



UNIVERSITY OF LEEDS

This is a repository copy of *A generalised Euler-Poincaré formula for associahedra*.

White Rose Research Online URL for this paper:
<http://eprints.whiterose.ac.uk/137910/>

Version: Accepted Version

Article:

Baur, K and Martin, PP (2019) A generalised Euler-Poincaré formula for associahedra. *Bulletin of the London Mathematical Society*, 51 (1). ISSN 0024-6093

<https://doi.org/10.1112/blms.12221>

© 2018 London Mathematical Society. This is the peer reviewed version of the following article: Baur, K and Martin, PP (2019) A generalised Euler-Poincaré formula for associahedra. *Bulletin of the London Mathematical Society*, 51 (1) which has been published in final form at <http://dx.doi.org/10.1112/blms.12221>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Self-Archiving.

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

A GENERALISED EULER–POINCARÉ FORMULA FOR ASSOCIAHEDRA.

KARIN BAUR AND PAUL P. MARTIN

ABSTRACT. We derive a formula for the number of flip-equivalence classes of tilings of an n -gon by collections of tiles of shape dictated by an integer partition λ . The proof uses the Euler–Poincaré formula; and the formula itself generalises the Euler–Poincaré formula for associahedra.

1. INTRODUCTION

In [11, 12, 13] Fomin and Zelevinsky introduce cluster algebras and in particular find a Lie theoretic manifestation of the associahedron or Stasheff polytope [28]. In [9] Chapoton investigates the combinatorics of the associahedron from the Lie theoretic perspective; while in [23], [25], Postnikov and Scott develop these constructions in the direction of total positivity and Grassmannians. (The associahedron also has a wider interconnected network of other uses. See for example [29], [14], [8], [22].) Baur et al. [3] develop Fomin–Zelevinsky’s and Scott’s original construction with emphasis on quiver mutation and cluster categories associated with the Grassmannian. In [4] we generalised Scott’s map ([25, Section 3]) to the whole associahedron. A partial result on counting tilings with prescribed tile sizes up to ‘flip equivalence’ (cf. [15]) is included in [4]. Based on this, Baur–Schiffler–Weyman conjectured a general formula [5]. In this paper, we note that the formula also generalises the Euler–Poincaré formula for the associahedron, and prove the conjecture. The proof uses the polytopal property of the associahedron and the Euler–Poincaré formula itself among other devices.

Let P be a convex polygon with n vertices, labeled $1, 2, \dots, n$. Here A_n denotes the set of tilings of such an n -gon, or equivalently the set of non-crossing subsets of diagonals $[i, j]$ of P . For example

$$A_4 = \left\{ \square, \begin{array}{|c|} \hline \diagdown \\ \hline \end{array}, \begin{array}{|c|} \hline \diagup \\ \hline \end{array} \right\} = \{ \emptyset, \{[1, 3]\}, \{[2, 4]\} \}$$

There are two ways of triangulating a 4-gon. The move between these two is called ‘flip’. Two tilings are *triangular equivalent*, or *flip equivalent*, if they are related by any sequence of flips (for example any two triangulations are equivalent [15]). The set of classes of tilings under equivalence is \mathcal{A}_n . These are enumerated in [4].

2010 Mathematics Subject Classification: 05B45, 51M20.

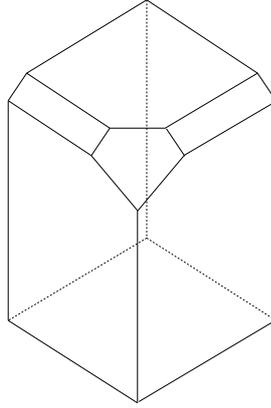


FIGURE 1. Associahedron K_5 drawn as a manifest convex polytope.

It is convenient to arrange A_n into a polytope, called the associahedron [17]. This is *ab initio* an abstract cell complex: each tiling t is a cell, and the boundary set of a cell is the set of tilings obtained from t by adding one more diagonal. Thus in particular the vertices (0-cells) are the tilings that are triangulations. Removing a diagonal from such a tiling gives a tiling with one quadrilateral tile. Starting now from this tiling, the quadrilateral tile can be triangulated in two ways. The tiling is an edge (1-cell) of the complex and its boundary is the two completions obtained by triangulating the quadrilateral. And so on.

Theorem 1.1 (Haiman, Lee [17]; see e.g. [10, Theorem 3.39]). *There exists a convex n -dimensional polytope K_{n+2} called the associahedron whose vertices and edges form the flip graph of a convex $(n+3)$ -sided polygon. The k -dimensional faces (k -cells) of this polytope are in one-to-one correspondence with the diagonalizations (tilings) of the polygon using exactly $n-k$ diagonals.*

See e.g. Figure 1.

The numbers $a_n = |A_n|$ are the little Schröder numbers (see e.g. [24]). Also of interest are the f -vectors [10] of these polytopes [9], see Sloane's OEIS number A001003 [27] and cf. (3).

(1.2) Let \mathcal{P}_n be the set of integer partitions of n , and \mathcal{P} be the set of all integer partitions. We use partition notation as in Macdonald [18, §I.1], as follows. For $\lambda \in \mathcal{P}$ let $m_i(\lambda)$ denote the number of parts i . We write $\lambda \in \mathcal{P}_n$ as $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ where the λ_i are non-negative integers in non-increasing order; or in *exponential* form, $\lambda = (1^{m_1}, 2^{m_2}, \dots, r^{m_r}, \dots)$, $m_i = |\{j : \lambda_j = i\}|$. We write $\lambda \cup \mu$ for the combination of partitions, thus $(3, 2) \cup (4, 2, 1) = (4, 3, 2, 2, 1)$. We write $l(\lambda)$ for the number of non-zero parts, the *length*; $|\lambda| = \sum_i \lambda_i = n$, the *weight* of λ ; and we may write $\lambda \in \mathcal{P}_n$ as $\lambda \vdash n$.

