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Contracting the Wigner kernel of a spin to the Wigner kernel of a particle

Jean-Pierre Amiet and Stefan Weigert*

Institut de Physique, Université de Neuchâtel, Rue A.-L. Breguet 1, CH-2000 Neuchâtel, Switzerland

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A general relation between the Moyal formalisms for a spin and a particle is established. Once the formalism has been set up for a spin, the phase-space description of a particle is obtained from contracting the group of rotations to the oscillator group. In this process, turn into a spin Wigner kernel turns into the Wigner kernel of a particle. In fact, only *one* out of 2^{2s} different possible kernels for a spin shows this behavior.

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I. INTRODUCTION

To represent quantum mechanics in terms of c-number valued functions has various appealing properties. It becomes possible to situate the quantum-mechanical description of a system in a familiar frame, namely the phase space of its classical analog. Similarities and differences of the two descriptions can be visualized particularly well in such an approach. Further, from a structural point of view, to calculate expectation values of operators by means of ''quasiprobabilities'' in phase space is strongly analogous to the determination of mean values in classical statistical mechanics [1]. The basic ingredient to set up such a *symbolic calculus* is a one-to-one correspondence between (self-adjoint) operators $\hat{A}$ acting on a Hilbert space $\mathcal{H}$ and (real) functions W_A defined on the phase space Γ of the classical system.

The quantum mechanics of spin and particle systems can be represented faithfully in terms of functions defined on the surface of a sphere with radius s, and on a plane, respectively. Intuitively, one expects these phase-space formulations to approach each other for increasing values of the spin quantum number since the surface of a sphere is then approximated by a plane with increasing accuracy. Therefore, appropriate Wigner functions of a spin, say, should go over smoothly into particle Wigner functions in the limit of large s. It will be shown how this transition can be performed in a rigorous and general way. The derivation is based on the group-theoretical technique of *contraction*. The group U(2) [containing the subgroup SU(2)] is contracted to the oscillator group having the Heisenberg-Weyl group HW_1 associated with the particle, as a subgroup. In this procedure, rotations go over into translations. Subsequently, the operator kernel which defines the spin Wigner formalism in a condensed manner will be shown, in the limit of infinite s, to contract to the operator kernel for a particle.

II. WIGNER KERNEL FOR A PARTICLE

Consider a particle on the real line $\mathbb{R}^1$, with position and momentum operators satisfying $[\hat{q},\hat{p}]=i\hbar$. The Stratonovich-Weyl correspondence, associating operators with functions in phase space, can be characterized elegantly by means of a *kernel* [2,3],

$$\hat{\Delta}(\alpha)=2\hat{T}(\alpha)\hat{\Pi}\hat{T}^{\dagger}(\alpha), \quad \alpha=\frac{1}{\sqrt{2}}(q+ip)\in\Gamma\equiv\mathbb{C}, \tag{1}$$

which has an interpretation as a *parity operator* displaced by α. The unitary [4]

$$\hat{T}(\alpha)=\exp[\alpha a^{+}-\alpha^{*}a] \tag{2}$$

effects translations in phase space Γ,

$$a\rightarrow\hat{T}(\alpha)a\hat{T}^{\dagger}(\alpha)=a-\alpha, \tag{3}$$

where $a^{-}\equiv a=(\hat{q}-i\hat{p})/\sqrt{2}$ and $a^{+}=a^{\dagger}$ are the standard annihilation and creation operators ($\hbar=1$). At the origin $\alpha=0$, the kernel equals (two times) the unitary, involutive parity operator $\hat{\Pi}$,

$$\hat{\Pi}a\hat{\Pi}^{\dagger}=-a, \tag{4}$$

corresponding to a reflection at the origin of Γ. Using the number operator $\hat{N}=a^{+}a$ and its eigenstates,

$$\hat{N}|n\rangle=n|n\rangle, \quad n=0,1,2,\ldots, \tag{5}$$

parity can be given a simple form which will be useful later,

$$\hat{\Pi}=\exp[i\pi\hat{N}]=\sum_{n=0}^{\infty}(-)^{n}|n\rangle\langle n|. \tag{6}$$

