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\mathcal{PT} -symmetry and its spontaneous breakdown explained by anti-linearity

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Abstract

The impact of an anti-unitary symmetry on the spectrum of non-hermitean operators is studied. Wigner's normal form of an anti-unitary operator is shown to account for the spectral properties of non-hermitean, \mathcal{PT} -symmetric Hamiltonians. Both the occurrence of single real or complex conjugate pairs of eigenvalues follows from this theory. The corresponding energy eigenstates span either one- or two-dimensional irreducible representations of the symmetry \mathcal{PT} . In this framework, the concept of a spontaneously broken \mathcal{PT} -symmetry is not needed.

Deep in their hearts, many quantum physicists will renounce hermiticity of operators only reluctantly. However, non-hermitean Hamiltonians are applied successfully in nuclear physics, biology and condensed matter, often modelling the interaction of a quantum system with its environment in a phenomenological way. Since 1998, non-hermitean Hamiltonians continue to attract interest from a conceptual point of view [1]: surprisingly, the eigenvalues of a one-dimensional harmonic oscillator Hamiltonian remain *real* when the *complex* potential $\hat{V} = i\hat{x}^3$ is added to it. Numerical, semiclassical, and analytic evidence [2] has been accumulated confirming that bound states with *real* eigenvalues exist for the vast class of *complex* potentials satisfying $V^\dagger(\hat{x}) = V(-\hat{x})$. In addition, pairs of complex conjugate eigenvalues occur systematically.

\mathcal{PT} -symmetry has been put forward to explain the observed energy spectra. The Hamiltonian operators \hat{H} under scrutiny are invariant under the combined action of parity \mathcal{P} and time reversal \mathcal{T} ,

$$[\hat{H}, \mathcal{PT}] = 0. \quad (1)$$

They act on the fundamental observables according to

$$\mathcal{P} : \begin{cases} \hat{x} \rightarrow -\hat{x}, \\ \hat{p} \rightarrow -\hat{p}, \end{cases} \quad \mathcal{T} : \begin{cases} \hat{x} \rightarrow \hat{x}, \\ \hat{p} \rightarrow -\hat{p}, \end{cases} \quad (2)$$

and \mathcal{T} *anti-commutes* with the imaginary unit,

$$\mathcal{T}i = i^*\mathcal{T} \equiv -i\mathcal{T}. \quad (3)$$

Whenever a \mathcal{PT} -symmetric Hamiltonian has a *real* eigenvalue E , the associated eigenstate $|E\rangle$ is found to be an eigenstate of the symmetry \mathcal{PT} ,

$$E = E^* : \quad \hat{H}|E\rangle = E|E\rangle, \quad \mathcal{PT}|E\rangle = +|E\rangle. \quad (4)$$

Occasionally, $\mathcal{PT}|E\rangle = -|E\rangle$ occurs [3] which is equivalent to (4) upon redefining the phase of the state: $\mathcal{PT}(i|E\rangle) = +(i|E\rangle)$. There is no difference between symmetry and anti-symmetry under \mathcal{PT} .

However, if the eigenvalue E is *complex*, the operator \mathcal{PT} does *not* map the corresponding eigenstate of \hat{H} to itself,

$$E \neq E^* : \quad \hat{H}|E\rangle = E|E\rangle, \quad \mathcal{PT}|E\rangle \neq \lambda|E\rangle, \text{ any } \lambda. \quad (5)$$

This situation is described as a ‘spontaneous breakdown’ of \mathcal{PT} -symmetry. No mechanism has been identified which would explain this breaking of the symmetry.

The \mathcal{PT} -symmetric square-well model provides a simple example for this behavior [4]. It describes a particle moving between reflecting boundaries at $x = \pm 1$, in the presence of a piecewise constant complex potential,

$$V_Z(x) = \begin{cases} iZ, & x < 0, \\ -iZ, & x > 0, \end{cases} \quad Z \in \mathbb{R}. \quad (6)$$

Acceptable solutions of Schrödinger’s equation must satisfy both the boundary conditions, $\psi(\pm 1) = 0$, and continuity conditions at the origin. As long as the value of the parameter Z is below a critical value, $Z < Z_0^c$, the eigenvalues E_n of the non-hermitean Hamiltonian $\hat{H} = -\partial_{xx} + V_Z(x)$ are real, and each eigenstate $|\psi_n\rangle$ satisfies the relations (4), with eigenvalues E_n and $+1$, respectively. Above the threshold, $Z > Z_0^c$, at least one pair of complex conjugate eigenvalues E_0 and E_0^* develops. One of the corresponding eigenstates has the form [4]

$$\psi_0(x) = \begin{cases} K_p \sinh \kappa(1-x), & x > 0, \\ K_n \sinh \lambda^*(1+x), & x < 0, \end{cases} \quad (7)$$

the complex parameters κ, λ, K_n , and K_p being determined by the boundary and continuity conditions. The state $\psi_0(x)$ is not invariant under \mathcal{PT} , i.e. (5) holds.