(1.3) For tiling $t \in A_n$ let $m_i(t)$ denote the number of $(i + 2)$ -gonal tiles. Define a ‘shape’ function $\mathbf{s} : A_n \rightarrow \mathcal{P}$ by $\mathbf{s}(t) = (1^{m_1(t)}, 2^{m_2(t)}, \dots)$. Let $A_n(\lambda) \subset A_n$ denote the subset of tilings of shape λ . Thus:

$$A_n = \bigcup_{\lambda \in \mathcal{P}_{n-2}} A_n(\lambda)$$

The first few of the numbers $a_n(\lambda) = |A_n(\lambda)|$ are computed in [4].

Note that shape is well-defined on triangular equivalence classes. Let $\mathbb{A}_n(\lambda)$ denote the set, and $\mathfrak{a}_n(\lambda)$ denote the number of tilings of P of a given shape up to triangular equivalence.

For $\mu \in \mathcal{P}$ define $\mu^+ \in \mathcal{P}$ by

$$(1) \quad \mu^+ = (2^{m_1(\mu)}, 3^{m_2(\mu)}, \dots, (i+1)^{m_i(\mu)}, \dots)$$

We extend the combination \cup of partitions as follows. Let $a > 0$, $\lambda \in \mathcal{P}$. Define $\lambda \cup (1^{-a})$ as $(1^{m_1(\lambda)-a}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \dots)$. For $m_1(\lambda) \geq a$, $\lambda \cup (1^{-a})$ is a partition. For example, $(1^3, 2^2, 3) \cup (1^{-2}) = (1, 2^2, 3)$. We will consider tilings of shape $\lambda \cup (1^{-a})$, for some λ . If $m_1(\lambda) < a$, $\lambda \cup (1^{-a})$ is not a partition, and $A_n(\lambda \cup (1^{-a})) = \emptyset$.

Theorem 1.4. For $\lambda, \mu \in \mathcal{P}$, let

$$\Pi_\mu^\lambda = \prod_{s \geq 1} \binom{m_{s+1}(\lambda) + m_s(\mu)}{m_s(\mu)} = \prod_{s \geq 2} \binom{m_s(\lambda \cup \mu^+)}{m_s(\mu^+)}$$

Then

$$(2) \quad \mathfrak{a}_n(\lambda) = \sum_{m=0}^{m_1(\lambda)-1} (-1)^m \sum_{\mu \vdash m} \Pi_\mu^\lambda a_n(\lambda \cup \mu^+ \cup 1^{-|\mu^+|})$$

Before proving the theorem we focus on one class of special cases: the cases where λ is of form 1^{n-2} . In these cases we know $\mathfrak{a}_n(\lambda) = 1$ by Hatcher’s theorem (a) in [15]. On the other hand the right hand side also takes a relatively simple form in these cases. All coefficients $\Pi_\mu^{(1^{n-2})}$ on the right hand side are 1 since $m_s(1^{n-2}) = 0$ for all $s > 1$. The right hand side is thus

$$(3) \quad \sum_{m=0}^{n-3} (-1)^m \sum_{\mu \vdash m} a_n(1^{n-2} \cup \mu^+ \cup 1^{-|\mu^+|})$$

Noting that $\sum_{\mu \vdash m} a_n(1^{n-2} \cup \mu^+ \cup 1^{-|\mu^+|})$ is the number of m -cells of K_{n-1} , since the signs alternate we actually have the Euler–Poincaré formula. Thus this special case of the theorem asserts:

$$1 = \text{Euler–Poincaré characteristic of } K_{n-1}$$

Consider the Schläfli–Poincaré Theorem:

Theorem 1.5 (Schläfli, Poincaré; see e.g. [30]). *Let f_i denote the number of i -cells of the d -dimensional polytope P . Then the Euler–Poincaré formula states that*

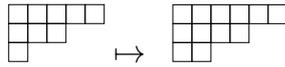
$$\sum_{i=0}^d (-1)^i f_i = 1.$$

By Theorem 1.1, K_{n-1} is a $n-3$ -dimensional polytope, so we can apply Theorem 1.5. We thus see on the one hand that Theorem 1.4 is verified in the case $\lambda = 1^{n-2}$; and on the other hand that the special cases of the theorem coincide with the Euler–Poincaré formulae for the various associahedra. In other words, the ‘count’ of triangulations of P up to flip equivalence computes the Euler–Poincaré characteristic of the corresponding associahedron.

2. PROOF OF MAIN THEOREM

2.1. Constructions and notation for proof: integer partitions.

(2.1) (To take arms against a sea of triangles [26].) Observe that the map $\mu \mapsto \mu^+$ from (1) corresponds to adding a leading column (of length $l(\mu)$) to the Young diagram of μ :



(NB $\emptyset^+ = \emptyset$). Assuming $\mu \neq \emptyset$ we also define $\mu \mapsto \mu^-$ by *removing* the first column:

$$(4) \quad \begin{array}{ccc} \square\square\square & \mapsto & \square\square \\ \square & & \square \\ \square & & \square \end{array} \quad \begin{array}{ccc} \square & \mapsto & \square \\ \square & & \square \end{array} \quad \begin{array}{ccc} \square & \mapsto & \emptyset \\ \square & & \square \end{array}$$

Note $\mu \mapsto \mu^+ \mapsto (\mu^+)^- = \mu$, but $\mu \mapsto \mu^- \mapsto (\mu^-)^+$ is not μ unless $m_1(\mu) = 0$.

At the level of *tilings* t of shape $\mathbf{s}(t) = \mu$ (as in (1.3)), $\mu \mapsto \mu^-$ ‘forgets’ all triangles. It nominally replaces every other i -gon with an $(i-1)$ -gon. Thus $(\mu^-)^+$ simply forgets all triangles and preserves other gons.

Note that if $\mu \in \mathcal{P}_n$ then μ^- lies in some \mathcal{P}_{n-m} (specifically in $\mathcal{P}_{n-l(\mu)}$). Indeed $\mu \mapsto \mu^-$ defines an injective map

$$(5) \quad \mathcal{P}_n \hookrightarrow \bigcup_{m=0}^{n-1} \mathcal{P}_m = \bigcup_{m=1}^n \mathcal{P}_{n-m}$$

(See (4) for the examples in \mathcal{P}_3 .)

At the level of i -cells t of associahedron K_{n-1} then cell dimension

$$i = |\mathbf{s}(t)^-|.$$

Note that this verifies the Euler–Poincaré formula assertion immediately after (3).

