Article:

Kong, D., Zhang, K., Lam, K. et al. (2018) Axially symmetric and latitudinally propagating nonlinear patterns in rotating spherical convection. Physical Review E, 98 (3). 031101. ISSN: 2470-0045

https://doi.org/10.1103/PhysRevE.98.031101

Axially symmetric and latitudinally propagating nonlinear waves in rotating spherical convection

Dali Kong,[1] Keke Zhang,[2] Kameng Lam,[3] and Ashley P. Willis[4]

[1]*Key Laboratory of Planetary Sciences, Shanghai Astronomical Observatory, Shanghai 200030, China*
[2]*College of Engineering, Mathematics and Physical Sciences, University of Exeter, Exeter EX4 4QE, UK*
[3]*Lunar and Planetary Science Laboratory, Macau University of Science and Technology, Macau*
[4]*School of Mathematics and Statistics, University of Sheffield, Sheffield, S3 7RH, UK*

(Dated: June 9, 2018)

We report a new nonlinear phenomenon discovered in the classical problem of thermal convection in rapidly rotating, self-gravitating, internally heated Boussinesq fluid spheres. When linear convective instability (the most unstable mode of convection) is in the form of an axially symmetric, equatorially antisymmetric torsional oscillation, its equatorial symmetry must be broken by nonlinear effects and, consequently, the key properties of the primary solution bifurcating from the instability cannot be predicted on the basis of linear solutions at the onset of convection. We reveal that, when the supercritical Rayleigh number is in the vicinity of its critical value, the primary nonlinear solution is in the form of an axially symmetric, equatorially nonsymmetric, latitudinally propagating wave whose amplitude varies periodically, representing a new nonlinear pattern of thermal convection in rotating fluid spheres.

PACS numbers: 47.54.+r,47.27.te,92.60.Ek

I. INTRODUCTION

A well-known classical problem in fluid dynamics is thermal convection in rapidly rotating, self-gravitating, internally heated Boussinesq fluid spheres or spherical shells [see, for example, 1–8]. The problem is characterized by the three physical parameters: the Rayleigh number Ra, the Prandtl number Pr and the Ekman number E. The Rayleigh number Ra is effectively the ratio of destabilizing buoyancy forces to the Coriolis and dissipative force, the Prandtl number Pr provides a measure of the relative importance of viscous and thermal diffusion, and the Ekman number E is related to the ratio of viscous forces to the Coriolis force. For applications to many planetary fluid systems like the Earth's liquid core, the Ekman number E is usually extremely small $E \ll 1$ and the Prandtl number Pr is moderately small while some astrophysical fluid systems like the solar convection zone are marked by extremely small sizes of the Prandtl number Pr [see, for example, 9].

It is well understood that the physically preferred mode of convective instability in rapidly rotating spheres for moderate values of Pr is in the form of axially nonsymmetric, equatorially symmetric and azimuthally drifting columnar rolls [2, 4] while the physically preferred convection mode for small Prandtl number is in the form of axially nonsymmetric, equatorially symmetric and azimuthally traveling thermal-inertial waves [3]. In both the cases, the flow, for example, its azimuthal component $\hat{\boldsymbol{\phi}} \cdot \mathbf{u}$ at the onset of convection with $Ra = (Ra)_c$, where $(Ra)_c$ denotes the critical Rayleigh number, can be expressed as

$$\hat{\boldsymbol{\phi}} \cdot \mathbf{u}(r, \theta, \phi, t) = F(r, \theta)\mathrm{e}^{\mathrm{i}(m\phi+\omega t)}, \tag{1}$$

where spherical polar coordinates (r, θ, ϕ) with unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ and $\theta = 0$ at the axis of rotation are adopted, m denotes the azimuthal wavenumber of convective instability with $m \geq 1$, the function $F(r, \theta)$ obeys the equatorial parity $F(r, \theta) = F(r, \pi - \theta)$, and ω denotes the frequency of the instability which is small for columnar rolls [2] but of order unity for thermal-inertial waves [3]. Near the threshold $0 < [Ra - (Ra)_c]/(Ra)_c \ll 1$, the key nonlinear property of convection is largely predictable on the basis of linear solutions: the primary solution bifurcating from the instability has constant amplitude with constant kinetic energy, is axially nonsymmetric and equatorially symmetric, and contains a weak zonal flow [see, for example, 5, 10].

