A = \emptyset$ for $\Pi_1^{\mathcal{P}}$ classes A).

Friedman [27] includes only Set Foundation in his formulation of a formal system $\mathbf{PAdm}^s$ appropriate to the concept of recursion in the power set operation $\mathcal{P}$.

Theorem 3.21 (Mathias, 2001) $\mathbf{KP}(\mathcal{P}) + V = L$ *is much stronger than* $\mathbf{KP}(\mathcal{P})$.

Proof: [51, Theorem 9]. $\square$

Finally we would like to connect $\mathbf{SCS}(\mathcal{P})$ and $\mathbf{KP}(\mathcal{P})$.

Theorem 3.22 $\mathbf{SCS}(\mathcal{P})+\mathbf{AC}_{global}$ *and* $\mathbf{KP}(\mathcal{P})+\mathbf{AC}_{global}$ *prove the same* $\Pi_2^{\mathcal{P}}$ *sentences. Thus, in view of Theorem 3.19, it follows that* $\mathbf{SCS}(\mathcal{P})$ *is much weaker than* $\mathbf{ZF}$.

Proof: $\mathbf{SCS}(\mathcal{P})+\mathbf{AC}_{global}$ has a realizability interpretation in $\mathbf{KP}(\mathcal{P})+\mathbf{AC}_{global}$ via codes for $\Sigma^{\mathcal{P}}$ definable partial functions. Moreover, this interpretation preserves the validity of $\Pi_2^{\mathcal{P}}$ sentences. Conversely, for each $\Pi_2^{\mathcal{P}}$ theorem of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ one can carry out the ordinal analysis of [68] within the theory $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global}$ in the relativized fashion of [10]. $\square$

4 Digression: The continuum problem and indeterminacy

Cantor sought to determine the size of the continuum among the infinite cardinals. The continuum hypothesis, CH, asserts that there are no cardinalities strictly between the cardinality of the natural numbers and that of the reals. On Hilbert's famous list of 23 mathematical problems, the continuum problem occupies place number 1. Hilbert, in his paper *Über das Unendliche*, [37], from 1925, sketched a 'proof' of CH. Instead of $\mathbb{R}$, he considers the set $\mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$. Hilbert's proof hinges on a remarkable and crucial assumption.

> *Wenn wir die Menge dieser Funktionen im Sinne des Kontinuumproblems ordnen wollen, so bedarf es dazu der Bezugnahme auf die* **Erzeugung** *der einzelnen Funktionen.* [37, p. 181][14]

Thus [37] can be seen as the birth place of an idea giving rise to Gödel's *constructible hierarchy*, L, which indeed validates CH.

Gödel [29] and Cohen [9], via L and forcing, respectively, provided results to the effect that **ZFC** does not determine CH. One could be forgiven for thinking that these negative results would completely discourage set theorists from talking about the truth of CH as such discussions seemingly belong to a former unenlightened age. But this is far from true. In particular Hugh Woodin's publications (cf. [79, 80, 81, 82]) and Peter Koellner's *Exploring the Frontiers of Incompleteness* project have injected new life into this debate (for more details also see [40]). Before looking at the different shades of more recent discussions of the nature of CH, let's briefly hark back to a more innocent state of mind for which Joel Hamkins' "Dream solution template" (cf. [36, p. 430]) provides a nice, if somewhat guileless, backcloth. The dream solution template for determining truth of a statement Θ (e.g. $\Theta = \text{CH}$) in the universe of sets, V, proceeds as follows:

[14]The emphasis is mine. In the English translation from [76, p. 385]: *"If we want to order the set of these functions in the way required by the problem of the continuum, we must consider how an individual function is* **generated**.*"*

- **Step 1**. Produce a set-theoretic assertion Φ expressing a natural and "*intuitively true*" set-theoretic principle.

- **Step 2**. Prove that Φ determines Θ. That is, prove

$$\Phi \Rightarrow \Theta$$

or prove that

$$\Phi \Rightarrow \neg\Theta.$$

At times, Gödel may have fostered hopes akin to to the dream solution template in what has been called the *intrinsic program* for extending **ZFC**. Chiefly the aim is to augment **ZFC** via new axioms embodying strong reflection principles. In 1964 Gödel wrote that:

> *"axioms of set theory* [**ZFC**] *by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of"."* [31, p. 260]

He mentions as examples the axioms asserting the existence of *inaccessible* and *Mahlo cardinals* and maintains that

> *"[t]hese axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set as explained above"* [31, p. 260–261].

