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Clock–Work Trade-Off Relation for Coherence in Quantum Thermodynamics

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In thermodynamics, quantum coherences—superpositions between energy eigenstates—behave in distinctly nonclassical ways. Here we describe how thermodynamic coherence splits into two kinds—"internal" coherence that admits an energetic value in terms of thermodynamic work, and "external" coherence that does not have energetic value, but instead corresponds to the functioning of the system as a quantum clock. For the latter form of coherence, we provide dynamical constraints that relate to quantum metrology and macroscopicity, while for the former, we show that quantum states exist that have finite internal coherence yet with zero deterministic work value. Finally, under minimal thermodynamic assumptions, we establish a clock–work trade-off relation between these two types of coherences. This can be viewed as a form of time-energy conjugate relation within quantum thermodynamics that bounds the total maximum of clock and work resources for a given system.

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Classical thermodynamics describes the physical behavior of macroscopic systems composed of large numbers of particles. Thanks to its intimate relationship with statistics and information theory, the domain of thermodynamics has recently been extended to include small systems, and even quantum systems. One particularly pressing question is how the existence of quantum coherences, or superpositions of energy eigenstates, impacts the laws of thermodynamics [1–4], in addition to quantum correlations [5–8].

We now have a range of results for quantum thermodynamics [9–21] that have been developed within the resource-theoretic approach. A key advantage of the resource theory framework is that it avoids highly problematic concepts such as "heat" or "entropy" as its starting point. Its results have been shown to be consistent with traditional thermodynamics [22] while allowing for the inclusion of coherence as a resource [23] in the quantum regime. Very recently, a framework for quantum thermodynamics with coherence was introduced in Ref. [21]. Remarkably, the thermodynamic structure (namely, which states $\hat{\sigma}$ are thermodynamically accessible from a given state $\hat{\rho}$) turns out to be fully describable in terms of a single family of entropies. This framework of "thermal processes" is defined by the following three minimal physical assumptions: (i) that energy is conserved microscopically, (ii) that an equilibrium state exists, and (iii) that quantum coherence has a thermodynamic value. These are described in more detail in Ref. [24]. Note that thermal processes contain thermal operations (TOs) [9,10] as a subset, and coincide with TOs on incoherent states; however, in contrast to TOs they admit a straightforward description for the evolution of states with coherences between energy eigenspaces.

In this Letter we work under the same thermodynamic assumptions (i)–(iii) as above and show that quantum coherence in thermodynamics splits into two distinct types: *internal* coherences between quantum states of the same energy and *external* coherences between states of different energies. This terminology is used because the external coherences in a system are only defined relative to an external phase reference frame, while internal coherences are defined within the system as relational coherences between its subcomponents.

We focus on the case of an N-partite system with noninteracting subsystems. The Hamiltonian is written as $\hat{H} = \sum_{i=1}^{N} \hat{H}_i$ and we assume that each ith local Hamiltonian $\hat{H}_i$ has an energy spectrum $\{E_i\}$ with local energy eigenstates $|E_i\rangle$. Then a quantum state of this system may be represented as

$$\hat{\rho} = \sum_{\boldsymbol{E},\boldsymbol{E}'} \rho_{\boldsymbol{E}\boldsymbol{E}'} |\boldsymbol{E}\rangle\langle\boldsymbol{E}'|,$$