Sanchez et al. [11] showed unexpectedly, via careful numerical simulation under a poloidal and toroidal decomposition, that the physically preferred mode of convection in a special regime of the Prandtl number Pr with the stress-free boundary condition is in the form of axially symmetric (invariant under rotation about the axis of rotation), equatorially antisymmetric, and temporally oscillatory (in the form of oscillatory fluid motion). This is referred to as the convective torsional instability. Zhang et al. [12] derived an asymptotic solution of the torsional instability in rapidly rotating fluid spheres, showing that its azimuthal component at the onset of convection can be expressed as

$$\hat{\boldsymbol{\phi}} \cdot \mathbf{u}(r, \theta, \phi, t) = G(r, \theta)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where the function G obeys the equatorial symmetry $G(r, \theta) = -G(r, \pi - \theta)$. The linear asymptotic solution [12] is in satisfactory quantitative agreement with the corresponding numerical solution [11]. It should be stressed that (2) is profoundly different from (1): (2) represents axially symmetric, equatorially antisymmetric, oscillatory flows while (1) describes axially nonsymmetric, equatorially symmetric, azimuthally traveling waves.

When linear convective instability (the most unstable mode) is in the form of axially symmetric and equatorially antisymmetric oscillation described by (2), its nonlinear developments near the threshold $0 < [Ra - (Ra)_c]/(Ra)_c \ll 1$ cannot be predicted on the basis of the result of linear stability analysis. This is because the equatorial symmetry at the onset of convection must be broken by nonlinear effects even near the threshold and the kinetic energy of the primary solution, in contrast to the equatorially symmetric wave described by (1), must be time-dependent. By performing careful numerical simulation near the threshold $0 < [Ra - (Ra)_c]/(Ra)_c \ll 1$, we reveal that the primary nonlinear solution near the threshold of the convective torsional instability is in the form of an axially symmetric, equatorially nonsymmetric and latitudinally propagating nonlinear wave whose amplitude varies periodically, representing a new nonlinear phenomenon in the classical problem of thermal convection in rapidly rotating fluid spheres. In what follows we begin by presenting the governing equations of the problem in §2 which is followed by the discussion of the result of nonlinear convection in §3. A summary and some remarks are presented in §4.

II. MATHEMATICAL FORMULATION

Consider the problem of thermal convection in a Boussinesq fluid sphere of radius r_o with constant thermal diffusivity κ, thermal expansion coefficient α and kinematic viscosity ν [see, for example, 1, 2, 4]. The fluid sphere rotates uniformly with constant angular velocity $\hat{\mathbf{z}}\Omega$ in the presence of its own gravitational field $-\gamma\mathbf{r}$, where γ is a positive constant and $\mathbf{r}$ is the position vector. The whole sphere is heated by a uniform distribution of heat sources, producing the unstable conducting temperature gradient $-\beta\mathbf{r}$, β being a positive constant. When β is sufficiently large, convective instability takes place and drives fluid motion in the sphere.

Upon employing the radius of the sphere r_o as the length scale, $1/\Omega$ as the unit of time and $\beta r_o^4\Omega/\kappa$ as the unit of temperature fluctuation, the problem of thermal convection is governed by the dimensionless equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\hat{\mathbf{z}}\times\mathbf{u} = -\boldsymbol{\nabla}p + Ra\Theta\mathbf{r} + E\boldsymbol{\nabla}^2\mathbf{u}, \quad (3)$$

$$(Pr/E)\left(\frac{\partial\Theta}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Theta\right) = \mathbf{u}\cdot\mathbf{r} + \boldsymbol{\nabla}^2\Theta, \quad (4)$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \quad (5)$$

where t is time, Θ represents the deviation of the temperature from its static distribution, p is the total pressure and $\mathbf{u}$ is the three-dimensional velocity field. All the variables are non-dimensional. The three non-dimensional