Gödel later refers to such axioms as having an *"intrinsic necessary"* status. He knew, of course, that "small" large cardinals such as inaccessible and Mahlo cardinals have no bearing on CH. It is not clear to this writer whether Gödel, at any time, thought that the existence of specific intrinsically justified large cardinals could settle the status of CH. Late in his life, Gödel wrote an unpublished note [34], wherein he attempted to determine the truth of CH from a "highly plausible axiom about orders of growth", referring to a principle about the scale of the orders of growth of the functions $\omega \to \omega$ that Emil Borel had considered as self-evident. This seems to indicate that Gödel, at least sometimes, thought that an intrinsic solution to the CH question might be possible.[15] Be this as it may, a serious obstacle for any intrinsic program aimed at settling CH emerged from the following results.

Theorem 4.1 (Cohen 1963 [9]; Levy and Solovay 1967 [45]) CH *is consistent with and independent of all "small" and "large" cardinal axioms that have been considered to date, provided they are consistent with* **ZF**.

[15]The referee of this paper, however, thinks that the 'dream solution' is a dream that no one has had, and that Gödel knew that this was just a pipe dream.

Proof: Via Cohen's method of forcing. □

Thus it became apparent that large cardinals could not be used to decide the truth value of CH. Another route to settling CH, also pursued by Gödel, is the so-called *extrinsic program*, where new set-theoretic principles, rather than through considerations of intrinsic necessity, acquire their status as axioms, as it were, *a posteriori* through their emanating fruitfulness.

> *"[E]ven disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its success" [...]*
>
> *There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline [...] that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory."* (Gödel 1947 [30] and 1964 [31])

These days one can roughly allocate the responses to Cantor's continuum problem to three camps:

- A universe view (e.g. Hugh Woodin and Tony Martin think of V as a definite totality): CH has a truth value in *the* universe V.
- A multiverse view (e.g. Hamkins): There are many universes that set theorists study. They all exist. CH has different truth values in different universes.
- An intrinsic indefiniteness view (e.g. Feferman): CH is not a definite mathematical problem.

Woodin's universe view is in keeping with Gödel's extrinsic program. In [79] from 1999 his aim was the identification of a "canonical" model in which CH is false. In the second revised edition from 2010, however, the direction of finding an answer to the CH problem appears to have reversed, favoring another canonical model as these quotes from [82, p. 19] indicate: "Ultimately of far more significance for this book is that recent results concerning the inner model program undermine the philosophical framework for this entire work." "I think the evidence now favors CH." "The picture that is emerging now [...] is as follows. The solution to the inner model problem for one supercompact cardinal yields the ultimate enlargement of L. This enlargement of L is compatible with all stronger large cardinal axioms and strong forms of covering hold relative to this inner model."

A semi-intuitionistic set theory that is germane to discussing CH is $\mathbf{SCS}^+$. At the end of [19], in order to support his claim of the indefiniteness of the CH problem, Feferman surmises the following.

Conjecture 4.2 (Feferman) $\mathbf{SCS}^+$ *does not prove* CH $\vee \neg$CH.

The conjecture is now a theorem.[16]

[16]For comments by Feferman on this result also see [24].

Theorem 4.3 (Rathjen, [65]) $\mathbf{SCS}^+$ *does not prove* CH $\vee$ $\neg$CH.

It is interesting that this result still holds in the presence of PD.

Theorem 4.4 *Assuming* $\mathbf{ZFC} + \mathrm{PD}$ *in the background,*

$$\mathbf{SCS}^+ + \mathrm{PD} \nvdash \mathrm{CH} \vee \neg\mathrm{CH}\,.$$

Proof: [65, Theorem 7.2]. $\square$

The above indeterminateness results also hold if one adds $\mathbf{AC}_{global}$ to $\mathbf{SCS}^+$ since the realizability interpretations used in [65] also validate $\mathbf{AC}_{global}$.

5 Feferman's second conjecture: Part 1

We will now address a second conjecture of Solomon Feferman stated in [19, 20] concerning the relationship between the provable Δ_1 predicates and those predicates for which the law of excluded middle can be proved. In this section we show that Δ_1 predicates satisfy LEM.