The kernel $\hat{\Delta}(\alpha)$ can be derived from the *Stratonovich-Weyl* postulates [6] which are natural conditions on a quantum-mechanical phase-space representation. The correspondence between a (self-adjoint) operator $\hat{A}$ and a (real) function is defined by

$$W_A(\alpha)=\mathrm{Tr}[\hat{\Delta}(\alpha)\hat{A}], \tag{7}$$

while its inverse reads

*Email address: stefan.weigert@unine.ch

$$\hat{A}=\int_{\Gamma} d\alpha W_A(\alpha)\hat{\Delta}(\alpha). \tag{8}$$

If $\hat{A}$ is the density operator of a pure state, $\hat{\rho}=|\psi\rangle\langle\psi|$, the symbol defined in Eq. (7) is the *Wigner function* of the state $|\psi\rangle$,

$$W_\psi(p,q)=\frac{2}{h}\int_{\Gamma} dx\,\psi^*(q+x)\psi(q-x)\exp[2ipx/\hbar]. \tag{9}$$

It is important to note that the kernel $\hat{\Delta}(\alpha)$ is entirely defined in terms of the operators $a^{\pm}$ and $\hat{N}$, forming a closed algebra under commutation if the identity is included

$$[a,a^+]=1, \quad [\hat{N},a^{\pm}]=\pm a^{\pm}. \tag{10}$$

The operators $a^{\pm}$ and the unity 1 generate the *Heisenberg-Weyl group* HW_1. The kernel $\hat{\Delta}(\alpha)$ in Eq. (1) is—apart from the factor of two— an element of the *oscillator group* [5] spanned by the generators of HW_1 *plus* the operator $\hat{N}$.

III. WIGNER KERNEL FOR A SPIN

For a quantum spin, the symbol associated with an operator is a continuous function defined on the *sphere* $\mathcal{S}^2$, being the phase space of the classical spin. When setting up a phase-space formalism, rotations take over the role of translations. The group SU(2) is generated by the components of the spin operator $\hat{\mathbf{S}}$. The three operators $\hat{S}^{\pm}=(\hat{S}^x\pm i\hat{S}^y)$ and $\hat{S}^z$, satisfy the commutation relations

$$[\hat{S}^+,\hat{S}^-]=2\hat{S}^z, \quad [\hat{S}^z,\hat{S}^{\pm}]=\pm\hat{S}^{\pm}, \tag{11}$$

while the algebra $u(2)$ contains the identity 1_s, in addition. The standard basis

$$\mathbf{n}_z\cdot\hat{\mathbf{S}}|s,m\rangle=m|s,m\rangle, \quad m=-s,\ldots,s, \tag{12}$$

is given by the eigenstates of the z component $\hat{S}^z$ of the spin.

For a quantum spin, it is natural to expect that the elements of the Wigner kernel will be labeled by points of the sphere $\mathcal{S}^2$, corresponding to unit vectors $\mathbf{n}=(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$, parametrized here by standard spherical coordinates. Replacing intuitively translations in Eq. (1) by rotations leads to the expression

$$\hat{\Delta}(\mathbf{n})=\hat{U}(\mathbf{n})\hat{\Pi}_s\hat{U}^{\dagger}(\mathbf{n}), \tag{13}$$

where

$$\hat{U}(\mathbf{n})=\exp[-i\vartheta\mathbf{k}\cdot\hat{\mathbf{S}}] \tag{14}$$

with a unit vector $\mathbf{k}=(-\sin\varphi,\cos\varphi,0)$ in the xy plane. Thus, $\hat{U}(\mathbf{n})$ represents a finite rotation which maps the operator $\hat{S}^z=\mathbf{n}_z\cdot\hat{\mathbf{S}}$ into $\mathbf{n}\cdot\hat{\mathbf{S}}$, i.e., $\mathbf{n}_z\to\mathbf{n}$. What are natural choices for the operator $\hat{\Pi}_s$?

