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RUIN PROBLEM OF A TWO-DIMENSIONAL FRACTIONAL BROWNIAN MOTION RISK PROCESS

LANPENG JI AND STEPHAN ROBERT

Abstract: This paper investigates ruin probability and ruin time of a two-dimensional fractional Brownian motion risk process. The net loss process of an insurance company is modeled by a fractional Brownian motion. The two-dimensional fractional Brownian motion risk process models the surplus processes of an insurance and a reinsurance company, where the net loss are divided between them in some specified proportions. The ruin problem considered is that of the two-dimensional risk process first entering the negative quadrant, that is, the simultaneous ruin problem. We derive both asymptotics of the ruin probability and approximations of the scaled conditional ruin time as the initial capital tends to infinity.

Key Words: Ruin probability; ruin time; asymptotics; two-dimensional risk process; fractional Brownian motion; reinsurance

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Let $\{B_H(t), t \in \mathbb{R}\}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, i.e., an H-self-similar centered Gaussian process with stationary increments and covariance function given by

$$Cov(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}.$$

Particularly, if $H = 1/2$, then $B_{1/2}$ is the standard Brownian motion (Bm).

In classical risk theory, the surplus process of an insurance company is modeled by the compound Poisson or the general compound renewal risk process. For both applied and theoretical investigations, calculation of the ruin probabilities for such models is of particular interest. In order to avoid technical issues and to allow for dependence among the claim sizes, these risk models are often approximated by the Bm (also called diffusion) (e.g., [1, 2, 3, 4]) or the fBm risk model (e.g., [5, 6]). The basic premise for the approximation is to let the number of claims grow in a unit time interval and to make the claim sizes smaller in such a way that the risk process converges to a self-similar process with drift. Calculations of ruin probabilities and related quantities for Bm, fBm and more general Gaussian risk models have been the subject of study of numerous contributions; see, e.g., [3, 4, 7, 8, 9, 10, 11, 12, 13].

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Consider the fBm risk process defined by

$$R(t) = x + pt - B_H(t), \quad t \geq 0,$$

where $x > 0$ is the *initial capital*, $p > 0$ is the *net profit rate*, and B_H models the *net loss process*. Roughly speaking, $B_H(t)$ is an approximation of the total claim amount process by time t minus its expectation, the latter is usually called the *pure premium* amount and calculated to cover the average payments of claims. The net profit, also called safety loading, is the component which protects the company from large deviations of claims from the average and also allows an accumulation of capital.

Motivated by [14, 15, 16] we shall consider in this paper a particular two-dimensional fBm risk model. In this model two insurance companies split the net loss in proportions $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = 1$, and receive net profit at rates $p_1, p_2 > 0$, respectively. Let R_i denote the risk process of the ith company

$$R_i(t) = x_i + p_i t - \delta_i B_H(t), \ t \geq 0, \quad i = 1, 2,$$

where $x_i > 0$ denotes the initial capital. Note that in the above two-dimensional model both claims and pure premiums (i.e., the net loss) are split between the two companies, which corresponds to proportional reinsurance dependence of the companies as discussed in the aforementioned contributions. We refer to [17, 18, 19, 20] and references therein for more motivations and discussions on multi-dimensional risk models.

As ruin time and ruin probability of the two-dimensional risk process that we are going to discuss do not change under a scaling of (R_1, R_2), we shall consider in the sequel the following scaled risk processes

$$U_i(t) = R_i(t)/\delta_i = u_i + c_i t - B_H(t), \quad t \geq 0, \quad i = 1, 2,$$

where $u_i := x_i/\delta_i$ and $c_i := p_i/\delta_i$.

We are interested in the simultaneous ruin time and the simultaneous ruin probability defined by

$$\tau(u_1, u_2) = \inf\{t \geq 0 : U_1(t) < 0, U_2(t) < 0\}$$

and, respectively,

$$\psi(u_1, u_2) = \mathbb{P}\left\{\tau(u_1, u_2) < \infty\right\}.$$

We shall obtain sharp approximations of the above quantities, as u_1, u_2 tend to infinity along a ray (i.e., u_1/u_2 is constant). For this purpose, we shall simply assume that

$$u_i = q_i u \tag{1}$$

with $q_i > 0, i = 1, 2$ fixed constants, and finally allow u to tend to infinity.

As indicated in [21], the consideration of large initial capitals is not just a mathematical assumption but also an economic necessity, which is reinforced by the supervisory authorities. In any civilized country it is not possible to start up an insurance business without a sufficiently large initial capital, which prevents the business from bankruptcy due to too many small or a few large claim sizes in the first period of its existence, before the premium income can balance the losses and the gains.

Notice that under the assumption (1) the quantities $\tau(u_1, u_2)$ and $\psi(u_1, u_2)$ are functions of a single variable $u > 0$, and thus we shall denote for any u

$$\tau(u) = \tau(u_1, u_2), \quad \psi(u) = \psi(u_1, u_2).$$

We shall give full analysis of approximations for the above simultaneous ruin time and the simultaneous ruin probability for large u. We point out that other different types of ruin can also be defined for the two-dimensional fBm risk process as in [14]. However, full analysis of the corresponding ruin problems is more complex and thus will be considered elsewhere.

Observe that under the assumption (1) the ruin time can be rewritten as

$$\tau(u) = \inf\{t \geq 0 : B_H(t) > \max(q_1 u + c_1 t, q_2 u + c_2 t)\}.$$

Thus, the two-dimensional problem may also be viewed as a one-dimensional crossing problem over a piece-wise linear barrier. If the two lines $q_1 u + c_1 t, q_2 u + c_2 t$ do not intersect over $[0, \infty)$, then the problems degenerate to the classical problems of one-dimensional fBm risk process, which has been discussed in the aforementioned contributions and thus will not be the focus of this paper. In consideration of that, we shall assume that

$$c_1 > c_2, \quad q_2 > q_1. \tag{2}$$

In Theorem 3.1 we derive exact asymptotics of $\psi(u)$ as u tends to infinity. Five different scenarios will be discussed; for two of them we show that the asymptotics are the same as those of the degenerated one-dimensional cases, for other two of them we show that the asymptotics are simply equivalent to the asymptotics of the one-dimensional cases multiplied by 1/2, whereas for the remaining scenario we obtain quite different asymptotics.

A related, interesting and vastly analyzed quantity of risk processes is the conditional ruin time, which in our setup is $\tau(u)|\tau(u) < \infty$. Approximation of this quantity will give us some idea of when ruin occurred knowing that it has occurred. We refer to [4, 10, 13, 22, 23, 24, 25] for related discussions on ruin time. In Theorem 3.2 we derive some approximation results for the scaled conditional simultaneous ruin time. Again five different scenarios will be discussed, and different approximations are obtained. Our results show that approximations rather than exponential and (truncated) normal are also possible.

Organization of the rest of the paper: In Section 2 we introduce some notation and present some results for the one-dimensional fBm risk process. The main results are displayed in Section 3, whereas the proofs are relegated to Section 4. Finally, we conclude with an Appendix containing some known results as well as some other technical results.

2. Preliminaries

We shall use the standard notation for asymptotic equivalence of two functions $f(\cdot)$ and $h(\cdot)$. Specifically, we write $f(x) = h(x)(1 + o(1))$ or simply $f(x) \sim h(x)$, if $\lim_{x\to a} f(x)/h(x) = 1$ $(a \in \mathbb{R} \cup \{\infty\})$. Further, write $f(x) = o(h(x))$, if $\lim_{x\to a} f(x)/h(x) = 0$.

Next we introduce the Pickands constant

$$\mathcal{H}_{2H} = \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_{2H}[0, T] \in (0,\infty),$$

where

$$\mathcal{H}_{2H}[0, T] = \mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_H(t) - t^{2H}\right)\right)\right), \qquad T \in (0,\infty).$$

It is known that $\mathcal{H}_1 = 1$ and $\mathcal{H}_2 = 1/\sqrt{\pi}$, see, e.g., [11, 26, 27, 28, 29, 30, 31, 32].

We define below another constant that will also appear in our main results, see Theorem 3.1 and Theorem 3.2. Recall that $\{B_{1/2}(t), t \in \mathbb{R}\}$ is a standard Brownian motion defined on $\mathbb{R}$. For any continuous function $d(\cdot)$ satisfying $d(0) = 0$, define

$$\mathcal{H}_1^d[S, T] = \mathbb{E}\left(\exp\left(\sup_{t\in[S,T]}(\sqrt{2}B_{1/2}(t) - |t| - d(t))\right)\right) \in (0,\infty), \quad -\infty < S < T < \infty.$$

It is known (see, e.g., [29, 33]) that for any $a > 0$ and $d(t) = a\,|t|\,, t \in \mathbb{R}$

$$\lim_{T\to\infty} \mathcal{H}_1^d[0, T] = \lim_{T\to\infty} \mathcal{H}_1^d[-T, 0] = 1 + \frac{1}{a}. \tag{3}$$

Furthermore, we define

$$\widetilde{\mathcal{H}}_1^d = \lim_{T\to\infty} \mathbb{E}\left(\exp\left(\sup_{t\in[-T,T]}(\sqrt{2}B_{1/2}(t) - |t| - d(t))\right)\right)$$

whenever the limit exists. As shown in Theorem 3.1 for special function $d(\cdot)$ the above constant is well-defined, positive and finite.