(2.2) We define another function from tilings to partitions (cf. (1.3)). A tiling t gives an integer partition $\mathbf{f}(t)$ as follows. First consider the partition $\mathbf{f}^+(t)$ of the number of its triangles into the unordered list of sizes of maximal triangulated regions. Then

$f(t) = f^+(t)^-$. Again this is well-defined on classes.

Examples:

$$\begin{aligned}
f^+ \left(\text{Diagram 1} \right) &= f^+ \left(\text{Diagram 2} \right) = f^+(\triangle \nabla \triangle) = (1, 1, 1); & f(\triangle \nabla \triangle) &= (0, 0, 0) = \emptyset. \\
f^+ \left(\text{Diagram 3} \right) &= (3, 1, 1); & f \left(\text{Diagram 4} \right) &= (2, 0, 0) = (2)
\end{aligned}$$

A further decomposition of $A_n(\lambda)$ is into subsets characterised as the inverse images of f . Thus, for ν in \mathcal{P} :

$$A_{\lambda, \nu} = f^{-1}(\nu) \cap A_n(\lambda)$$

and we have $\mathbb{A}_{\lambda, \nu}$ similarly. We write $a_{\lambda, \nu}$ and $\mathfrak{a}_{\lambda, \nu}$ for the sizes of these sets.

(2.3) The utility of this decomposition for us lies in the fact that the ‘overcount factor’ in counting A_n instead of \mathbb{A}_n is uniform in $A_{\lambda, \nu}$ — since every tiling in this subset belongs to a class of size

$$(6) \quad c^\nu = \prod_i C_{\nu_i}$$

where C_n is the n -th Catalan number: $C_1 = 1$, $C_2 = 2$ and so on. That is

$$\mathfrak{a}_{\lambda, \nu} = \frac{a_{\lambda, \nu}}{c^\nu}$$

In order to explain the proof it will be helpful to write out the first few terms explicitly. To this end let

$$\lambda \wr \mu = \lambda \cup \mu^+ \cup 1^{-|\mu^+|}$$

To obtain shape $\lambda \wr \mu$ we start with λ , but move to a shape with fewer triangular tiles, replacing multiple triangles with larger tiles such that the overall weight is unchanged. Here we note that on the level of tilings, the operation $\cup(1^{-a})$ corresponds to removing a triangles.

Then:

$$\mathfrak{a}_n(\lambda) = a_n(\lambda) \Pi_{\emptyset}^\lambda - a_n(\lambda \cup (2) \cup (1^{-2})) \Pi_{(1)}^\lambda + a_n(\lambda \cup (3) \cup (1^{-3})) \Pi_{(2)}^\lambda - \dots$$

becomes

$$\mathfrak{a}_n(\lambda) = a_n(\lambda) - a_n(\lambda \wr (1)) \Pi_{(1)}^\lambda + a_n(\lambda \wr (2)) \Pi_{(2)}^\lambda + a_n(\lambda \wr (1^2)) \Pi_{(1^2)}^\lambda - a_n(\lambda \wr (3)) \Pi_{(3)}^\lambda + \dots$$

We interpret each of the terms as counting a certain subset of $A_n(\lambda)$. The claim is that the multiplicity of the intersection of this subset with $f^{-1}(\nu)$ is, for each ν , uniform in $\mathbb{A}_n(\lambda)$, in the sense of our discussion of $A_{\lambda, \nu}$ in (2.2) above.

(2.4) The set counted by $\Pi_\mu^\lambda a_n(\lambda \wr \mu)$ is the subset of $A_n(\lambda)$ where the triangulated regions include regions corresponding to triangulations of the tiles arising from μ — specifically the construction is to count tilings of $\lambda \wr \mu$ but with the tiles with sizes

given by μ^+ marked (hence the multiplicity Π_μ^λ). The marked regions can then simply be triangulated. In this way we have tilings of the right shape, but where there are triangulated regions of size at least given by μ^+ (they might also be bigger). Let us write B_μ^λ for this subset of $A_n(\lambda)$. (Note that tilings here have only one triangulation in each triangulated polygon from μ — and indeed we do not actually specify one.) That is

$$b_\mu^\lambda := |B_\mu^\lambda| = \Pi_\mu^\lambda a_n(\lambda \wr \mu)$$

2.2. Tilings organised by maximal triangulated regions. As discussed in (2.3), if we organise tilings by taking the fibre of f over some partition $\nu \in \mathcal{P}$, the overcount factor in counting A_n instead of \mathbb{A}_n is uniform. So we define the overcount factor

$$(7) \quad OF_{\mu,\nu} := \frac{|f^{-1}(\nu) \cap B_\mu^\lambda|}{|\mathbb{A}_{\lambda,\nu}|}$$

By definition, $OF_{\nu,\nu} = 1$. We will need the sizes of these factors, but the first thing to note is that they do not depend on λ , since this dependence is removed by the quotient. With this, fibre-wise the claim of Theorem 1.4 becomes

$$|\mathbb{A}_{\lambda,\nu}| \stackrel{\text{Thm. 1.4}}{=} \sum_{m=0}^{m_1(\lambda)-1} (-1)^m \sum_{\mu \vdash m} |f^{-1}(\nu) \cap B_\mu^\lambda|$$

Dividing by $|\mathbb{A}_{\lambda,\nu}|$ on both sides, this translates to

$$(8) \quad 1 \stackrel{\text{Thm. 1.4}}{=} \sum_{m=0}^{m_1(\lambda)-1} (-1)^m \sum_{\mu \vdash m} OF_{\mu,\nu}$$

So we will have proved Theorem 1.4 if we can show that every *signed* column sum as in (8) equals 1.

Let us start with a table illustrating the first few cases of the numbers $OF_{\mu,\nu}$ — see Figures 2,3. We unpack what this table is saying in a few cases before treating the general case.