where $\boldsymbol{E} \coloneqq (E_1, E_2, \ldots, E_N)$ and $|\boldsymbol{E}\rangle \coloneqq |E_1 E_2 \cdots E_N\rangle$. We also define the total energy of the string $\boldsymbol{E}$ as $\mathcal{E}_{\boldsymbol{E}} \coloneqq \sum_{i=1}^{N} E_i$. Classical thermodynamic properties are determined by the probability distribution of the local energies, including their correlations. This information is contained in the diagonal terms of the density matrix $P(\boldsymbol{E}) \coloneqq \mathrm{Tr}[\hat{\Pi}_{\boldsymbol{E}} \hat{\rho}] = \rho_{\boldsymbol{E}\boldsymbol{E}}$, where $\hat{\Pi}_{\boldsymbol{E}} \coloneqq |\boldsymbol{E}\rangle\langle\boldsymbol{E}|$. Corresponding classical states could have degeneracies in energy, but we still have a distinguished orthonormal basis set $\{|\boldsymbol{E}\rangle\}$. The probability distribution of the total energy is $p_{\mathcal{E}} \coloneqq \sum_{\boldsymbol{E}:\mathcal{E}_{\boldsymbol{E}}=\mathcal{E}} P(\boldsymbol{E})$. So every state has a corresponding classical state defined via the projection $\Pi(\hat{\rho}) \coloneqq \sum_{\boldsymbol{E}} P(\boldsymbol{E}) |\boldsymbol{E}\rangle\langle\boldsymbol{E}|$.

However, a quantum system is defined by more than its classical energy distribution—it may have coherence in the energy eigenbasis. This coherence is associated with nonzero off-diagonal elements in the density matrix, namely, $|\boldsymbol{E}\rangle\langle\boldsymbol{E}'|$ for $\boldsymbol{E} \neq \boldsymbol{E}'$. The internal coherence corresponds to off-diagonal terms of the same total energy (where $\mathcal{E}_{\boldsymbol{E}} = \mathcal{E}_{\boldsymbol{E}'}$) and external coherence corresponds to terms with different energies ($\mathcal{E}_{\boldsymbol{E}} \neq \mathcal{E}_{\boldsymbol{E}'}$). For any state $\hat{\rho}$, we denote the corresponding state in which all external coherence is removed by $\mathcal{D}(\hat{\rho}) \coloneqq \sum_{\mathcal{E}} \hat{\Pi}_{\mathcal{E}} \hat{\rho} \hat{\Pi}_{\mathcal{E}}$, where $\hat{\Pi}_{\mathcal{E}} \coloneqq \sum_{\boldsymbol{E}:\mathcal{E}_{\boldsymbol{E}}=\mathcal{E}} \hat{\Pi}_{\boldsymbol{E}}$ is the projector onto the eigenspace of total energy $\mathcal{E}$.

As illustrated in Fig. 1, internal coherence may be used to extract work; however, it has been shown that external coherences obey a superselection rule (called work locking) that forbids work extraction, and is unavoidable if one wishes to explicitly account for all sources of coherence in thermodynamics [13]. We study this phenomenon by defining the process of extracting work purely from the coherence, without affecting the classical energy statistics. We find the conditions under which work can be deterministically extracted in this way from a pure state. Next, we show that external coherence is responsible for a system's ability to act as a clock. The precision of the clock may be quantified by the quantum Fisher information (QFI) [28]; we show that the QFI satisfies a second-law-like condition, stating that it cannot increase under a thermal process. Finally, we derive a fundamental trade-off inequality between the QFI and the extractable work from coherence demonstrating how a system's potential for producing work is limited by its ability to act as a clock and vice versa.

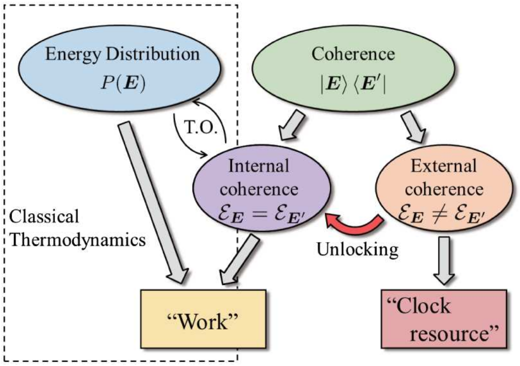

FIG. 1. Thermodynamic resources for many-body quantum systems. Coherences between energy levels provide coherent oscillations and are resources for the composite system to act as a quantum clock. At the other extreme, projective energy measurements on the individual systems provide the classical energy statistics, which may display classical correlations. Intermediate between these two cases are quantum coherences that are internal to energy eigenspaces. Partial interconversions are possible between these three aspects.