Moreover, let $\Psi(x)$ denote the tail distribution function of a standard normal random variable, and denote by $\Phi(x) = 1 - \Psi(x)$ the corresponding distribution function.

We conclude this section with some results on the one-dimensional fBm risk process

$$U(t) = qu + ct - B_H(t), \quad t \geq 0,$$

with $q, c, u > 0$. Define the corresponding ruin time and ruin probability by

$$\tau_{q,c}(u) = \inf\{t \geq 0 : U(t) < 0\}, \qquad \psi_{q,c}(u) = \mathbb{P}\left\{\tau_{q,c}(u) < \infty\right\}.$$

The following result, concerning the ruin probability and scaled conditional ruin time of the above one-dimensional fBm risk process, follows directly from [9, 10] or [29, 34]. For the sake of completeness, we present a sketch of the proof in Appendix.

Proposition 2.1. *We have, as $u \to \infty$*

$$\psi_{q,c}(u) \sim 2^{\frac{1}{2}-\frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \mathcal{H}_{2H} \left(\frac{c^H q^{1-H} u^{1-H}}{H^H (1-H)^{1-H}} \right)^{1/H-1} \Psi \left(\frac{c^H q^{1-H} u^{1-H}}{H^H (1-H)^{1-H}} \right), \tag{4}$$

and with

$$A(u) = \frac{H^{H+1/2}}{(1-H)^{H+1/2} c^{H+1}} u^H, \qquad t_0 = \frac{H}{c(1-H)},$$

it holds that

$$\lim_{u\to\infty} \mathbb{P}\left\{ \frac{\tau_{q,c}(u) - t_0 qu}{A(qu)} \le x \middle| \tau_{q,c}(u) < \infty \right\} = \Phi(x), \quad x \in \mathbb{R}. \tag{5}$$

3. Main Results

By the self-similarity of fBm, we can rewrite the ruin probability as

$$\begin{aligned} \psi(u) &= \mathbb{P}\left\{\exists_{t\ge 0} \ (B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u)\right\} \\ &= \mathbb{P}\left\{\exists_{t\ge 0} \ \left(X_1(t) > u^{1-H}, X_2(t) > u^{1-H}\right)\right\}, \end{aligned}$$

where

$$X_i(t) = \frac{B_H(t)}{q_i + c_i t}, \quad t \ge 0, \quad i = 1, 2. \tag{6}$$

Define

$$\sigma_i(t) = \sqrt{Var(X_i(t))} = \frac{t^H}{q_i + c_i t}, \quad t \ge 0, \quad i = 1, 2.$$

Elementary calculations show that for any $i = 1, 2$

$$t_i = \frac{H q_i}{c_i (1-H)}$$

is the unique maximum point of the function $\sigma_i(t), t \ge 0$. Additionally, by the assumption (2) it holds that

$$t_1 < t_2.$$

Consider two lines $y_i(t) = q_i + c_i t, t \ge 0, i = 1, 2$. It turns out that the unique intersection point of these two lines

$$t^* = \frac{q_2 - q_1}{c_1 - c_2} > 0$$

shall play a crucial rule in our analysis.

Denote

$$a_i = \frac{|H(q_i + c_i t^*) - c_i t^*|}{t^*(q_i + c_i t^*)}, \quad i = 1, 2, \tag{7}$$

and denote by $I_{(\cdot)}$ the indicator function.

Below we present our results for the simultaneous ruin probability. Recall that the ruin probability of the one-dimensional fBm risk process has been given in Proposition 2.1.

Theorem 3.1. *Assume that* (2) *is satisfied. We have, as* $u \to \infty$

1) if $t_1 > t^*$, *then*

$$\psi(u) \sim \psi_{q_1,c_1}(u).$$

2) if $t_2 < t^*$, *then*

$$\psi(u) \sim \psi_{q_2,c_2}(u).$$

3) if $t_1 < t^* < t_2$, *then*

$$\psi(u) \sim \Psi\left(\frac{(q_1 + c_1 t^*)u^{1-H}}{t^{*H}}\right) \times \begin{cases} \frac{a_1+a_2}{2^{1/(2H)} t^* a_1 a_2} \mathcal{H}_{2H} \left(\frac{(q_1+c_1 t^*)u^{1-H}}{t^{*H}}\right)^{1/H-2}, & H < 1/2 \\ \widetilde{\mathcal{H}}_1^d, & H = 1/2 \\ 1, & H > 1/2, \end{cases}$$

with $d(t) = 2t^* a_2 \left|t\right| I_{(t<0)} + 2t^* a_1 \left|t\right| I_{(t\geq 0)}, t \in \mathbb{R},$ *and*

$$\widetilde{\mathcal{H}}_1^d \in (0, \infty).$$

4) if $t^* = t_1$, *then*

$$\psi(u) \sim \frac{1}{2}\psi_{q_1,c_1}(u).$$

5) if $t^* = t_2$, *then*

$$\psi(u) \sim \frac{1}{2}\psi_{q_2,c_2}(u).$$

Define

$$D(u) = \frac{t^{*2H}}{(q_1 + c_1 t^*)^2} u^{2H-1},$$

and recall $A(u)$ given in Proposition 2.1 and $a_i, i = 1, 2$ given in (7).

The approximations for the scaled conditional simultaneous ruin times are given in the following theorem.

Theorem 3.2. *Assume that* (2) *is satisfied. For any* $x \in \mathbb{R}$, *we have, as* $u \to \infty$

1) if $t_1 > t^*$, *then*

$$\lim_{u\to\infty} \mathbb{P}\left\{\frac{\tau(u) - t_1 u}{A(q_1 u)} \leq x \middle| \tau(u) < \infty\right\} = \Phi(x).$$

2) if $t_2 < t^*$, *then*

$$\lim_{u\to\infty} \mathbb{P}\left\{\frac{\tau(u) - t_2 u}{A(q_2 u)} \leq x \middle| \tau(u) < \infty\right\} = \Phi(x).$$

3) if $t_1 < t^* < t_2$, then

$$\lim_{u\to\infty} \mathbb{P}\left\{ \frac{\tau(u)-t^*u}{D(u)} \le x \Big| \tau(u)<\infty \right\} = \begin{cases} \frac{a_1}{a_1+a_2} e^{a_2 x} I_{(x<0)} + \left(1-\frac{a_2}{a_1+a_2}e^{-a_1 x}\right) I_{(x\ge 0)}, & H<1/2 \\ \frac{\mathcal{H}_1^d[-\infty,1/(2t^*)x]}{\widetilde{\mathcal{H}}_1^d}, & H=1/2 \\ e^{a_2 x} I_{(x<0)} + I_{(x\ge 0)}, & H>1/2, \end{cases}$$

with $d(\cdot)$ the same as in Theorem 3.1, and for any $x\in\mathbb{R}$

$$\mathcal{H}_1^d[-\infty,1/(2t^*)x] := \lim_{S\to\infty} \mathcal{H}_1^d[-S,1/(2t^*)x] \in (0,\infty).$$

4) if $t^* = t_1$, then

$$\lim_{u\to\infty} \mathbb{P}\left\{ \frac{\tau(u)-t_1u}{A(q_1u)} \le x \Big| \tau(u)<\infty \right\} = (1-2\Psi(x))\ I_{(x\ge 0)}.$$

5) if $t^* = t_2$, then

$$\lim_{u\to\infty} \mathbb{P}\left\{ \frac{\tau(u)-t_2u}{A(q_2u)} \le x \Big| \tau(u)<\infty \right\} = 2\Phi(x)\ I_{(x<0)} + I_{(x\ge 0)}.$$

Remark 3.3. *We observe that, as $u\to\infty$, when $t_1<t^*<t_2$ and $H>1/2$, or when $t^*=t_2$ it holds that*

$$\tau(u) \le t^*u \Big| \tau(u)<\infty \quad \text{almost surely,}$$

and when $t^*=t_1$ it holds that

$$\tau(u) \ge t^*u \Big| \tau(u)<\infty \quad \text{almost surely.}$$

4. Proofs of Main Results

This section consists of the proofs of Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1: The five different scenarios will be treated separately.

Proof for Case 1). First observe that

$$\begin{aligned} \psi(u) &= \mathbb{P}\left\{\exists_{t\ge 0}\ (B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u)\right\} \\ &\le \mathbb{P}\left\{\exists_{t\ge 0}\ B_H(t)-c_1t>q_1u\right\} = \psi_{q_1,c_1}(u). \end{aligned}$$

Furthermore, for any $\varepsilon>0$ such that $t_1-\varepsilon>t^*$ we have

$$\begin{aligned} \psi(u) &\ge \mathbb{P}\left\{\exists_{t\in[t_1-\varepsilon,t_1+\varepsilon]}\ (X_1(t)>u^{1-H}, X_2(t)>u^{1-H})\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[t_1-\varepsilon,t_1+\varepsilon]}\ X_1(t)>u^{1-H}\right\} \\ &\sim \psi_{q_1,c_1}(u) \end{aligned}$$

as $u\to\infty$, where the last asymptotic equivalence follows from the classical Piterbarg theorem (see Theorem 5.5 in Appendix). Consequently, the claim of Case 1) follows.