2.3. Overcount factor columnwise: small ν .

First column: The μ -th entry in the first column is the overcount factor for the subset of B_μ^λ (any λ) with no two triangular tiles together. Zero means that in fact the intersection is empty. In the first column we are looking only at tilings in which the triangulation has no clusters of size bigger than 1. Since the equivalence classes are singletons in this case, the count is the same in $\mathbb{A}_n(\lambda)$:

$$OF_{0,0} = 1$$

All of the remaining combinatorics count sets in which higher tiles are triangulated, so there are larger clusters, and so the intersection in the first column is empty. This means that the theorem counts these classes correctly.

| $\mu \setminus \nu$ | \emptyset | (1) | (2) | (1 ²) | (3) | (21) | (1 ³) |
|---------------------|-----------------|-------------------|-------------------|---|---------------------|--|--|
| \emptyset | $a_3(1)$ = 1 | $a_4(1^2)$ = 2 | $a_5(1^3)$ = 5 | $a_4(1^2)^2$ = 2 ² | $a_6(1^4)$ = 14 | $a_5(1^3)a_4(1^2)$ = 5 · 2 | $a_4(1^2)^3$ = 2 ³ |
| (1) | 0 | $a_4(2)$ = 1 | $a_5(21)$ = 5 | $a_4(1^2)a_4(2)$ + $a_4(2)a_4(1^2)$ = 2 + 2 | $a_6(21^2)$ = 21 | $a_5(1^3)a_4(2)$ + $a_5(21)a_4(1^2)$ = 5 · 1 + 5 · 2 | $a_4(1^2)a_4(1^2)a_4(2)$ + $a_4(1^2)a_4(2)a_4(1^2)$ + $a_4(2)a_4(1^2)a_4(1^2)$ = 3(2 · 2 · 1) |
| (2) | 0 | 0 | $a_5(3)$ = 1 | 0 | $a_6(31)$ = 6 | $a_5(3)a_4(1^2)$ = 1 · 2 | 0 |
| (1 ²) | 0 | 0 | 0 | $a_4(2)^2$ = 1 | $a_6(2^2)$ = 3 | $a_5(21)a_4(2)$ = 5 · 1 | $a_4(1^2)a_4(2)a_4(2)$ + $a_4(2)a_4(2)a_4(1^2)$ + $a_4(2)a_4(1^2)a_4(2)$ = 3 · 2 |
| (3) | 0 | 0 | 0 | 0 | $a_6(4) = 1$ | 0 | 0 |
| (21) | 0 | 0 | 0 | 0 | 0 | $a_5(3)a_4(2) = 1$ | 0 |
| (1 ³) | 0 | 0 | 0 | 0 | 0 | 0 | $a_4(2)^3 = 1$ |
| (4) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

 FIGURE 2. Tabulating the $OF_{\mu,\nu}$ function.

Second column: In the second column, $\nu = (1)$, we are intersecting with tilings having a quadrilateral triangulated region and then any other triangles singletons. Such tilings are present in $A_n(\lambda)$ (the first row) and, since the equivalence classes are flip pairs in this case, the count is 2x that in $\mathcal{A}_n(\lambda)$. In the second row (the case of $B_{(1)}^\lambda$) we assemble tilings with a marked quadrilateral (which we consider to then be triangulated). However since we only count one tiling here, we count for $\mathcal{A}_n(\lambda)$ correctly. (In this set there are tilings where the triangulated quadrilateral is adjacent to other triangles, but these do not lie in the intersection in this column.) All of the remaining combinatorics count sets in which higher tiles are triangulated, so there are larger clusters, and so the intersection in the second column, for μ all other μ , is empty. This means that the theorem again counts these classes correctly overall.

Column $\nu = (2)$: Here we have tilings with a pentagonal triangulated region and any other triangles singletons. Thus the overcount in $A_n(\lambda)$ is 5x. In $B_{(1)}^\lambda$ such pentagons arise when the triangulated quadrilateral is adjacent to one other triangle. Indeed note that a given triangulated pentagonal subregion of the tiling could arise in 5 different ways in $B_{(1)}^\lambda$ — we are counting the number of ways of tiling a pentagon with a quadrilateral and a triangle, which number is $a_5(21)$. In $B_{(2)}^\lambda$ such pentagons arise when we triangulate a pentagon, with no other adjacent triangles. The construction only counts one such triangulation, so the count is the same as for

| $\mu \setminus \nu$ | (4) | (31) | (2 ²) | (21 ²) |
|---------------------|---------------------|---|--|---|
| 0 | $a_7(1^5)$ = 42 | $a_6(1^4)a_4(1^2)$ 14 · 2 | $a_5(1^3)a_5(1^3)$ | $a_5(1^3)a_4(1^2)a_4(1^2)$ 5 · 2 · 2 |
| (1) | $a_7(21^3)$ = 84 | $a_6(21^2)a_4(1^2) + a_6(1^4)a_4(2)$ 21 · 2 + 14 · 1 | $a_5(21)a_5(1^3)$ + $a_5(1^3)a_5(21)$ | $a_5(21)a_4(1^2)a_4(1^2)$ + $a_5(1^3)a_4(2)a_4(1^2)$ + $a_5(1^3)a_4(1^2)a_4(2)$ |
| (2) | $a_7(31^2)$ = 28 | $a_6(31)a_4(1^2)$ 6 · 2 | $a_5(3)a_5(1^3)$ + $a_5(1^3)a_5(3)$ | $a_5(3)a_4(1^2)a_4(1^2)$ |
| (1 ²) | $a_7(2^21)$ = 28 | $a_6(2^2)a_4(1^2) + a_6(21^2)a_4(2)$ 3 · 2 + 21 · 1 | $a_5(21)a_5(21)$ | $a_5(21)a_4(2)a_4(1^2)$ + $a_5(21)a_4(1^2)a_4(2)$ |
| (3) | $a_7(41)$ = 7 | $a_6(4)a_4(1^2)$ = 1 · 1 | 0 | 0 |
| (21) | $a_7(32)$ = 7 | $a_6(31)a_4(2)$ = 6 · 1 | $a_5(3)a_5(21)$ + $a_5(21)a_5(3)$ | $a_5(3)a_4(2)a_4(1^2)$ + $a_5(3)a_4(1^2)a_4(2)$ |
| (1 ³) | 0 | $a_6(2^2)a_4(2)$ = 3 · 1 | 0 | $a_5(21)a_4(2)a_4(2)$ |
| (4) | $a_7(5)$ = 1 | 0 | 0 | 0 |
| (31) | 0 | $a_6(4)a_4(2)$ = 1 | 0 | 0 |
| (2 ²) | 0 | 0 | 1 | 0 |
| (21 ²) | 0 | 0 | 0 | 1 |

FIGURE 3. Continuing the $OF_{\mu,\nu}$ table.

the class in $\mathbb{A}_n(\lambda)$. The remaining intersections in this column are empty. So again the count in the Theorem is correct.

