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Marstrand's Theorem revisited: projecting sets of dimension zero

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With an Appendix: The gap of uncertainty

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Abstract

We establish a refinement of Marstrand's projection theorem for Hausdorff dimension functions finer than the usual power functions, including an analogue of Marstrand's Theorem for logarithmic Hausdorff dimension.

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1 Introduction

1.1 Motivation

Given $0 \leq \theta < \pi$, let L_θ denote the line through the origin of $\mathbb{R}^2$ that forms an angle θ with the horizontal axis. Let proj_θ denote orthogonal projection onto the

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line L_θ and $\dim A$ denote the Hausdorff dimension of a set $A \subseteq \mathbb{R}^2$. Then proj_θ is a Lipschitz mapping; indeed for all θ,

$$|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y| \leq |x - y| \quad \forall x, y \in \mathbb{R}^2 . \tag{1}$$

This together with the trivial fact that $\mathrm{proj}_\theta A$ is a subset of a line, implies

$$\dim \mathrm{proj}_\theta A \leq \min \{1, \dim A\} , \tag{2}$$

see for example [6, Proposition 3.3]. The famous projection theorem of Marstrand [10], dating from 1954, tells us that equality holds in (2) for almost almost all directions θ with respect to Lebesgue measure $\mathcal{L}$. Equivalently, the exceptional values of $\theta \in [0, \pi)$ for which the inequality (2) is strict, form a set of one-dimensional Lebesgue measure zero.

Theorem (Marstrand). *Let $A \subseteq \mathbb{R}^2$ be a Borel set.*

(i) If $\dim A \leq 1$ then $\dim \mathrm{proj}_\theta A = \dim A$ for almost all $\theta \in [0, \pi)$.

(ii) If $\dim A > 1$ then $\mathcal{L}(\mathrm{proj}_\theta A) > 0$ for almost all $\theta \in [0, \pi)$.

Observe that the measure conclusion of (ii) is significantly stronger than the corresponding dimension statement; it trivially implies that $\dim \mathrm{proj}_\theta A = 1$ for almost all $\theta \in [0, \pi)$. Marstrand's proof depends heavily on delicate and, in places, complicated geometric and measure theoretic arguments. Subsequently, Kaufman [9] gave a slick, two page, proof that made natural use of the potential theoretic characterization of Hausdorff dimension and Fourier transform methods.

Here we will be concerned with the case when $\dim A \leq 1$. Indeed, to motivate our investigation, consider the extreme situation when $\dim A = 0$. Then Marstrand's Theorem implies no more than the trivial statement that $\dim \mathrm{proj}_\theta A = 0$ for all θ. Thus, to obtain non-trivial information in such situations, it is natural to ask whether a version of Marstrand's Theorem remains valid for finer notions of Hausdorff dimension. One consequence of Theorem 2, our main result, is the following analogue of Marstrand's Theorem for logarithmic Hausdorff dimension, that is where the Hausdorff measures are defined with respect to dimension or gauge functions $(-\log r)^{-s}$ (for small r) and $s \geq 0$, and $\dim_{\log} A$ is the critical value of s at which these measures jump from ∞ to 0, see §1.2.1 for the full definitions.

Theorem 1. *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Then*

(i) $\dim_{\log} \mathrm{proj}_\theta A \leq \dim_{\log} A$ for all $\theta \in [0, \pi)$,

(ii) $\dim_{\log} \mathrm{proj}_\theta A = \dim_{\log} A$ for $\mathcal{L}$-almost all $\theta \in [0, \pi)$.

Remark 1. By considering the size of sets of exceptional angles, see §3, we are further able to conclude that

$$\dim_{\log}\{\theta \in [0,\pi) : \dim_{\log} \mathrm{proj}_\theta A < \dim_{\log} A\} \;\leq\; \dim_{\log} A\,. \tag{3}$$

Of course, the interesting case is when $\dim_{\log} A$ is finite. Then, by definition $\dim A = 0$ and so (3) is significantly stronger than Theorem 1.

Before moving onto our main result, Theorem 2, which is an analogue of Marstrand's Theorem for a general class of dimension functions, we consider an explicit class of sets that has motivated our work and which illustrates and clarifies the need for statements such as Theorem 1.

The motivating example. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function. A point $(y_1,\ldots,y_k) \in \mathbb{R}^k$ is called *simultaneously ψ–approximable* if there are infinitely many $q \in \mathbb{N}$ and $(p_1,\ldots,p_k) \in \mathbb{Z}^k$ such that

$$\left| y_i - \frac{p_i}{q} \right| \;<\; \psi(q) \qquad 1 \leq i \leq k\,. \tag{4}$$

The set of simultaneously ψ–approximable points in $\mathrm{I}^k := [0,1]^k$ will be denoted by $W_k(\psi)$. For convenience, we work within the unit cube I^k rather than $\mathbb{R}^k$ as it makes full measure results easier to state and avoids ambiguity. This is not at all restrictive as the set of simultaneously ψ-approximable points is invariant under translations by integer vectors. The following statement provides a beautiful and simple criterion for the 'size' of $W_k(\psi)$ in terms of Hausdorff measures with respect to a dimension function f, see §1.2 for the full definition of these measures.

Theorem (Khintchine–Jarník). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function. Let f be a dimension function such that $r^{-k} f(r)$ is monotonic. Then*

$$\mathcal{H}^f\left(W_k(\psi)\right) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q^k f\left(\psi(q)\right) < \infty\,, \\ \mathcal{H}^f(\mathrm{I}^k) & \text{if} \quad \sum_{q=1}^{\infty} q^k f\left(\psi(q)\right) = \infty\,. \end{cases}$$

This theorem unifies the fundamental results of Khintchine and Jarník in the classical theory of metric Diophantine approximation. Khintchine's Theorem (1924) corresponds to the situation in which $f(r) = r^k$ when $\mathcal{H}^f$ is equivalent to k-dimensional Lebesgue measure. Jarník's Theorem (1931) corresponds to the situation in which $r^{-1} f(r) \to \infty$ as $r \to 0$ and $r^{-k} f(r)$ is decreasing in which case $\mathcal{H}^f(\mathrm{I}^k) = \infty$. For background and further details see [1, 2] and references therein.

For all $\tau > 0$, let ψ_τ be the 'approximating' function given by $\psi_\tau(q) := \exp(-q^\tau)$. Then by definition, when $k = 1$ the corresponding set $W_k(\psi_\tau)$ is a

subset of the set of Liouville numbers which is well-known to be of Hausdorff dimension zero. In fact $\dim W_k(\psi_\tau) = 0$ for all positive integers k. To see this, note that for any dimension function $f_s(r) = r^s$ $(s > 0)$,

$$\sum_{q=1}^{\infty} q^k f_s(\psi_\tau(q)) = \sum_{q=1}^{\infty} \exp\big(-(s\,q^\tau - k\log q)\big) < \infty$$

for all $\tau > 0$ and $k \in \mathbb{R}$. Hence, it follows from the Khintchine-Jarník Theorem and the definition of Hausdorff dimension that $\dim W_k(\psi_\tau) = 0$ for all $\tau > 0$ and $k \geq 1$. The upshot of this is that by (2), for all $\theta \in [0, \pi)$

$$\dim\big(\mathrm{proj}_\theta\, W_2(\psi_\tau)\big) = 0$$

and Marstrand's Theorem is not particularly informative. The problem is that the dimension functions f_s given by $f_s(r) = r^s$ are not delicate enough to differentiate between sets of dimension zero. Instead, for $s > 0$ consider the logarithmic dimension function f_s given by $f_s(r) = (-\log r)^{-s}$ for $0 < r < 1$. Then, for $\tau > 0$ and $k \geq 1$, it is easily verified that

$$\sum_{q=1}^{\infty} q^k f_s(\psi_\tau(q)) = \sum_{q=1}^{\infty} q^{-(\tau s - k)} \quad \begin{cases} < \infty & \text{if} \quad s > s_0 \\ = \infty & \text{if} \quad s < s_0 \end{cases} ,$$

where

$$s_0 := \frac{k+1}{\tau} .$$

It then follows from the Khintchine-Jarník Theorem and the definition of logarithmic Hausdorff dimension, see §1.2.1, that $\dim_{\log} W_k(\psi_\tau) = s_0$ for all $\tau > 0$ and $k \geq 1$. In turn Theorem 1 implies the non-trivial statement that for almost all $\theta \in [0, \pi)$,

$$\dim_{\log}\big(\mathrm{proj}_\theta\, W_k(\psi_\tau)\big) = s_0 .$$

1.2 The main result

We first recall the definition of f-dimensional Hausdorff measure. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a *dimension* or *gauge function*, that is a function that is increasing and continuous with $f(r) \to 0$ as $r \to 0$. Let A be a non–empty subset of $\mathbb{R}^n$. For $\rho > 0$, let

$$\mathcal{H}^f_\rho(A) := \inf\Big\{ \sum_i f(|U_i|) : A \subseteq \bigcup_i U_i,\ |U_i| \leq \rho \Big\} ,$$

where $|U|$ denotes the diameter of a set U and the infimum is over countable covers $\{U_i\}$ of A by sets of diameter at most ρ. The *Hausdorff f-measure* of A is defined by

$$\mathcal{H}^f(A) := \lim_{\rho \to 0} \mathcal{H}^f_\rho(A) .$$

When $f(r) = r^s$ $(s > 0)$, the measure $\mathcal{H}^f$ is the usual *s-dimensional Hausdorff measure* $\mathcal{H}^s$.