Proof for Case 2). The proof is similar as in Case 1). We have the upper bound

$$\psi(u) \le \psi_{q_2,c_2}(u).$$

For the lower bound, we have for any $\varepsilon > 0$ such that $t_2 + \varepsilon < t^*$

$$\begin{aligned}\psi(u) &\geq \mathbb{P}\left\{\exists_{t\in[t_2-\varepsilon,t_2+\varepsilon]}\ (X_1(t) > u^{1-H}, X_2(t) > u^{1-H})\right\}\\ &= \mathbb{P}\left\{\exists_{t\in[t_2-\varepsilon,t_2+\varepsilon]}\ X_2(t) > u^{1-H}\right\}\\ &\sim \psi_{q_2,c_2}(u)\end{aligned}$$

holds as $u \to \infty$. Thus, the claim of Case 2) is established.

Below, we continue with Cases 3), 4), 5). Denote

$$Z(t) = \frac{B_H(t)}{g(t)}, t \geq 0, \quad \text{with} \quad g(t) = \max(q_1 + c_1 t, q_2 + c_2 t), t \geq 0, \tag{8}$$

and let $\sigma_Z(t) = \sqrt{Var(Z(t))} = t^H/g(t), t \geq 0$. We have

$$\psi(u) = \mathbb{P}\left\{\sup_{t\geq 0} Z(t) > u^{1-H}\right\}.$$

We first analyze the standard deviation function $\sigma_Z(t), t \geq 0$. Elementary calculations show that, for all the three cases, the maxima of $\sigma_Z(t)$ over $[0, \infty)$ is attained uniquely at t^*. Furthermore, for Case 3) we have

$$\sigma_Z(t) - \sigma_Z(t^*) = (\sigma_2'(t^*)(t-t^*)I_{(t<t^*)} + \sigma_1'(t^*)(t-t^*)I_{(t\geq t^*)})(1+o(1)), \quad t \to t^*, \tag{9}$$

for Case 4) we have

$$\sigma_Z(t) - \sigma_Z(t^*) = (\sigma_2'(t^*)(t-t^*)I_{(t<t^*)} + \frac{1}{2}\sigma_1''(t^*)(t-t^*)^2 I_{(t\geq t^*)})(1+o(1)), \quad t \to t^*, \tag{10}$$

and for Case 5) we have

$$\sigma_Z(t) - \sigma_Z(t^*) = (\frac{1}{2}\sigma_2''(t^*)(t-t^*)^2 I_{(t<t^*)} + \sigma_1'(t^*)(t-t^*)I_{(t\geq t^*)})(1+o(1)), \quad t \to t^*.$$

Moreover, for the correlation function $r_Z(s,t) = Corr(Z(s), Z(t))$ we have

$$r_Z(s,t) = 1 - \frac{1}{2t^{*2H}}\left|t-s\right|^{2H}(1+o(1)), \quad s,t \to t^* \tag{11}$$

holds for all the Cases 3), 4), 5).

Next, for any sufficiently small $\rho > 0$ denote $D_\rho = [t^* - \rho, t^* + \rho]$. We have

$$\pi_\rho(u) := \mathbb{P}\left\{\sup_{t\in D_\rho} Z(t) > u^{1-H}\right\} \leq \psi(u) \leq \pi_\rho(u) + \mathbb{P}\left\{\sup_{t\in[0,\infty)\setminus D_\rho} Z(t) > u^{1-H}\right\}.$$

Since $\lim_{t\to\infty} Z(t) = 0$, the process $\{Z(t), t \geq 0\}$ has bounded sample paths, and thus by the Borell-TIS inequality (see Lemma 5.3 in Appendix) we have

$$\mathbb{P}\left\{\sup_{t\in[0,\infty)\setminus D_\rho} Z(t) > u^{1-H}\right\} \leq \exp\left(-\frac{1}{2\sup_{t\in[0,\infty)\setminus D_\rho}\sigma_Z^2(t)}(u^{1-H} - C_0)^2\right) \tag{12}$$

holds for all sufficiently large u, where $C_0 = \mathbb{E}\left(\sup_{t\in[0,\infty)\setminus D_\rho} Z(t)\right) < \infty$. Note that $\sup_{t\in[0,\infty)\setminus D_\rho}\sigma_Z^2(t) < \sigma_Z^2(t^*)$ since t^* is the unique maximum point of $\sigma_Z(t), t \geq 0$.

Hereafter, we shall focus on the asymptotics of $\pi_\rho(u)$ as $u \to \infty$ for all the three Cases 3), 4), 5). For simplicity, we denote $\hat{u} = u^{1-H}/\sigma_Z(t^*) = u^{1-H}/\sigma_i(t^*), i = 1, 2$.

Proof for Case 3). It follows from (9) that for the small chosen ρ there exists some small $\varepsilon > 0$ such that

$$
\begin{aligned}
&1 + (a_2 - \varepsilon) |t - t^*| I_{(t<t^*)} + (a_1 - \varepsilon) |t - t^*| I_{(t \ge t^*)} \\
&\le \frac{\sigma_Z(t^*)}{\sigma_Z(t)} \le \\
&1 + (a_2 + \varepsilon) |t - t^*| I_{(t<t^*)} + (a_1 + \varepsilon) |t - t^*| I_{(t \ge t^*)}
\end{aligned}
\tag{13}
$$

holds for all $t \in D_\rho$, where $a_i = \frac{|\sigma_i'(t^*)|}{\sigma_Z(t^*)} = \frac{|H(q_i + c_i t^*) - c_i t^*|}{t^*(q_i + c_i t^*)}, i = 1, 2$. Set $\delta_u = \left(\frac{\log(\hat{u})}{\hat{u}}\right)^2$. We have

$$
\begin{aligned}
p(u) &:= \mathbb{P}\left\{ \sup_{t \in [t^* - \delta_u, t^* + \delta_u]} Z(t) > u^{1-H} \right\} \\
&\le \pi_\rho(u) \\
&\le p(u) + \mathbb{P}\left\{ \sup_{t \in [t^* - \rho, t^* - \delta_u] \cup [t^* + \delta_u, t^* + \rho]} Z(t) > u^{1-H} \right\} =: p(u) + r(u).
\end{aligned}
$$

It follows from (13) that for all $t \in [t^* - \rho, t^* - \delta_u] \cup [t^* + \delta_u, t^* + \rho]$

$$
\frac{\sigma_Z(t^*)}{\sigma_Z(t)} \ge 1 + a_- \delta_u
$$

with $a_- = \min(a_1, a_2) - \varepsilon > 0$ for the chosen sufficiently small ε. Thus, denoting $\overline{Z}(t) = Z(t)/\sigma_Z(t)$ we have

$$
r(u) \le \mathbb{P}\left\{ \sup_{t \in [t^* - \rho, t^* - \delta_u] \cup [t^* + \delta_u, t^* + \rho]} \overline{Z}(t) > \hat{u}(1 + a_- \delta_u) \right\}.
$$

Next we have from (11) that for the small chosen ρ

$$
\mathbb{E}\left((\overline{Z}(t) - \overline{Z}(s))^2 \right) \le Q |t - s|^{2H}
$$

holds for all $t \in D_\rho$, with some positive constant Q. Thus, by the Piterbarg inequality (see Lemma 5.4 in Appendix) we conclude that

$$
r(u) \le C u^{1/H - 1} \exp\left(-\frac{\hat{u}^2}{2} (1 + a_- \delta_u)^2 \right) \tag{14}
$$

holds for all large u, with some positive constant C which does not depend on u.

In the following, we shall derive the asymptotics of $p(u)$ as $u \to \infty$ which will imply that

$$
\mathbb{P}\left\{ \sup_{t \in [0, \infty) \setminus D_\rho} Z(t) > u^{1-H} \right\} = o(p(u)), \quad r(u) = o(p(u)), \quad u \to \infty, \tag{15}
$$

and thus

$$
\psi(u) \sim p(u), \quad u \to \infty.
$$

Note that from (11) we have for the small chosen ρ, ε

$$
b_- |t - s|^{2H} \le 1 - r_Z(s, t) \le b_+ |t - s|^{2H}
$$

for all $s,t\in D_\rho$, where $b_\pm = \frac{1\pm\varepsilon}{2t^{*2H}}$. Let $\{Y_\pm(t), t\in\mathbb{R}\}$ be two continuous centered stationary Gaussian processes with unit variance and correlation functions $r_{Y_\pm}(\cdot)$ given by

$$r_{Y_\pm}(t) = e^{-b_\pm |t|^{2H}}, \quad t\in\mathbb{R}.$$

Consequently, by (13) and the Slepian inequality (see Lemma 5.2 in Appendix) we have

$$\mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_2(t) > \hat{u}\right\} \le p(u) \le \mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_1(t) > \hat{u}\right\}, \tag{16}$$

where

$$\begin{aligned} W_1(t) &= \frac{Y_+(t)}{1+(a_2-\varepsilon)\,|t|\,I_{(t<0)}+(a_1-\varepsilon)\,|t|\,I_{(t\ge 0)}}, \quad t\in\mathbb{R},\\ W_2(t) &= \frac{Y_-(t)}{1+(a_2+\varepsilon)\,|t|\,I_{(t<0)}+(a_1+\varepsilon)\,|t|\,I_{(t\ge 0)}}, \quad t\in\mathbb{R}. \end{aligned}$$

In order to estimate the above bounds, we introduce the following notation. Let $q=q(u)=\hat{u}^{-1/H}$ and set for any $T>0$

$$\Delta_i = [iTq,(i+1)Tq]\,,\ i\in\mathbb{Z}, \quad \text{and} \quad N_u = \left\lfloor T^{-1}\delta_u q^{-1}\right\rfloor,$$

where $\lfloor\cdot\rfloor$ is the ceiling function. We shall investigate separately the following three cases:

Case 3.i) $H<1/2$, Case 3.ii) $H=1/2$, Case 3.iii) $H>1/2$.

