How to determine a quantum state by measurements: The Pauli problem for a particle with arbitrary potential

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The problem of reconstructing a pure quantum state $|\psi\rangle$ from measurable quantities is considered for a particle moving in a one-dimensional potential $V(x)$. Suppose that the position probability distribution $|\psi(x,t)|^2$ has been measured at time t, and let it have M nodes. It is shown that after measuring the time evolved distribution at a short-time interval Δt later, $|\psi(x,t+\Delta t)|^2$, the set of wave functions compatible with these distributions is given by a smooth manifold $\mathscr{M}$ in Hilbert space. The manifold $\mathscr{M}$ is isomorphic to an M-dimensional torus, $\mathscr{T}^M$. Finally, M additional expectation values of appropriately chosen nonlocal operators fix the quantum state uniquely. The method used here is the analog of an approach that has been applied successfully to the corresponding problem for a spin system.

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I. INTRODUCTION

In this paper progress is reported concerning a deceptively simple question known as the *Pauli problem*: does the measurement of the probability densities for position and momentum of a particle determine its quantum state? Originating from a footnote in Pauli's article in *Handbuch der Physik* [1], this question has led, in a more general setting, to a number of investigations over the past decades: the expectation values of which sets of operators characterize uniquely a (pure or mixed) state of a quantum system? Apparently, one important early motivation for dealing with this problem has been to demystify the concept of the wave function [2]: being a complex quantity it seems impossible to directly observe it in experiments. However, if an appropriate set of expectation values provides the same information about a quantum system as does the wave function itself, then it is reasonable to consider the wave function just as a particularly convenient description of the system.

Various works investigating the Pauli problem have been reviewed in [3(a)]. There are many possibilities to approach the problem in its general form since one is free to choose the set of operators to be measured at will. It has been shown by various authors [2,4–6] that knowledge of position and momentum distributions alone (being equivalent to knowledge of the expectation values of all powers of position and momentum operators, respectively) does *not* single out one specific state. This result is obtained from explicitly constructing states with identical probability distributions of both position and momentum. It is not known, in general, how large such a set of "Pauli partners" actually is, and which supplementary expectation values would allow one to distinguish between them. For more details and a list of references the reader is referred to [3(a)].

A novel approach is based on tomographic methods in order to determine the Wigner function of a quantum state [7,8]. In its sequel, successful experimental realizations have been reported. The quantum state of an electromagnetic field mode has been reconstructed by using a method called "optical homodyne tomography" [9]. It relies on the possibility of measuring probability distributions of the quadrature amplitude $\hat{x}_\varphi = \hat{x}\cos\varphi + \hat{p}\sin\varphi$, obtained from rotating the position operator in the phase plane by an angle φ. In a similar vein, the vibrational quantum state of a molecule has been determined through the measurement of a time-dependent spectrum of fluorescence [10]. The information obtained in this way from "molecular emission tomography" can be shown to encode a quasiprobability function, and it allows one thus to reconstruct the sought-after quantum state. A related proposal has been made to experimentally determine the state of both a scalar light wave or a particle wavefunction [11]. It exploits the transformations induced on the Wigner function when the state under investigation passes through well-defined lenses while propagating along some direction in space.

It is straightforward to pose this problem for a quantum spin of length s. The setting in a *finite-dimensional* Hilbert space turns out to be an important modification. Using a specific version of a Stern-Gerlach apparatus (in which individual beams after the separation can be shielded and the remaining ones can be brought together again) allows one to measure directly the intensities and relative phases of the splitted beams [12]. It is not clear, however, whether it is actually possible to perform this experiment without destroying the phase relations. It has been shown in [3(a)] that measuring the intensities of the components of a spin state along two (infinitesimally) close directions in space is compatible with 2^{2s} spin states. In this setup the familiar version of the Stern-Gerlach apparatus is sufficient. It turns out that all Pauli partners can be exhibited explicitly, and it is sufficient to know the expectation value of one additional well-defined operator in order to uncover the actual quantum-mechanical state. This procedure is not constructive: *uniqueness* of the state compatible with the measurements is shown—its explicit determination is another matter.