Extractable work from coherence.—Here we demonstrate that, in a single-shot setting, work may be extracted from coherence without changing the classical energy distribution $P(\boldsymbol{E})$ of the system. We consider the following type of work extraction process,

$$\hat{\rho} \otimes |0\rangle\langle 0|_B \xrightarrow{\text{thermal process}} \Pi(\hat{\rho}) \otimes |W\rangle\langle W|_B,$$

in which the energy of a work qubit [29] B with Hamiltonian $\hat{H}_B = W|W\rangle\langle W|_B$ $(W \geq 0)$ is raised from $|0\rangle_B$ to $|W\rangle_B$.

For energy-block-diagonal states $\hat{\rho} = \mathcal{D}(\hat{\rho})$ and $\hat{\sigma} = \mathcal{D}(\hat{\sigma})$, the work distance is defined as $D_{\text{work}}(\hat{\rho} \succ \hat{\sigma}) = \inf_\alpha [F_\alpha(\hat{\rho}) - F_\alpha(\hat{\sigma})]$ [12], where $F_\alpha(\hat{\rho}) = k_B T S_\alpha(\hat{\rho}||\hat{\gamma}) - k_B T \log Z$ is a generalized free energy based on the Rényi divergence

$$S_\alpha(\hat{\rho}||\hat{\sigma}) = \begin{cases} \frac{1}{\alpha-1} \log \mathrm{Tr}[\hat{\rho}^\alpha \hat{\sigma}^{1-\alpha}] & \alpha \in [0,1) \\ \frac{1}{\alpha-1} \log \mathrm{Tr}[(\hat{\sigma}^{(1-\alpha)/2\alpha} \hat{\rho} \hat{\sigma}^{(1-\alpha)/2\alpha})^\alpha] & \alpha > 1. \end{cases}$$

Here, $Z = \mathrm{Tr} e^{-\hat{H}/(k_B T)}$ is a partition function of the system. The work distance is the maximum extractable work by a thermal process by taking $\hat{\rho}$ to $\hat{\sigma}$ [12].

Even when the initial state $\hat{\rho}$ is not block diagonal in the energy basis, the extractable work is still given by $D_{\text{work}}(\mathcal{D}(\hat{\rho}) \succ \hat{\sigma})$, so external coherence cannot be used to extract additional work [13]. In order to exploit external coherence for work, one needs multiple copies of $\hat{\rho}$ [4,11] or ancilliary coherent resources [1,11,13]. Thus, the single-shot extractable work purely from coherence is given by

$$W_{\text{coh}} = \inf_\alpha [F_\alpha(\mathcal{D}(\hat{\rho})) - F_\alpha(\Pi(\hat{\rho}))]. \quad (1)$$

For example, consider extracting work from coherence in the pure two-qubit state,

$$|\psi\rangle = \sqrt{p_0}|00\rangle + \sqrt{p_1}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) + \sqrt{p_2}|11\rangle,$$

where each qubit has local Hamiltonain $H_i = \omega_0 |1\rangle\langle 1|$. As shown in Fig. 2 using the concept of thermomajorization [10], we have $W_{\text{coh}} > 0$ only for sufficiently large p_1. In this case, we have the necessary condition $p_1 > (1 + e^{\beta\omega_0} + e^{-\beta\omega_0})^{-1}$ and the sufficient condition $p_1 > e^{\beta\omega_0}/(1 + e^{\beta\omega_0})$ for $W_{\text{coh}} > 0$, independent of p_0 and p_2 at an inverse temperature $\beta = (k_B T)^{-1}$. See Ref. [24] for details.

We generalize this statement to the internal coherence of an arbitrary pure state. Since external coherences cannot contribute to work, we need only consider the properties of the state dephased in the energy eigenbasis.