<u>Case 3.i) $H<1/2$</u>. We have by the Bonferroni inequality (see Lemma 5.1 in Appendix)

$$\mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_1(t) > \hat{u}\right\} \le \sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t)>\hat{u}\right\}, \tag{17}$$

$$\mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_2(t) > \hat{u}\right\} \ge \sum_{i=-N_u}^{N_u-1} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t)>\hat{u}\right\} - \Theta(u), \tag{18}$$

where

$$\Theta(u) = \sum_{-N_u\le i<j\le N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t)>\hat{u}, \sup_{t\in\Delta_j} W_2(t)>\hat{u}\right\}.$$

Continuing (17) we have

$$\begin{aligned} \sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t)>\hat{u}\right\} &\le \sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} Y_+(t)>\hat{u}B(i,T,u)\right\}\\ &= \sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t)>\hat{u}B(i,T,u)\right\} \end{aligned}$$

with

$$B(i,T,u) = 1+(a_2-\varepsilon)I_{(i<0)}\,|(i+1)Tq| + (a_1-\varepsilon)I_{(i\ge 0)}\,|iTq|\,.$$

Next, by Lemma 5.6 (see Appendix) we have

$$\mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t)>\hat{u}B(i,T,u)\right\} = \mathcal{H}_{2H}[0,b_+^{\frac{1}{2H}}T]\Psi(\hat{u}B(i,T,u))(1+o(1)), \quad u\to\infty,$$

where $o(1)$ is chosen uniformly for i such that $-N_u \le i \le N_u$. Noting that $\Psi(u) \sim \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}$, we have as $u \to \infty$

$$\begin{aligned}
&\sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t) > \hat{u}B(i,T,u)\right\}\\
&= \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}}T]}{\sqrt{2\pi}\hat{u}} \left(\sum_{i=-N_u-1}^{-1} e^{-\frac{\hat{u}^2}{2}(1+(a_2-\varepsilon)|(i+1)Tq|)^2} + \sum_{i=0}^{N_u} e^{-\frac{\hat{u}^2}{2}(1+(a_1-\varepsilon)|iTq|)^2}\right)(1+o(1))\\
&= \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}}T]}{T} \hat{u}^{\frac{1}{H}-2}\Psi(\hat{u}) \left(\sum_{i=-N_u-1}^{-1} e^{(a_2-\varepsilon)((i+1)T\hat{u}^{2-\frac{1}{H}})} + \sum_{i=0}^{N_u} e^{-(a_1-\varepsilon)(iT\hat{u}^{2-\frac{1}{H}})}\right)(T\hat{u}^{2-\frac{1}{H}})(1+o(1))\\
&= \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}}T]}{T} \hat{u}^{\frac{1}{H}-2}\Psi(\hat{u}) \left(\int_{-\infty}^{0} e^{(a_2-\varepsilon)x}dx + \int_0^\infty e^{-(a_1-\varepsilon)x}dx\right)(1+o(1))\\
&= \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}}T]}{T} \frac{a_1+a_2-2\varepsilon}{(a_1-\varepsilon)(a_2-\varepsilon)} \hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})(1+o(1)).
\end{aligned}$$

Consequently, as $u \to \infty$

$$\sum_{i=-N_u-1}^{N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t) > \hat{u}\right\} \le \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}}T]}{T} \frac{a_1+a_2-2\varepsilon}{(a_1-\varepsilon)(a_2-\varepsilon)} \hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})(1+o(1)). \tag{19}$$

Similarly, we can show as $u \to \infty$

$$\sum_{i=-N_u}^{N_u-1} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t) > \hat{u}\right\} \ge \frac{\mathcal{H}_{2H}[0, b_-^{\frac{1}{2H}}T]}{T} \frac{a_1+a_2+2\varepsilon}{(a_1+\varepsilon)(a_2+\varepsilon)} \hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})(1+o(1)). \tag{20}$$

Next we consider $\Theta(u)$. Set $a_+ = \min(a_1, a_2) + \varepsilon$. It follows that

$$\Theta(u) \le \sum_{-N_u \le i < j \le N_u} \mathbb{P}\left\{\sup_{t\in\Delta_i} \frac{Y_-(t)}{1+a_+|t|} > \hat{u}, \sup_{t\in\Delta_j} \frac{Y_-(t)}{1+a_+|t|} > \hat{u}\right\} =: \Theta_1(u)$$

We split the above sum into two by distinguishing $j = i+1$ and $j > i+1$, i.e., $\Theta_1(u) = \Theta_{11}(u) + \Theta_{12}(u)$, with $\Theta_{11}(u)$ being the sum over indexes $j = i+1$ and $\Theta_{12}(u)$ being the sum over indexes $j > i+1$. For $\Theta_{11}(u)$ we have

$$\begin{aligned}
\Theta_{11}(u) &= \sum_{-N_u \le i \le N_u} \left(\mathbb{P}\left\{\sup_{t\in\Delta_i} \frac{Y_-(t)}{1+a_+|t|} > \hat{u}\right\} + \mathbb{P}\left\{\sup_{t\in\Delta_{i+1}} \frac{Y_-(t)}{1+a_+|t|} > \hat{u}\right\} - \mathbb{P}\left\{\sup_{t\in\Delta_i\cup\Delta_{i+1}} \frac{Y_-(t)}{1+a_+|t|} > \hat{u}\right\}\right)\\
&=: S_1(u) + S_2(u) - S_3(u).
\end{aligned}$$

Then using similar arguments as the derivation of (19) and (20) for $S_i(u), i = 1,2,3$, we have

$$\begin{aligned}
\lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty} \frac{\Theta_{11}(u)}{\hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})} &= \lim_{T\to\infty}\lim_{\varepsilon\to 0} \frac{2}{a_+}\left(2\frac{\mathcal{H}_{2H}[0, b_-^{\frac{1}{2H}}T]}{T} - \frac{\mathcal{H}_{2H}(2b_-^{\frac{1}{2H}}T)}{T}\right)\\
&= 0,
\end{aligned} \tag{21}$$

where the last equation follows since $\lim_{T\to\infty} \mathcal{H}_{2H}[0,T]/T = \mathcal{H}_{2H} \in (0,\infty)$. Next, we consider $\Theta_{12}(u)$. It follows that

$$\Theta_{12}(u) \le \sum_{-N_u \le i \le N_u} \sum_{\substack{-N_u \le j \le N_u\\ j > i+1}} \mathbb{P}\left\{\sup_{t\in\Delta_i} Y_-(t) > \hat{u}\left(1 + a_+\left|\hat{i}Tq\right|\right), \sup_{t\in\Delta_j} Y_-(t) > \hat{u}\right\}$$

where $\hat{i} = (i+1)I_{(i<0)} + iI_{(i\geq 0)}$. Furthermore, by Lemma 5.7 (see Appendix)

$$\Theta_{12}(u) \leq CT^2 \sum_{k\geq 1} e^{-G(kT)^{2H}} \sum_{-N_u\leq i\leq N_u} \Psi\left(\hat{u}(1+\frac{a_+}{2}\left|\hat{i}Tq\right|)\right)$$

for all u large, with some positive constants C, G independent of i, k, T, u. Using again similar arguments as the derivation of (19) we obtain

$$\sum_{-N_u\leq i\leq N_u} \Psi\left(\hat{u}(1+\frac{a_+}{2}\left|\hat{i}Tq\right|)\right) = \frac{4}{a_+T}\hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})(1+o(1)), \quad u\to\infty.$$

Thus,

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty}\frac{\Theta_{12}(u)}{\hat{u}^{\frac{1}{H}-2}\Psi(\hat{u})} \leq \lim_{T\to\infty}\lim_{\varepsilon\to 0}\frac{4}{a_+}CT\sum_{k\geq 1}e^{-G(kT)^{2H}} = 0. \tag{22}$$

Consequently, by letting $\varepsilon\to 0, T\to\infty$ (in this order) we conclude from (16)-(22) that as $u\to\infty$

$$\begin{aligned} p(u) &\sim \frac{1}{2^{\frac{1}{2H}}t^*}\mathcal{H}_{2H}\frac{a_1+a_2}{a_1a_2}\hat{u}^{\frac{1}{H}-2}\Psi(\hat{u}) \\ &\sim \frac{a_1+a_2}{2^{\frac{1}{2H}}t^*a_1a_2}\mathcal{H}_{2H}\left(\frac{(q_1+c_1t^*)u^{1-H}}{t^{*H}}\right)^{1/H-2}\Psi\left(\frac{(q_1+c_1t^*)u^{1-H}}{t^{*H}}\right). \end{aligned}$$