We will also use centred Hausdorff measure. Here we consider covers by a countable collection of balls $\{B(x_i, r_i)\}$ of radii $r_i \le \rho$ with centres in A. Thus, for $\rho > 0$ we set

$$\mathcal{H}^f_{C,\rho}(A) := \inf\Big\{ \sum_i f(r_i) : A \subseteq \bigcup_i B(x_i, r_i),\ x_i \in A,\ r_i \le \rho \Big\},$$

and define the *centred Hausdorff f-measure* of A by

$$\mathcal{H}^f_C(A) := \lim_{\rho \to 0} \mathcal{H}^f_{C,\rho}(A)\ .$$

These two measures are equivalent, in the sense that for all $A \subseteq \mathbb{R}^n$

$$\mathcal{H}^f_C(A) \le \mathcal{H}^f(A) \le m_n\, \mathcal{H}^f_C(A), \tag{5}$$

where m_n depends only on n. This follows easily from the definitions, noting that every set U that intersects A is contained in a ball with centre in A and diameter $|U|$, and that every ball $B \subseteq \mathbb{R}^n$ of radius r is contained in a finite number m_n of balls of radius $\frac{1}{2}r$, that is diameter r; in particular $m_2 = 7$.

Note that f-Hausdorff measure only depends on $f(r)$ for $r \in [0, r_0]$ for arbitrarily small r_0, so changing the dimension function f outside a neighbourhood of 0 does not affect the measure.

The *Hausdorff dimension* $\dim A$ of a set A is defined by

$$\dim A := \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}\,.$$

It follows from (5) that we get the same value for Hausdorff dimension if we replace $\mathcal{H}^s$ by $\mathcal{H}^s_C$ in this definition. For further discussion of Hausdorff measures and dimensions, see [6, 11, 16].

Defining Hausdorff measures for general dimension functions allows a more precise notion of dimension than just a numerical value. For example, a set A may have Hausdorff dimension s but with $\mathcal{H}^s(A) = 0$. However, it may be that $0 < \mathcal{H}^f(A) < \infty$ where, say $f(r) = r^s \log(1/r)$, in which case we think of A having dimension 'logarithmically smaller' than s. Introducing a partial order $\prec$ on the set of dimension functions by $f \prec g$ if $\lim_{r\to 0} g(r)/f(r) = 0$, which implies that $\mathcal{H}^g(A) = 0$ whenever $\mathcal{H}^f(A) < \infty$, allows a much finer notion of dimension, see [16]. It is also worth noting that there are sets $A \subseteq \mathbb{R}^n$ for which there is no dimension function f such that $0 < \mathcal{H}^f(A) < \infty$, see [4].

In order to state our main theorem we need the notion of doubling. A dimension function f is said to be *doubling* if there exist constants $c > 1$ and $r_0 > 0$ such that

$$f(2r) \le c f(r) \quad \forall\, 0 < r < r_0\,. \tag{6}$$

The number c is called a *doubling constant*. Note that if f is given by $f(r) = r^s$ $(s > 0)$ then $f(2r) = 2^s f(r)$ and so $c = 2^s$ is a doubling constant for f.

We are now in the position to state our main result.

Theorem 2. *Let $A \subseteq \mathbb{R}^2$ be a Borel set.*

(i) Let f be a dimension function. Then $\mathcal{H}^f(\mathrm{proj}_\theta A) \leq \mathcal{H}^f(A)$ for all $\theta \in [0, \pi)$. In particular if $\mathcal{H}^f(A) = 0$ then $\mathcal{H}^f(\mathrm{proj}_\theta A) = 0$ for all $\theta \in [0, \pi)$.

(ii) Let f be a dimension function such that $\mathcal{H}^f(A) > 0$. Suppose g is a dimension function that is doubling with constant $c < 2$ and such that

$$-\int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(r)}\right) < \infty\,. \tag{7}$$

Then, $\mathcal{H}^g(\mathrm{proj}_\theta A) = \mathcal{H}^g(A) = \infty$ for almost all $\theta \in [0, \pi)$.

Several remarks are in order.

Remark 2. Part (i) of Theorem 2 is an immediate consequence of the Lipschitz condition (1) and the definition of $\mathcal{H}^f$, see [6, Proposition 3.1] where the case of $f(r) = r^s$ is given. Thus the main substance of the theorem is part (ii) when $\mathcal{H}^f(A) > 0$.

Remark 3. Note that the conclusion of (ii) remains true if the range of integration in (7) is an interval $[0, r_0]$ for any $r_0 > 0$. Moreover, if g is differentiable, or at least differentiable except at finitely many points, then

$$-\int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(r)}\right) = \int_0^1 f(r)\frac{g'(r)}{g^2(r)}\ \mathrm{d}r\ . \tag{8}$$

In particular, if f and g are dimension functions satisfying (7) then

$$\lim_{r\to 0}\frac{f(r)}{g(r)} = 0. \tag{9}$$

For suppose not. Then there exists $a > 0$ and a sequence $r_n \searrow 0$ such that $f(r_n)/g(r_n) \geq a$ for all n. Let $r_n' > r_n$ be the least number such that $g(r_n') = 2g(r_n)$; such an r_n' exists by continuity and monotonicity of g provided that the sequence is chosen taking r_1 sufficiently small. Then

$$\int_{r_n}^{r_n'} f(r)\frac{g'(r)}{g^2(r)}\ \mathrm{d}r \ \geq\ \int_{r_n}^{r_n'} \frac{f(r_n)}{2g(r_n)}\frac{g'(r)}{g(r)}\ \mathrm{d}r \ \geq\ \tfrac{1}{2}a\log\left[\frac{g(r_n')}{g(r_n)}\right] \ =\ \tfrac{1}{2}a\log 2.$$

Since $0 < r_n < r_n' \to 0$, the integrals in (8) and (7) cannot be finite.

When contemplating an extension of Marstrand's theorem to general dimension functions, it is not unreasonable to suspect a statement along the lines of Theorem 2 with condition (7) replaced by (9). The latter condition is natural and it initially appears to avoid known examples (see for instance [11, 9.2 Example]) of s-sets $A \subseteq \mathbb{R}^2$ with $0 < s \leq 1$ for which $\mathcal{H}^s(\mathrm{proj}_\theta A) = 0$ for all $\theta \in [0, \pi)$. As we shall see in the appendix the construction of these sets can be adapted to show that we can not in general replace condition (7) by (9) in Theorem 2. The following statement is easily deduced from the theorem proved in the appendix for codoubling dimension function. A dimension function f is said to be *codoubling* if there exist constants $c > 1$ and $r_0 > 0$ such that

$$f(2r) \geq c\, f(r) \quad \forall\, 0 < r < r_0\,. \tag{10}$$

Theorem A′. *Let f, g be dimension functions such that f is doubling with constant $c \leq 2$ and codoubling, and such that*

$$g(r) \leq M\ f\left(r \log(r^{-1})\right) \quad \forall\, 0 < r < r_0 \tag{11}$$

where $r_0 > 0$ and $M > 0$ are constants. Then there exists a set $A \subseteq \mathbb{R}^2$ with $0 < \mathcal{H}^f(A) < \infty$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

It is easily seen that Theorem A′ is not a converse to Theorem 2. Namely, fix some $\delta \in (0, 1)$ and for $s > 0$ consider dimension functions g_s given by

$$g_s(r) = r^\delta \log^s(r^{-1})\,.$$

Clearly, g_s is doubling with constant $c < 2$. Also, let f be given by $f(r) = r^\delta$. Clearly, f is doubling with constant $c \leq 2$ and codoubling. Then for any $s \leq \delta$, condition (11) is satisfied and Theorem A′ implies that there exists a set A_s with positive δ-dimensional Hausdorff measure such that $\mathcal{H}^{g_s}(\mathrm{proj}_\theta A_s) = 0$ for all $0 \leq \theta < \pi$. On the other hand, for any $s \geq 1$, condition (7) is satisfied and Theorem 2 implies that there does not exist such a set A_s. But for $s \in (\delta, 1)$, we do not know whether such a set exists. Nevertheless, Theorem A′ shows that we can not in general replace condition (7) by (9) and thus there is a gap of uncertainty associated with Theorem 2 where we do not know what happens (for further discussion see §4). This gap of uncertainty will be explicitly highlighted in §1.2.2 when we return to our motivating example. The fact that condition (7) shows up is very much a consequence of the approach taken to prove the theorem. Concerning this the reader is especially directed towards Remark 7 at the end of §2.2 in the proof.

Not surprisingly, condition (11) implies that the integral convergence condition (9) is violated – see the appendix for the details; namely Remark 12.

Remark 4. It is easily verified that if f and g are dimension functions satisfying (9) and $\mathcal{H}^f(A) > 0$ then $\mathcal{H}^g(A) = \infty$. Thus, the main substance of part (ii) of Theorem 2 is the statement that $\mathcal{H}^g(\text{proj}_\theta A) \geq \mathcal{H}^g(A)$ for almost all $\theta \in [0, \pi)$. For further relations between measures with respect to different gauge functions, see [16, Section 4].