Definition 5.1 For a predicate given by a formula $A(x)$ with at most x free we say that $A(x)$ is Δ_1 with respect to a theory $T \supseteq \mathbf{SCS}$ if there exist a Σ_1-formula $B(x)$ and a Π_1-formula $C(x)$ such that

$$T \vdash \forall x\,[A(x) \leftrightarrow B(x) \leftrightarrow C(x)].$$

$A(x)$ is said to satisfy the law of excluded middle, *LEM*, with respect to T if

$$T \vdash \forall x\,[A(x) \vee \neg A(x)].$$

In constructive mathematics, the latter property is also know as decidability (with respect to T).

We need a preliminary result about realizing **SCS** in Kripke-Platek set theory with global choice. Kripke-Platek set theory, **KP**, is assumed to include the axiom of Infinity. Recall from Definition 3.5 that by **KP** with global choice, i.e., $\mathbf{KP} + \mathbf{AC}_{global}$, we mean an extension of **KP** where the language contains a new binary relation symbol R, the axiom schemes of Δ_0-Separation and Δ_0-Collection are formulated for Δ_0-formulas of the extended language, and the following axioms pertaining to R are added:

$$(i) \quad \forall x \forall y \forall z[\mathsf{R}(x,y) \wedge \mathsf{R}(x,z) \rightarrow y = z]$$
$$(ii) \quad \forall x[x \neq \emptyset \rightarrow \exists y \in x\, \mathsf{R}(x,y)].$$

Theorem 5.2 $\mathbf{SCS} + \mathbf{AC}_{global}$ *has a realizability interpretation in* $\mathbf{KP} + \mathbf{AC}_{global}$*, where the realizers are codes for partial* Σ_1*-definable (in parameters) functions. To be precise, here* Σ_1*-definability refers to the extended language with* R.

Theorem 5.3 *(i)* $\mathbf{KP}+\mathbf{AC}_{global}$ *and* $\mathbf{SCS}+\mathbf{AC}_{global}$ *prove the same* Π_2*-sentences of the language with just* $\in$*.*

(ii) $\mathbf{SCS}+\mathbf{AC}_{global}$ *and* $\mathbf{SCS}$ *prove the same* Π_2*-sentences of the language with just* $\in$*.*

Proof: (i) If A is a Π_2 sentence and $\mathbf{SCS}+\mathbf{AC}_{global} \vdash A$, then A is realizable in $\mathbf{KP}+\mathbf{AC}_{global}$, but this entails that A is provable in $\mathbf{KP}+\mathbf{AC}_{global}$.

For the converse direction, to show that if $\mathbf{KP}+\mathbf{AC}_{global} \vdash A$ then $\mathbf{SCS} \vdash A$, one can use a relativized ordinal analysis of $\mathbf{KP}+\mathbf{AC}_{global}$. The details are presented in [10].

(ii) The relativized ordinal analysis of $\mathbf{KP}+\mathbf{AC}_{global}$ can be already be carried out in $\mathbf{SCS}$, yielding that every Π_2-theorem of $\mathbf{KP}+\mathbf{AC}_{global}$ is provable in $\mathbf{SCS}$. □

Theorem 5.4 *Suppose*

$$\mathbf{SCS}+\mathbf{AC}_{global} \vdash \forall\vec{x}\,[\exists y A(\vec{x},y) \leftrightarrow \forall z B(\vec{x},z)] \tag{4}$$

where A and B are Δ_0. Then

$$\mathbf{SCS}+\mathbf{AC}_{global} \vdash \forall\vec{x}\,[\exists y A(\vec{x},y) \vee \neg\exists y A(\vec{x},y)].$$

Proof: By Theorem 5.2, the statement $\forall\vec{x}\,[\exists y A(\vec{x},y) \leftrightarrow \forall z B(\vec{x},z)]$ is provably realizable in $\mathbf{KP}+\mathbf{AC}_{global}$. But then it's actually provable in $\mathbf{KP}+\mathbf{AC}_{global}$ as this flavor of realizability happens to coincide with truth for Σ_1 as well as Π_1 statements.