The purpose of the present contribution is a group-theoretical analysis of \mathcal{PT} -symmetry. The properties of \mathcal{PT} -symmetric systems are explained in a natural way by taking into account that \mathcal{PT} is not a unitary but an *anti-unitary* symmetry of a *non-hermitean* operator. The argument proceeds in three steps. First, Wigner’s normal form of anti-unitary operators is reviewed, i.e. their (irreducible) representations are identified. Second, the properties of non-hermitean operators with anti-unitary symmetry are derived. These results are then shown to account for the characteristic features of \mathcal{PT} -symmetric systems.

Wigner develops a normal form of anti-unitary operators \hat{A} in [5]. Anti-unitarity of \hat{A} is defined by the relation

$$\langle \hat{A}\chi | \hat{A}\psi \rangle = \langle \psi | \chi \rangle \quad (8)$$

and it implies anti-linearity,

$$\hat{A}(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha^* \hat{A}|\psi\rangle + \beta^* \hat{A}|\chi\rangle. \quad (9)$$

which is equivalent to (3). The representation theory of \hat{A} relies on the fact that the square of an anti-unitary operator is *unitary*:

$$\langle \hat{A}^2\chi | \hat{A}^2\psi \rangle = \langle \hat{A}\psi | \hat{A}\chi \rangle = \langle \chi | \psi \rangle. \quad (10)$$

Therefore, the operator \hat{A}^2 has a complete, orthonormal set of eigenvectors $|\Omega\rangle$ with eigenvalues Ω of modulus one,

$$\hat{A}^2|\Omega\rangle = \Omega|\Omega\rangle, \quad |\Omega| = 1. \quad (11)$$

It plays the role of a Casimir-type operator labelling different representations of \hat{A} . Wigner distinguishes three different types of representations corresponding to the eigenvalues of \hat{A}^2 : complex Ω ($\neq \Omega$), $\Omega = +1$, or $\Omega = -1$, summarized in Table (1).

1. An eigenstate $|\Omega\rangle$ of \hat{A}^2 with eigenvalue Ω ($\neq \Omega^*$) is not invariant under \hat{A} . Instead, the states $|\Omega\rangle$ and $|\Omega^*\rangle \equiv \hat{A}|\Omega\rangle$ constitute a ‘flipping pair’ with complex ‘flipping value’ ω (and ω^*), where $\omega^2 = \Omega$. They span a two-dimensional space which is closed under the action of \hat{A} . Therefore, it carries a two-dimensional representation of \hat{A} , denoted by Γ_* , which is *irreducible*: due to the anti-linearity of \hat{A} , no (non-zero) linear combination of the flipping states exist which is invariant under \hat{A} .
2. Similarly, if \hat{A}^2 has an eigenvalue $\Omega = -1$, then the operator \hat{A} flips the states $|-\rangle$ and $|-*\rangle \equiv \hat{A}|-\rangle$. The flipping value is i , and the associated two-dimensional representation Γ_- is not reducible.
3. Two different situations arise if there is an eigenstate $|1\rangle$ of \hat{A}^2 with eigenvalue $+1$. The state $\hat{A}|1\rangle$ is either a multiple of itself or not. In the first case, the space spanned by $|1\rangle$ is invariant under \hat{A} and hence carries a one-dimensional representation γ_+ of \hat{A} . When redefining the phase of the state appropriately, one obtains an eigenstate $|1\rangle$ of \hat{A} with eigenvalue $+1$. In the second case, the two states $|+\rangle \equiv |1\rangle$ and $|+*\rangle \equiv \hat{A}|1\rangle$ provide a flipping pair with flipping value $\omega = +1$, and hence a representation Γ_+ . This representation, however, is *reducible* due to the reality of the flipping value: the linear combinations $|1_r\rangle = |+\rangle + |+*\rangle$ and $|1_i\rangle = i(|+\rangle - |+*\rangle)$ are both eigenstates of \hat{A} with eigenvalue $+1$.

$\Omega \equiv \omega^2$	Γ	action of \hat{A}	$\dim \Gamma$
$\Omega \neq \Omega^*$	Γ_*	$\hat{A} \Omega\rangle = \omega^* \Omega^*\rangle$ $\hat{A} \Omega^*\rangle = \omega \Omega\rangle$	2
-1	Γ_-	$\hat{A} -\rangle = -i -*\rangle$ $\hat{A} -*\rangle = +i -\rangle$	2
+1	Γ_+	$\hat{A} +\rangle = + +^*\rangle$ $\hat{A} +^*\rangle = + +\rangle$	2
+1	γ_+	$\hat{A} 1\rangle = + 1\rangle$	1