<u>Case 3.ii) $H = 1/2$</u>. We have

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_1(t) > \hat{u}\right\} &\leq \mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} W_1(t)>\hat{u}\right\} + \sum_{\substack{-N_u-1\leq i\leq N_u \\ i\neq -1,0}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t)>\hat{u}\right\}, \\ \mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_2(t) > \hat{u}\right\} &\geq \mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} W_2(t)>\hat{u}\right\}. \end{aligned}$$

It follows from Corollary 2.2 in [35] that

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} W_1(t)>\hat{u}\right\} &= \mathbb{P}\left\{\sup_{t\in[-b_+Tq,b_+Tq]} W_1(b_+^{-1}t)>\hat{u}\right\} \\ &\sim \mathcal{H}_1^{d_1}[-b_+T, b_+T]\Psi(\hat{u}), \quad u\to\infty, \end{aligned}$$

where $d_1(t) = (a_2-\varepsilon)b_+^{-1}\left|t\right|I_{(t<0)} + (a_1-\varepsilon)b_+^{-1}\left|t\right|I_{(t\geq 0)}$. Similarly,

$$\mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} W_2(t)>\hat{u}\right\} \sim \mathcal{H}_1^{d_2}[-b_-T, b_-T]\Psi(\hat{u}), \quad u\to\infty,$$

where $d_2(t) = (a_2+\varepsilon)b_-^{-1}\,|t|\,I_{(t<0)} + (a_1+\varepsilon)b_-^{-1}\,|t|\,I_{(t\geq 0)}$. Next, we have by Lemma 5.6

$$\begin{aligned}
&\sum_{\substack{-N_u-1\le i\le N_u\\ i\neq -1,0}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t)>\hat{u}\right\} \le \sum_{\substack{-N_u-1\le i\le N_u\\ i\neq -1,0}} \mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t)>\hat{u}B(i,T,u)\right\}\\
&= \frac{\mathcal{H}_1[0,b_+T]}{\sqrt{2\pi}\hat{u}}\left(\sum_{i=-N_u}^{-2} e^{-\frac{\hat{u}^2}{2}(1+(a_2-\varepsilon)|(i+1)Tq|)^2} + \sum_{i=1}^{N_u} e^{-\frac{\hat{u}^2}{2}(1+(a_1-\varepsilon)|iTq|)^2}\right)(1+o(1))\\
&\le \mathcal{H}_1[0,b_+T]\Psi(\hat{u})\left(\sum_{i=-\infty}^{-1} e^{-(a_2-\varepsilon)|iT|} + \sum_{i=1}^{\infty} e^{-(a_1-\varepsilon)|iT|}\right)(1+o(1))\\
&\le 2b_+T\sum_{i=1}^{\infty} e^{-a_-|iT|}\Psi(\hat{u})(1+o(1)),
\end{aligned}$$

where the last inequality follows since $\mathcal{H}_{2H}[0,T]$ is sub-additive for any $H\in(0,1)$ (see, e.g., [26] or [35]). Consequently, we have from the above formulas and (16) that, for any $S,T>0$

$$\begin{aligned}
&\mathcal{H}_1^d[-bS,bS]\\
&\le \lim_{\varepsilon\to 0}\liminf_{u\to\infty}\frac{p(u)}{\Psi(\hat{u})} \le \lim_{\varepsilon\to 0}\limsup_{u\to\infty}\frac{p(u)}{\Psi(\hat{u})}\\
&\le \mathcal{H}_1^d[-bT,bT] + 2bT\sum_{i=1}^{\infty} e^{-a|iT|},
\end{aligned}\tag{23}$$

where $b=\frac{1}{2t^*}, a=\min(a_1,a_2)$ and

$$d(t) = a_2 b^{-1}\,|t|\,I_{(t<0)} + a_1 b^{-1}\,|t|\,I_{(t\geq 0)},\quad t\in\mathbb{R}.$$

Letting $S\to\infty$ in (23) we get $\widetilde{\mathcal{H}}_1^d<\infty$ and letting $T\to\infty$ we obtain $\widetilde{\mathcal{H}}_1^d>0$. Therefore, we conclude that

$$p(u)\sim\widetilde{\mathcal{H}}_1^d\Psi(\hat{u}),\quad u\to\infty.$$

Case 3.iii) $H>1/2$. We have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_1(t)>\hat{u}\right\} &\le \mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} Y_+(t)>\hat{u}\right\}\\
\mathbb{P}\left\{\sup_{t\in[-\delta_u,\delta_u]} W_2(t)>\hat{u}\right\} &\ge \mathbb{P}\left\{Y_-(0)>\hat{u}\right\}.
\end{aligned}$$

Further, since

$$\mathbb{P}\left\{\sup_{t\in\Delta_{-1}\cup\Delta_0} Y_+(t)>\hat{u}\right\}\sim\mathcal{H}_{2H}[0,2b_+^{\frac{1}{2H}}T]\Psi(\hat{u}),\quad u\to\infty,$$

and

$$\mathbb{P}\left\{Y_-(0)>\hat{u}\right\}=\Psi(\hat{u}),$$

by letting $T\to 0$ we conclude that

$$p(u)\sim\Psi(\hat{u}),\quad u\to\infty.$$

Consequently, a comparison of the above asymptotics for Cases 3.i)-3.iii) with the formulas in (12) and (14) shows that (15) holds, and thus the the claim for Case 3) follows.

Proof for Case 4). It follows from (10) that for the sufficiently small ρ there exists some small $\varepsilon > 0$ such that

$$\begin{aligned}
&1 + (a_2 - \varepsilon)\,|t - t^*|\, I_{(t<t^*)} + (\widehat{a}_1 - \varepsilon)(t - t^*)^2 I_{(t \ge t^*)} \\
&\le \frac{\sigma_Z(t^*)}{\sigma_Z(t)} \le \\
&1 + (a_2 + \varepsilon)\,|t - t^*|\, I_{(t<t^*)} + (\widehat{a}_1 + \varepsilon)(t - t^*)^2 I_{(t \ge t^*)}
\end{aligned} \tag{24}$$

holds for all $t \in D_\rho$, where $\widehat{a}_1 = \frac{\left|\sigma_1''(t^*)\right|}{2\sigma_Z(t^*)} = \frac{H(1-H)}{2t_1^2}$ and $a_2 = \frac{\left|\sigma_2'(t^*)\right|}{\sigma_Z(t^*)}$. Set $\delta_{1,u} = \frac{\log(\hat{u})}{\hat{u}}$ and $\delta_{2,u} = \left(\frac{\log(\hat{u})}{\hat{u}}\right)^2$. Then for all $t \in [t^* - \rho, t^* - \delta_{2,u}] \cup [t^* + \delta_{1,u}, t^* + \rho]$ we have

$$\frac{\sigma_Z(t^*)}{\sigma_Z(t)} \ge 1 + a_- \delta_{2,u}$$

with $a_- = \min(\widehat{a}_1, a_2) - \varepsilon$. Thus, similarly to (14) we have

$$\mathbb{P}\left\{ \sup_{t \in [t^*-\rho, t^*-\delta_{2,u}] \cup [t^*+\delta_{1,u}, t^*+\rho]} Z(t) > \hat{u} \right\} \le C u^{1/H-1} \exp\left(-\frac{\hat{u}^2}{2}(1 + a_- \delta_{2,u})^2 \right) \tag{25}$$

holds for all large enough u, with some constant $C > 0$ which does not depend on u.

Now we consider

$$p(u) = \mathbb{P}\left\{ \sup_{t \in [t^* - \delta_{2,u}, t^* + \delta_{1,u}]} Z(t) > u^{1-H} \right\}.$$

Let $\{Y_\pm(t), t \ge 0\}$ be two continuous centered stationary Gaussian processes defined the same as in Case 3). By the Slepian inequality and (24) we have

$$\mathbb{P}\left\{ \sup_{t \in [-\delta_{2,u}, \delta_{1,u}]} W_2(t) > \hat{u} \right\} \le p(u) \le \mathbb{P}\left\{ \sup_{t \in [-\delta_{2,u}, \delta_{1,u}]} W_1(t) > \hat{u} \right\},$$

where

$$\begin{aligned}
W_1(t) &= \frac{Y_+(t)}{1 + (a_2 - \varepsilon)\,|t|\, I_{(t<0)} + (\widehat{a}_1 - \varepsilon) t^2 I_{(t \ge 0)}}, \quad t \in \mathbb{R}, \\
W_2(t) &= \frac{Y_-(t)}{1 + (a_2 + \varepsilon)\,|t|\, I_{(t<0)} + (\widehat{a}_1 + \varepsilon) t^2 I_{(t \ge 0)}}, \quad t \in \mathbb{R}.
\end{aligned}$$

Similar to Case 3) we introduce the following notation. Recall $q = q(u) = \hat{u}^{-1/H}$, and set for any $T > 0$

$$\Delta_i = [iTq, (i+1)Tq]\,,\ i \in \mathbb{Z}, \quad \text{and} \quad N_{k,u} = \left\lfloor T^{-1}\delta_{k,u} q^{-1} \right\rfloor,\ k = 1, 2.$$

We shall investigate separately the following two cases:

Case 4.i) $H < 1/2$, Case 4.ii) $H \ge 1/2$.