Now, in the classical theory $\mathbf{KP}+\mathbf{AC}_{global}$, (4) can be rendered in prenex normal form giving

$$\mathbf{KP}+\mathbf{AC}_{global} \vdash \forall\vec{x}\,\forall y\forall z\exists u\exists v[(A(\vec{x},y) \rightarrow B(\vec{x},z)) \wedge (B(\vec{x},u) \rightarrow A(\vec{x},v))].$$

Since the latter statement is Π_2, Theorem 5.3 yields

$$\mathbf{SCS} \vdash \forall\vec{x}\,\forall y\forall z\exists u\exists v[(A(\vec{x},y) \rightarrow B(\vec{x},z)) \wedge (B(\vec{x},u) \rightarrow A(\vec{x},v))].$$

Now, arguing in $\mathbf{SCS}+\mathbf{AC}_{global}$ take arbitrary sets $\vec{x},y,z$. By the foregoing statement, we can pick u,v such that

$$A(\vec{x},y) \rightarrow B(\vec{x},z) \text{ and } B(\vec{x},u) \rightarrow A(\vec{x},v).$$

If $\neg B(\vec{x},u)$ holds then $\neg\forall z B(\vec{x},z)$ and hence $\neg\exists p A(\vec{x},p)$ by (4). If $B(\vec{x},u)$ holds then $A(\vec{x},v)$, thus $\exists p A(\vec{x},p)$. As excluded middle obtains for the Δ_0 statement $B(\vec{x},u)$ we are done. □

5.1 Counterexample

One might think that there is a more general result to the effect that a theory with decidable atomic formulae and Collection has the property that Δ_1 predicates provably satisfy excluded middle. We show that this is not generally the case. This also provides a counterexample to Proposition 3 in [46]. Note that $\mathbf{HA}$ proves a version of $\mathbf{AC}_{global}$ where R is the predecessor relation on $\mathbb{N}$.

Proposition 5.5 *There is an extension T of* **HA** *such that there is a formula A that is Δ_1 relative to T but T does not prove $A \vee \neg A$.*

Proof: By [28, Theorem 3.14] we can pick Π_1 statements $A = \forall x\, A_0(x)$ and $B = \forall y\, B_0(y)$ of the language of **PA** with $A_0(x)$ and $B_0(y)$ Δ_0 such that

$$\mathbf{PA} + A \nvdash B \quad \text{and} \quad \mathbf{PA} + B \nvdash A$$

and, moreover, $\neg A$ as well as $\neg B$ imply $\neg\mathrm{Con}(\mathbf{PA})$, i.e. the arithmetized inconsistency of **PA**, and therefore do not hold in the standard model. As a result, the theories $T_1 := \mathbf{PA} + A + \exists y\, \neg B_0(y)$ and $T_2 := \mathbf{PA} + B + \exists x\, \neg A_0(x)$ are (classically) consistent and thus have non-standard models $\mathfrak{A} \models T_1$ and $\mathfrak{B} \models T_2$, respectively. We would now like to study the following Kripke model $\mathcal{K}$ with three nodes.

$$\begin{array}{ccc} \mathfrak{A} & & \mathfrak{B} \\ \nwarrow & & \nearrow \\ & \mathbb{N} & \end{array}$$

where $\mathbb{N}$ signifies the standard model of number theory and the arrows indicate the canonical embeddings of $\mathbb{N}$ into the respective non-standard model.

We claim that $\mathcal{K}$ is a Kripke model of **HA**. This is obvious for the basic axioms of **HA** pertaining to $0, +1, +, \cdot$, where $+1$ denotes the successor function $x \mapsto x + 1$. All instances of the induction scheme hold at the end nodes of $\mathcal{K}$ by virtue of being models of **PA**. So it remains to check that they hold at the bottom node. Let $a, b, 0$ be the nodes of $\mathcal{K}$ associated with $\mathfrak{A}, \mathfrak{B}, \mathbb{N}$, respectively. Suppose

$$0 \Vdash F(0) \wedge \forall u\, (F(u) \to F(u+1)) \tag{5}$$

By monotonicity of Kripke-forcing the latter entails for $\sigma \in \{a, b\}$ that $\sigma \Vdash F(0) \wedge \forall u\, (F(u) \to F(u+1))$, and thus $\sigma \Vdash \forall u\, F(u)$ since σ is an end node and the pertaining structures $\mathfrak{A}$ and $\mathfrak{B}$ are models of **PA**. By meta-induction on $n \in \mathbb{N}$ it follows from (5) that $0 \Vdash F(n)$ holds for all $n \in \mathbb{N}$. Thus $0 \Vdash \forall x\, F(x)$.