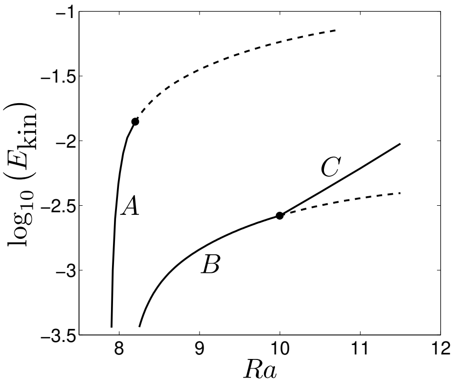

FIG. 1: Average kinetic energy density $E_{\rm kin}$ as a function of Ra for $E = 10^{-3}$ and $Pr = 0.01$. Three different branches are identified: Branch A for axially symmetric and latitudinally propagating waves, Branch B for axially nonsymmetric, equatorially symmetric and azimuthally traveling waves, and Branch C for the latitudinally propagating waves modulated by the azimuthally traveling waves.

parameters, the Rayleigh number Ra, the Prandtl number Pr and the Ekman number E, are defined as

$$Ra = \frac{\alpha\beta\gamma r_o^4}{\Omega\kappa},\ Pr = \frac{\nu}{\kappa},\ E = \frac{\nu}{\Omega r_o^2}.$$

We focus on perfectly conducting, impenetrable and stress-free boundary conditions given by

$$\frac{\partial(\hat{\boldsymbol{\phi}}\cdot\mathbf{u}/r)}{\partial r} = \frac{\partial(\hat{\boldsymbol{\theta}}\cdot\mathbf{u}/r)}{\partial r} = \hat{\mathbf{r}}\cdot\mathbf{u} = \Theta = 0. \quad (6)$$

Our numerical analysis, for the purpose of simulating nonlinear convection in the whole sphere, adopts a spherical shell marked by a very small inner core whose radius r_i is given by $r_i/r_o = 0.001$.

III. LATITUDINALLY PROPAGATING WAVES

According to both the numerical analysis [11] and the asymptotic analysis [12], the torsional convective instability (which is axially symmetric, equatorially antisymmetric, and temporally oscillatory) is physically preferred in rapidly rotating spheres in a special regime of the Prandtl number. It is found in the asymptotic analysis [12] that $E \leq 10^{-3}$ is sufficiently small to be in the asymptotic regime in the sense that an asymptotic solution with $E \leq 10^{-3}$ shows a satisfactory agreement with the corresponding fully numerical solution. Our numerical simulation is therefore to concentrate on a moderately small Ekman number $E = 10^{-3}$ that would be sufficiently small for the purpose of illustrating a new nonlinear convection phenomenon in rotating spheres. Several computations at $E = 10^{-4}$ are also performed to confirm a similar behavior to that for $E = 10^{-3}$.

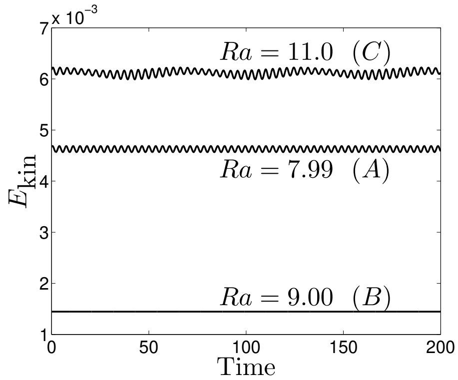

FIG. 2: Kinetic energy density $E_{\rm kin}(t)$ of nonlinear convection as a function of time for three typical values of the Rayleigh number, $Ra = 7.99, 9.0, 11.0$ for $E = 10^{-3}$ and $Pr = 0.01$.