Two possibilities come to one's mind. First, try to transfer the concept of reflection about some point in phase space. Introduce canonical coordinates $(q,p)=(\varphi,\cos\vartheta)$ on the sphere. Then, ''parity'' would correspond to the map $(\varphi,\cos\vartheta)\to(-\varphi,-\cos\vartheta)$, or $(\varphi,\vartheta)\to(2\pi-\varphi,\pi-\vartheta)$. This is just a rotation by π about the x axis. Since all points of the sphere are equivalent, one could also choose a rotation by π about the z axis as candidate for parity. Second, $\hat{\Pi}_s$ might be considered to generate reflections about the center of the sphere, $\mathbf{n}\to-\mathbf{n}$, that is, $(\varphi,\vartheta)\to(\varphi+\pi,\pi-\vartheta)$. It can be shown that *both* possibilities do *not* give rise to a symbolic calculus on the sphere [6], violating bijectivity between operators and phase-space functions, for example.

Nevetheless, acceptable operator kernels $\hat{\Delta}_\varepsilon(\mathbf{n})$ *do* exist as shown by Stratonovich [7], Agarwal [8], Várilly and Gracia-Bondía [9], and by Amiet and Cibils [10]. For example, the condition that the kernel should satisfy appropriate Stratonovich-Weyl postulates implies [9] that

$$\hat{\Delta}_\varepsilon(\mathbf{n})=\sum_{m,m'=-s}^{s} Z^{\varepsilon}_{mm'}(\mathbf{n})|s,m\rangle\langle s,m'|. \tag{15}$$

The coefficients

$$Z^{\varepsilon}_{mm'}(\mathbf{n})=\frac{\sqrt{4\pi}}{2s+1}\sum_{l=0}^{2s}\varepsilon_l\sqrt{2l+1}\left\langle\begin{matrix} s & l \\ m & m'-m\end{matrix}\middle|\begin{matrix} s\\ m'\end{matrix}\right\rangle \times Y_{l,m'-m}(\mathbf{n}), \tag{16}$$

where $\varepsilon_0=1$ and $\varepsilon_l=\pm1, l=1,\ldots,2s$, are linear combinations of Clebsch-Gordan coefficients multiplied by spherical harmonics $Y_{l,m}(\mathbf{n}), l=0,1,\ldots,2s$, $m=-l,\ldots,l$. Note that Eq. (16) does not provide a unique kernel but, due to the factors ε_l, one can define 2^{2s} different Stratonovich-Weyl correspondence rules.

Unfortunatley, the expression (15) does not admit a simple interpretation of the operator in analogy to Eq. (1). It follows from an independent derivation [11] of $\hat{\Delta}(\mathbf{n})$ that Eq. (15) can be written in the form (13) where

$$\hat{\Pi}_s=\hat{\Delta}_\varepsilon(\mathbf{n}_z)=\sum_{m=-s}^{s}\Delta_\varepsilon(m)|s,m\rangle\langle s,m|, \tag{17}$$

with coefficients

$$\Delta_\varepsilon(m)=\sum_{l=0}^{2s}\varepsilon_l\frac{2l+1}{2s+1}\left\langle\begin{matrix} s & l\\ m & 0\end{matrix}\middle|\begin{matrix} s\\ m\end{matrix}\right\rangle. \tag{18}$$

Still, the operator $\hat{\Pi}_s$ does not have an obvious interpretation but a new strategy to render plausible its form emerges. Consider a plane tangent to the sphere at its north pole. For increasing radius, the sphere is approximated locally better and better by the plane. Therefore, one might expect that for $s\to\infty$ objects defined on the sphere turn into objects defined on the plane. It has been conjectured in [11] that in this limit the Wigner kernel of a spin goes over into the kernel for a particle. It is the purpose of this paper to show that

$$\lim_{s\to\infty}\hat{U}(\mathbf{n})\hat{\Delta}(\mathbf{n}_z)\hat{U}^{\dagger}(\mathbf{n})=\hat{\Delta}(\alpha) \tag{19}$$

is indeed true for the kernel $\hat{\Delta}_\varepsilon(\mathbf{n}_z)$ with parameters $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{2s} = 1$, denoted by $\hat{\Delta}(\mathbf{n}_z)$ for short. Thus, while the rotations $\hat{U}(\mathbf{n})$ go over into translations, the operator $\hat{\Delta}(\mathbf{n}_z)$ corresponds, in one way or another, to parity for a spin. A convenient framework to prove Eq. (19) is the *contraction* of groups [5] as is explained in the next section.