A constructive solution of the Pauli Problem has been obtained for a restricted set of states in a one-particle Hilbert space [13]. Imagine that the system is known to be prepared in a state which is made up of a finite but arbitrarily large number of energy eigenstates of a harmonic oscillator. With

this additional information one is able to reconstruct the state from the knowledge of position and momentum distributions over the real line. Another proposition [14] expresses the quantum state in terms of expectation values of simple projection operators, and a quantum optical realization of this approach seems feasible.

It is the goal of the present paper to investigate the Pauli problem for a particle in analogy to the method developed in [3(a)]. To do so, the method to solve the Pauli problem for a spin is, in Sec. II, briefly reviewed, and then it is reinterpreted in terms of particle dynamics. In Sec. III the main result is established: it is possible to enumerate all pure quantum states compatible with the spatial probability densities at two times t and t', separated by a short-time interval, Δt. Finally, in Sec. IV it is shown how to distinguish between these Pauli partners through appropriate additional measurements.

II. REINTERPRETATION OF THE PAULI PROBLEM FOR A SPIN

In this section a solution of the Pauli problem for a spin of length s is briefly reviewed. Then, the method of solution is reinterpreted in such a way that an analogous treatment of the Pauli problem for a particle becomes possible. It has been shown in [3(a)] that the measurement of the intensities of a spin state $|\chi\rangle$ along two neighboring axes of quantization, z and z', by means of a Stern-Gerlach apparatus is compatible with a discrete set $\mathcal{N}(s)$ of states. An additional measurement of the expectation value of one well-defined operator allows one to discriminate between the elements of the set $\mathcal{N}(s)$.

More specifically, one proceeds as follows. It is assumed that there is a beam of particles propagating along the x axis, say, each of which is prepared in one and the same pure spin state $|\chi\rangle$ in the $(2s+1)$-dimensional Hilbert space $\mathcal{H}$. They enter a Stern-Gerlach apparatus that defines the axis of quantization to be along the z direction. The associated eigenstates of the z component of the spin, $\{|m,z\rangle\}$, constitute a basis in Hilbert space, and the quantum number m takes on all (half-) integer values between $\pm s$. In a first series of measurements all intensities $|\chi_m(z)|^2 \equiv |\langle m,z|\chi\rangle|^2$ are determined with respect to the z basis. The states compatible with these measurements are located on a $2s$-dimensional manifold $\mathcal{A}(s)$: the phase of each state with respect to the basis $|m,z\rangle$ is undetermined, but the overall phase of the state $|\psi\rangle$does not have physical meaning since a state of the system is associated with a *ray* in $\mathcal{H}$. The set of states corresponding to the elements of the manifold $\mathcal{A}(s)$ will be denoted by $|\chi(\boldsymbol{\gamma})\rangle$, where the label $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_{2s})$ parameterizes the manifold $\mathcal{A}(s)$. It is assumed that the phase conventions are chosen in such a way that the actual state of the system corresponds to $\boldsymbol{\gamma}=0$: $|\chi(0)\rangle \equiv |\chi\rangle$.

Consider now an infinitesimal coordinate transformation,

$$\hat{U}(\boldsymbol{\epsilon}) = \exp\left[-\frac{i\boldsymbol{\epsilon}}{\hbar}\hat{S}_x\right] = 1 - \frac{i\boldsymbol{\epsilon}}{\hbar}\hat{S}_x + \mathcal{O}(\boldsymbol{\epsilon}^2), \tag{1}$$

corresponding to a rotation of the Stern-Gerlach apparatus by $\boldsymbol{\epsilon}$ about the x axis. It defines a new direction of quantization, z', which is infinitesimally close to z and perpendicular to the x axis. The measurement of the intensities $|\chi_m(z')|^2$ along z' represents $2s$ more conditions on the possible states:

$$|\langle m,z'|\chi(\boldsymbol{\gamma})\rangle|^2 = |\chi_m(z')|^2, \quad m=-s,-s+1,\ldots,+s. \tag{2}$$

It can be shown that 2^{2s} states on the $2s$-dimensional manifold $\mathcal{A}(s)$ are compatible with the measurements (2); they represent the discrete set of Pauli partners, $\mathcal{N}(s)$. Finally, by measuring in addition the expectation value of the operator $\hat{S}_x$, one can distinguish between the individual Pauli partners that lead to identical intensities along the z and z' axis.