Remark 5. Regarding the dimension function g, the condition that $c < 2$ on the doubling constant is necessary. To see this, we derive the dimension aspect of Marstrand's Theorem from our result. With this in mind, assume without loss of generality that $\dim A > 0$ and let s_1, s_2 be arbitrary real numbers satisfying $0 < s_1 < s_2 < \dim A$. Now let g and f be dimension functions given by $g(r) := r^{s_1}$ and $f(r) := r^{s_2}$. It follows from the definition of Hausdorff dimension that $\mathcal{H}^{s_1}(A) = \mathcal{H}^{s_2}(A) = \infty$. Also it is easily checked that condition (7) is satisfied and thus, modulo the condition on the doubling constant, part (ii) of Theorem 2 implies that $\mathcal{H}^{s_1}(\text{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. In turn, it follows (from the definition of Hausdorff dimension) that

$$\dim \text{proj}_\theta A \geq s_1 \tag{12}$$

for almost all $\theta \in [0, \pi)$. The application of Theorem 2 is legitimate as long as the doubling constant $c = 2^{s_1}$ associated with g satisfies $s_1 < 1$. Now with reference to (12) this restriction on s_1 makes perfect sense since $\dim \text{proj}_\theta A \leq 1$ regardless of the size of A. By continuity, we can replace s_1 in (12) by $\dim A$. The complementary upper bound can easily be deduced via part (i) of Theorem 2 but inequality (2) gives it directly.

Remark 6. Even if $\mathcal{H}^f(A) = \infty$, the conclusion of part (ii) of Theorem 2 is not in general valid for the dimension function f. Indeed, if f is given by $f(r) := r$ so that $\mathcal{H}^f$ is simply 1-dimension Lebesgue measure, it is known [6, Section 6.4] that there are sets A for which $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^f(\text{proj}_\theta A) = 0$ for almost all $\theta \in [0, \pi)$.

As alluded to in Remark 1, in §3 we will investigate the size of the set of exceptional angles θ for which the conclusion of part (ii) of Theorem 2 fails. In short, by replacing the integral convergence condition (7) by a suitable rate of convergence condition we are able to conclude that the exceptional set of $\theta \in [0, \pi)$ for which $\mathcal{H}^g(\text{proj}_\theta A) < \infty$ is of $\mathcal{H}^f$-measure 0, see Theorem 9 for the precise statement.

1.2.1 The logarithmic dimension result

In terms of dimension theory, when we are confronted with sets of Hausdorff dimension 0 it is natural to change the usual 'r^s-scale' in the definition of Hausdorff dimension to a logarithmic scale. For $s > 0$, let f_s be the dimension function

given by $f_s(r) := (-\log^* r)^{-s}$, where

$$\log^* r := \begin{cases} \log r & \text{for } r \in (0, \frac{1}{2}) \\ \log \frac{1}{2} & \text{for } r \geq \frac{1}{2}\,. \end{cases}$$

[The form of $\log^* r$ for $r \geq \frac{1}{2}$ is to ensure that it is defined for $r \geq 1$; as remarked earlier, the particular form for $r \geq \frac{1}{2}$ is of no consequence.] The *logarithmic Hausdorff dimension* $\dim_{\log} A$ of a set A is given by

$$\dim_{\log} A := \inf\left\{s : \mathcal{H}^{f_s}(A) = 0\right\} = \sup\left\{s : \mathcal{H}^{f_s}(A) = \infty\right\}\,. \tag{13}$$

It is easily verified that if $\dim A > 0$ then $\dim_{\log} A = \infty$, precisely as one would expect.

Armed with Theorem 2 it is straightforward to prove Theorem 1.

Proof of Theorem 1. Part (i) is immediate from Theorem 2(i) and (13).

For part (ii), without loss of generality, assume that $\dim_{\log} A > 0$ and let s_1, s_2 be real numbers satisfying $0 < s_1 < s_2 < \dim_{\log} A$. Let g and f be dimension functions given by $g(r) := (-\log^* r)^{-s_1}$ and $f(r) := (-\log^* r)^{-s_2}$. It follows from (13) that $\mathcal{H}^g(A) = \mathcal{H}^f(A) = \infty$. It is easily verified that condition (7) is satisfied and that g is doubling with constant $c < 2$ for $r \in (0, r_0)$ for some r_0. Thus, Theorem 2(ii) implies that $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. In turn, it follows from (13) that for almost all $\theta \in [0, \pi)$, $\dim_{\log} \mathrm{proj}_\theta A \geq s_1$ and thus $\dim_{\log} \mathrm{proj}_\theta A \geq \dim_{\log} A$. □

1.2.2 Explicitly exposing the gap of uncertainty

With reference to our motivating example, for all $\tau > 0$ let ψ_τ be the 'approximating' function given by

$$\psi_\tau(q) := q^{-\tau}(\log q)^{-\tau} \qquad (q \geq 1).$$

It follows, via the Khintchine-Jarník Theorem and the definition of Hausdorff dimension, that for all $\tau \geq (1+k)/k$

$$\delta = \delta(\tau) := \dim W_k(\psi_\tau) = \frac{k+1}{\tau}\,.$$

In fact, the Khintchine-Jarník Theorem implies a much finer conclusion. Fix $\tau > (1+k)/k$ and consider the family of dimension functions $(f_{\delta,s})_{s>0}$ given by

$$f_{\delta,s}(r) := r^\delta(-\tfrac{1}{\tau}\log^* r)^s\,.$$

It is easily verified that

$$\sum_{q=2}^{\infty} q^k f_{\delta,s}(\psi_\tau(q)) \asymp \sum_{q=2}^{\infty} \frac{1}{q(\log q)^{1+k-s}},$$

in the sense that the series either both converge or diverge, and so the Khintchine-Jarník Theorem implies that

$$\mathcal{H}^{f_{\delta,s}}(W_k(\psi_\tau)) = \begin{cases} 0 & \text{if} \quad s < k\,, \\ \infty & \text{if} \quad s \geq k\,. \end{cases}$$

Loosely speaking, the set $W_k(\psi_\tau)$ has "$\delta(\tau)$-logarithmic dimension" equal to k.

Now let $k = 2$ and with reference to Theorem 2, put $f = f_{\delta,2}$ and $g = f_{\delta,s}$. Suppose that $\tau > 3$ so that $\delta(\tau) < 1$. This ensures that g is doubling with constant $c < 2$. Theorem 2 then implies that for almost all $\theta \in [0, \pi)$

$$\mathcal{H}^{f_{\delta,s}}(\mathrm{proj}_\theta W_2(\psi_\tau)) = \begin{cases} 0 & \text{if} \quad s < 2\,, \\ \infty & \text{if} \quad s > 3\,. \end{cases}$$

Of course, part (i) of Theorem 2 implies that the zero measure statement associated with $s < 2$ is true for all $\theta \in [0, \pi)$. Regarding the application of part (ii), we need that $s > 3$ in order to satisfy the integral convergence condition (7). Thus the latter gives rise to a gap of uncertainty; namely $s \in (2, 3)$ in the specific example under consideration. We suspect that the infinity measure statement for $s > 3$ is actually true for $s > 2$.

Problem: Show that $\mathcal{H}^{f_{\delta,s}}(\mathrm{proj}_\theta W_2(\psi_\tau)) = \infty$ if $s > 2$.

The fact of the matter is that it is highly unlikely that any set $W_2(\psi)$ of simultaneously ψ–approximable points will have the necessary 'dense rotational' structure that underpins the construction of the sets associated with Theorem A$'$.

2 Proof of main result

Our proof of Theorem 2 will follow Kaufman's potential theoretic proof [9] of Mastrand's Theorem. We adapt the proof that he gave for the specific functions $f(r) = r^s$ $(s > 0)$ to general dimension functions.

2.1 Preliminaries: doubling revisited and Frostman

We start by stating an equivalent form of the doubling condition (6).

Lemma 3. *Let f be a dimension function. Then f is doubling if and only if there exist constants $s > 0$, $\kappa > 0$ and $r_1 > 0$ such that*

$$f(r\lambda) \geq \kappa\, \lambda^s f(r) \qquad \forall\, 0 < \lambda < 1 \;\text{ and }\; 0 < r < r_1\,. \tag{14}$$

Moreover, if f has a doubling constant $c > 1$ then (14) holds with $\kappa = c^{-1}$ and $s = \log_2 c$.

This equivalence is essentially 'folklore' and the exponent s appearing in (14) is referred to as the *doubling exponent* of f. Nevertheless, for the sake of completeness we include the short proof.

Proof. Suppose f has a doubling constant $c > 1$. For each positive integer n, applying (6) n times gives that

$$f\,(r) \;\leq\; c^n\, f(2^{-n}r) \quad \forall\, r < 2^n r_0\,.$$

Put $s := \log_2 c > 0$. For each $0 < \lambda < 1$, let $m \geq 0$ be the unique integer such that $2^m \leq \lambda^{-1} < 2^{m+1}$. Then

$$f(\lambda r) \geq f\left(2^{-(m+1)}r\right) \;\geq\; c^{-(m+1)}\, f(r) \;\geq c^{-1}\lambda^s f(r)\,.$$

For the converse implication simply put $\lambda = \frac{1}{2}$ in (14). □

The following statement is a generalisation of Frostman's fundamental lemma to arbitrary dimension functions f . Throughout, given a Borel set $A \subseteq \mathbb{R}^2$ we denote by $\mathcal{M}_1(A)$ the set of Radon probability measures μ with compact support in A.