IV. CONTRACTING $U(2)$

We introduce three new operators $\hat{A}^\pm$ and $\hat{A}^z$ defined as linear combinations of the elements of the algebra $u(2)$ in polar form,

$$\hat{A}^\pm = c\hat{S}^\mp, \quad \hat{A}^z = -\hat{S}^z + \frac{1_s}{2c^2}, \tag{20}$$

while leaving the identity 1_s unchanged. This transformation is invertible for each value of the parameter $c>0$. The nonzero commutators of these operators are given by

$$[\hat{A}^-,\hat{A}^+] = 1_s - 2c^2\hat{A}^z, \quad [\hat{A}^z,\hat{A}^\pm] = \pm\hat{A}^\pm, \tag{21}$$

and the identity 1_s commutes with $\hat{A}^\pm$ and $\hat{A}^z$. These relations have a well-defined limit if $c\to 0$, nonwithstanding that the transformation (20) is not invertible for $c=0$. In fact, they reproduce the commutation relations of the algebra in (10) after identifying

$$\lim_{c\to 0}\hat{A}^\pm = a^\pm, \quad \lim_{c\to 0}\hat{A}^z = \hat{N}, \quad \lim_{c\to 0} 1_s = 1. \tag{22}$$

How do rotations behave in this limit? Any finite rotation $\hat{U}(\mathbf{n}) \in SU(2)$ in Eq. (14) can be written in the form

$$\hat{U}(\mathbf{n}) = \exp[\xi_-\hat{S}^- - \xi_+\hat{S}^+], \quad \xi_- = \frac{\vartheta}{2}e^{i\varphi}, \quad \xi_+ = \xi_-^*, \tag{23}$$

or, expressed in terms of the operators (20),

$$\hat{U}(\mathbf{n}) = \exp[(\xi_-\hat{A}^+ - \xi_+\hat{A}^-)/c]. \tag{24}$$

Let the coefficients $\xi_\pm$ shrink with the parameter c according to

$$\lim_{c\to 0}\frac{\xi_-}{c} = \lim_{c\to 0}\frac{\vartheta e^{i\varphi}}{2c} = \alpha, \quad \lim_{c\to 0}\frac{\xi_+}{c} = \lim_{c\to 0}\frac{\vartheta e^{-i\varphi}}{2c} = \alpha^*, \tag{25}$$

which requires ever smaller rotations (ϑ decreasing linearly with c) on an ever larger sphere (s going to infinity) as detailed in [5]. Then, a rotation $\hat{U}(\mathbf{n})$ tends to a well-defined element of the Heisenberg-Weyl group, Eq. (2):

$$\lim_{c\to 0}\hat{U}(\mathbf{n}) = \hat{T}(\alpha). \tag{26}$$

For consistency, the limit $c\to 0$ must correctly reproduce the eigenvalues of the operator $\hat{N}$, given by the non-negative integers. This is achieved if the limits $c\to 0$ and $s\to\infty$ are performed simultaneously in such a way that

$$\lim_{c\to 0}[1 - 2c^2 s(c)] = 0. \tag{27}$$

Consequently, the radius of the sphere s increases with decreasing values of c. Let us now look at the fate of the eigenvalue equation (12) which implies

$$\lim_{c\to 0}\left[\left(-\hat{S}^z + \frac{1_s}{2c^2}\right)\left|s,s-n\right\rangle\right] = \lim_{c\to 0}\left[(s-m) + \left(\frac{1}{2c^2} - s\right)\right]\lim_{c\to 0}|s,s-n\rangle. \tag{28}$$