To help understand the new nonlinear phenomenon reported in this paper, it is desirable to provide a brief review of the torsional convective instability at the onset of convection. The leading-order linear solution, described by its pressure p_0, its velocity $\mathbf{u}_0$ and its temperature Θ_0 in rapidly rotating spheres, is given by

$$p_0 = \mathcal{A}\left(\frac{3}{2} - 3r^2 + \frac{5}{2}r^2\cos^2\theta\right) r\cos\theta\cos\frac{2t}{\sqrt{5}}, \quad (7)$$

$$\hat{\mathbf{r}}\cdot\mathbf{u}_0 = -\mathcal{A}\frac{3\sqrt{5}}{4}\left(1-r^2\right)\cos\theta\ \sin\frac{2t}{\sqrt{5}}, \quad (8)$$

$$\hat{\boldsymbol{\theta}}\cdot\mathbf{u}_0 = -\mathcal{A}\frac{3\sqrt{5}}{4}\left(2r^2-1\right)\sin\theta\ \sin\frac{2t}{\sqrt{5}}, \quad (9)$$

$$\hat{\boldsymbol{\phi}}\cdot\mathbf{u}_0 = -\mathcal{A}\frac{15}{8}r^2\sin2\theta\ \cos\frac{2t}{\sqrt{5}}, \quad (10)$$

$$\Theta_0 = \mathcal{A}\sum_{l,q}\frac{2\pi P_l(\cos\theta)j_l(\beta_{lq}r)}{\left[(\beta_{lq})^2 + \mathrm{i}(2/\sqrt{5})Pr/E\right]} \times\int_0^\pi\int_0^1 \mathbf{r}\cdot\mathbf{u}_0 P_l(\cos\theta) j_l\left(\beta_{lq}r\right) r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta, \quad (11)$$

where $\mathcal{A}$ represents a small amplitude, j_l denotes the spherical Bessel function of the first kind, P_l is the Legendre function, β_{lq} with $q = 1, 2, 3, \ldots$ are solutions of $j_l\left(\beta_{lq}\right) = 0$ and ordered such that $0 < \beta_{l1} < \beta_{l2} < \beta_{l3} < \ldots$. Note that the convective instability is axially symmetric ($\partial/\partial\phi = 0$), equatorially antisymmetric $\hat{\boldsymbol{\phi}}\cdot\mathbf{u}_0(r,\theta,t) = -\hat{\boldsymbol{\phi}}\cdot\mathbf{u}_0(r,\pi-\theta,t)$, and temporally oscillatory ($\partial/\partial t \neq 0$). Note also that the solution (7)–(11) represents the most unstable mode of convection in some regime of the parameters [11]. For $E = 10^{-3}$ and $Pr = 0.01$, the torsional convective instability, according to the asymptotic analysis, takes place at the critical Rayleigh number $(Ra)_c = 7.8$ with the critical frequency $\omega_c = 0.89$.

We are mainly concerned with the property of the primary nonlinear solution when the Rayleigh number Ra is slightly supercritical, *i.e.*, $0 < (Ra - 7.8)/7.8 \ll 1$, where E and Pr are fixed. It is significant to notice that the linear solution (7)–(11) is equatorially antisymmetric but this symmetry is not allowed by the governing nonlinear equations (3)–(5). This is why the primary nonlinear solution at $0 < (Ra - 7.8)/7.8 \ll 1$ must break the equatorial symmetry of the linear solution (7)–(11) and why weakly nonlinear convection in connection with the torsional convective instability can be spatially complicated. To measure the amplitude of convection, we introduce the kinetic energy density $E_{\rm kin}$ of the flow $\boldsymbol{u}$ defined as $E_{\rm kin}(t) = \frac{3}{8\pi}\int_0^{2\pi}\int_0^\pi\int_0^1 |\boldsymbol{u}(t)|^2 r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$. An extensive numerical simulation for $E = 10^{-3}$ and $Pr = 0.01$ is carried out for many different values of Ra. The results of the simulation are summarized in Figure 1 where the three different branches of nonlinear solutions are identified. We focus on Branch A, the primary nonlinear solution bifurcating from the torsional convective instability. Kinetic energy densities $E_{\rm kin}$ of a typical primary solution as a function of time obtained for $Ra = 7.99$ – which belongs to Branch A in Figure 1 – are presented in Figure 2. It can be seen that the kinetic energy $E_{\rm kin}(t)$ of the primary solution changes periodically with a period of about $T = 7.1$ as suggested by (7)–(11). The corresponding spatial structure as a function of time is depicted in Figure 3 in a meridional plane. The primary nonlinear solution is marked by the four key features: (i) the convective flow is still axially symmetric; (ii) it is neither equatorially antisymmetric nor equatorially symmetric; (iii) it represents latitudinally propagating nonlinear waves; and (iv) its amplitude varies periodically. Given that the linear azimuthal flow is in the form of simple oscillatory flow, $\hat{\boldsymbol{\phi}}\cdot\mathbf{u}_0 \sim r^2\sin2\theta\cos\frac{2t}{\sqrt{5}}$ for $Ra = 7.8$, the complicated latitudinally propagating waves depicted in Figure 3 for $Ra = 7.99$ are unexpected.