Theorem 4 (Frostman's Lemma)**.** *Let $A \subseteq \mathbb{R}^2$ be a Borel set and f be a dimension function. Then $\mathcal{H}^f(A) > 0$ if and only if there exist a measure $\mu \in \mathcal{M}_1(A)$ and a constant $c_1 > 0$ such that*

$$\mu(B(x,r)) \leq c_1 f(r) \qquad \forall\, x \in \mathbb{R}^2 \;\text{ and }\; r > 0\,.$$

Proof. Two very different proofs for the case where $f(r) = r^s$ $(s > 0)$ are given in [11, Theorem 8.8], where is explicitly pointed out that both proofs are valid for general dimension functions. Alternatively, for the harder of the implications, namely that suitable measures exist, the result in Rogers [16, Theorem 57], that for general dimension functions there exists a compact subset A' of A with $0 < \mathcal{H}^f(A') < \infty$, followed by a density argument akin to [6, Proposition 4.11], also gives the conclusion. □

2.2 Energies and capacities

We first generalise the standard notions of $s-$energy and $s-$capacity of a measure, see for example [6, Section4.3] and [11, Chapter 8]. As usual, let f be a dimension function.

Definition 5. *The* f-energy *of* $\mu \in \mathcal{M}_1(A)$ *is defined as*

$$I_f(\mu) := \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{f(|x-y|)}.$$

Alternatively, we could have defined the f-energy via the f*-potential* at a point $x \in \mathbb{R}^2$, that is

$$\phi_f(x) := \int \frac{\mathrm{d}\mu(y)}{f(|x-y|)} \quad \text{and so} \quad I_f(\mu) = \int \phi_f(x)\ \mathrm{d}\mu(x).$$

Definition 6. *The* f-capacity *of a Borel set* $A \subseteq \mathbb{R}^2$ *is defined as*

$$C_f(A) := \sup\left\{\frac{1}{I_f(\mu)} : \mu \in \mathcal{M}_1(A)\right\}$$

with the interpretation that $C_f(\emptyset) = 0$.

Naturally, when f is given by $f(r) = r^s$ $(s > 0)$ we recover the familiar notions of s-energy and s-capacity.

We now establish the connection between the Hausdorff measure $\mathcal{H}^f(A)$ and the capacity $C_f(A)$ of a set A with respect to a general dimension function f. These results stated below have a long history: apart from notational differences they appear as Theorems 1 and 2 in [17], though versions for the dimension functions of the form $f(r) = r^s$ date back to the 1930s. The paper [17] discusses the historical development to increasingly general dimension functions and includes further references. Proofs for dimension functions $f(r) = r^s$ may be found in several more recent accounts of fractal geometry, for example [5, 11]. Even for general dimension functions the proofs are relatively short, so for the sake of clarity, consistency of notation and completeness we include the proofs.

Proposition 7. *Let* $A \subseteq \mathbb{R}^2$ *be a Borel set and* f *be a dimension function. If* $\mathcal{H}^f(A) < \infty$ *then* $C_f(A) = 0$.

Proof. Assume $C_f(A) > 0$. By definition, the set A supports a Radon probability measure μ such that $I_f(\mu) < \infty$. Thus

$$\int \frac{\mathrm{d}\mu(y)}{f(|x-y|)} < \infty \quad \text{for } \mu\text{-almost all } x \in A.$$

For such $x \in A$,

$$\lim_{r \to 0} \int\limits_{B(x,r)} \frac{\mathrm{d}\mu(y)}{f(|x-y|)} = 0.$$

By Egorov's theorem, for all $\varepsilon > 0$ there exist $\delta > 0$ and a Borel set $K \subseteq A$ such that $\mu(K) > \frac{1}{2}$ and

$$\begin{aligned} \mu(B(x,r)) &\leq f(r) \int\limits_{B(x,r)} \frac{\mathrm{d}\mu(y)}{f(|x-y|)} \\ &\leq \varepsilon f(r) \quad \text{for all } x \in K \text{ and } \ 0 < r \leq \delta. \end{aligned}$$

Now let $\big(B(x_i, r_i)\big)_{i=1}^{\infty}$ be a cover of K by balls with $x_i \in K$ and $r_i \leq \delta$ such that

$$\sum_{i=1}^{\infty} f(r_i) \ < \ \mathcal{H}_C^f(K) + 1$$

Then

$$\frac{1}{2} < \ \mu(K) \leq \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \leq \varepsilon \sum_{i=1}^{\infty} f(r_i) \leq \varepsilon \left(\mathcal{H}_C^f(A) + 1\right),$$

where $\mathcal{H}_C^f$ is centred Hausdorff measure. Since $\varepsilon > 0$ can be made arbitrarily small, we conclude that $\mathcal{H}^f(A) = \mathcal{H}_C^f(A) = \infty$, using (5). This contradicts our hypothesis that $\mathcal{H}^f(A)$ is finite. □

Proposition 8. *Let $A \subseteq \mathbb{R}^2$ be a Borel set and let f and g be dimension functions satisfying the integral convergence condition* (7)*. If $\mathcal{H}^f(A) > 0$ then $C_g(A) > 0$.*

Proof. By Frostman's lemma, Theorem 4, the Borel set A supports a Radon probability measure μ such that

$$\mu\left(B(x,r)\right) \leq c_1 f(r) \quad \forall \ x \in \mathbb{R}^2 \ \text{ and } \ r > 0 \tag{15}$$

for some constant $c_1 > 0$. Fix $x \in \mathbb{R}^2$ and let

$$m(r) := \mu\left(B(x,r)\right) .$$

Using (15) and that $\mu(\mathbb{R}^2) = \mu(K) = 1$ and integrating by parts,

$$\begin{aligned}
\int \frac{\mathrm{d}\mu(y)}{g(|x-y|)} &= \int\limits_{|x-y|\le 1} \frac{\mathrm{d}\mu(y)}{g(|x-y|)} + \int\limits_{|x-y|>1} \frac{\mathrm{d}\mu(y)}{g(|x-y|)} \\
&\le \int_0^1 \frac{1}{g(r)}\,\mathrm{d}m(r) + \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&= \left[\frac{m(r)}{g(r)}\right]_0^1 - \int_0^1 m(r)\,\mathrm{d}\left(\frac{1}{g(r)}\right) + \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&\le \frac{m(1)}{g(1)} - \lim_{r\to 0^+}\frac{m(r)}{g(r)} - \int_0^1 f(r)\,\mathrm{d}\left(\frac{1}{g(r)}\right) + \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&< \infty,
\end{aligned}$$

noting that $m(r)/g(r) \le c_1 f(r)/g(r) \to 0$ by (9). This bound is uniform for all $x \in \mathbb{R}$ and so

$$I_g(\mu) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|x-y|)} < \infty$$

giving $C_g(A) > 0$ by Definition 6. □

Remark 7. Fix $0 < \delta < 1$ and consider the family of dimension functions $(f_{\delta,s})_{s>0}$ given by

$$f_{\delta,s}(r) := r^\delta(-\log^* r)^s\,,$$

to within constants the same as those considered in §1.2.2. Let $A \subseteq \mathbb{R}^2$ be a Borel set and $\alpha > 0$. Then, by Propositions 7 and 8,

(i) if $s \le \alpha$ and $\mathcal{H}^{f_{\delta,\alpha}}(A) < \infty$ then $C_{f_{\delta,s}}(A) = 0$,

(ii) if $s > \alpha + 1$ and $\mathcal{H}^{f_{\delta,\alpha}}(A) > 0$ then $C_{f_{\delta,s}}(A) > 0$.

The upshot is that if $\alpha < s \le \alpha + 1$, condition (7) is not satisfied and the propositions provide no information. The main aim of the paper [17] is to expose this gap of uncertainty. So for example, by [17, Theorem 3], if f and g are dimension functions not satisfying condition (7), then there exist Borel sets A with $0 < \mathcal{H}^f(A) < \infty$ but $C_g(A) = 0$.

2.3 Proof of Theorem 2

(i) As pointed out in Remark 2, this is a trivial consequence of the definition of the Hausdorff measures that projection is a Lipschitz mapping.

(ii) From Remark 4, $\mathcal{H}^g(A) = \infty$. Thus it suffices to show that $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$.

Since $\mathcal{H}^f(A) > 0$, it follows via Proposition 8 and the definition of capacity, that A supports a Radon probability measure μ such that $I_g(\mu) < \infty$. For each $\theta \in [0, \pi)$, projecting μ onto the line L_θ gives a measure μ_θ supported on $\mathrm{proj}_\theta A$ defined by the requirement that $\mu_\theta(K) = \mu(\mathrm{proj}_\theta^{-1}(K))$ for each Borel set $K \subseteq L_\theta$. For each $x \in \mathbb{R}^2$, let $\phi(x)$ denote the angle that x (viewed as a vector) forms with the horizontal axis. Then, by Lemma 3 and using the fact that g is doubling with constant $c < 2$, it follows that

$$\begin{aligned}
\int_0^\pi I_g(\mu_\theta)\mathrm{d}\theta &= \int_0^\pi \iint \frac{\mathrm{d}\mu_\theta(x)\mathrm{d}\mu_\theta(y)}{g(|x-y|)}\mathrm{d}\theta \\
&= \int_0^\pi \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y|)}\mathrm{d}\theta \\
&\leq \iint \left(\int_0^\pi \frac{c}{g(|x-y|)|\cos(\phi(x-y)-\theta)|^s}\mathrm{d}\theta \right) \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\leq c_1 \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|x-y|)} \qquad (\text{because } s = \log_2 c < 1) \\
&= c_1 I_g(\mu) \ < \ \infty\,.
\end{aligned}$$

This implies that $I_g(\mu_\theta) < \infty$ for almost all $\theta \in [0, \pi)$. From the definition of capacity, $C_g(\mathrm{proj}_\theta A) > 0$ for such θ, so by Proposition 7, $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. □

3 Exceptional projections

Marstrand's Theorem trivially implies that the set of exceptional angles

$$E(A) := \{\theta \in [0, \pi) \,:\, \dim \mathrm{proj}_\theta A < \dim A\}\ ,$$

is a set of (one-dimensional) Lebesgue measure zero. Kaufman also showed [9] that

$$\dim\ E(A) \ \leq\ \min\{1, \dim A\} \tag{16}$$

(see also Remark 8 below). Clearly, when $\dim A < 1$, this bound on the size of the set of exceptional angles is significantly stronger than the measure zero statement of Marstrand's Theorem. It is natural to attempt to extend Theorem 2 in a similar fashion. With this in mind, let $E_g(A)$ denote the exceptional set of $\theta \in [0, \pi)$ for which the conclusion of part (ii) of Theorem 2 fails; that is

$$E_g(A) := \{\theta \in [0, \pi) : \mathcal{H}^g(\mathrm{proj}_\theta A) < \infty\} \ . \tag{17}$$

By replacing the integral convergence condition (7) by a rate of convergence condition we are able to establish the following strengthening of Theorem 2. It is easily verified that condition (18) below implies condition (7) of Theorem 2.