Column $\nu = (1^2)$: Here we have tilings with two triangulated quadrilaterals and any other triangles singletons. The overcount factor in $A_n(\lambda)$ is 2^2 . In $B_{(1)}^\lambda$ tilings of the required clustering arise when the triangulated quadrilateral is not adjacent to any triangle, and there are two other triangles that are adjacent (possibly along with further singletons). There are two ways the quadrilateral could be assigned to one of the triangulated quadrilaterals; and there are two triangulations of the other triangulated quadrilateral, so we have a 2x2x overcount here. In $B_{(2)}^\lambda$ there is a triangulated pentagon, and hence no intersection. In $B_{(1^2)}^\lambda$ there are two marked quads. Each is given a triangulation, so the count is the same as for classes. There are no more non-empty intersections in this column. Again the theorem is verified.

Column $\nu = (3)$: Here we have tilings with one triangulated hexagon. Such tilings are overcounted in $A_n(\lambda)$ by the appropriate Catalan number, 14. In $B_{(1)}^\lambda$ we have

the triangulated hexagon formed by a quadrilateral and two triangles. The number of ways of doing this is $a_6(21^2)$.

2.4. Overcount factor columnwise: general ν .

Let $\mathcal{P}^* = \mathcal{P} \setminus \{\emptyset\}$. We write \mathcal{P}^+ for the image of \mathcal{P} under $\mu \mapsto \mu^+$. That is $\mathcal{P}^+ = \{(0), (2), (3), (2^2), (4), (32), (2^3), \dots\}$. Recall that $\mu \mapsto \mu^-$ acts as the inverse of $\mu \mapsto \mu^+$ on \mathcal{P}^{*+} .

(2.5) For $s \geq 1$ consider the set of multipartitions $\mathcal{P}^s = \{\gamma = (\gamma^1, \gamma^2, \dots, \gamma^s) \mid \gamma^i \in \mathcal{P}\}$. Define a map $\cup_s : \mathcal{P}^s \rightarrow \mathcal{P}$ by $\gamma \mapsto \cup_{i=1}^s \gamma^i$.

Since \cup_s is surjective, the fibres $\{\cup_s^{-1}(\mu) \mid \mu \in \mathcal{P}\}$ are a partition of \mathcal{P}^s .

Define $\overset{s}{\vdash}\mu = \cup_s^{-1}(\mu^+)$. For $\nu \in \mathcal{P}^*$ we write $\overset{\nu}{\vdash}\mu$ for $\overset{l(\nu)}{\vdash}\mu$. Thus

$$(9) \quad \bigsqcup_{\mu \in \mathcal{P}} \overset{\nu}{\vdash}\mu = (\mathcal{P}^+)^{l(\nu)}$$

is a partition of $(\mathcal{P}^{l(\nu)})^+ = (\mathcal{P}^+)^{l(\nu)}$.

We write $\gamma \overset{s}{\vdash}\mu$ for $\gamma \in \overset{s}{\vdash}\mu$.

Examples: $\overset{2}{\vdash}(21) = \{((32), \emptyset), ((3), (2)), ((2), (3)), (\emptyset, (32))\}$. And for $\nu = (21^2)$ the partitioning of $(\mathcal{P}^{l(\nu)})^+$ starts with

$$\overset{3}{\vdash}\emptyset = \{(0, 0, 0)\}; \quad \overset{3}{\vdash}(1) = \{((2), 0, 0), (0, (2), 0), (0, 0, (2))\};$$

$$\overset{3}{\vdash}(2) = \{((3), 0, 0), (0, (3), 0), (0, 0, (3))\};$$

$$\overset{3}{\vdash}(1^2) = \{((2^2), 0, 0), (0, (2^2), 0), (0, 0, (2^2)), ((2), (2), 0), ((2), 0, (2)), (0, (2), (2))\}; \text{ etc.}$$

Definition 2.6. For $m \in \mathbb{N}$ and $\gamma \in \mathcal{P}$ define $\bar{\gamma}^m \in \mathcal{P}_{\geq m} = \cup_{r=m}^{\infty} \mathcal{P}_r$ by

$$\bar{\gamma}^m = \begin{cases} \gamma \cup 1^{m-|\gamma|} & \text{if } |\gamma| \leq m \\ \gamma & \text{if } |\gamma| > m \end{cases}.$$

We understand $a_n(\bar{\gamma}) = a_n(\bar{\gamma}^{n-2})$ (completing the partial shape γ with triangles).

Lemma 2.7. For arbitrary $\mu \in \mathcal{P}$, $\nu \in \mathcal{P}^*$, the overcount function $OF_{\mu, \nu}$ obeys

$$(10) \quad OF_{\mu, \nu} = \sum_{\substack{l(\nu) \\ \gamma \overset{\nu}{\vdash} \mu}} \prod_{i=1}^{l(\nu)} a_{\nu_i+3}(\bar{\gamma}^i)$$

Example 2.8. For a full example we consider $\mu = (21)$, $\nu = (31)$ (so $l(\nu) = 2$):

$$OF_{(21), (31)} = a_6(\bar{32})a_4(\bar{\emptyset}) + a_6(\bar{3})a_4(\bar{2}) + a_6(\bar{2})a_4(\bar{3}) + a_6(\bar{\emptyset})a_4(\bar{32}) = a_6(31)a_4(2)$$

1. $a_6(\bar{32}) = 0$ as the partition (32) has size $5 \geq 6 - 1$, there is no way to tile a hexagon with a pentagon and a quadrilateral, while $a_4(\bar{\emptyset}) = a_4(1^2)$.
2. $a_6(\bar{3}) = a_6(31)$ — we complete the tiling with a triangle.