This new form of nonlinear convection occurs only in a range of small supercritical Rayleigh numbers. When the Rayleigh number increases further, the axially symmetric convection in Figure 3 is replaced by the axially nonsymmetric, equatorially symmetric and retrogradely traveling wave with azimuthal wavenumber $m = 1$, which is labeled as Branch B in Figure 2. The property of Branch B is well understood: it represents non-axisymmetric and azimuthally traveling waves whose dominant spatial structure is essentially the same as that of the linear solution at the onset of convection; the flow has constant kinetic energy (which is shown in Figure 2 for $Ra = 9.00$); and its spatial structure is time-independent in a drifting frame. When the Rayleigh number increases even further, the secondary solution is to be replaced by the tertiary solution, labeled by Branch C in Figure 1, which is neither axially symmetric nor equatorially symmetric. The tertiary solution is, as expected, characterized by a mixed feature of the primary and secondary solutions: a typical tertiary solution is presented in Figure 2 for $Ra = 11.00$.

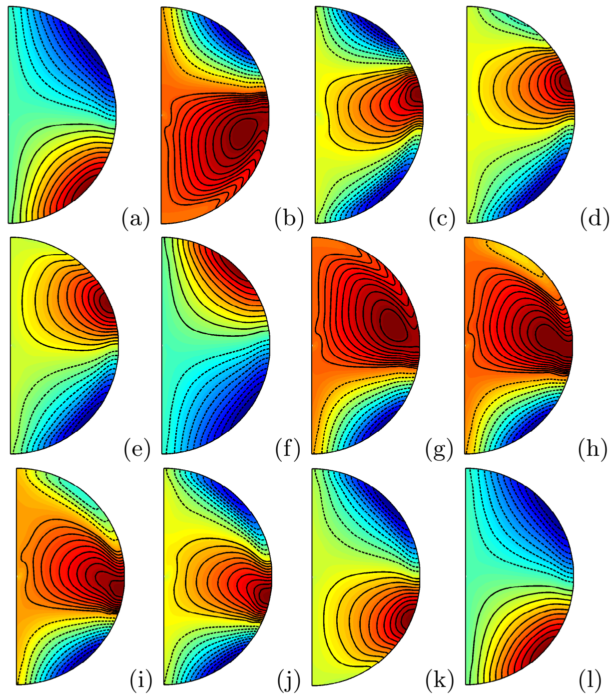

FIG. 3: Contours of $\hat{\boldsymbol{\phi}} \cdot \mathbf{u}$ in a meridional plane at 12 different instants in one period for $E = 10^{-3}$ and $Pr = 0.01$ at $Ra = 7.99$, showing the axially symmetric ($\partial/\partial\phi = 0$) and latitudinally propagating nonlinear waves.