This gives

$$\hat{N}|n\rangle = n|n\rangle, \tag{29}$$

using Eq. (22) and a relation from [5],

$$\lim_{c\to 0}|s,m\rangle = \lim_{c\to 0}|s,s-n\rangle = |n\rangle, \quad n = s-m \in \mathbb{N}_0. \tag{30}$$

Thus, the state $|s,s\rangle$ turns into the "ground state" associated with the operator $\hat{N}$, and it becomes obvious why one needs to associate the *creation* operator $\hat{S}^+$ with the *annihilation* operator a [cf. (20)]: the eigenstates with *maximal* s are linked to the oscillator ground state with *minimal* $n=0$. In [5], the opposite convention has been used. Nevertheless, it remains true that not only spin eigenstates are mapped into number eigenstates but many other expressions related to the group U(2) turn into an equivalent expression for the oscillator group.

This is good news for the present purpose to establish a relation between the Moyal formalism of a particle and a spin. Consider the limit of the kernel (13) under contraction using Eq. (26),

$$\lim_{c\to 0}\hat{\Delta}(\mathbf{n}) = \hat{T}(\alpha)(\lim_{c\to 0}\hat{\Pi}_s)\hat{T}^\dagger(\alpha). \tag{31}$$

The middle term can be written as [cf. (30)]

$$\lim_{c\to 0}\hat{\Pi}_s = \lim_{c\to 0}\sum_{m=-s}^{s}\Delta_\varepsilon(m)|s,m\rangle\langle s,m| = \sum_{n=0}^{\infty}[\lim_{c\to 0}\Delta_\varepsilon(s-n)]|n\rangle\langle n|. \tag{32}$$

Upon comparison with Eq. (6), the Wigner kernel of a spin is seen to turn into the Wigner kernel of the particle if

$$\lim_{s\to\infty}\sum_{l=0}^{2s}\varepsilon_l\left(\frac{2l+1}{2s+1}\right)^{1/2}\left\langle \begin{matrix} s & s \\ s-n & n-s \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle = 2 \tag{33}$$

holds for all non-negative integers n. In the next section, this will be shown to be true for the choice $\varepsilon_l=+1$, $l=1,\ldots,2s$.

V. SUMMING THE SERIES

Evaluating the sum (33) in the limit $s\to\infty$ proceeds in two steps. First, the asymptotic form of the terms

$$\Delta^s_{l,n}=\left(\frac{2l+1}{2s+1}\right)^{1/2}\left\langle \begin{matrix} s & s \\ s-n & n-s \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle \tag{34}$$

to be summed is determined with the help of a recurrence formula for Clebsch-Gordan coefficients. Then, the sums are transformed into integrals which can be evaluated. All approximations drop terms of the order $1/s$ at least, hence the result is *exact* in the limit of infinite s.

Clebsch-Gordan coefficients satisfy the following recursion relation [12]:

$$\begin{aligned}&[l(l+1)-2s(s+1)+2m^2]\left\langle \begin{matrix} s & s \\ m & -m \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle \\ &\quad=[s(s+1)-m(m+1)]\left\langle \begin{matrix} s & s \\ m+1 & -(m+1) \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle \\ &\quad\quad+[s(s+1)-m(m-1)]\left\langle \begin{matrix} s & s \\ m-1 & -(m-1) \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle,\end{aligned} \tag{35}$$

implying that

$$\begin{aligned}&(n+1)\left(1-\frac{n+1}{2s+1}\right)\Delta^s_{l,n+1}+\left(2n+1-\frac{2n^2+2n+1}{2s+1}\right)\Delta^s_{l,n} \\ &\quad+n\left(1-\frac{n}{2s+1}\right)\Delta^s_{l,n-1}=\frac{l(l+1)}{2s+1}\Delta^s_{l,n}.\end{aligned} \tag{36}$$