We set

$$T := \mathbf{HA} + A \leftrightarrow \exists y\, \neg B_0(y).$$

We claim that $0 \Vdash A \leftrightarrow \exists y\, \neg B_0(y)$. Note first that (i) $0 \nVdash A$, (ii) $0 \nVdash \exists y\, \neg B_0(y)$, (iii) $a \Vdash A \wedge \exists y\, \neg B_0(y)$, and (iv) $b \Vdash \neg A \wedge \neg\exists y\, \neg B_0(y)$ hold. (i) follows since $\mathfrak{B} \models \neg A$ and A is a universal statement. (ii) follows since $0 \Vdash \exists y\, \neg B_0(y)$ implies that there is a proof of an inconsistency of **PA** in the standard model (which we don't believe). (iii) and (iv) follow by choice of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. (i)–(iv) yield the claim.

In the theory T, A is trivially provably Δ_1. We want to show that T does not prove (intuitionistically) $A \vee \neg A$. If T were to prove $A \vee \neg A$, then, since $\mathcal{K}$ is a Kripke model of T, $0 \Vdash A \vee \neg A$, and thus $0 \Vdash A$ or $0 \Vdash \neg A$. The former is is ruled out by (i) and the latter is ruled out since $a \Vdash A$. $\square$

6 Feferman's second conjecture: Part 2

This section deals with the reverse direction of Feferman's second conjecture. We want to show that provable LEM predicates are provably Δ_1. Together with the previous section one obtains a a plethora of statements that are indeterminate with respect to semi-intuitionistic set theories.

Assume that $\mathbf{SCS} \vdash \forall x\,[A(x) \vee \neg A(x)]$. We want to show that the predicate $A(x)$ is provably Δ_1 in **SCS**. In a first step we show that $A(x)$ is provably Δ_1 in the extension of **SCS** with global choice. The main technical tool is a realizability with truth interpretation of **SCS** in $\mathbf{SCS} + \mathbf{AC}_{global}$ along the lines of [64, Definition 3.1]. However, this is actually a much simpler context than the one in [64] as it can be done without sets of witnesses. The notion of E-recursive function from [64, 2.9] has to be augmented, though, to take care of R. So in addition there is a clause

$$[\mathsf{r}](x) \simeq z \quad \text{iff} \quad \mathsf{R}(x,z).$$

We call them the E'-recursive functions. We adopt the conventions and notations from [64, §3]. We write $\jmath_0 e$ and $\jmath_1 e$ rather than $(e)_0$ and $(e)_1$, respectively, and instead of $[\,a\,](b) \simeq c$ we shall write $a \bullet b \simeq c$.

Definition 6.1 Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers.

We define a relation $a \Vdash_{\mathfrak{wt}} B$ between sets and set-theoretic formulae. $a \bullet f \Vdash_{\mathfrak{wt}} B$ will be an abbreviation for $\exists x[a \bullet f \simeq x \wedge x \Vdash_{\mathfrak{wt}} B]$.

$$
\begin{array}{rll}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & [\jmath_0 a = 0 \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} A] \ \vee\ [\jmath_0 a = 1 \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} B] \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \ \wedge\ \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \to B & \text{iff} & (A \to B) \ \wedge\ \forall c\,[c \Vdash_{\mathfrak{wt}} A \ \to\ a \bullet c \Vdash_{\mathfrak{wt}} B] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & \jmath_0 a \in b \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} A[x/\jmath_0 a] \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c\ \ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & \jmath_1 a \Vdash_{\mathfrak{wt}} A[x/\jmath_0 a] \\
 & & \\
\Vdash_{\mathfrak{wt}} B & \text{iff} & \exists a\, a \Vdash_{\mathfrak{wt}} B.
\end{array}
$$

Corollary 6.2 $\mathbf{SCS} + \mathbf{AC}_{global} \vdash (\Vdash_{\mathfrak{wt}} B \to B)$.

Proof: This is immediate by induction on the complexity of B. ☐

Theorem 6.3 *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{SCS} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an E'-recursive function $\mathfrak{g}$ such that

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall a_1, \ldots, a_r\, \mathfrak{g}(a_1, \ldots, a_r) \Vdash_{\mathfrak{wt}} D(a_1, \ldots, a_r)\,.$$

Proof: The proof is similar to that in [64]. □

One can strengthen this as follows.