Case 4.i) $H < 1/2$. We have by the Bonferroni inequality

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},\delta_{1,u}]} W_1(t) > \hat{u}\right\} &\le \sum_{i=-N_{2,u}-1}^{N_{1,u}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t) > \hat{u}\right\},\\
\mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},\delta_{1,u}]} W_2(t) > \hat{u}\right\} &\ge \mathbb{P}\left\{\sup_{t\in[0,\delta_{1,u}]} W_2(t) > \hat{u}\right\}\\
&\ge \sum_{i=0}^{N_{1,u}-1} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t) > \hat{u}\right\} - \Theta(u),
\end{aligned}$$

where

$$\Theta(u) = \sum_{0\le i<j\le N_{1,u}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t) > \hat{u}, \sup_{t\in\Delta_j} W_2(t) > \hat{u}\right\}.$$

For the upper bound, we have

$$\begin{aligned}
\sum_{i=-N_{2,u}-1}^{N_{1,u}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t) > \hat{u}\right\} &\le \sum_{i=0}^{N_{1,u}} \mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t) > \hat{u}B_1(i,T,u)\right\}\\
&\quad + \sum_{i=-N_{2,u}-1}^{-1} \mathbb{P}\left\{\sup_{t\in\Delta_0} Y_+(t) > \hat{u}B_2(i,T,u)\right\}
\end{aligned}$$

with

$$\begin{aligned}
B_1(i,T,u) &= 1 + (\widehat{a}_1 - \varepsilon)(iTq)^2, \quad i \ge 0,\\
B_2(i,T,u) &= 1 + (a_2 - \varepsilon)\left|(i+1)Tq\right|, \quad i < 0.
\end{aligned}$$

Similar arguments as in (19) yield that, as $u \to \infty$

$$\begin{aligned}
\sum_{i=-N_{2,u}-1}^{N_{1,u}} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_1(t) > \hat{u}\right\} &\le \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}} T]}{T}\left(\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1 - \varepsilon}}\hat{u}^{\frac{1}{H}-1} + \frac{1}{a_2 - \varepsilon}\hat{u}^{\frac{1}{H}-2}\right)\Psi(\hat{u})(1+o(1))\\
&= \frac{\mathcal{H}_{2H}[0, b_+^{\frac{1}{2H}} T]}{T}\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1 - \varepsilon}}\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u})(1+o(1))
\end{aligned}$$

and

$$\sum_{i=0}^{N_{1,u}-1} \mathbb{P}\left\{\sup_{t\in\Delta_i} W_2(t) > \hat{u}\right\} \ge \frac{\mathcal{H}_{2H}[0, b_-^{\frac{1}{2H}} T]}{T}\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1 + \varepsilon}}\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u})(1+o(1)).$$

For $\Theta(u)$, we can use the same arguments as for Case 3.i) and show that

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty} \frac{\Theta(u)}{\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u})} = 0.$$

Consequently, we conclude that

$$p(u) \sim \frac{1}{2^{\frac{1}{2H}} t^*}\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1}}\mathcal{H}_{2H}\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u}) \sim \frac{1}{2}\psi_{q_1,c_1}(u), \quad u \to \infty.$$

Case 4.ii) $H \geq 1/2$. We have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},\delta_{1,u}]} W_1(t) > \hat{u}\right\} &\leq \mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},0]} W_1(t) > \hat{u}\right\} + \mathbb{P}\left\{\sup_{t\in[0,\delta_{1,u}]} W_1(t) > \hat{u}\right\}\\
\mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},\delta_{1,u}]} W_2(t) > \hat{u}\right\} &\geq \mathbb{P}\left\{\sup_{t\in[0,\delta_{1,u}]} W_2(t) > \hat{u}\right\}.
\end{aligned}$$

Similarly to Case 3.ii) and Case 3.iii) we have

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[-\delta_{2,u},0]} W_1(t) > \hat{u}\right\}}{\Psi(\hat{u})} = \left(1+\frac{1}{2t^* a_2}\right) I_{(H=1/2)} + I_{(H>1/2)},$$

where we used (3) for $H = 1/2$. Furthermore, we have from Case 4.i) that

$$\begin{aligned}
\lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[0,\delta_{1,u}]} W_1(t) > \hat{u}\right\}}{\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u})} &= \lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[0,\delta_{1,u}]} W_2(t) > \hat{u}\right\}}{\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u})}\\
&= \frac{1}{2^{\frac{1}{2H}} t^*}\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1}}\mathcal{H}_{2H}.
\end{aligned}$$

Thus,

$$p(u) \sim \frac{1}{2^{\frac{1}{2H}} t^*}\frac{\sqrt{\pi}}{2\sqrt{\widehat{a}_1}}\mathcal{H}_{2H}\hat{u}^{\frac{1}{H}-1}\Psi(\hat{u}) \sim \frac{1}{2}\psi_{q_1,c_1}(u), \quad u\to\infty.$$

Consequently, the claim for Case 4) follows from a comparison of the above asymptotics for Cases 4.i)-4.ii) with (12) and (25).

Proof for Case 5). The proof is the same as for Case 4) and thus omitted. □

Proof of Theorem 3.2: Define the following passage time

$$\widehat{\tau}(u) := \inf\{t \geq 0 : Z(t) > u^{1-H}\},$$

with Z defined in (8). Clearly, we have $\tau(u) = u\widehat{\tau}(u)$ for any $u > 0$.

In the following, we discuss the five difference scenarios, separately.

Proof for Case 1). It follows that

$$\mathbb{P}\left\{\frac{\tau(u) - t_1 u}{A(q_1 u)} \leq x \,\middle|\, \tau(u) < \infty\right\} = \frac{\mathbb{P}\left\{\widehat{\tau}(u) \leq t_1 + u^{-1}A(q_1 u)x\right\}}{\mathbb{P}\left\{\tau(u) < \infty\right\}}, \tag{26}$$

where

$$\mathbb{P}\left\{\widehat{\tau}(u) \leq t_1 + u^{-1}A(q_1 u)x\right\} = \mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1 u)x]} Z(t) > u^{1-H}\right\}.$$

For any fixed $x\in\mathbb{R}$ we have $t_1 + u^{-1}A(q_1 u)x > t_1 - \log(u)u^{H-1} > t^*$ when u is sufficiently large. Thus, (recall X_1 defined in (6))

$$\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[t_1-\log(u)u^{H-1},t_1+u^{-1}A(q_1 u)x]} X_1(t) > u^{1-H}\right\} &\leq \mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1 u)x]} Z(t) > u^{1-H}\right\}\\
&\leq \mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1 u)x]} X_1(t) > u^{1-H}\right\}.
\end{aligned}$$

It is known from the proof of Proposition 2.1 that the above upper and lower bounds are asyptotically equivalent when $q_1 = 1$. For general $q_1 > 0$ we can also establish, as in the proof of Theorem 1.1 of [34] (let $\gamma = 0$ therein), the same equivalence result, and further

$$\mathbb{P}\left\{\widehat{\tau}(u) \le t_1 + u^{-1} A(q_1 u)x\right\} \sim \psi_{q_1,c_1}(u)\Phi(x), \quad u \to \infty.$$

Consequently, the claim follows from the above combined with (26) and Case 1) of Theorem 3.1.

Proof for Case 2). The proof follows similarly as in Case 1), and thus omitted here.

Proof for Case 3). Recall $\hat{u} = u^{1-H}/\sigma_Z(t^*)$. Similarly as above we have

$$\mathbb{P}\left\{\frac{\tau(u) - t^* u}{D(u)} \le x \middle| \tau(u) < \infty\right\} = \frac{\mathbb{P}\left\{\exists_{t\in[0,t^*+x\hat{u}^{-2}]} Z(t) > u^{1-H}\right\}}{\mathbb{P}\left\{\tau(u) < \infty\right\}}.$$

Note that the asymptotics of $\mathbb{P}\{\tau(u) < \infty\}$ as $u \to \infty$ have been obtained in Case 3) of Theorem 3.1. Next we shall derive the asymptotics for the numerator. Finally, the claims follow by comparing these two asymptotics. Consider first Case 3.i) where $H < 1/2$. Using similar arguments as in the proof of Case 3.i) in Theorem 3.1, we can conclude that

$$\mathbb{P}\left\{\exists_{t\in[0,t^*+x\hat{u}^{-2}]} Z(t) > u^{1-H}\right\} \sim \frac{1}{2^{\frac{1}{2H}} t^*} \mathcal{H}_{2H} \int_{-\infty}^{x} \left(e^{a_2 y} I_{(y<0)} + e^{-a_1 y} I_{(y\ge 0)}\right) dy \ \hat{u}^{\frac{1}{H}-2} \Psi(\hat{u})$$

holds as $u \to \infty$. The key difference comes out by checking the formula before (19).