Theorem 9. *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Let f be a dimension function such that $\mathcal{H}^f(A) > 0$ and let g be a dimension function that is doubling. Suppose that there exist constants t_0 and $c_2 > 0$ such that*

$$-\int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(tr)}\right) < c_2 \frac{1}{g(t)} \qquad \textit{for all} \quad 0 < t < t_0 \, . \tag{18}$$

Then, $\mathcal{H}^f(E_g(A)) = 0$.

Proof. In view of Proposition 7,

$$E_g(A) \subseteq E_* := \{\theta \in [0, \pi) : C_g(\mathrm{proj}_\theta A) = 0\} \ .$$

Thus, it suffices to show that $\mathcal{H}^f(E_*) = 0$. Suppose this is not the case. Then $\mathcal{H}^f(E_*) > 0$ and by Theorem 4 the set E_* supports a probability measure $\nu \in \mathcal{M}_1(E_*)$ such that

$$\nu\left(B(x, r)\right) \leq c_1 f(r) \ \text{ for all } \ x \in \mathbb{R}^2 \ \text{ and } \ r > 0$$

where $c_1 > 0$ is an absolute constant. On the other hand, since $\mathcal{H}^f(A) > 0$ and condition (18) implies condition (7), it follows via Proposition 8 and the definition of capacity, that A supports a probability measure $\mu \in \mathcal{M}_1(A)$ such that

$$I_g(\mu) < \infty. \tag{19}$$

For each $\theta \in [0, \pi)$, let μ_θ be the projection of μ onto the line L_θ supported on $\mathrm{proj}_\theta A$ such that $\mu_\theta(K) = \mu(\mathrm{proj}_\theta^{-1}(K))$ for each Borel set $K \subseteq L_\theta$ – as in the proof of Theorem 2. Let us assume for the moment that

$$\int_{E_*} I_g(\mu_\theta) \mathrm{d}\nu(\theta) < \infty \, . \tag{20}$$

This implies that $I_g(\mu_\theta) < \infty$ for ν-almost all $\theta \in E_*$. By the definition of capacity, $C_g(\mathrm{proj}_\theta A) > 0$ for such θ, contradicting that $C_g(\mathrm{proj}_\theta A) = 0$ if $\theta \in E_*$. This completes the proof of the theorem modulo establishing (20).

To establish (20), we first observe that for all $x \in \mathbb{R}^2 \setminus \{0\}$ and $d > 0$, the set

$$\{\theta \in [0, \pi) : |\mathrm{proj}_\theta x| \leq d\}$$

is a union of at most two intervals each of diameter at most $\pi\, d/|x|$. The upshot is that

$$\nu\left(\{\theta \in [0, \pi) : |\mathrm{proj}_\theta x| \leq d\}\right) \;\leq\; 2\, c_1 f\Big(\pi \frac{d}{|x|}\Big)\,.$$

This, together with the fact that g is doubling, implies that

$$\begin{aligned}
\int_{E_*} \frac{1}{g(|\mathrm{proj}_\theta x|)}\, \mathrm{d}\nu(\theta) \;&=\; \int_0^\infty \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \geq r\right\}\right) \mathrm{d}r \\
&=\; \int_0^{1/g(|x|)} \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \geq r\right\}\right) \mathrm{d}r \\
&\quad +\; \int_{1/g(|x|)}^\infty \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \geq r\right\}\right) \mathrm{d}r \\
&\leq\; \frac{1}{g(|x|)} \;+\; \int_{1/g(|x|)}^\infty 2\, c_1 f\left(\frac{\pi}{|x|}\, g^{-1}\left(\tfrac{1}{r}\right)\right) \mathrm{d}r \\
&\leq\; \frac{1}{g(|x|)} \;-\; 2c_1 \int_0^\pi f(u)\, \mathrm{d}\Big(\frac{1}{g(|x|u/\pi)}\Big) \\
&=\; \frac{1}{g(|x|)} \;+\; 2c_1 \int_0^1 f(u)\, \mathrm{d}\Big(\frac{-1}{g(|x|u/\pi)}\Big) \\
&\quad +\; 2c_1 \int_1^\pi f(u)\, \mathrm{d}\Big(\frac{-1}{g(|x|u/\pi)}\Big) \\
&\overset{(18)}{\leq}\; \frac{1}{g(|x|)} \;+\; 2c_1 c_2\, \frac{1}{g(|x|/\pi)} \\
&\quad +\; 2c_1\, f(\pi)\, \left(\frac{1}{g(|x|/\pi)} - \frac{1}{g(|x|)}\right) \\
&\leq\; c_3\, \frac{1}{g(|x|)}
\end{aligned}$$

for some c_3 and for all $x \neq 0$ with $|x| < t_0$. Hence, using Fubini's theorem,

$$\begin{aligned}
\int_{E_*} I_g(\mu_\theta)\ \mathrm{d}\nu(\theta) &= \int_{E_*} \iint \frac{\mathrm{d}\mu_\theta(x)\,\mathrm{d}\mu_\theta(y)}{g(|x-y|)}\ \mathrm{d}\nu(\theta) \\
&= \int_{E_*} \iint \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{g(|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y|)}\ \mathrm{d}\nu(\theta) \\
&= \iint \int_{E_*} \frac{\mathrm{d}\nu(\theta)}{g(|\mathrm{proj}_\theta(x-y)|)}\ \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\leq c_3 \iint \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{g(|x-y|)} \ < \ \infty
\end{aligned}$$

by (19). This establishes (20) and completes the proof. □

Remark 8. The above proof of Theorem 9 is based on the proof of the dimension inequality (16) presented in [12, Theorem 5.1]. Note that it is easy to deduce (16) from Theorem 9. Indeed, to see that this is the case, without loss of generality assume that $0 < \dim A < 1$ and let s_1, s_2 be real numbers satisfying $0 < s_1 < s_2 < \dim A$. Let

$$E(A, s_1) := \{\theta \in [0, \pi) : \dim \mathrm{proj}_\theta A < s_1\}.$$

Let g and f be dimension functions given by $g(r) := r^{s_1}$ and $f(r) := r^{s_2}$. It follows that $\mathcal{H}^{s_2}(A) = \infty$ and that $\mathcal{H}^{s_1}\big(\mathrm{proj}_\theta A\big) = 0$ for all $\theta \in E(A, s_1)$. Thus

$$E(A, s_1) \ \subseteq \ E_g(A)\,,$$

with $E_g(A)$ as in (17). Clearly, the function g is doubling and it is easily checked that f and g satisfy condition (18). Theorem 9 implies that $\mathcal{H}^{s_2}(E_g(A)) = 0$ and so $\dim\big(E(A, s_1)\big) \leq s_2$, and (16) follows on taking s_1, s_2 arbitrarily close to $\dim A$.

Remark 9. The above proof of Theorem 9 is based on the proof of the special case (16) presented in [12, Theorem 5.1].

Armed with Theorem 9 it is straightforward to prove (3) which we formally state as a corollary.

Corollary 10. *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Then,* $\dim_{\log} E_{\log}(A) \leq \dim_{\log} A$ *where*

$$E_{\log}(A) := \{\theta \in [0, \pi) : \dim_{\log} \mathrm{proj}_\theta(A) < \dim_{\log} A\}.$$

Proof. Without loss of generality, assume that $0 < \dim_{\log} A < \infty$ and let s_1, s_2 be real numbers satisfying $0 < s_1 < s_2 < \dim_{\log} A$. Let

$$E_{\log}(A, s_1) := \{\theta \in [0, \pi) : \dim_{\log} \mathrm{proj}_\theta(A) < s_1\}.$$

As in the proof of Theorem 1, let g and f be the dimension functions $g(r) := (-\log^* r)^{-s_1}$ and $f(r) := (-\log^* r)^{-s_2}$. Then $\mathcal{H}^f(A) = \infty$ and $\mathcal{H}^g\big(\mathrm{proj}_\theta(A)\big) = 0$ for all $\theta \in E_{\log}(A, s_1)$ so $E_{\log}(A, s_1) \subseteq E_g(A)$. Clearly g is doubling. Assume for the moment that f and g satisfy condition (18). Then Theorem 9 implies that $\mathcal{H}^f(E_g(A)) = 0$ so from the definition of logarithmic Hausdorff dimension (13),

$$\dim_{\log}\big(E_{\log}(A, s_1)\big) \le s_2\,.$$

The conclusion now follows on taking s_1, s_2 arbitrarily close to $\dim_{\log} A$.