Theorem 6.4 *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_{\in}(\mathsf{R})$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an E'-recursive function $\mathfrak{g}$ such that

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall a_1, \ldots, a_r\, \mathfrak{g}(a_1, \ldots, a_r) \Vdash_{\mathfrak{wt}} D(a_1, \ldots, a_r)\,.$$

Proof: The proof is basically the same. □

Theorem 6.5 *Suppose* $\mathbf{SCS} \vdash \forall x\,[A(x) \vee \neg A(x)]$*. Then there exist a Σ_1-formula $B(x)$ and a Π_1-formula $C(x)$ in the language of* $\mathbf{SCS} + \mathbf{AC}_{global}$*, $\mathcal{L}_{\in}(\mathsf{R})$, such that*

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \leftrightarrow B(x) \leftrightarrow C(x)].$$

Proof: Using the Realizability Theorem 6.3, $\mathbf{SCS} \vdash \forall x\,[A(x) \vee \neg A(x)]$ implies that we can construct an index of an E'-recursive function $\mathfrak{g}$ such that

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall x\, \mathfrak{g}(x) \Vdash_{\mathfrak{wt}} [A(x) \vee \neg A(x)].$$

Unraveling this yields that, provably in $\mathbf{SCS} + \mathbf{AC}_{global}$, we have

$$\forall x\,[(\jmath_0\mathfrak{g}(x) = 0 \wedge \jmath_1\mathfrak{g}(x) \Vdash_{\mathfrak{wt}} A(x)) \vee (\jmath_0\mathfrak{g}(x) = 1 \wedge \jmath_1\mathfrak{g}(x) \Vdash_{\mathfrak{wt}} \neg A(x))].$$

In view of Corollary 6.2 we thus have

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \leftrightarrow \jmath_0\mathfrak{g}(x) = 0]$$

furnishing the desired Σ_1 characterization $\jmath_0\mathfrak{g}(x) = 0$ of $A(x)$ in the language with the predicate R. For the Π_1 characterization, note that because of the (provable) totality of $\mathfrak{g}$, $\jmath_0\mathfrak{g}(x) = 0$ is equivalent to the statement $C(x)$ expressing that every halting computation σ of $\mathfrak{g}(x)$ yields an output z with $\jmath_0 z = 0$. Note that the predicate "σ encodes a halting computation of $\mathfrak{g}(x)$" is Δ_0. □

The previous result can also be slightly strengthened.

Theorem 6.6 *Suppose* $\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \vee \neg A(x)]$*. Then there exist a Σ_1-formula $B(x)$ and a Π_1-formula $C(x)$ in the language of* $\mathbf{SCS} + \mathbf{AC}_{global}$*, $\mathcal{L}_{\in}(\mathsf{R})$, such that*

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \leftrightarrow B(x) \leftrightarrow C(x)].$$

Remark 6.7 One would like to get rid of global choice in the interpreting theory in Theorem 6.5, i.e., weaken global choice to just choice. One idea is to use class forcing (as in the case of **ZFC**) to find an interpretation of $\mathbf{SCS} + \mathbf{AC}_{global}$ in **SCS** that preserves the formulas of $\mathcal{L}_\in$. In the case of **ZFC** the class of forcing conditions consists of local choice functions h, i.e., functions such that for all x in the domain of h, $h(x) \in x$ if $x \neq \emptyset$ and $h(x) = \emptyset$ if $x = \emptyset$. While forcing can be defined in **SCS**, the problem we encountered is that **SCS** seems to be too weak to be able to show that every theorem of **SCS** is forced, the particular culprit being Collection.

Theorem 6.8 *There are simplified realizability interpretations of* **SCS** *in* $L(b, \prec)$ *and admissible sets of the form* $L_\kappa(b, \prec)$*, where* b *is a transitive set,* $\prec$ *is well-ordering on* b*, and* $\kappa > \omega$*. Note that such structures have a* Δ_1*-definable global well-ordering in parameters* b *and* $\prec$*.*

Let $D(u_1, \ldots, u_r)$ *be a formula of* $\mathcal{L}_\in(\mathsf{R})$ *all of whose free variables are among* $u_1, \ldots, u_r$*. If*

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of a partial Σ_1 *function* $\mathfrak{g}$ *such that*

$$L_\kappa(b, \prec) \models \forall a_1, \ldots, a_r\, \mathfrak{g}(a_1, \ldots, a_r, b, \prec) \Vdash_{\mathfrak{wt}} D(a_1, \ldots, a_r)$$

and

$$L(b, \prec) \models \forall a_1, \ldots, a_r\, \mathfrak{g}(a_1, \ldots, a_r, b, \prec) \Vdash_{\mathfrak{wt}} D(a_1, \ldots, a_r),$$

where $\mathsf{R}(y, x)$ *is interpreted to mean that* x *is the least element of* y *with respect to the global well-ordering if* $y \neq \emptyset$ *and* $x = \emptyset$ *otherwise.*