As it stands, this method to determine a quantum-mechanical state from measurements cannot be applied directly to perform the same task for a particle wave-function. However, one can rephrase this procedure in such a way that it becomes possible to transfer the method to the particle problem. Change from the passive view of the rotation to an active point of view. Then the Stern-Gerlach apparatus defines once and for all an axis of quantization pointing along z; the measurements, however, are now performed with respect to the states $|\psi\rangle$ and $\hat{U}(-\boldsymbol{\epsilon})|\psi\rangle$, having been rotated by $(-\boldsymbol{\epsilon})$ about the x axis relative to the apparatus. Consider $\boldsymbol{\epsilon}$ as a time parameter, then the operator $\hat{U}$ of Eq. (1) can be conceived as the time evolution operator of a spin in a constant magnetic field $\mathbf{B}\|\mathbf{e}_x$ of appropriate strength, generated by the Hamiltonian

$$\hat{H} = -\mathbf{B}\cdot\hat{\mathbf{m}} = \hat{S}_x, \tag{3}$$

$\hat{\mathbf{m}}$ being the magnetic moment of the spin. In other words, the measurement of the intensities along two different spatial directions, z and z', is equivalent to a measurement of the intensities associated with the state $|\chi(t)\rangle$ and its time evolution $|\chi(t+\Delta t)\rangle$ for $\Delta t=\boldsymbol{\epsilon}$ with respect to one fixed direction, z. This point of view motivates an investigation of the problem for a particle with Pauli data given by the position probability densities $|\psi(x,t)|^2$ and $|\psi(x,t+\Delta t)|^2$ of a state $|\psi\rangle$ at two nearby times.

III. DETERMINATION OF PAULI PARTNERS FOR A PARTICLE

A particle with mass m is assumed to move in a one-dimensional potential $V(\hat{x})$, described by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \equiv \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)|\psi(t)\rangle. \tag{4}$$

Suppose that the system is prepared at time t in a normalized pure state $|\psi(t)\rangle$ with finite energy, $\langle\psi|\hat{H}|\psi\rangle<\infty$; *a fortiori*, the expectation value of the kinetic energy $\hat{T}$

$$\langle\psi|\hat{T}|\psi\rangle = \langle\psi|\hat{p}^2/2m|\psi\rangle < \infty, \tag{5}$$

is finite.

The Hamiltonian $\hat{H}$ defined in Eq. (4) generates the time evolution of the particle state $|\psi\rangle$ just as the Hamiltonian

(3) did for the spin. By analogy, in this section the Pauli partners for a particle will be determined that are compatible with the measurement of the probability distribution of position at time t and at a short time later, $t+\Delta t$; in the meantime, the state is assumed to evolve according to (4).

In the position representation one can decompose the wave function as

$$\langle x|\psi(t)\rangle = |\psi(x,t)|e^{i\phi(x,t)}, \tag{6}$$

with positive modulus $|\psi(x,t)|$ and a real phase $\phi(x,t)$, which is assumed to be a continuous function of x. Discontinuities of the phase $\phi(x,t)$ at points where the amplitude $|\psi(x,t)|$ of the wave function vanishes (i.e., at its nodes), however, are *not* excluded.

Under these assumptions the measurement of the probability distributions $|\psi(x,t)|^2$ and $|\psi(x,t+\Delta t)|^2$ at two times t and $t+\Delta t$ with $\Delta t \ll 1$ will be seen to be compatible with an M-dimensional manifold $\mathscr{M}$ of states, M being the number of nodes of the wave function under consideration. The set of Pauli partners turns out to be isomorphic to an M-dimensional torus, $\mathscr{T}^M$.

Suppose that at time t the probability distribution $|\psi(x,t)|^2$ has been determined experimentally. All states compatible with this distribution can be written

$$\psi(x,t;\xi(x,t)) \equiv \psi(x,t)e^{i\xi(x,t)} = |\psi(x,t)|e^{i[\phi(x,t)+\xi(x,t)]}, \tag{7}$$

$\xi(x,t)$ being a real function. For later convenience, the function $\xi(x,t)$ is defined as the *deviation* from the true (but yet unknown) phase $\phi(x,t)$; the strategy will be to eliminate the "freedom" in the choice of the function ξ by constraints imposed by additional measurements. The relation between the amplitudes of the actual state $|\psi\rangle$ at times t and $t'=t+\Delta t$ follows from Schrödinger's equation, Eq. (4),

$$\begin{aligned}\psi(x,t+\Delta t) &= \psi(x,t) + \frac{\partial\psi(x,t)}{\partial t}\Delta t + \mathscr{O}(\Delta t^2) \\ &= \psi(x,t) - \frac{i}{\hbar}\hat{H}\psi(x,t)\Delta t + \mathscr{O}(\Delta t^2).\end{aligned} \tag{8}$$