For any finite n the terms subtracted on the left-hand side become less and less important if $s\to\infty$. Assume now that one can factorize the terms with large values of n in the form

$$\Delta^s_{l,n}(x_l)=\Lambda_n(x_l)\Delta^s_{l,0}, \quad \Lambda_0(x_l)=1, \quad x_l=\frac{l(l+1)}{2s+1}. \tag{37}$$

Due to Eq. (36), the polynomial $\Lambda_n(x_l)$ of order n in x_l must satisfy a three-term recursion relation,

$$(n+1)\Lambda_{n+1}(x_l)+(2n+1)\Lambda_n(x_l)+n\Lambda_{n-1}(x_l)=x_l\Lambda_n(x_l), \tag{38}$$

where terms of order $1/s$ have been dropped in Eq. (38). Its solutions [13] are proportional to the Laguerre polynomials, $L_n(x_l)$, and the ‘‘normalization’’ condition $\Lambda_0(x_l)=1$ implies that for all $n=0,1,2,\ldots$,

$$\Lambda_n(x_l)=(-)^nL_n(x_l)=(-)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-x_l)^k}{k!}. \tag{39}$$

The value of the term $\Delta^s_{l,0}$ in Eq. (37) can be determined in the following way. If s is large, one has for each finite k

$$\left(1-\frac{k}{2s+1}\right)^{2s+1}\sim\exp[-k], \tag{40}$$

which leads to the approximation

$$\begin{aligned}\Delta^s_{l,0}&=\left(\frac{2l+1}{2s+1}\right)^{1/2}\left\langle \begin{matrix} s & s \\ s & -s \end{matrix}\middle| \begin{matrix} l \\ 0 \end{matrix}\right\rangle \\ &=\frac{2l+1}{2s+1}\left(\frac{(2s)!}{(2s-l)!}\frac{(2s)!}{(2s+l+1)!}\right)^{1/2} \\ &=\frac{2l+1}{2s+1}\left(\frac{\prod_{k=0}^{l}[1-k/(2s+1)]}{\prod_{k=0}^{l}[1+k/(2s+1)]}\right)^{1/2} \\ &\sim\frac{2l+1}{2s+1}\exp\left[-\frac{1}{2}\frac{l(l+1)}{2s+1}\right].\end{aligned} \tag{41}$$

Collecting the results, one has

$$\lim_{s\to\infty}\sum_{l=0}^{2s}\Delta^s_{l,n}\sim(-)^n\lim_{s\to\infty}\sum_{l=0}^{2s}\Delta x_lL_n(x_l)e^{-x_l/2}, \tag{42}$$

where $\Delta x_l=(x_{l+1}-x_l)=(2l+1)/(2s+1)+\mathcal{O}(1/s)$. Transforming now the Riemann sum into an integral, one obtains the result announced in Eq. (33),

$$\lim_{s\to\infty}\sum_{l=0}^{2s}\Delta^s_{l,n}=(-)^n\int_0^\infty dxL_n(x)e^{-x/2}=2, \tag{43}$$

using the formula

$$\int_0^\infty dxL_n(x)e^{-x/t}=t(1-t)^n \tag{44}$$

for $t=2$. This identity is proven easily by means of the expansion in Eq. (39).

VI. DISCUSSION

Starting from a new form of the kernel defining the familiar Wigner formalism for a spin, its limit for infinite values of s has been shown to be the Wigner kernel of a particle. As the kernel defines entirely a phase-space representation, this result guarantees that the Moyal formalism for a particle is reproduced automatically and *in toto*, if the limit $s\to\infty$ of the spin Moyal formalism is taken. In [14], a similar idea has been worked out leading to an equivalent result.

In fact, slightly more has been shown in the present paper. The result removes an ambiguity of the Moyal formalism for a spin: the Stratonovich-Weyl postulates are compatible with a discrete *family* of 2^{2s} distinct kernels $\hat{\Delta}_\varepsilon(\mathbf{n})$. However, only *one* of these kernels turns into the particle kernel. This kernel had been singled out before for other reasons [10]. In summary, the group-theoretical contraction shows that the phase-space representations à la Wigner for spin and particle systems are structurally equivalent.

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