Theorem 6.9 *Suppose* A *is a* Π_2 *statement of* $\mathcal{L}_\in(\mathsf{R})$ *that holds in* $L(b, \prec)$*, where* b *is a transitive set and* $\prec$ *is well-ordering on* b*. If* $\mathbf{SCS} + \mathbf{AC}_{global} \vdash A \vee \neg A$*, then*

$$L_\kappa(b, \prec) \models A$$

holds for all admissible sets $L_\kappa(b, \prec)$ *with* $\kappa > \omega$*.*

Proof: By Theorem 6.6, there exists a Π_1-sentence C such that

$$\mathbf{SCS} + \mathbf{AC}_{global} \vdash A \leftrightarrow C. \tag{6}$$

Now since A is true in $L(b, \prec)$ there exists a realizer in $L(b, \prec)$ for this statement on account of being of Π_2-form. Thus, by Theorem 6.8, there exists a realizer in $L(b, \prec)$ for C, entailing that C is true in $L(b, \prec)$. Therefore we have $L_\kappa(b, \prec) \models C$ since C is Π_1. From this we can deduce that C is realized in $L_\kappa(b, \prec)$ and so A is realized in $L_\kappa(b, \prec)$ which implies that A holds in $L_\kappa(b, \prec)$. □

Corollary 6.10 *We say that a statement* D *is indeterminate relative to* $\mathbf{SCS} + \mathbf{AC}_{global}$ *if* $\mathbf{SCS} + \mathbf{AC}_{global} \nvdash D \vee \neg D$*.*

One can now take, e.g., any of the statements that are equivalent to $\mathbf{ATR}_0$ *over* $\mathbf{ACA}_0$ *(cf. [72, Chap. V]) and conclude that they are indeterminate relative to* $\mathbf{SCS} + \mathbf{AC}_{global}$ *as they are of* Π_2 *form, hold in* L *but fail to hold in* $L_{\omega_1^{ck}}$*. Here are two examples.*

1. *Comparability of well-orderings on subsets of* ω.

2. Δ_0*-determinacy.*

One can also apply the foregoing machinery to show indeterminateness with regard to $\mathbf{SCS} + \mathbf{AC}_{global}$ of the statement *"Every real is constructible"* since there are admissible sets of the form $L_\kappa(x)$, where x is a constructible real but x fails to be constructible in $L_\kappa(x)$: Working in L add a new real x to L_κ via forcing. The result can also be shown by the techniques of [65] as observed by Koellner and Woodin, [42], [41].

Theorem 6.11 *Let* NCR *be the statement* "There is a non-constructible real". *Then*

$$\mathbf{SCS} + \mathbf{AC}_{global} \nvdash \mathsf{NCR} \ \vee \ \neg\mathsf{NCR}.$$

We will now look at extensions of Theorem 6.5 to the stronger theories $\mathbf{SCS}^+ + \mathbf{AC}_{global}$ and $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global}$.

To be precise, below we shall assume that the language of $\mathbf{SCS}^+$ has a constant $\mathbb{R}$ and $\mathbf{SCS}^+$ has an axiom that asserts that $\mathbb{R}$ is the set of reals. In the same vein as Theorem 5.3 one has partial conservativity when one adds $\mathbf{AC}_{global}$.

Theorem 6.12 $\mathbf{SCS}^+ + \mathbf{AC}_{global}$ *is conservative over* $\mathbf{SCS}^+$ *for* Π_2*-sentences of the language of* $\mathbf{SCS}^+$ *(which may contain the constant* $\mathbb{R}$*).*

The analogue of Theorem 6.6 also holds for $\mathbf{SCS}^+ + \mathbf{AC}_{global}$.