Observation 1.—Consider any pure state state $|\psi\rangle$ with $\mathcal{D}(|\psi\rangle\langle\psi|) = \sum_k p_{\mathcal{E}} |\psi_{\mathcal{E}}\rangle\langle\psi_{\mathcal{E}}|$, where $|\psi_{\mathcal{E}}\rangle$ is an energy $\mathcal{E}$ eigenstate. Nonzero work can be extracted deterministically from the internal coherence of $\hat{\rho}$ at an inverse

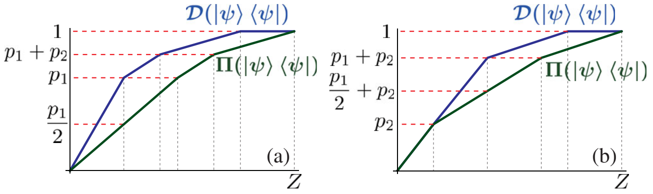

FIG. 2. Thermomajorization graph for the energy-block-diagonal state $\mathcal{D}(|\psi\rangle\langle\psi|)$ and its projection to an incoherent state $\Pi(|\psi\rangle\langle\psi|)$ for a two-qubit state $|\psi\rangle$ studied in the text with coefficients p_1 and p_2. Z represents the partition function of the system. (a) When $p_1 e^{\beta\omega_0}$ is the maximum among $p_i e^{\beta E_i}$, $W_{\rm coh}$ is positive, but (b) if another energy (e.g., $p_2 e^{2\beta\omega_0}$ in the plot) obtains the maximum, $W_{\rm coh}=0$.

temperature β if and only if $\Pi(|\psi_{\mathcal{E}^*}\rangle\langle\psi_{\mathcal{E}^*}|)\neq|\psi_{\mathcal{E}^*}\rangle\langle\psi_{\mathcal{E}^*}|$ for $\mathcal{E}^*=\underset{\mathcal{E}}{\rm argmax}\, p_{\mathcal{E}}e^{\beta\mathcal{E}}$.

Internal coherence has some overlap with nonclassical correlations, namely, quantum discord [31]. Consider the following quantity which quantifies the sharing of free energy between subsystems:

$$C_\alpha(\hat\rho_{1:2:\cdots:N}) \coloneqq \beta\left(F_\alpha(\hat\rho)-\sum_{i=1}^{N}F_\alpha(\hat\rho_i)\right),$$

where $\hat\rho_i$ is the local state of the ith subsystem. For nondegenerate local Hamiltonians, the extractable work from coherence can be written as

$$W_{\rm coh}=k_B T\inf_\alpha[C_\alpha(\mathcal{D}(\hat\rho))-C_\alpha(\Pi(\hat\rho))],$$

noting that the local free energies are the same for $\mathcal{D}(\hat\rho)$ and $\Pi(\hat\rho)$. This is of the same form as discord defined by Ollivier and Zurek [32], expressed as a difference between total and classical correlations. Note that the classical correlations are defined here with respect to the energy basis, instead of the usual maximization over all local basis choices. This free-energy correlation is also related to "measurement-induced disturbance" [33] by considering $\Pi(\hat\rho)$ as a classical measurement with respect to the local energy bases.

Unlike previous related studies [4,8], our result requires that only coherence is consumed in the work extraction processes, leaving all energy statistics unchanged. We may also consider the "incoherent" contribution to the extractable work, $W_{\rm incoh}\coloneqq\inf_\alpha[F_\alpha(\Pi(\hat\rho))-F_\alpha(\hat\gamma)]=F_0(\Pi(\hat\rho))+k_BT\log Z$, which is the achievable work from an incoherent state $\Pi(\hat\rho)$ ending with a Gibbs state $\hat\gamma$. The sum of the coherent and incoherent terms cannot exceed the total extractable work from $\hat\rho$ to $\hat\gamma$, i.e., $W_{\rm coh}+W_{\rm incoh}\leq W_{\rm tot}=D_{\rm work}(\mathcal{D}(\hat\rho)\succ\hat\gamma)$. The equality holds when $W_{\rm coh}$ in Eq. (1) is given at $\alpha=0$. We also point out that this type of work extraction process operates without any measurement or information storage as in Maxwell's demon [34,35] or the Szilard engine [5] in the quantum regime.