Next, we consider Case 3.ii) where $H = 1/2$. Again similar arguments as in the proof of Case 3.ii) in Theorem 3.1 yield that

$$\mathbb{P}\left\{\exists_{t\in[0,t^*+x\hat{u}^{-2}]} Z(t) > u^{1-H}\right\} \sim \mathcal{H}_1^d[-\infty, 1/(2t^*)x]\Psi(\hat{u})$$

holds as $u \to \infty$, and additionally $\mathcal{H}_1^d[-\infty, 1/(2t^*)x] \in (0,\infty)$ for any $x \in \mathbb{R}$.

Now we consider Case 3.iii) where $H > 1/2$. If $x \ge 0$, then similarly as in the proof of Case 3.iii) in Theorem 3.1 we have

$$\mathbb{P}\left\{\exists_{t\in[0,t^*+x\hat{u}^{-2}]} Z(t) > u^{1-H}\right\} \sim \Psi(\hat{u}), \quad u \to \infty.$$

For $x < 0$, letting $T_x(u) = t^* + x\hat{u}^{-2}$ we have

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,T_x(u)]} Z(t) > u^{1-H}\right\} &= \mathbb{P}\left\{\exists_{t\in[0,T_x(u)]} X_2(t) > u^{1-H}\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[0,1]} X_2(T_x(u)t) > u^{1-H}\right\}.\end{aligned}$$

Then the same arguments as in the proof of Theorem 2.4 in [22] yield that

$$\mathbb{P}\left\{\exists_{t\in[0,T_x(u)]} Z(t) > u^{1-H}\right\} \sim \Psi\left(\frac{u^{1-H}}{\sigma_2(T_x(u))}\right), \quad u \to \infty.$$

Since further

$$\lim_{u\to\infty} \frac{\Psi\left(\frac{u^{1-H}}{\sigma_2(T_x(u))}\right)}{\Psi(\hat{u})} = e^{a_2 x}$$

we conclude that the claim follows.

Proof for Case 4). We see from the proof of Case 1) that it is sufficient to derive the asymptotics of $\mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1u)x]}Z(t)>u^{1-H}\right\}$ as $u\to\infty$. If $x>0$, then using similar ideas as in Case 4.i) of Theorem 3.1 we can show that, as $u\to\infty$

$$
\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1u)x]}Z(t)>u^{1-H}\right\} &\sim \mathbb{P}\left\{\exists_{t\in[t_1-(\log(\hat{u})/\hat{u})^2,t_1+u^{-1}A(q_1u)x]}Z(t)>u^{1-H}\right\}\\
&\sim \psi_{q_1,c_1}(u)\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{y^2}{2}}dy \qquad (27)\\
&= \psi_{q_1,c_1}(u)(\frac{1}{2}-\Psi(x)).
\end{aligned}
$$

If $x\le 0$, then we have

$$
\mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1u)x]}Z(t)>u^{1-H}\right\}\le \mathbb{P}\left\{\exists_{t\in[0,t_1]}Z(t)>u^{1-H}\right\}
$$

Again similar arguments as in the proof of Case 4) in Theorem 3.1 yields that

$$
\mathbb{P}\left\{\exists_{t\in[0,t_1]}Z(t)>u^{1-H}\right\}\sim\Psi(\hat{u})\times\begin{cases}\frac{1}{2^{\frac{1}{2H}}t^*a_2}\mathcal{H}_{2H}\hat{u}^{1/H-2}, & H<1/2\\ 1+\frac{1}{2t^*a_2}, & H=1/2\\ 1, & H>1/2,\end{cases}
$$

implying that

$$
\mathbb{P}\left\{\exists_{t\in[0,t_1+u^{-1}A(q_1u)x]}Z(t)>u^{1-H}\right\}=o(\psi_{q_1,c_1}(u)),\quad u\to\infty. \tag{28}
$$

Consequently, the claim follows from a comparison of (27) and (28) with Case 4) of Theorem 3.1.

Proof for Case 5). Similarly, it is sufficient to derive the asymptotics of $\mathbb{P}\left\{\exists_{t\in[0,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\}$ as $u\to\infty$. If $x<0$, then using similar ideas as in Case 4.i) of Theorem 3.1 we can show that, as $u\to\infty$

$$
\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\} &\sim \mathbb{P}\left\{\exists_{t\in[t_2-\log(\hat{u})/\hat{u},t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\}\\
&\sim \psi_{q_2,c_2}(u)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{y^2}{2}}dy \qquad (29)
\end{aligned}
$$

If $x\ge 0$, then we have

$$
\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,t_2]}Z(t)>u^{1-H}\right\} &\le \mathbb{P}\left\{\exists_{t\in[0,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\}\\
&\le \mathbb{P}\left\{\exists_{t\in[0,t_2]}Z(t)>u^{1-H}\right\}+\mathbb{P}\left\{\exists_{t\in[t_2,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\}
\end{aligned}
$$

Again similar arguments as in the proof of Case 4) in Theorem 3.1 yields that, as $u\to\infty$

$$
\mathbb{P}\left\{\exists_{t\in[0,t_2]}Z(t)>u^{1-H}\right\}\sim\frac{1}{2}\psi_{q_2,c_2}(u)
$$

and

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[t_2,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\} &\le \mathbb{P}\left\{\exists_{t\in[t_2,\infty)}Z(t)>u^{1-H}\right\}\\ &\sim \Psi\left(\hat{u}\right)\times\begin{cases}\frac{1}{2^{\frac{1}{2H}}t^*a_1}\mathcal{H}_{2H}\hat{u}^{1/H-2}, & H<1/2\\ 1+\frac{1}{2t^*a_1}, & H=1/2\\ 1, & H>1/2,\end{cases}\end{aligned}$$

implying that

$$\mathbb{P}\left\{\exists_{t\in[0,t_2+u^{-1}A(q_2u)x]}Z(t)>u^{1-H}\right\}\sim\frac{1}{2}\psi_{q_2,c_2}(u),\quad u\to\infty. \tag{30}$$

Consequently, the claim follows from a comparison of (29) and (30) with Case 5) of Theorem 3.1. □

5. Appendix

This appendix consists of some known results, a skeleton of proof of Proposition 2.1, and some technical results which were used in Section 4.

Lemma 5.1. *(Bonferroni inequality) Let $(\Omega,\mathcal{S},\mathbb{P})$ be a probability space and $A_1,A_2,\cdots,A_n\in\mathcal{S}$ for $n\ge 2$. Then*

$$\sum_{i=1}^n\mathbb{P}\left\{A_i\right\}-\sum_{1\le i<j\le n}\mathbb{P}\left\{A_i\cap A_j\right\}\ \le\ \mathbb{P}\left\{\cup_{i=1}^nA_i\right\}\ \le\ \sum_{i=1}^n\mathbb{P}\left\{A_i\right\}.$$

A complete proof of the Bonferroni inequality can be found, e.g., in [36].

We write below $\mathcal{T}$ for a subinterval of $\mathbb{R}$.

Lemma 5.2. *(Slepian inequality) Let $\{Y(t),t\in\mathcal{T}\}$ and $\{Z(t),t\in\mathcal{T}\}$ be two centered almost surely continuous Gaussian processes, almost surely bounded on $\mathcal{T}$. If for all $t,s\in\mathcal{T}$*

$$\mathbb{E}\left(Y^2(t)\right)=\mathbb{E}\left(Z^2(t)\right),\quad \mathbb{E}\left(Y(t)Y(s)\right)\ge\mathbb{E}\left(Z(t)Z(s)\right),$$

then for any $u\in\mathbb{R}$ we have

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}Y(t)>u\right\}\ \le\ \mathbb{P}\left\{\sup_{t\in\mathcal{T}}Z(t)>u\right\}.$$

Lemma 5.3. *(Borell-TIS inequality) Let $\{X(t),t\in\mathcal{T}\}$ be a centered almost surely continuous Gaussian process, almost surely bounded on $\mathcal{T}$, with variance function $\sigma^2(\cdot),1\le i\le n$. Then*

$$\mu:=\mathbb{E}\left(\sup_{t\in\mathcal{T}}X(t)\right)<\infty.$$

Furthermore, if $\sigma_{\mathcal{T}}:=\sup_{t\in\mathcal{T}}\sigma(t)>0$, then for all $u>\mu$

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}X(t)>u\right\}\le\exp\left(-\frac{(u-\mu)^2}{2\sigma_{\mathcal{T}}^2}\right).$$

Complete proofs of the Slepian inequality and the Borell-TIS inequality can be found in [37].

Lemma 5.4. *(Piterbarg inequality) Under the conditions of Lemma 5.3, if further $mes(\mathcal{T})<\infty$ and*

$$\mathbb{E}\left((X(t)-X(s))^2\right)\le G\left|t-s\right|^{\gamma}$$

holds for all $s,t\in\mathcal{T}$ with some constants $\gamma,G>0$, then, for all u large

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}X(t)>u\right\}\le Cmes(\mathcal{T})u^{\frac{2}{\gamma}-1}\exp\left(-\frac{u^2}{2\sigma_{\mathcal{T}}^2}\right),$$

where C is some positive constant not depending on u.