3. $a_4(\overline{3}) = 0$ similar to 1., while $a_6(\overline{2}) = a_6(21^2)$.
 4. $a_6(\overline{\emptyset}) = a_6(1^4)$, again completing with triangles; while $a_4(\overline{32}) = 0$ similar to 1.

Proof of Lemma 2.7. Observe that in each product $\prod_{i=1}^{l(\nu)} a_{\nu_i+3}(\overline{\gamma^i})$ each factor addresses one maximal triangulated region (for these regions their sizes, as polygons, are the $\nu_i + 3$). We have to take the maximal subregions prescribed by ν (note that we can, and this does, ignore those that are already simple triangles) and populate them according to the polygons available from μ . The contribution to the overcount factor is $a_{\nu_i+3}(\overline{\gamma^i})$ since this counts the number of ways of tiling the subregion as indicated by γ^i (after this, in our construction, recall from (2.4), we simply triangulate to any one triangulation). Note that the sum over $\gamma \models \mu$ constructively exhausts all possibilities. Indeed it does not check against internal details of ν , so also formally includes some impossibilities, in general — the cases where $|\gamma^i| > \nu_i + 1$ for some i and the factor $a_{\nu_i+3}(\overline{\gamma^i}) = a_{\nu_i+3}(\overline{\gamma^i})$ is zero. (Since the extra terms are zero this does not affect the identity.) \square

Our next step in proving (8) is Lemma 2.13. We will need a couple of preliminaries.

(2.9) Note that for any m and $\gamma \in \mathcal{P}$

$$(11) \quad \gamma \mapsto \overline{\gamma^{+m}} \mapsto (\overline{\gamma^{+m}})^- = \gamma$$

Meanwhile $\gamma \mapsto \gamma^-$ is defined and injective on \mathcal{P}_r for $r > 0$ by (5), and then

$$(12) \quad \gamma \mapsto \gamma^- \mapsto (\overline{\gamma^-})^+ = \gamma$$

Lemma 2.10. *For $m \in \mathbb{N}$ the map $\gamma \mapsto \overline{\gamma^{+m}}$ is an injection $\mathcal{P} \rightarrow \mathcal{P}_{\geq m}$; and the image includes \mathcal{P}_m .*

Proof. First note that $\overline{\gamma^{+m}} \in \mathcal{P}_{\geq m}$ by construction. Then note that injectivity follows from (11). Finally the image property follows from (12). \square

(2.11) For $\nu \in \mathcal{P}$ define $\mathcal{P}_\nu = \times_i \mathcal{P}_{\nu_i}$; and $\mathcal{P}_{\geq \nu} = \{\delta \in \mathcal{P}^{l(\nu)} \mid |\delta^i| \geq \nu_i \forall i\}$.

Given $\nu \in \mathcal{P}$ and a multipartition $\gamma \in \mathcal{P}^{l(\nu)}$ we define $\overline{\gamma}^\nu = (\overline{\gamma^1}^{\nu_1}, \overline{\gamma^2}^{\nu_2}, \dots)$. Note that $\overline{\gamma}^\nu \in \mathcal{P}_{\geq \nu}$.

Lemma 2.12. *Fix $\nu \in \mathcal{P}^*$. The map $\gamma \mapsto \overline{\gamma}^{\nu^+}$ is an injection $\bigsqcup_{\mu \in \mathcal{P}} \overset{\nu}{\models} \mu \rightarrow \mathcal{P}_{\geq \nu^+}$; and the image includes \mathcal{P}_{ν^+} .*

Proof. We write the maps from 2.10 as $\gamma \mapsto \gamma^+ \mapsto \overline{\gamma^{+m}}$ taking $\mathcal{P} \rightarrow \mathcal{P}^+ \xrightarrow{\alpha_m} \mathcal{P}_{\geq m}$. By (9) we have $\bigsqcup_{\mu \in \mathcal{P}} \overset{\nu}{\models} \mu = (\mathcal{P}^+)^{l(\nu)}$. Thus the map here is the Cartesian product $\times_{i=1}^{l(\nu)} \alpha_{\nu_i^+}$. Thus it is injective. Since the image of a Cartesian product is the product of images, the image bound of Lemma 2.10 also implies the image bound here. \square

Lemma 2.13. *For every $\nu \in \mathcal{P}$ we have*

$$\sum_{\mu \in \mathcal{P}} (-1)^{|\mu|} OF_{\mu, \nu} = \prod_i F_{\nu_i}$$

where

$$(13) \quad F_r := \sum_{m=0}^r (-1)^m \sum_{\rho \vdash m} a_{r+3}(\overline{\rho^+})$$

Proof. Note that $a_{r+3}(\rho) = 0$ when $|\rho| > r + 1$. Applying Lemma 2.10 to (13) then gives

$$F_r = \sum_{\delta \in \mathcal{P}_{r+1}} (-1)^{|\delta^-|} a_{r+3}(\delta).$$

Expanding $\prod_{i=1}^s F_{\nu_i} = (\sum_{\delta^1 \in \mathcal{P}_{\nu_1+1}} (-1)^{|\delta^1|-1} a_{\nu_1+3}(\delta^1)) \cdots (\sum_{\delta^s \in \mathcal{P}_{\nu_s+1}} (-1)^{|\delta^s|-1} a_{\nu_s+3}(\delta^s))$ we obtain

$$(14) \quad \prod_{i=1}^s F_{\nu_i} = \sum_{(\delta^1, \dots, \delta^s) \in \mathcal{P}_{\nu_1+1} \times \cdots \times \mathcal{P}_{\nu_s+1}} (-1)^{\sum_i |\delta^{i-}|} a_{\nu_1+3}(\delta^1) \cdots a_{\nu_s+3}(\delta^s)$$

where $s = l(\nu)$. But, noting the vanishing condition again, applying Lemma 2.12 to $\sum_{\mu} \pm OF_{\mu, \nu}$ yields the same expression up to signs. So it remains to check the signs.