It remains to verify (18). For all sufficiently small $t > 0$,

$$\begin{aligned} -\int_0^1 f(r)\,\mathrm{d}\left(\frac{1}{g(tr)}\right) &= \left[\frac{f(r)}{g(tr)}\right]_1^0 + \int_0^1 \frac{\mathrm{d}f(r)}{g(tr)} \\ &= -\frac{f(1)}{g(t)} + \int_0^t \frac{\mathrm{d}f(r)}{g(tr)} + \int_t^1 \frac{\mathrm{d}f(r)}{g(tr)}. \end{aligned}$$

For the first integral, $r \le t$ implies that

$$\frac{1}{g(tr)} \le 2^{s_1}(-\log r)^{s_1} = 2^{s_1}\frac{1}{g(r)}$$

and hence it follows that

$$\begin{aligned} \int_0^t \frac{\mathrm{d}f(r)}{g(tr)} &\le 2^{s_1}\int_0^t \frac{\mathrm{d}f(r)}{g(r)} \\ &\le 2^{s_1}\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} \\ &= \frac{2^{s_1}s_2(s_2 - s_1)}{(\log 2)^{s_2 - s_1}}. \end{aligned}$$

For the second integral, $r \ge t$ implies that

$$\frac{1}{g(tr)} < 2^{s_1}\frac{1}{g(t)}$$

and hence it follows that

$$\int_t^1 \frac{\mathrm{d}f(r)}{g(tr)} < 2^{s_1} \int_0^1 \frac{\mathrm{d}f(r)}{g(t)} = 2^{s_1} \frac{f(1)}{g(t)}.$$

On combining these estimates, we obtain that

$$\begin{aligned} -\int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(tr)}\right) &= (2^{s_1}-1)f(1)\frac{1}{g(t)} + \frac{2^{s_1}s_2(s_2-s_1)}{(\log 2)^{s_2-s_1}} \\ &\leq c_2 \frac{1}{g(t)} \end{aligned}$$

for some constant c_2, as desired. □

4 Final comments

Apart from working in higher dimensions, there are several other directions in which one could attempt to strengthen/generalize the main theorem. We concentrate on just a few of them.

The gap of uncertainty. Theorem A′ shows that we can not in general replace condition (7) by (9) in Theorem 2. Thus there is a gap of uncertainty associated with Theorem 2. It would be highly desirable to know whether or not condition (7) is really necessary. Namely, if f and g are dimension functions such that (7) is not satisfied, then does there exist a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for almost all $0 \leq \theta < \pi$? Theorem A′ provides sufficient conditions on f and g for the existence of such a set A.

Brownian paths. Brownian motion sample paths, see [6, Chapter 16] for a general introduction, illustrate the sort of situation that can arise for projections of sets in $\mathbb{R}^3$, and which perhaps may occur in $\mathbb{R}^2$, though there is no direct analogue. Let $B[0,1] \subseteq \mathbb{R}^3$ be (random) Brownian motion path over the unit time interval. Then, almost surely, the Hausdorff dimension of $B[0,1]$ is logarithmically smaller than 2, more precisely $0 < \mathcal{H}^f(B[0,1]) < \infty$ where f is the dimension function $f(r) = r^2 \log\log(1/r)$ (for small r), see [3]. However, the projection $\mathrm{proj}_P(B[0,1])$ of $B[0,1]$ onto any given plane P has exactly the same distribution as a Brownian motion in the plane, which is almost surely of Hausdorff dimension 2, or precisely, $0 < \mathcal{H}^g(\mathrm{proj}_P(B[0,1])) < \infty$ where g is the dimension function $g(r) = r^2 \log(1/r) \log\log\log(1/r)$, see [18]. This example, where the exact dimension functions of a set and of almost all its projections onto a plane can be identified, illustrates the sort of change in exact dimension that may occur under projection.

Sets with no exceptional projections. The dimension result (16) for the set of exceptional projections has been extended in various ways – see [7, 12] and

references within. We highlight a result concerning sets A for which there are no exceptional projections; that is, sets A for which $E(A) = \emptyset$.

Theorem (Peres–Shmerkin). *Let $A \subseteq \mathbb{R}^2$ be a self-similar set with dense rotations. Then*

$$\dim \mathrm{proj}_\theta A = \min\{\dim A, 1\} \quad \textit{for all} \quad \theta \in [0, \pi). \tag{21}$$

This theorem was proved by Peres and Shmerkin [14] and subsequently generalized by Hochman and Shmerkin [8]. Now suppose A is self-similar set with dense rotations and f and g are dimension functions as in Theorem 2. It is natural to ask whether or not the conclusion of part (ii) of Theorem 2 is actually valid for all θ rather than just almost all $\theta \in [0, \pi)$.

Lengths of projections. It is natural to seek a finer version of part (ii) of Marstrand's theorem which gives a criterion for almost all projections of a set to have positive length. One aspect of this was investigated by Peres and Solomyak [15], who considered dimension functions f such that $f(r)/r^2$ is decreasing for $r > 0$ (a condition that holds in virtually all cases of interest). The following statement constitutes parts (i) and (ii) of their main result [15, Theorem 1.1].

Theorem (Peres-Solomyak). *Let f be a dimension function such that $f(r)/r^2$ is decreasing. Then $\int_0^1 r^{-2} f(r) \mathrm{d}r < \infty$ if and only if for any Borel set $A \subseteq \mathbb{R}^2$ with $\mathcal{H}^f(A) > 0$ one has that $\mathcal{L}(\mathrm{proj}_\theta A) > 0$ for almost all $\theta \in [0, \pi)$.*

Note that the integral convergence condition in the above theorem is exactly condition (7) in Theorem 9 with g given by $g(r) = r$ and so $\mathcal{H}^g$ is Lebesgue measure $\mathcal{L}$.

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Appendix: The gap of uncertainty

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As discussed in Remark 3 and explicitly demonstrated in §1.2.2, the integral convergence condition (7) gives rise to a gap of uncertainty. It is natural to ask whether this condition is really necessary. Namely, if f and g are dimension functions such that (7) is not satisfied, then does there exist a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for almost all $0 \leq \theta < \pi$? In this appendix we partially answer this question by providing sufficient conditions on f and g for the existence of such a set A. Our construction of this set is a generalization of a construction of Martin and Mattila [2], in which they proved that for every $0 < s \leq 1$ there exists a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^s(A) > 0$ but $\mathcal{H}^s(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$. By making a careful quantitative analysis of their construction, we are able to improve their result and establish the following statement.

Theorem A. *Let f, g be dimension functions such that f is doubling with exponent $s_1 \leq 1$ and codoubling with exponent $s_2 > 0$, and such that*

$$g(r) \leq M \; f\left(r \log(r^{-1})\right) \qquad \forall\, 0 < r < r_0 \tag{22}$$

where $r_0 > 0$ and $M > 0$ are constants. Then there exists a set $A \subseteq \mathbb{R}^2$ with $0 < \mathcal{H}^f(A) < \infty$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

Recall, a dimension function f is called *doubling with exponent s* if there exist constants $\kappa > 0$ and $r_1 > 0$ such that

$$f(r\lambda) \;\geq\; \kappa\, \lambda^s f(r) \qquad \forall\; 0 < \lambda < 1 \;\text{ and }\; 0 < r < r_1\,. \tag{23}$$

Moreover, f is called *codoubling with exponent s* if there exist constants $\kappa > 0$ and $r_1 > 0$ such that

$$f(r\lambda) \;\leq\; \kappa\, \lambda^s f(r) \qquad \forall\; 0 < \lambda < 1 \;\text{ and }\; 0 < r < r_1\,. \tag{24}$$

Note that these definitions of doubling and codoubling with exponent s are different from the definitions of doubling and codoubling (with constant $c > 1$) appearing in §1.2. It is easily seen that if a function is doubling (resp. codoubling) in the sense of (6) (resp. (10)) with constant $c > 1$, then it is also doubling (resp. codoubling) in the sense of (23) (resp. (24)) with exponent $s = \log_2 c$. (See Lemma 3 for the case of doubling functions.) However, observe that if f is doubling with exponent $s \leq 1$ then it is not necessarily doubling with constant $c \leq 2$. Also, if a function is codoubling in the sense of (24) (with exponent $s > 0$) then it is not necessarily codoubling in the sense of (10) (with constant $c > 1$). Thus Theorem A is strictly stronger than Theorem A′ appearing in Remark 3.

Remark 10. The assumption that the function f is doubling with exponent $s \leq 1$ restricts our attention to subsets $A \subseteq \mathbb{R}^2$ of Hausdorff dimension at most 1, which is expected given the nature of the problem (cf. Remark 5).

Remark 11. The growth condition (22) can be replaced by any condition of the form

$$g(r) \ll f\left(r \log(r^{-1}) \log_3(r^{-1}) \log_4(r^{-1}) \cdots \log_p(r^{-1})\right) \tag{25}$$

where $p \geq 3$ is a positive integer and $r > 0$ is sufficiently small – see Remark 14 below. Here we write $\log_2 t = \log\log t,\ \log_3 t = \log\log\log t,$ etc .