Apart from the above example, a significant case is the so-called coherent Gibbs state [13], defined for a single subsystem as $|\gamma\rangle\coloneqq\sum_i\sqrt{(e^{-\beta E_i})/Z}\,|E_i\rangle$. No work can be extracted from this state, as $\mathcal{D}(|\gamma\rangle\langle\gamma|)=\hat\gamma$—an instance of work locking. However, nonzero work can be unlocked [13] from multiple copies $|\gamma\rangle^{\otimes N}, N>1$. In fact, from Observation 1, we see that $N=2$ is always sufficient to give $W_{\rm coh}>0$. This is because $p_{\mathcal{E}}e^{\beta\mathcal{E}}$ is proportional to the degeneracy of the $\mathcal{E}$ subspace—there always exists a degenerate subspace for $N\geq 2$, and this is guaranteed to have coherence.

Coherence as a clock resource.—Having discussed the thermodynamical relevance of internal coherence, we now turn to external coherence. Suppose we have an initial state $\hat\rho_0=\sum_{\boldsymbol{E},\boldsymbol{E}'}\rho_{\boldsymbol{E}\boldsymbol{E}'}|\boldsymbol{E}\rangle\langle\boldsymbol{E}'|$. After free unitary evolution for time t, this becomes $\hat\rho_t=\sum_{\boldsymbol{E},\boldsymbol{E}'}\rho_{\boldsymbol{E}\boldsymbol{E}'}e^{-i\Delta\omega_{\boldsymbol{E}\boldsymbol{E}'}t}|\boldsymbol{E}\rangle\langle\boldsymbol{E}'|$, where each off-diagonal component $|\boldsymbol{E}\rangle\langle\boldsymbol{E}'|$ rotates at frequency $\Delta\omega_{\boldsymbol{E}\boldsymbol{E}'}=(\mathcal{E}_{\boldsymbol{E}}-\mathcal{E}_{\boldsymbol{E}'})/\hbar$. Internal coherences do not evolve ($\Delta\omega_{\boldsymbol{E}\boldsymbol{E}'}=0$), while external coherences with larger energy gaps, and hence higher frequencies, can be considered as providing more sensitive quantum clocks [36,37].

By comparing $\hat\rho_0$ with $\hat\rho_t$, one can estimate the elapsed time t. More precisely, the resolution of a quantum clock can be quantified by $(\Delta t)^2=\langle(\hat t-t)^2\rangle$, where $\hat t$ is the time estimator derived from some measurement on $\hat\rho_t$. The resolution is limited by the quantum Cramér-Rao bound [28], $(\Delta t)^2\geq 1/I_F(\hat\rho,\hat H)$, where $I_F(\hat\rho,\hat H)=2\sum_{i,j}(\lambda_i-\lambda_j)^2/(\lambda_i+\lambda_j)|\langle i|\hat H|j\rangle|^2$ is the QFI, and λ_i, $|i\rangle$ are the eigenvalues and eigenstates of $\hat\rho$, respectively. For the optimal time estimator $\hat t$ saturating the bound, the larger the QFI, the higher the clock resolution. The maximum value of QFI for a given Hamiltonian $\hat H$ is $(E_{\rm max}-E_{\rm min})^2$, which can be obtained by the equal superposition $|E_{\rm min}\rangle+|E_{\rm max}\rangle$ between the maximum ($E_{\rm max}$) and minimum ($E_{\rm min}$) energy eigenstates. The Greenberger–Horne–Zeilinger (GHZ) state in an N-particle two-level system is a state of this form.