Multiplying this equation with the corresponding expansion of $\psi^*(x,t+\Delta t)$ leads to

$$\begin{aligned}|\psi_{t'}|^2 &= |\psi_t|^2 + \frac{i}{\hbar}(\psi_t\hat{H}\psi_t^* - \psi_t^*\hat{H}\psi_t)\Delta t + \mathscr{O}(\Delta t^2) \\ &= |\psi_t|^2 - \frac{i\hbar}{2m}(\psi_t\partial_{xx}\psi_t^* - \psi_t^*\partial_{xx}\psi_t)\Delta t + \mathscr{O}(\Delta t^2),\end{aligned} \tag{9}$$

showing that, to first order in Δt, the change of the modulus of the wave function does not depend on the potential $V(x)$. From now on, the time argument of the wave function will be given as a lower index, and the dependence on x is suppressed; furthermore, $\partial_x \equiv \partial/\partial x$, etc. The corresponding relation for the states given in Eq. (7) reads

$$\begin{aligned}|\psi_{t'}(\xi)|^2 = |\psi_t(\xi)|^2 - \frac{i\hbar}{2m}&(\psi_t(\xi)\partial_{xx}\psi_t^*(\xi) \\ &- \psi_t^*(\xi)\partial_{xx}\psi_t(\xi))\Delta t + \mathscr{O}(\Delta t^2).\end{aligned} \tag{10}$$

The second term on the right-hand-side of this equation can be written as

$$\begin{aligned}-\frac{i\hbar}{2m}&(\psi_t\partial_{xx}\psi_t^* - \psi_t^*\partial_{xx}\psi_t - 2i(\psi_t\partial_x\psi_t^* + \psi_t^*\partial_x\psi_t)\partial_x\xi \\ &- 2i|\psi_t|^2\partial_{xx}\xi),\end{aligned} \tag{11}$$

with the explicit form of the second derivative of the wave function $\psi_t(\xi)$ given by

$$\partial_{xx}\psi_t(\xi) = (\partial_{xx}\psi_t + 2i\partial_x\psi_t\partial_x\xi + i\psi_t\partial_{xx}\xi - \psi_t(\partial_x\xi)^2)e^{i\xi}. \tag{12}$$

The assumption of identical probability distributions of position for the states $|\psi\rangle$ and $|\psi(\xi)\rangle$ at both times t and t' requires the expressions in Eqs. (9) and (10) to be equal, which by using Eq. (11) implies

$$(\psi_t\partial_x\psi_t^* + \psi_t^*\partial_x\psi_t)\partial_x\xi + |\psi_t|^2\partial_{xx}\xi = 0, \tag{13}$$

a condition that also can be written as

$$\partial_x(|\psi_t|^2\partial_x\xi) = 0; \tag{14}$$

this equation will be referred to as the *phase equation*. Every solution $\xi(x)$ of this equation defines a wave function compatible with the observed probability distributions at times t and t', to first-order in Δt. Formally, this equation is identical to the amplitude transport equation known from semiclassical quantum mechanics [15]. In correspondence with the result for the spin system only information about points close to x is required: only first- and second-order derivatives occur, to be compared with the occurrence of at most, second-order differences in the corresponding equation for the spin [Eq. (19) of 3(a)].

It will be shown in the sequel that the nodal structure of the wave function determines the manifold of solutions of the phase equation. Strictly speaking, there is a number of phase equations with solutions $\xi(x)$: it is *not* necessary to smoothly continue the functions $\xi(x)$ on the left and on the right of a zero of the amplitude, since the phase of the wave function is undetermined at its nodes. Therefore, it is reasonable to consider the solutions of Eq. (14) separately in each "compartment," defined as a region between two zeros of the amplitude $|\psi_t(x)|^2$. Suppose that there are M nodes, apart from those at $x=\pm\infty$. Label the zeros of the amplitude $|\psi_t(x)|^2$ from the left to the right by $x_-, x_1, \ldots, x_M, x_+$, starting with 0 at $x_- = -\infty$ and ending at $M+1$ at $x_+ = +\infty$. The μth compartment $\mathscr{C}_\mu$ is defined as that one on the *right* of the zero x_μ; the compartment extending to $-\infty$ will be referred to as $\mathscr{C}_0$. A wave function with M nodes defines $M+1$ compartments; in particular, for a state without a node there is just one single compartment, $\mathscr{C}_0$.