IV. SUMMARY AND REMARKS

We have unveiled a new nonlinear phenomenon in the classical problem of thermal convection in rapidly rotating, self-gravitating, internally heated Boussinesq fluid spheres. In a parameter regime marked by $Pr/E \approx 10$ and $E \ll 1$ [11, 12], the most unstable mode of convection is characterized by axially symmetric, equatorially antisymmetric torsional oscillation. We have shown that, when the Rayleigh number Ra is in the vicinity of its critical value, weakly nonlinear convection is in the form of axially symmetric, equatorially nonsymmetric and latitudinally propagating waves depicted in Figure 3. The latitudinally propagating wave represents a new pattern of nonlinear convection in the problem of thermal convection rapidly rotating fluid spheres.

It was P.H. Roberts [1] who first discussed the possibility of equatorially antisymmetric convective instability in rapidly rotating, self-gravitating, internally heated Boussinesq fluid spheres. After nearly fifty years, both the numerical analysis [11] and the asymptotic analysis [12] have revealed that the equatorially antisymmetric mode can be indeed physically preferred in some regime of the physical parameters. In contrast to the equatorially symmetric mode [see, for example, 2, 4] whose primary bifurcation solution has the same equatorial symmetry, we have demonstrated that the equatorially antisymmetric torsional oscillation [11, 12] merely represents a linear state of thermal convection: even weakly nonlinear effects in real physical systems would destroy the equatorial symmetry described by (7)–(11) and lead to the latitudinally propagating waves in rotating fluid spheres depicted in Figure 3.

Acknowledgments

This work was first reported in the conference " Fifty years after P. H. Roberts' MHD: Dynamos and Planetary Flows Today" at the Royal Astronomical Society, London, 16–17 November, 2017. KZ would like to take this opportunity to thank P.H. Roberts for his helpful discussions over the past several decades. KZ is supported by Leverhulme Research Project Grant RPG-2015-096, by STFC Grant ST/R000891/1, by Macau FDCT grants 119/2017/A3, 007/2016/A1 and 001/2016/AFJ, and by the CAS grant XDB18010203.

---

[1] Roberts, P. H. 1968. On the thermal instability of a rotating–fluid sphere containing heat sources. *Phil. Trans. R. Soc. Lond.* **A263**, 93-117.

[2] Busse, F. H. 1970. Thermal instabilities in rapidly rotating systems. *J. Fluid Mech.* **44**, 441-460.

[3] Zhang, K. 1994. On coupling between the Poincaré equation and the heat equation. *J. Fluid Mech.* **268**, 211-229.

[4] Jones, C.A., Soward, A.M. and Mussa, A.I. 2000. The onset of thermal convection in a rapidly rotating sphere. *J. Fluid Mech.* **405**, 157-179.

[5] Christensen, U.R. 2002. Zonal flow driven by strongly supercritical convection in rotating spherical shells. *J. Fluid Mech.* **470**, 115–133.

[6] Garcia, F., Sanchez, J., and Net, M. 2008. Antisymmetric polar modes of thermal convection in rotating spherical fuid shells at high Taylor numbers. *Phys. Rev. Lett.* **101**, 194501(1–4).

[7] Guervilly, C. and Cardin, P. 2016. Subcritical convection of liquid metals in a rotating sphere using a quasi-geostrophic model. *J. Fluid Mech.* **808**, 61–89.

[8] Garcia, F., Chambers, F. R. N., and Watts, A. L. Onset of low Prandtl number thermal convection in thin spherical shells *Phys. Rev. Fluids* **3**, 024801

[9] Miesch, M. S. 2005. Large-scale dynamics of the convection zone and tachocline. *Living Reviews in Solar Physics*, 2(1)

[10] Zhang, K. 1992. Spiralling columnar convection in rapidly rotating spherical fluid shells. *J. Fluid Mech.* **236**, 535-556.

[11] Sanchez, J., Garcia, F. and Net, M. 2016. Critical torsional modes of convection in rotating fluid spheres at high Taylor numbers. *J. Fluid Mech.* **791**, R1.

[12] Zhang, K., Lam, K. and Kong, D. 2017. Asymptotic theory for torsional convection in rotating fluid spheres. *J. Fluid Mech.* **813**, R2.