Remark 12. Under the assumptions of Theorem A the integral

$$\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} \;=\; \frac{f(1)}{g(1)} - \int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(r)}\right)$$

diverges, hence the integral convergence condition (7) is not satisfied and in turn Theorem 2 is not violated. To see this, observe that since f is doubling with exponent $s_1 \leq 1$, we have

$$f\left(r \log \frac{1}{r}\right) \ll f(r)\ \log \frac{1}{r} \qquad (r \to 0^+).$$

Now since f is codoubling with exponent $s_2 > 0$, we have

$$f(r) \ll r^{s_2}$$

and thus

$$\log \frac{1}{r} \ll \log \frac{1}{f(r)} \quad (r \to 0^+).$$

Together the above estimates yield

$$f\left(r \log \frac{1}{r}\right) \ll f(r) \log \frac{1}{f(r)}\,.$$

It then follows that for any g satisfying condition (22), we have

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} \;&\gg\; \int_0^1 \frac{\mathrm{d}f(r)}{f\left(r \log \frac{1}{r}\right)} \\
&\gg\; \int_0^1 \frac{\mathrm{d}f(r)}{f(r) \log \frac{1}{f(r)}} \\
&=\; \int_0^{f^{-1}(1)} \frac{\mathrm{d}x}{x \log \frac{1}{x}} \;=\; \infty\,.
\end{aligned}$$

A.1 Construction of an f-set

Given a dimension function f, a set $A \subseteq \mathbb{R}^n$ is called an *f-set* if $0 < \mathcal{H}^f(A) < \infty$. Here we present the construction of an f-set $A \subseteq \mathbb{R}^2$ for a given function f, which is similar to the one presented by Martin and Mattila in [2, Section 5.3] for dimension functions of the form $r \mapsto r^s$. In the next section, we will show that by choosing the parameters of the construction appropriately, the resulting f-set A will satisfy $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

Throughout, $(r_k)_{k=0}^\infty$ is a decreasing sequence of positive real numbers tending to 0, $(N_k)_{k=1}^\infty$ is a sequence of positive integers ≥ 2, and $(\theta_k)_{k=1}^\infty$ is a sequence of angles $0 \leq \theta_k < \pi,\ k \geq 1$. The sequences $(r_k)_{k=0}^\infty$ and $(N_k)_{k=1}^\infty$ will be assumed to satisfy the inequalities

$$a \leq N_1 \cdots N_k f(r_k) \leq 2a \tag{26}$$

and

$$N_{k+1} r_{k+1} < r_k \tag{27}$$

for all $k \geq 0$, for some constant $a > 0$.

Let A_0 be the closed disc of radius r_0 centered at the origin. In the first step, inside A_0 we consider N_1 subdiscs of radius r_1, denoted $C_1, \ldots, C_{N_1}$ and defined as follows: their centers are equally spaced, lying on the diameter of A_0 which forms angle θ_1 (measured counterclockwise) with the horizontal axis, and the boundaries of first and last subdisc are tangent to the boundary of A_0. Condition (27) guarantees that these subdiscs are disjoint. Let $d_1 = \theta_1$, and set

$$A_1 = \bigcup_{i=1}^{N_1} C_i.$$

Now inductively assume that for some $k \geq 1$ we have defined the discs $C_{i_1 \ldots i_k},\ 1 \leq i_j \leq N_j,\ 1 \leq j \leq k$, each of radius r_k.

At the $(k+1)$st step, inside each disc $C_{i_1 \ldots i_k}$ we consider N_{k+1} subdiscs $C_{i_1 \ldots i_k 1}, \ldots, C_{i_1 \ldots i_k N_{k+1}}$, each of radius r_{k+1}, defined as follows: their centers are equally spaced along the diameter of $C_{i_1 \ldots i_k}$ which forms angle θ_{k+1} with the line containing the centers of the discs of the kth step, and the boundaries of the first and last subdiscs are tangent to the boundary of $C_{i_1 \ldots i_k}$. Again, condition (27) guarantees that these subdiscs are disjoint. Let $d_{k+1} \equiv \theta_1 + \ldots + \theta_{k+1} \pmod{\pi}$, so that for any disc $C_{i_1 \ldots i_k}$ of A_k, d_{k+1} is the angle between the diameter of $C_{i_1 \ldots i_k}$ used to define the subdiscs of $C_{i_1 \ldots i_k}$ and the horizontal axis. Set

$$A_{k+1} = \bigcup_{\substack{1 \leq i_j \leq N_j \\ (j=1,\ldots,k+1)}} C_{i_1 \ldots i_{k+1}}.$$

We complete the construction by setting

$$A = \bigcap_{k=1}^{\infty} A_k.$$

We show that under certain conditions on f and appropriate choices of the sequences $(r_k)_{k=0}^{\infty}$ and $(N_k)_{k=1}^{\infty}$, the set A is an f-set.

Proposition A. *Let f be a dimension function which is doubling with exponent $s \leq 1$, and let $(r_k)_{k=0}^{\infty}$ be a sequence satisfying the inequalities*

$$f(r_{k+1}) < \frac{1}{4} f(r_k) \tag{28}$$

and

$$\frac{f(r_{k+1})}{r_{k+1}} > 3 \frac{f(r_k)}{r_k} \tag{29}$$

for all $k \geq 0$. Let $(\theta_k)_{k=1}^{\infty}$ be any sequence of real numbers. Then the parameter $a > 0$ and the sequence $(N_k)_{k=1}^{\infty}$ can be chosen so as to satisfy (26) *and* (27) *for all $k \geq 0$. The resulting set $A \subseteq \mathbb{R}^2$ constructed as above is an f-set.*

Proof. Let $a = f(r_0)$, so that (26) automatically holds when $k = 0$. Now inductively assume that for some $k \geq 0$ we have chosen $N_1, \ldots, N_k \geq 2$ such that (26) holds. Since

$$\begin{aligned} \frac{2a}{N_1 \cdots N_k f(r_{k+1})} - \frac{a}{N_1 \cdots N_k f(r_{k+1})} &= a \frac{f(r_k)}{f(r_{k+1})} \frac{1}{N_1 \cdots N_k \; f(r_k)} \\ &\overset{(26)}{\geq} \frac{1}{2} \frac{f(r_k)}{f(r_{k+1})} \\ &\overset{(28)}{>} 2, \end{aligned}$$

the interval

$$\left[\frac{2a}{N_1 \cdots N_k f(r_{k+1})}, \frac{a}{N_1 \cdots N_k f(r_{k+1})} \right]$$

contains a positive integer $N_{k+1} \geq 2$. Thus, the inequality

$$a \;\leq\; N_1 \cdots N_k N_{k+1} \, f(r_{k+1}) \;\leq\; 2a \tag{30}$$

is satisfied. This completes the inductive step, thus demonstrating that the sequence $(N_k)_{k=1}^{\infty}$ can be chosen so that (26) holds for all $k \geq 0$.

To demonstrate (27), we note that

$$N_{k+1} \overset{(30)}{\leq} \frac{2a}{N_1 \cdots N_k f(r_{k+1})} \overset{(26)}{\leq} 2 \cdot \frac{f(r_k)}{f(r_{k+1})} \overset{(29)}{<} \frac{2}{3} \cdot \frac{r_k}{r_{k+1}} \tag{31}$$

and in particular $N_{k+1} r_{k+1} < r_k$.

For each $k \in \mathbb{N}$, the set A_k is a cover of A consisting of discs of radius r_k. The number of balls in this cover is $N_1 \cdots N_k$, hence for $k \in \mathbb{N}$ we have

$$\mathcal{H}^f_{r_k}(A) \;\leq\; N_1 \cdots N_k f(r_k) \;\leq\; 2a$$

and thus

$$\mathcal{H}^f(A) \;=\; \sup_{k>0} \mathcal{H}^f_{r_k}(A) \;\leq\; 2a \;<\; \infty.$$

Now consider the probability measure μ supported on A which is defined by assigning each of the discs $C_{i_1 \ldots i_k}$ of A_k the same measure, i.e. by setting

$$\mu(C_{i_1 \ldots i_k}) = \frac{1}{N_1 \cdots N_k}.$$

We claim that for all $x \in A$ and $r > 0$ small enough,

$$\mu\left(B(x,r)\right) \leq C f(r) \tag{32}$$

for some constant $C > 0$. By the Mass Distribution Principle (see for example [1, Lemma 3]), this will imply that $\mathcal{H}^f(A) > 0$ and thus complete the proof.