The general solution of the phase equation in compartment $\mathscr{C}_\mu, \mu = 0,1,\ldots,M$ is given by

$$\xi_\mu(x)=\alpha_\mu\int_{x_\mu^0}^{x}\frac{dy}{|\psi_t(y)|^2}+\beta_\mu,\quad x\in\mathscr{C}_\mu,\quad \alpha_\mu,\beta_\mu\in\mathbb{R},\tag{15}$$

where x_μ^0 is an arbitrary but fixed point in compartment $\mathscr{C}_\mu$. Since the denominator approaches the value zero at the boundaries of the compartment $\mathscr{C}_\mu$, nonzero values α_μ would imply that the solutions $\xi(x)$ go to infinity at the nodes of the wave function. Such a behavior of the phase, however, is not compatible with the assumption of Eq. (5), as will be shown momentarily: a finite expectation value of the kinetic energy, $\hat{T}$, can be ensured only by all constants α_μ being equal to zero.

First, the behavior of the integral in Eq. (15) stemming from points close to a node is calculated approximately. Then, the expectation value of the kinetic energy will be shown to diverge whenever α_μ is different from zero. Consider, for definiteness, the left endpoint x_μ of the region $\mathscr{C}_\mu$; the following argument can be repeated analogously for right endpoints. The expansion of the amplitude $P(x)\equiv|\psi_t(x)|$ in the neighborhood to the right of the point x_μ yields

$$\psi_t(x)=\{\partial_x P(x_\mu)(x-x_\mu)+\mathscr{O}((x-x_\mu)^2)\}e^{i\phi(x)},\tag{16}$$

using $P(x_\mu)=0$, and the term $\partial_x P(x)$ in this expansion is assumed to be different from zero. It turns out that its vanishing, corresponding to a nongeneric coincidence of a node and and an extremum of the modulus, would make things even worse (cf. below). Therefore, the probability density near the point x_μ is approximately given by

$$|\psi_t(x)|^2=(\partial_x P(x_\mu)(x-x_\mu))^2+\mathscr{O}((x-x_\mu)^3).\tag{17}$$

Consider now a wave function $\psi_{t_0}(x)$ with phase $\xi_\mu(x)$, obtained from using (15) in (17); the expectation value of the kinetic energy in a small interval $D(\delta)=(x_\mu+\delta,x_\mu+\delta_0)$, with $0<\delta,\delta_0\ll 1$ and δ_0 fixed, is given by

$$\langle\psi(\xi)|\hat{T}|\psi(\xi)\rangle_{D(\delta)}$$

$$=\frac{-\hbar^2}{2m}\int_{x_\mu+\delta}^{x_\mu+\delta_0}dx\ \psi_t^* e^{-i\xi_\mu}\partial_{xx}(\psi_t e^{i\xi_\mu})\tag{18}$$

$$=\frac{-\hbar^2}{2m}\int_{x_\mu+\delta}^{x_\mu+\delta_0}dx(\psi_t^*\partial_{xx}\psi_t-|\psi_t|^2(\partial_x\xi_\mu)^2$$

$$+i|\psi_t|^2\partial_{xx}\xi_\mu+2i\psi_t^*\partial_x\psi_t\partial_x\xi_\mu).\tag{19}$$

In the limit $\delta\to 0$ the main contribution is due to the second term on the right-hand side being proportional to

$$\alpha_\mu^2\int_{x_\mu+\delta}^{x_\mu+\delta_0}\frac{dx}{(x-x_\mu)^2}\sim\frac{-\alpha_\mu^2}{x-x_\mu}\Big|_{x_\mu+\delta}^{x_\mu+\delta_0}$$

$$=\alpha_\mu^2\left(\frac{1}{\delta}-\frac{1}{\delta_0}\right)\to\infty\ \text{ if }\delta\to 0.\tag{20}$$