Let $b := (-1)^{\sum_i |\delta^{i-}|} a_{\nu_1+3}(\delta^1) \cdots a_{\nu_s+3}(\delta^s)$ be a summand of $\prod_i F_{\nu_i}$. Note that b is a non-zero term of $OF_{\mu, \nu}$ for $\mu = (\bigcup_i \delta^i)^-$ (the partition with $m_j(\mu) = m_{j+1}(\bigcup_i \delta^i)$ for $j \geq 1$), but with sign $(-1)^{|\mu|}$. Now $|\mu| = \sum_{j \geq 1} jm_j(\mu) = \sum_{j \geq 1} jm_{j+1}(\bigcup_i \delta^i)$. Let us compare this with $(-1)^{\sum_i |\delta^{i-}|}$. We have

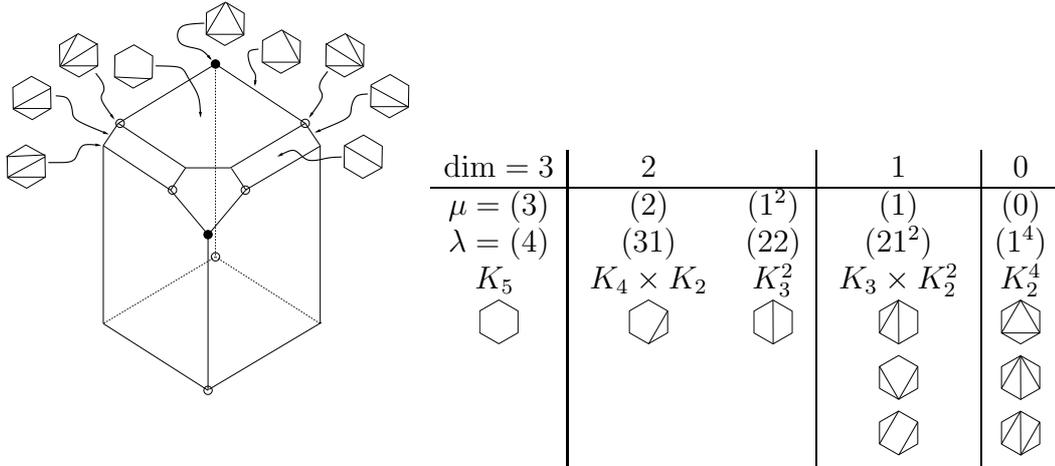
$$|\delta^{i-}| = \sum_{j \geq 1} jm_j(\delta^{i-}) = \sum_{j \geq 1} jm_{j+1}(\delta^i)$$

so the signs match up. \square

Lemma 2.14. *For each r , $F_r = 1$.*

Proof. Observe from (3) that F_r is the Euler–Poincaré characteristic. Now use Theorem 1.1 and Theorem 1.5. \square

Proof of Theorem 1.4. Combining Lemma 2.14 and Lemma 2.13 gives (8) as required. \square

FIGURE 4. Facets and shapes in K_5 .

3. ON COMBINATORIAL ISOMETRIES AND OTHER DISCUSSION POINTS

This section takes the form of some extended remarks, on open questions and possible connections.

Here we have drawn attention to the classification of cells of the associahedron by mechanisms such as ‘shapes’ of tilings as defined in (1.3). One question thus raised is how (literal) shapes of cells in the associahedron are related to shapes of tilings.

Consider the associahedron as a combinatorial complex (see for example [2]). We call a self-map of the (combinatorial) complex a combinatorial isometry. This should be distinguished from an isometry of one of the concrete polytopal realisations, since these depend on the realisation. A typical realisation has *no* true isometries. On the other hand at least some non-trivial self-maps of the complex exist. That is, the ones induced by rotating the polygon P . We illustrate by considering K_5 and K_6 .

3.1. Associahedron K_5 . Consider Figure 4 (interpolating between the polytopal and tiling complex realisations of K_5 , by an extension of Brown’s figure in [8]). Here we see that there are various sub-classifications of i -cells of K_n available. Firstly there is the evident and well-known intrinsic classification by products of smaller associahedra. This essentially coincides with our tiling shape classification via $\lambda \rightsquigarrow \times_i K_{\lambda_i^+}$. This corresponds to ignoring all triangles in tilings, since K_2 is trivial. What about extrinsic properties?

Firstly consider the case of 0-cells. In terms of their intrinsic physical shape these are obviously all the same — points. (Compare this with their equivalence under flip in the tiling realisation.) From another perspective, 0-cells are ‘tristate’ points (points belonging to three cells). They can be sorted into black circles (‘tristate’ points for 3 5-gonal faces); white circles (tristate points for two 5-gonal and one 4-gonal face that are adjacent to 5^3 tristate points); the rest (tristate points of 5^{24} type that are

not adjacent to 5^3 tristate points). It will be clear from the figure that these classes coincide (in this case) with the classes induced under rotation of P .

What about 1-cells? Again these all have the same shape. (They might have different lengths in a concrete realisation, but this is not canonical.) We leave it as an exercise to consider extrinsic properties.

In preparation for looking briefly at K_6 (where our 4d sketching skills fail) we include a tabulation of cells in Figure 4 in a format that does lift to K_6 . The table is organised by dimension of cell. The μ label runs through the integer partitions of the dimension and $\lambda = \overline{\mu^-}$ (let us generally exclude those μ s that give an impossible λ). The final component gives representative tilings, up to polygon isometries.

3.2. Associahedron K_6 . Here drawing a picture is hard. But we have the tabulated form as follows.

| dim = 4 | 3 | | | 2 | | 1 | 0 |
|--|--|--|-------------------|---|--|--|--|
| $\mu = (4)$ | (3) | (21) | (1 ³) | (2) | (1 ²) | (1) | (0) |
| $\lambda = (5)$ | (41) | (32) | | (31 ²) | (221) | (21 ³) | (1 ⁵) |
| K_6 | $K_5 \times K_2$, | $K_4 \times K_3$, | K_3^3 | $K_4 \times K_2^2$, | $K_3^2 \times K_2$ | $K_3 \times K_2^3$ | K_2^5 |
|  |  |  | – |  |  |  |  |
| | | | |  |  | ⋮ | ⋮ |
| | | | |  |  | | |

NB although we include the column here, $\mu = (1^3)$ is not possible in this rank, since $|\lambda| = |(2^3)| = 6 > n - 1$. On the other hand some μ labels again correspond to multiple isometry classes in the tiling realisation, as indicated.