Another family of relevant measures of the clock resolution is the skew information $I_\alpha(\hat\rho,\hat H)={\rm Tr}(\hat\rho\hat H)-{\rm Tr}(\hat\rho^\alpha\hat H\hat\rho^{1-\alpha}\hat H)$ for $0\leq\alpha\leq 1$ [28,38]. For pure states, both the QFI and skew information reduce to the variance: $\frac{1}{4}I_F(|\psi\rangle,\hat H)=I_\alpha(|\psi\rangle,\hat H)=V(|\psi\rangle,\hat H)\coloneqq\langle\psi|\hat H^2|\psi\rangle-\langle\psi|\hat H|\psi\rangle^2$. In particular, the skew information of $\alpha=1/2$ has been studied in the context of quantifying coherence [39] and quantum macroscopicity [40]. We also remark that a similar approach to "time references" in quantum thermodynamics has been recently suggested using an entropic clock performance quantifier [21].

We first note that, even though a quantum state might be very poor at providing work, it can still function as a good time reference. The coherent Gibbs state is a canonical example. As mentioned earlier, no work may be extracted from $|\gamma\rangle$; however, such a state does allow time measurements, since $I_F(|\gamma\rangle,\hat H)=4(\partial^2/\partial\beta^2)\log Z$, which is proportional to the heat capacity $k_B\beta^2(\partial^2/\partial\beta^2)\log Z$ [41].

Furthermore, the QFI and skew information are based on monotone metrics [42,43], and monotonically decrease under time-translation-covariant operations [44,45]. It follows that the resolution of a quantum clock gives an additional constraint of a second-law type.

Observation 2.—Under a thermal process, the quantum Fisher (skew) information $I_{F(\alpha)}$ of a quantum system cannot increase; i.e.,

$$\Delta I_{F(\alpha)} \leq 0. \tag{2}$$

We highlight that this condition is independent from those obtained previously, based on a family of entropy asymmetry measures $A_\alpha(\hat{\rho}_S) = S_\alpha(\hat{\rho}_S||\mathcal{D}(\hat{\rho}_S))$ [13] and modes of asymmetry $\sum_{\mathcal{E}_E - \mathcal{E}_{E'} = \omega} |\rho_{\boldsymbol{EE}'}|$ [14,46]. In Ref. [24], we give an example of a state transformation that is forbidden by Eq. (2) but not by previous constraints.

Importantly, condition Eq. (2) remains significant in the many-copy or independent and identically distributed limit. This follows from the additivity of the QFI and skew information, namely, $I_{F(\alpha)}(\hat{\rho}^{\otimes N}, \hat{H})/N = I_{F(\alpha)}(\hat{\rho}, \hat{H}_1)$ for all N, where $\hat{H} = \sum_{i=1}^{N} \hat{H}_i$. In contrast, the measure A_α is negligible in this limit: $\lim_{N\to\infty} A_\alpha(\hat{\rho}^{\otimes N})/N = 0$ for all α [13]. The free energy $F_\alpha(\hat{\rho})$ with $\alpha = 1$ has been stated to be the unique monotone for asymptotic transformations [11]. However, this is true *only* if one is allowed to use a catalyst containing external coherence between every energy level, $|M\rangle = |M|^{-1/2} \sum_{m\in M} |m\rangle$, where $M = \{0, \ldots, 2N^{2/3}\}$, which contains a superlinear amount of clock resources $I_F = O(N^{4/3})$, so I_F/N is unbounded. Thus, Eq. (2) is the first known nontrivial coherence constraint on asymptotic transformations under thermal processes without additional catalytic coherence resources.