The origin of the divergence is obvious: if the phase ξ_μ goes to infinity for $\delta\to 0$, then the wave function $\psi(x)$ acquires a more and more rapidly oscillating phase factor, $\xi_\mu(x)$. Such oscillations correspond to large values of the energy. Consequently, the constants α_μ are necessarily equal to zero for all compartments $\mathscr{C}_\mu$, and the resulting dependence of the phase ξ on x is strongly limited: it has to be *constant* within each compartment $\mathscr{C}_\mu$:

$$\xi(x)=\beta(x)\equiv\sum_{\mu=0}^{M}\beta_\mu\chi_\mu(x),\quad \boldsymbol{\beta}=(\beta_0,\beta_1,\ldots,\beta_M),$$

$$\beta_\mu\in[0,2\pi),\tag{21}$$

where the characteristic function of the compartment $\mathscr{C}_\mu$ has been introduced: $\chi_\mu(x)=1$ if $x\in\mathscr{C}_\mu$, and 0 else. A similar argument also applies to compartment $\mathscr{C}_0$ or, equivalently, to a state without any zero, such as the ground state of the potential $V(x)$; in this case the point $x_-=-\infty$ is considered to be the node. In retrospect, it is now obvious that the situation does not change qualitatively if the first nonzero derivative of the expansion of the wave function $|\psi_t|^2$ in Eq. (17) were of higher order: the oscillations would become even stronger.

Thus, after having measured the probability densities of position at times t and t', one is able to define the underlying state up to $(M+1)$ constant phases in the $(M+1)$ compartments. The absolute value of the phase of the wave function is arbitrary so that there remains an M-dimensional manifold $\mathscr{M}$ of states that is compatible with the experimentally determined data. Since the values β_μ are restricted to the interval $[0,2\pi)$ the manifold of Pauli partners is seen to to coincide with an M-dimensional torus: $\mathscr{M}=\mathscr{T}^M$.

IV. TELLING PAULI PARTNERS APART ON THE TORUS $\mathscr{T}^M$

The measurement of the probability distributions $|\psi(x,t)|^2$ and $|\psi(x,t+\Delta t)|^2$ is sufficient to determine the wave function $\psi(x,t)$ up to M relative phases, where M is the number of nodes of the wave function under consideration. This set of states is conveniently described in the form

$$\psi(x,\boldsymbol{\beta})=\psi(x)e^{i\beta(x)}=\sum_{\mu=0}^{M}\psi_\mu(x)e^{i\beta_\mu},\tag{22}$$

where $\psi_\mu(x)=\chi_\mu(x)\psi(x)$ is a function identical to $\psi(x)$ in the compartment $\mathscr{C}_\mu$ and equal to zero elsewhere. Note that both, modulus $|\psi_\mu(x)|$ and phase $\phi_\mu(x)$, have already been determined, at least implicitly: for the time being, *only* the $(M+1)$ real numbers $\boldsymbol{\beta}$ remain unknown [or, equivalently, the piecewise constant function $\beta(x)$ in (21)]. Thus, the measurement of M appropriate expectation values is expected to allow one to single out the actual state of the system.

Measuring quantities that refer to one single point in configuration space only, i.e., measuring *local* operators, will not provide the required information, since all unknown phases drop out immediately. Being the generator of spatial translations, the momentum operator $\hat{p}$ will be involved in any nonlocal quantity, in one way or another. For example, imagine shifting the wave function $\psi(x,\boldsymbol{\beta})$ by a (finite) amount Δx and consider the scalar product of the original state and the shifted one, $\psi(x+\Delta x,\boldsymbol{\beta})$. Close to each node there will

be a region where the multiplication of the wave functions (originating from the compartments $\mathscr{C}_\mu$ and $\mathscr{C}_{\mu-1}$, respectively) produces a factor $\exp[i\beta_{\mu\mu-1}]$, with $\beta_{\mu\mu-1} \equiv \beta_\mu - \beta_{\mu-1}$. Repeating this procedure M times with different shifts $\Delta_\mu x$, one obtains a system of M linear equations for the phase differences that, in general, will allow one to determine the unknowns $\beta_{\mu\mu-1}$. In the following, a different approach is proposed, that does not require one to measure the (real or imaginary part of the) translation operator.