Whenever r, r' are sufficiently small and $r < r'$, since f is doubling with exponent $s \leq 1$, for some constant $\kappa > 0$ we have

$$\frac{f(r)}{r} \geq \kappa \frac{1}{r} \left(\frac{r}{r'}\right)^s f(r') = \kappa \left(\frac{r'}{r}\right)^{1-s} \frac{f(r')}{r'} \geq \kappa \, \frac{f(r')}{r'},$$

which gives

$$\frac{f(r')}{r'} \;\leq\; \kappa^{-1} \frac{f(r)}{r}. \tag{33}$$

Now fix $x \in A$ and $r > 0$, and let $k \in \mathbb{N}$ be maximal such that $B(x,r) \cap A$ is contained in only one disc of A_k. Consider the following cases:

Case 1: $r < r_k$. Let s_{k+1} be the common distance between any two consecutive subdiscs of A_{k+1}. Then by subdividing the appropriate diameter of $C_{i_1 \ldots i_k}$ into intervals consisting of its intersections with discs $C_{i_1 \ldots i_k j}$ as well as the gaps between them, we find that

$$2N_{k+1} r_{k+1} + (N_{k+1} - 1) s_{k+1} = 2r_k. \tag{34}$$

Now in any sequence of n consecutive subdiscs of A_{k+1}, the distance between the first and last subdiscs in this sequence is

$$(n-1)s_{k+1} + (n-2)2r_{k+1} > (n-2)(s_{k+1} + 2r_{k+1}).$$

Since the diameter of $B(x,r)$ is $2r$, if n is the number of subdiscs of A_{k+1} that intersect $B(x,r)$, then the distance given above must be less than $2r$. It follows that

$$n \leq 2 + \frac{2r}{2r_{k+1} + s_{k+1}}. \tag{35}$$

On the other hand, we have

$$\begin{aligned} N_{k+1}(s_{k+1} - r_{k+1}) &> (N_{k+1} - 1)s_{k+1} - N_{k+1}r_{k+1} \\ &\overset{(34)}{=} 2r_k - 3N_{k+1}r_{k+1} \\ &\overset{(31)}{>} 0, \end{aligned}$$

which implies that

$$s_{k+1} > r_{k+1}\,. \tag{36}$$

On the other hand, by the maximality of k, $B(x,r)$ intersects at least 2 discs of A_{k+1}, including the disc containing x, and thus it follows that $r > s_{k+1}$. This together with (36) implies that $r > r_{k+1}$ and so

$$\frac{2r}{2r_{k+1} + s_{k+1}} \geq \frac{2}{3}\,.$$

Hence,

$$\begin{aligned} n &\overset{(35)}{\leq} 4\left(\frac{2r}{2r_{k+1} + s_{k+1}}\right) \\ &\overset{(34)}{=} 8r\left(2r_{k+1} + 2\frac{r_k - N_{k+1}r_{k+1}}{N_{k+1} - 1}\right)^{-1} \\ &\leq 4r\left(r_{k+1} + \frac{r_k - N_{k+1}r_{k+1}}{N_{k+1}}\right)^{-1} \\ &= 4\frac{N_{k+1}}{r_k}r \\ &\overset{(31)}{\leq} 8\frac{f(r_k)}{f(r_{k+1})} \cdot \frac{r}{r_k} \\ &\overset{(33)}{\leq} \frac{8}{\kappa}\,\frac{f(r)}{f(r_{k+1})} \end{aligned}$$

Since each subdisc of A_{k+1} has measure $\frac{1}{N_1 \cdots N_{k+1}}$, it follows that

$$\mu\left(B(x,r)\right) \leq \frac{8}{\kappa} \frac{f(r)}{f(r_{k+1})} \cdot \frac{1}{N_1 \cdots N_{k+1}} \overset{(30)}{\leq} \frac{8}{a\kappa} f(r).$$

Case 2: $r \geq r_k$. Let $C_{i_1 \ldots i_k}$ be the unique disc of A_k intersecting $B(x,r)$, which exists by the definition of k. Then

$$\mu(B(x,r)) \leq \mu(C_{i_1 \ldots i_k}) = \frac{1}{N_1 \cdots N_k} \overset{(26)}{\leq} \frac{1}{a} f(r_k) \leq \frac{1}{a} f(r),$$

where in the last inequality, we have used the fact that f is increasing.

Thus in either case, (32) holds with $C = \max\left\{\frac{8}{a\kappa}, \frac{1}{a}\right\} > 0$. □

Remark 13. Note that Proposition A applies to any possible sequence of angles $(\theta_k)_{k=1}^{\infty}$, indicating that varying the sequence of angles may cause the quantity $\mathcal{H}^f(A)$ to change slightly but will not affect the fact that it is finite and positive. The role of the sequence $(\theta_k)_{k=1}^{\infty}$ will become apparent in the next section.

A.2 Proof of Theorem A

We show that if g satisfies the growth condition (22) relative to f, the sequences $(r_k)_{k=0}^{\infty}$, $(N_k)_{k=1}^{\infty}$ and $(\theta_k)_{k=1}^{\infty}$ in the aforementioned construction can be suitably selected so that the corresponding f-set $A \subseteq \mathbb{R}^2$ satisfies $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

First, we claim that the sequence $(r'_k)_{k=k_0}^{\infty}$ defined by the formula

$$r'_k = (k \log k \log\log k)^{-k} \tag{37}$$

satisfies (28) and (29) for all sufficiently large k. To prove this, we first observe that

$$\frac{r'_{k+1}}{r'_k} \asymp \frac{1}{k \log k \log\log k} \to 0 \qquad \text{as} \qquad k \to \infty. \tag{38}$$

On the other hand, by the doubling and codoubling hypotheses imposed on f, there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\kappa_1 \lambda^{s_1} f(r) \leq f(\lambda r) \leq \kappa_2 \lambda^{s_2} f(r)$$

for all $0 < \lambda < 1$ and $r > 0$ sufficiently small. Since $r'_k > r'_{k+1}$ for all k sufficiently large, we have that

$$f(r'_{k+1}) \leq \kappa_2 \left(\frac{r'_{k+1}}{r'_k}\right)^{s_2} f(r_k)$$

and

$$\begin{aligned}\frac{f(r'_{k+1})}{r'_{k+1}} &\geq \kappa_1 \frac{1}{r'_{k+1}} \left(\frac{r'_{k+1}}{r'_k}\right)^{s_1} f(r'_k) \\ &= \kappa_1 \left(\frac{r'_{k+1}}{r'_k}\right)^{s_1-1} \frac{f(r'_k)}{r'_k}.\end{aligned}$$

Thus by (38), the inequalities $(28)_{r_k=r'_k}$ and $(29)_{r_k=r'_k}$ are satisfied for all k large enough. Let $k_1 \geq k_0$ be chosen so that $(28)_{r_k=r'_k}$ and $(29)_{r_k=r'_k}$ are satisfied for all $k \geq k_1$.

Now consider the sequence $(r_k)_{k=0}^{\infty}$ defined by the formula

$$r_k = r'_{k+k_1},$$

and note that (28) and (29) are satisfied for all $k \geq 0$. Thus by Proposition A, we can choose a sequence $(N_k)_{k=1}^{\infty}$ such that (26) and (27) hold for all $k \geq 0$. Also note that by (38) we have that

$$\frac{r_{k+1}}{r_k} \asymp \frac{1}{k \log k \log\log k}. \tag{39}$$

Let the sequence of angles be defined by

$$\theta_{k+1} = \frac{r_{k+1}}{r_k}, \qquad k \geq 0.$$

Then

$$\sum_{k=1}^{\infty} \theta_k = \infty.$$

Take an arbitrary $0 \leq \theta < \pi$. Let d_θ denote the direction perpendicular to L_θ, i.e. the direction of projection, $d_\theta \equiv \theta + \pi/2 \pmod{\pi}$. Since the series $\sum_{k=1}^{\infty} \theta_k$ diverges, there are infinitely many $k \in \mathbb{N}$ such that d_θ lies between d_k and d_{k+1}. For each of these values of k, the angle between d_θ and d_{k+1} is at most θ_{k+1}, and thus for each disc $C_{i_1 \ldots i_k}$ of A_k, the distances from the centers of all subdisc $C_{i_1 \ldots i_k j}$ of $C_{i_1 \ldots i_k}$ from the diameter of $C_{i_1 \ldots i_k}$ in the direction d_θ are at most

$$r_k \sin \theta_{k+1} \leq r_k \theta_{k+1} = r_{k+1}.$$

This is because all of these centers lie on the diameter of $C_{i_1 \ldots i_k}$ in the direction d_{k+1}. This means that within each disc of A_k, when we project the union of the subdiscs of A_{k+1} onto L_θ we get an interval of length at most $4r_{k+1}$, which we can think of as the union of at most 4 intervals of length at most r_{k+1}. The number

of such intervals is equal to the number of discs of A_k, that is, $N_1 \cdots N_k \leq \frac{2a}{f(r_k)}$.

We have shown that for infinitely many values of k there is a cover of $\mathrm{proj}_\theta A$ which consists of $4N_1 \cdots N_k$ intervals of length at most r_{k+1}, hence for such k we obtain that

$$\begin{aligned} \mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) &\leq 8a\frac{g(r_{k+1})}{f(r_k)} \\ &\leq 8aM\frac{1}{f(r_k)} f\left(r_{k+1}\log(r_{k+1}^{-1})\log\log(r_{k+1}^{-1})\right). \end{aligned} \tag{40}$$

Now (37) implies that

$$\log(r_k^{-1}) \asymp \log(r_k'^{-1}) \asymp k\log(k\log k\log\log k) \asymp k\log k,$$

$$\log\log(r_k^{-1}) \asymp \log\log(r_k'^{-1}) \asymp \log(k\log k) \asymp \log k$$

as $k \to \infty$. Combining this estimate with (39), (40), and the fact that f is codoubling shows that

$$\mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) \leq \frac{M_1}{(\log k)^{s_2}},$$

where $M_1 > 0$ is some absolute constant. This implies that

$$\mathcal{H}^g(\mathrm{proj}_\theta A) = \lim_{k\to\infty} \mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) = 0$$

and thereby completes the proof of Theorem A.

Remark 14. As mentioned in Remark 11, the growth condition (22) in Theorem A can be replaced by any condition of the form (25). The corresponding set in that case is constructed using the sequence $(r_k')_{k\geq k_0}$ defined by the formula

$$r_k' = \left(k\log k\log_2 k\cdots\log_p k\right)^{-k}.$$

The proof is nearly identical and we leave it to the interested reader.

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