We can illustrate the physical implications of this condition in an N-particle two-level system with a local Hamiltonian $\hat{H}_i = 0|0\rangle_i\langle 0| + \omega_0 |1\rangle_i \langle 1|$ for every ith particle. As noted above, for a product state $\hat{\rho}^{\otimes N}$, the QFI and skew information scale linearly with N. On the other hand, the GHZ state $|\psi_{\mathrm{GHZ}}\rangle = 2^{-1/2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ has quadratic scaling, $I_{F(\alpha)}(|\psi_{\mathrm{GHZ}}\rangle, \hat{H}) = O(N^2)$. Thus, the restriction given by Eq. (2) indicates that a thermal process cannot transform a product state into a GHZ state. More generally, it is known that $I_F(\hat{\rho}, \hat{H}) \leq kN$ for k-producible states in N-qubit systems [47,48], so genuine multipartite entanglement is necessary to achieve a high clock precision of $I_F = O(N^2)$. Also note that the QFI has been used to quantify "macroscopicity," the degree to which a state displays quantum behavior on a large scale [45,49].

Trade-off between work and clock resources.—Having examined the two types of thermodynamic coherence independently, it is natural to ask if there is a relation between them. Here, we demonstrate that there is always a trade-off between work and clock coherence resources. We first give the following bound in an N-particle two-level system.

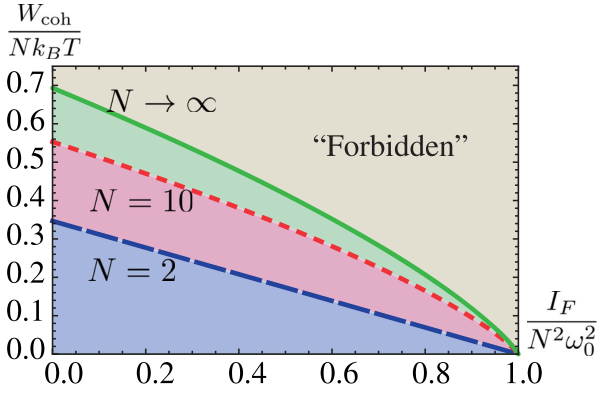

FIG. 3. Trade-off between work and clock coherences. The solid line refers to Eq. (3), the dashed line refers to Eq. (4), and the dotted line refers the tighter bound for $N = 10$.

Theorem 1: Clock–work trade-off for two-level subsystems.—For a system composed of N two-level particles with energy level difference ω_0, the coherent work and clock resources satisfy

$$W_{\mathrm{coh}} \leq Nk_BT(\log 2) H_b\left(\frac{1}{2}\left[1 - \sqrt{\frac{I_F(\hat{\rho}, \hat{H})}{N^2\omega_0^2}}\right]\right), \tag{3}$$

where $\hat{H}$ is the total Hamiltonian and $H_b(r) = -r\log_2 r - (1-r)\log_2(1-r)$ is the binary entropy.

This shows that a quantum state cannot simultaneously contain maximal work and clock resources. When the clock resource is maximal, $I_F = N^2\omega_0^2$, no work can be extracted from coherence, $W_{\mathrm{coh}} = 0$. Conversely, if the extractable work form coherence is maximal, $W_{\mathrm{coh}} = Nk_BT\log 2$, the state cannot be utilized as a quantum clock as $I_F = 0$. For $N = 2$ we derive a tighter inequality:

$$W_{\mathrm{coh}} + (k_BT\log 2)\left(\frac{I_F(\hat{\rho}, \hat{H})}{4\omega_0^2}\right) \leq k_BT\log 2. \tag{4}$$