Theorem 6.13 *Suppose* $\mathbf{SCS}^+ + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \vee \neg A(x)]$. *Then there exist a* Σ_1*-formula* $B(x,y)$ *and a* Π_1*-formula* $C(x,y)$ *in the language* $\mathcal{L}_\in(\mathsf{R})$, *such that*

$$\mathbf{SCS}^+ + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \leftrightarrow B(x,\mathbb{R}) \leftrightarrow C(x,\mathbb{R})].$$

Proof: One has to retrace the steps leading to Theorem 6.5 with $\mathbf{SCS}^+ + \mathbf{AC}_{global}$ in $\mathbf{SCS} + \mathbf{AC}_{global}$'s stead. So we have to prove the pertaining versions of Theorems 5.2, 5.3, 5.4, 6.3, 6.4. □

The foregoing Theorem can be utilized in establishing indeterminateness results for $\mathbf{SCS}^+ + \mathbf{AC}_{global}$. However, we will exhibit one such result that follows from rather deep set-theoretic results and the methods of [65].

Theorem 6.14 (Koellner, Woodin [42], [41]) *Let* DWOR *be the statement "there is a well-ordering of* $\mathbb{R}$ *in* $L(\mathbb{R})$*". Assume* $\mathsf{Con}(\mathsf{ZFC}+$ *"There are* ω*-many Woodin cardinals")*. *Then*

$$\mathbf{SCS}^+ + \mathbf{AC}_{global} \nvdash \mathsf{DWOR} \ \vee \ \neg\mathsf{DWOR}.$$

Thus from the point of view of $\mathbf{SCS}^+ + \mathbf{AC}_{global}$, DWOR is indefinite. The importance of this result lies in the fact that $\mathbf{SCS}^+ + \mathbf{AC}_{global}$ is not in alignment with the community of modern descriptive set theorists who consider $\neg$DWOR to be true.

Finally, let's consider the much stronger theory $\mathbf{SCS}(\mathcal{P})$.

Theorem 6.15 *Suppose* $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \vee \neg A(x)]$. *Then there exist a* $\Sigma_1^{\mathcal{P}}$*-formula* $B(x)$ *and a* $\Pi_1^{\mathcal{P}}$*-formula* $C(x)$ *such that*

$$\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global} \vdash \forall x\,[A(x) \leftrightarrow B(x) \leftrightarrow C(x)].$$

Proof: Again, one has to retrace the steps leading to Theorem 6.5 with $\mathbf{SCS}(\mathcal{P})+\mathbf{AC}_{global}$ in $\mathbf{SCS} + \mathbf{AC}_{global}$'s stead. So we have to prove the pertaining versions of Theorems 5.2, 5.3, 5.4, 6.3, 6.4. □

We will finish this section by exhibiting a statement that is indefinite from the point of view of $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global}$. Let $V = L$ be the statement that every set lies in Gödel's constructible hierarchy.

Theorem 6.16 *Assume* $\mathsf{Con}(\mathsf{ZF})$. *Then* $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global} \nvdash V = L \vee V \neq L$.

Proof: From $\mathsf{Con}(\mathsf{ZF})$ we get $\mathsf{Con}(\mathsf{ZF} + V = L)$. So without loss of generality we may assume $\mathbf{ZF} + V = L$. Aiming at a contradiction, suppose $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global} \vdash V = L \vee V \neq L$. By Theorem 6.15 there is a $\Pi_1^{\mathcal{P}}$ statement C such that $\mathbf{SCS}(\mathcal{P}) + \mathbf{AC}_{global} \vdash V = L \leftrightarrow C$. Since the universe is a model of $V = L$ and all axioms of $\mathbf{SCS}(\mathcal{P})$ are true in V, we conclude that $V \models C$. Invoking Theorem 3.22, we conclude that $\mathbf{KP}(\mathcal{P})+\mathbf{AC}_{global} \vdash C \to V = L$. In particular we have

$$\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash C \to \forall x \in \mathbb{R} \exists \alpha\; x \in L_\alpha$$

and hence

$$\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash \exists \beta\,[C \to \forall x \in \mathbb{R}\; x \in L_\beta].$$

The latter statement is $\Sigma_1^{\mathcal{P}}$. Thus it follows from [68, Theorem 3.2] and [68, Theorem 3.3] that

$$V_\tau \models \exists \beta\,[C \to \forall x \in \mathbb{R}\; x \in L_\beta]$$

where τ denotes the countable Bachmann-Howard ordinal. Since C is true in V and of $\Pi_1^{\mathcal{P}}$ form it follows that $V_\tau \models C$. Thus

$$V_\tau \models \exists \beta\, \forall x \in \mathbb{R}\; x \in L_\beta.$$

But this is impossible as $\mathbb{R}$ is not countable. □

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