Consider the self-adjoint operators

$$\hat{K}_{x^0_\mu x^0_{\mu-1}} \equiv \hat{K}_{\mu\mu-1} = \tfrac{1}{2}\,(|x^0_\mu\rangle\langle x^0_{\mu-1}| + |x^0_{\mu-1}\rangle\langle x^0_\mu|), \quad (23)$$

where, as before, x^0_μ denotes a point in compartment $\mathscr{C}_\mu$; similar operators have been introduced in [16]. The M equations

$$\langle\psi(\boldsymbol{\beta})|\hat{K}_{\mu\mu-1}|\psi(\boldsymbol{\beta})\rangle = |\psi(\bar{x}_\mu)||\psi(\bar{x}_{\mu-1})|\cos(\beta_{\mu\mu-1} + \phi_{\mu\mu-1}), \quad \mu = 1,\ldots,M, \quad (24)$$

with $\phi_{\mu\mu-1} \equiv \phi_\mu - \phi_{\mu-1}$, can be solved immediately for the unknowns $\beta_{\mu\mu-1}$, expressing them in terms of already known quantities and the expectation values of the $\hat{K}$ operators:

$$\beta_{\mu\mu-1} = -\phi_{\mu\mu-1} + \arccos\left(\frac{|\psi(\bar{x}_\mu)||\psi(_{\mu-1})|}{\langle\psi(\boldsymbol{\beta})|\hat{K}_{\mu\mu-1}|\psi(\boldsymbol{\beta})\rangle}\right). \quad (25)$$

Fixing, for example, the value of $\beta_0 = 0$, one can determine the value of β_1 from Eq. (25) for $\mu = 1$; then, β_2 follows from considering (25) for $\mu = 2$; etc. In this way, one sees that the wave function $\psi(x,t)$ is indeed completely specified by the measurement of the expectation values of the operators $\hat{K}_{\mu\mu-1}$.

V. DISCUSSION

In summary, it has been shown that the measurement of two position probability densities at subsequent times t and $t + \Delta t$ plus the expectation values of a finite number of nonlocal operators is equivalent to the knowledge of the pure state (ray) in Hilbert space. Just as in the case of a spin, the method is not constructive; the *uniqueness* of the determined state, however, is ensured.

These results have to be compared to earlier ones obtained along similar lines by Feenberg [17], and to their generalization to three spatial dimensions given in Kemble's book [18]. The basic idea has been to measure the position probability distribution and its time derivative at one instant of time; the resulting three-dimensional version of the phase equation (14) is contained in Kemble's book. According to a critical evaluation given in Appendix A of [19], Kemble's claim that these quantities would determine uniquely a quantum state is not correct. In other words, the analysis of the solutions of the phase equation is erroneous. Furthermore, in these works the form of the manifold $\mathscr{M}$, i.e., the manifold of Pauli partners, has not been stated.

The original Pauli problem, namely, determination of the quantum state from enunciating the position and momentum distributions, $|\psi(x)|^2$ and $|\psi(p)|^2$, has not yet been solved. The result of the present work, however, makes it plausible that the knowledge of two continuous functions over the real line might actually be sufficient, again except for an ambiguity corresponding to points on a finite-dimensional torus. Known sets of Pauli partners usually require the (real-valued) state under consideration to possess specific discrete symmetries. For example, an antisymmetric state with one single node at the origin and the state that is obtained by flipping the sign of the wave function in one of the two compartments are known to have the same position and momentum probability distributions [6]; in this case, the Pauli partners are finite in number.

The result of the present work may actually provide a step towards the solution of the original Pauli problem. If the potential $V(x)$ were chosen to be harmonic, then the role of position and momentum operators would be exchanged (apart from a minus sign) after a quarter of the oscillator's period, $T = 2\pi/\omega$. In other words, a measurement of the spatial probability distribution at two times, t and $t' = t + T/4$, is closely related to the measurement of position and momentum distributions at one given instant of time. (According to [20] it is possible to extract the unknown phases of the wave function from these measurements by means of a known algorithm.) In order to exploit this observation in the presence of a general potential, one has to extend the considerations to measurements separated by a finite time interval, which, for the time being, is not obvious. As it stands, the present approach is, effectively, not sensitive to the potential since the short-time evolution of the wave function is governed completely by the kinetic energy, but for longer times the potential will intervene in a nontrivial manner.

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