We demonstrate that the GHZ state $|\psi_{\mathrm{GHZ}}\rangle = 2^{-1/2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ and Dicke states $|N, k\rangle = \binom{N}{k}^{-1/2} \sum_P P(|1\rangle^k |0\rangle^{N-k})$, summing over all permutations P of subsystems, are limiting cases that saturate this trade-off relation. For a Dicke state, the extractable work from coherence is given by $W_{\mathrm{coh}} = k_BT\log\binom{N}{k} \approx Nk_BT(\log 2)H_b(k/N)$. However, each Dicke state has $I_F = 0$ since it has support on a single energy eigenspace with $\mathcal{E} = k\omega_0$. In particular, when $k = N/2$, $W_{\mathrm{coh}} = Nk_BT\log 2$, attaining the maximal value and saturating the bounds Eqs. (3) and (4). The GHZ state behaves in the opposite way: $|\psi_{\mathrm{GHZ}}\rangle$ has maximal QFI, $I_F(|\psi_{\mathrm{GHZ}}\rangle, \hat{H}) = N^2\omega_0^2$, while having no internal coherence; thus, $W_{\mathrm{coh}} = 0$. Figure 3 shows the clock-work trade-off relation and the saturation of both bounds Eqs. (3) and (4).

Furthermore, our two-level trade-off relation can be generalized for an arbitrary noninteracting N-particle system.

Theorem 2: Clock–work trade-off for arbitrary subsystems.—Let $\hat{H}$ be a noninteracting Hamiltonian of N subsystems, where the ith subsystem has an arbitrary (possibly degenerate) $d^{(i)}$-level spectrum $\{E_1^{(i)} \leq E_2^{(i)} \leq \cdots \leq E_{d^{(i)}}^{(i)}\}$. Also define $\Delta_E^2 = \sum_{i=1}^N (\Delta_E^{(i)})^2$, with $\Delta_E^{(i)} = E_{d^{(i)}}^{(i)} - E_1^{(i)}$. Then,

$$W_{\mathrm{coh}} + k_B T \left(\frac{I_F(\hat{\rho}, \hat{H})}{2\Delta_E^2} \right) \leq k_B T \sum_{n=1}^{N} \log d^{(n)}. \qquad (5)$$

This is more generally applicable than Eq. (4), but is weaker for two-level subsystems—maximal I_F does not imply $W_{\mathrm{coh}} = 0$ via Eq. (5).

For systems with identical local d-level Hamiltonians, Eq. (5) reduces to

$$\bar{w}_{\mathrm{coh}} + k_B T \left(\frac{I_F(\hat{\rho}, \hat{H})}{2N^2 \Delta_0^2} \right) \leq k_B T \log d, \qquad (6)$$

where $\bar{w}_{\mathrm{coh}} = W_{\mathrm{coh}}/N$ is extractable work per particle and $\Delta_E^2 = N\Delta_0^2$, where Δ_0 is the maximum energy difference between the local energy eigenvalues. Our bounds do not limit the extractable work in the independent and identically distributed limit, since $I_F(\hat{\rho}^{\otimes N}, \hat{H})/N^2 \to 0$ for $N \gg 1$.

We can also describe how one extends our analysis into the regime of weak interactions between local systems. We note that interactions can break degeneracies in energy eigenspaces, and energy eigenstates of the free Hamiltonian may not be expressed as a product of local states. However, breaking of degeneracies in energy should only be treated above a finite experimental width ϵ in energy resolution. Weak perturbations admit a similar analysis in terms of work extraction up to ϵ fluctuations. This ϵ-energy window allows us to use external coherences with energy gaps less than ϵ for work extraction by effectively treating them as internal coherences in the same energy levels. In this case, we can calculate how work and clock resources are perturbed and note that the trade-off relation Eq. (5) still holds with an $\mathcal{O}(\epsilon)$ correction. We also discuss an example for transverse Ising models in which the trade-off relation can be resolved using a quasiparticle picture (see Ref. [24] for details). The more general interacting case is nontrivial and we leave this for future study.

Remarks.—We have found that thermodynamic coherence in a many-body system can be decomposed into time- and energy-related components. Many-body coherence contributing to the thermodynamic free energy has been shown to be convertible into work by a thermal process, without changing the classical energy statistics. We have illustrated that this work-yielding resource comes from correlations due to coherence in a multipartite system. We have also shown that coherence may take the form of a clock resource, and we have quantified this with the quantum Fisher (skew) information. Our main result is a trade-off relation between these two different thermodynamic coherence resources.

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