3.3. Brief remarks on connections with other areas. General connections of associahedra to several areas are already mentioned in §1. More specifically here, there are a number of areas of representation theory where the map $\mu \mapsto \mu^-$ plays an interesting role. See for example the partition algebra [19, 16] (and hence geometric complexity [20]); and Gamba’s formula [6, CH. VI §4]. For connections to moduli spaces see e.g. [10, 8]. For cyclic sieving see e.g. [21]. For Baur et al.’s tiling/Temperley–Lieb correspondence see [1]. Finally here we mention that there are potential connections to quantum codes via 2d surface tilings (for a review see e.g. [7]).

Acknowledgements. KB thanks Max Glick and Lukas Andritsch for discussions. KB acknowledges support from FWF grants P30549-N26 and DK W1230. PM thanks Robert Marsh and Joao Faria Martins for discussions; and acknowledges support from EPSRC grant EP/I038683/1 “Algebraic, Geometric and Physical underpinnings of Topological Quantum Computation”.

REFERENCES

1. Oswin Aichholzer, Lukas Andritsch, Karin Baur, and Birgit Vogtenhuber, *Perfect k -colored matchings and $k+2$ -gonal tilings*, arxiv (2017), arXiv:1710.06757.
2. Tathagata Basak, *Combinatorial cell complexes and Poincaré duality*, *Geom. Dedicata* **147** (2010), 357–387. MR 2660584
3. Karin Baur, Alastair D. King, and Robert J. Marsh, *Dimer models and cluster categories of Grassmannians*, *Proc. Lond. Math. Soc. (3)* **113** (2016), no. 2, 213–260. MR 3534972
4. Karin Baur and Paul P. Martin, *The fibres of the Scott map on polygon tilings are the flip equivalence classes*, *Monatsh. Math.* **187** (2018), no. 3, 385–424, (with an appendix joint with Max Glick). MR 3858423
5. Karin Baur, Ralf Schiffler, and Jerzy Weyman, *private correspondence*.
6. Hermann Boerner, *Representations of groups*, North Holland, 1970.
7. Nikolas P. Breuckmann and Barbara M. Terhal, *Constructions and noise threshold of hyperbolic surface codes*, *IEEE Transactions on Information Theory* **62** (2016), 3731–3744, DOI: 10.1109/TIT.2016.2555700; arXiv: <https://arxiv.org/abs/1506.04029>.
8. Francis C. S. Brown, *Multiple zeta values and periods of moduli spaces $\overline{\mathcal{M}}_{0,n}$* , *Ann. Sci. Éc. Norm. Supér. (4)* **42** (2009), no. 3, 371–489. MR 2543329
9. Frédéric Chapoton, *Enumerative properties of generalized associahedra*, *Sém. Lothar. Combin.* **51** (2004/05), Art. B51b, 16. MR 2080386
10. Satyan L. Devadoss and Joseph O’Rourke, *Discrete and computational geometry*, Princeton University Press, Princeton, NJ, 2011. MR 2790764
11. Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, *J. Amer. Math. Soc.* **15** (2002), no. 2, 497–529. MR 1887642
12. ———, *Cluster algebras. II. Finite type classification*, *Invent. Math.* **154** (2003), no. 1, 63–121. MR 2004457
13. ———, *Y -systems and generalized associahedra*, *Ann. of Math. (2)* **158** (2003), no. 3, 977–1018. MR 2031858
14. Israel M. Gelfand, Michail M. Kapranov, and Andrei V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition. MR 2394437
15. Allen Hatcher, *On triangulations of surfaces*, *Topology Appl.* **40** (1991), no. 2, 189–194. MR 1123262
16. Vaughan F. R. Jones, *The Potts model and the symmetric group*, in *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras*, World Scientific, Singapore, 1994.
17. Carl W. Lee, *The associahedron and triangulations of the n -gon*, *European J. Combin.* **10** (1989), no. 6, 551–560. MR 1022776
18. Ian G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015. MR 3443860
19. Paul P. Martin and Hubert Saleur, *Algebras in higher dimensional statistical mechanics — the exceptional partition algebras*, *Lett. Math. Phys.* **30** (1994), 179–185, (hep-th/9302095).
20. Ketan D. Mulmuley and Milind Sohoni, *Geometric complexity theory I: An approach to the P vs. NP and related problems*, *SIAM J. Comput.* **31(2)** (2006), 496526.
21. T.K. Petersen, P. Pylyavskyy, and B. Rhoades, *Promotion and cyclic sieving via webs*, *J. Algebraic Combin.* **30** (2009), no. 1, 19–41.
22. Alexander Postnikov, *Permutohedra, associahedra, and beyond*, *Int. Math. Res. Not. IMRN* (2009), no. 6, 1026–1106. MR 2487491

23. Alexander Postnikov, *Total positivity, Grassmannians, and networks*, preprint (arXiv:math/0609764).
24. Jozef H. Przytycki and Adam S. Sikora, *Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers*, J. Combin. Theory Ser. A **92** (2000), no. 1, 68–76, arXiv:math/9811086v1.
25. Jeanne Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. **92** (2006), no. 2, 345–380.
26. William Shakespeare, *Hamlet*, 1602.
27. N. J. A. Sloane, *The on-line encyclopedia of integer sequences*.
28. James Dillon Stasheff, *Homotopy associativity of H -spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275–292; *ibid.* **108** (1963), 293–312. MR 0158400
29. Andy Tonks, *Relating the associahedron and the permutohedron*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math., vol. 202, Amer. Math. Soc., Providence, RI, 1997, pp. 33–36. MR 1436915
30. Günter M. Ziegler and Christian Blatter, *Euler’s polyhedron formula—a starting point of today’s polytope theory*, Elem. Math. **62** (2007), no. 4, 184–192. MR 2357757

DEPARTMENT OF MATHEMATICS AND SCIENTIFIC COMPUTING, UNIVERSITY OF GRAZ, NAWI GRAZ, 8010 GRAZ, AUSTRIA

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UK