We refer to Lemma 8.1 of [26] for a proof of the Piterbarg inequality.

Next, let $\{X(t),t\in\mathcal{T}\}$ be a centered Gaussian process with almost surely continuous sample paths, standard deviation function $\sigma(\cdot)$ and correlation function $r(\cdot,\cdot)$ satisfying the following assumptions:

A1: The function $\sigma(t),t\in\mathcal{T}$ attains its maximum at a unique point t_0 which is an inner point of $\mathcal{T}$, with $\sigma(t_0)=1$, and further

$$\sigma(t_0+t)=1-a\left|t\right|^{\beta}(1+o(1)),\quad t\to 0$$

holds for some $a,\beta>0$.

A2: It holds that

$$1-r(s,t)=b\left|t-s\right|^{\alpha}(1+o(1)),\quad s,t\to t_0$$

for some $b>0$ and $\alpha\in(0,2)$.

A3: For all $s,t\in\mathcal{T}$ there exist some constants $\gamma,G>0$ such that

$$\mathbb{E}\left((X(t)-X(s))^2\right)\le G\left|t-s\right|^{\gamma}.$$

Now we present a result of Piterbarg, see, e.g., Theorem 8.2 or Theorem D.3 of [26].

Theorem 5.5. *(Piterbarg theorem) Let $\{X(t),t\in\mathcal{T}\}$ be defined as above which satisfies* **A1**–**A3** *with $mes(\mathcal{T})<\infty$. Then, as $u\to\infty$,*

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}X(t)>u\right\}\sim\Psi(u)\times\begin{cases}2b^{\frac{1}{\alpha}}a^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\mathcal{H}_{\alpha}u^{\frac{2}{\alpha}-\frac{2}{\beta}}, & \alpha<\beta\\ \widetilde{\mathcal{H}}_{\alpha}^{a/b}, & \alpha=\beta\\ 1, & \alpha>\beta,\end{cases}$$

where $\Gamma(\cdot)$ is the gamma function and, for any $d>0$,

$$\widetilde{\mathcal{H}}_{\alpha}^{d}=\lim_{T\to\infty}\mathbb{E}\left(\exp\left(\sup_{t\in[-T,T]}(\sqrt{2}B_{\alpha/2}(t)-(1+d)\left|t\right|^{\alpha})\right)\right)\in(0,\infty).$$

In the literature, the constant $\widetilde{\mathcal{H}}_{\alpha}^{d}$ defined above is called Piterbarg constant. We refer to [33] for the study of Piterbarg and related constants.

Proof of Proposition 2.1: We give a skeleton of the proof; we refer to [9, 10] or [29, 34] for more details.

By the self-similarity of fBm, we can rewrite the ruin probability as

$$\begin{aligned}\psi_{q,c}(u) &= \mathbb{P}\left\{\sup_{t\geq 0}\ (B_H(t)-ct) > qu\right\}\\ &= \mathbb{P}\left\{\sup_{t\geq 0}\ X(t) > u^{1-H}\right\},\end{aligned}$$

where

$$X(t)=\frac{B_H(t)}{q+ct},\quad t\geq 0.$$

Define

$$\sigma(t)=\sqrt{Var(X(t))}=\frac{t^H}{q+ct},\quad t\geq 0.$$

Elementary calculations show that

$$t_0=\frac{Hq}{c(1-H)}$$

is the unique maximum point of the function $\sigma(t), t\geq 0$. Let $\theta>0$ be a small constant. We have

$$\begin{aligned}\pi(u):=\mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]} X(t)>u^{1-H}\right\} &\leq \psi_{q,c}(u)\\ &\leq \pi(u)+\mathbb{P}\left\{\sup_{t\in[0,t_0-\theta]\cup[t_0+\theta,\infty)} X(t)>u^{1-H}\right\}.\end{aligned}\tag{31}$$

It can be shown that for fixed θ small enough, assumptions **A1**–**A3** are satisfied by $\{X(t), t\in[t_0-\theta,t_0+\theta]\}$ with $\beta=2>\alpha=2H$. Thus, by Theorem 5.5

$$\pi(u)\sim 2^{\frac{1}{2}-\frac{1}{2H}}\frac{\sqrt{\pi}}{\sqrt{H(1-H)}}\mathcal{H}_{2H}\left(\frac{c^Hq^{1-H}u^{1-H}}{H^H(1-H)^{1-H}}\right)^{1/H-1}\Psi\left(\frac{c^Hq^{1-H}u^{1-H}}{H^H(1-H)^{1-H}}\right).\tag{32}$$

Furthermore, since $\lim_{t\to\infty}X(t)=0$, by Lemma 5.3 we have for u large enough

$$\mathbb{P}\left\{\sup_{t\in[0,t_0-\theta]\cup[t_0+\theta,\infty)} X(t)>u^{1-H}\right\}\leq \exp\left(-\frac{(u^{1-H}-\mu)^2}{2\sigma_m^2}\right),\tag{33}$$

where $\mu=\mathbb{E}\left(\sup_{t\in[0,t_0-\theta]\cup[t_0+\theta,\infty)}X(t)\right)<\infty$ and $\sigma_m^2=\sup_{t\in[0,t_0-\theta]\cup[t_0+\theta,\infty)}\sigma^2(t)<\sigma^2(t_0)$. Consequently, we conclude from (31)–(33) that

$$\psi_{q,c}(u)\sim\pi(u),\qquad u\to\infty$$

implying thus (4).

Now we consider (5). Without loss of generality, we assume that $q=1$. It follows that for any $x\in\mathbb{R}$

$$\mathbb{P}\left\{\frac{\tau_{1,c}(u)-t_0u}{A(u)}\leq x\Big|\tau_{1,c}(u)<\infty\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,t_0u+A(u)x]}(B_H(t)-ct)>u\right\}}{\psi_{1,c}(u)}.$$

Furthermore, we have, as $u \to \infty$,

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,t_0u+A(u)x]}(B_H(t)-ct)>u\right\} &= \mathbb{P}\left\{\sup_{t\in[0,t_0+u^{-1}A(u)x]}\frac{B_H(t)}{1+ct}>u^{1-H}\right\}\\ &\sim \psi_{1,c}(u)\Phi(x),\end{aligned}$$

where the last asymptotic equivalence follows similarly as the proof of Theorem 1 in [34] (where we take $\gamma = 0$), see also the proof of Theorem 2 in [10]. This completes the proof. □

Next, let $\{X(t), t \ge 0\}$ be a centered stationary Gaussian process with almost surely continuous sample paths, unit variance and correlation function $r(\cdot)$ satisfying

$$1 - r(t) = b\,|t|^{\alpha}\,(1 + o(1)), \quad t \to 0$$

for some $b > 0$ and $\alpha \in (0, 2)$.

The following lemma shows the uniformity of the tail asymptotics of the supremum taking over the Pickands' interval, which is crucial for the derivation of the single sum in the proof of Theorem 3.1. A general result has been shown in Theorem 2.1 of [38].

Lemma 5.6. *Let $\{X(t), t \ge 0\}$ be defined as above, and let $f_i(u), i \in K_u$, be a family of functions. If*

$$\lim_{u\to\infty}\sup_{i\in K_u}\left|\frac{f_i(u)}{u}-1\right|=0$$

then for any $T > 0$

$$\lim_{u\to\infty}\sup_{i\in K_u}\left|\frac{\mathbb{P}\left\{\sup_{t\in[0,Tu^{-2/\alpha}]}X(t)>f_i(u)\right\}}{\Psi(f_i(u))}-\mathcal{H}_\alpha[0,b^{\frac{1}{\alpha}}T]\right|=0.$$

Let $f_i(u), i \in K_u$, $g_i(u), i \in K_u$ be two families of functions. Denote, for any $T > 0$ and $k \in \mathbb{N}$

$$p_{i,j}(k,u)=\mathbb{P}\left\{\sup_{t\in[0,Tu^{-2/\alpha}]}X(t)>f_i(u),\ \sup_{t\in[(k+1)Tu^{-2/\alpha},(k+2)Tu^{-2/\alpha}]}X(t)>g_j(u)\right\},\quad i,j\in K_u.$$

For the approximation of the double sum term in the proof of Theorem 3.1, the following lemma concerning uniform bounds of $p_{i,j}(k, u)$ is crucial. We refer to Theorem 3.1 and Remarks 3.4 in [38] for more general versions of this result.

Lemma 5.7. *Let $\{X(t), t \ge 0\}$ and $f_i(u), i \in K_u$, $g_i(u), i \in K_u$, be defined as above. If*

$$\lim_{u\to\infty}\sup_{i\in K_u}\left|\frac{f_i(u)}{u}-1\right|=0,\qquad \lim_{u\to\infty}\sup_{i\in K_u}\left|\frac{g_i(u)}{u}-1\right|=0$$

then, for any $k \in \mathbb{N}$ such that $k = o(u^{2/\alpha}), u \to \infty$ we have

$$\sup_{j\in K_u}\sup_{i\in K_u}\frac{p_{i,j}(k,u)}{T^2e^{-G(kT)^\alpha}\Psi((f_i(u)+g_j(u))/2)}\le C$$

holds for all large u, with some positive constants G, C which are independent of i, j, k, T and u.

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