Table 1: Representations Γ of the operator \hat{A}

Consequently, a Hilbert space \mathcal{H} naturally decomposes into a direct product of invariant subspaces, each invariant under the action of the anti-unitary operator \hat{A} ,

$$\mathcal{H} = \Gamma_*^{\otimes N_*} \otimes \Gamma_-^{\otimes N_-} \otimes \Gamma_+^{\otimes N_+} \otimes \gamma_+^{\otimes n_+}; \quad (12)$$

the nonnegative integers N_* , N_{\pm} and n_+ are related to the degeneracies of the eigenvalues $\Omega (\neq \Omega^*)$ and $\Omega = \pm 1$ of the operator \hat{A}^2 . The corresponding decomposition of a vector $|\psi\rangle \in \mathcal{H}$ is the closest analog of an expansion into the eigenstates of a hermitean (or unitary) operator. Surprisingly, *two-dimensional* irreducible representations of \hat{A} exist although there is only one generator, \hat{A} . No ‘good quantum number’ exists which would label the states spanning these representations.

A (diagonalizable) *non-hermitean* Hamiltonian \hat{H} with a discrete spectrum [6] and its adjoint \hat{H}^\dagger each have a complete set of eigenstates:

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle, \quad \hat{H}^\dagger|\psi^n\rangle = E^n|\psi^n\rangle, \quad (13)$$

with complex conjugate eigenvalues related by $E^n = E_n^*$. They form a *bi-orthonormal* basis in \mathcal{H} , as they provide two resolutions of unity,

$$\sum_n |\psi^n\rangle\langle\psi_n| = \sum_n |\psi_n\rangle\langle\psi^n| = \hat{I}, \quad (14)$$

and satisfy orthogonality relations,

$$\langle\psi_m|\psi^n\rangle = \delta_m^n. \quad (15)$$

Let the non-hermitean operator \hat{H} have an anti-unitary symmetry \hat{A} ,

$$[\hat{H}, \hat{A}] = 0. \quad (16)$$

Then the unitary operator \hat{A}^2 commutes with \hat{H} , and it has eigenvalues Ω of modulus one. Consequently, there are simultaneous eigenstates $|n, \Omega\rangle$ of \hat{H} and \hat{A}^2 :

$$\hat{H}|n, \Omega\rangle = E_n|n, \Omega\rangle, \quad \hat{A}^2|n, \Omega\rangle = \Omega|n, \Omega\rangle, \quad E_n \in \mathbb{C}. \quad (17)$$

For simplicity, the eigenvalues Ω are assumed discrete and not degenerate. Wigner's normal form of anti-unitary operators suggests to consider three cases separately: complex $\Omega (\neq \Omega^*)$ and $\Omega = \pm 1$.

$\Omega \neq \Omega^*$ The state

$$|n, \Omega^*\rangle \equiv \omega \hat{A}|n, \Omega\rangle, \quad \omega^2 = \Omega, \quad (18)$$

is a second eigenstate of \hat{A}^2 , with eigenvalue Ω^* . The states $\{|n, \Omega\rangle, |n, \Omega^*\rangle\}$ provide a *flipping pair* under the action of the operator \hat{A} ,

$$\hat{A}|n, \Omega\rangle = \omega^*|n, \Omega^*\rangle, \quad \hat{A}|n, \Omega^*\rangle = \omega|n, \Omega\rangle, \quad (19)$$

carrying the representation Γ_* . No degeneracy of the eigenvalue E_n is implied by the anti-unitary \hat{A} -symmetry of \hat{H} . However, the non-hermitean Hamiltonian has a second eigenstate $|n, \Omega^*\rangle$ with eigenvalue E_n^* ,

$$\hat{H}|n, \Omega^*\rangle = E_n^*|n, \Omega^*\rangle, \quad (20)$$

as follows from multiplying the first equation of (17) with \hat{A} and ω .

$\Omega = -1$ Formally, the results for the representation Γ_- are obtained from the previous case by setting $\omega = i$. Again, a pair of complex conjugate eigenvalues is found, and the associated flipping pair spans a two-dimensional representation space.

$\Omega = +1$ This case is conceptually different from the previous ones as two possibilities arise. Consider the state $|n, +\rangle$, an eigenvector of both \hat{H} and \hat{A}^2 with eigenvalues E_n and $+1$, respectively. It satisfies Eqs. (17) with $\Omega \rightarrow +$. If, on the one hand, applying \hat{A} to $|n, +\rangle$ results in $e^{i\phi}|n, +\rangle$, then the state $|n, 1\rangle \equiv e^{-i\phi/2}|n, +\rangle$ is an eigenstate of \hat{A} with eigenvalue $+1$,

$$\hat{A}|n, 1\rangle = |n, 1\rangle. \quad (21)$$

This occurrence of the one-dimensional representation γ_+ forces the associated eigenvalue E_n of \hat{H} to be real since

$$E_n|n, 1\rangle = \hat{H}\hat{A}|n, 1\rangle = \hat{A}\hat{H}|n, 1\rangle = E_n^*|n, 1\rangle. \quad (22)$$

If, on the other hand, $|n, +^*\rangle \equiv \hat{A}|n, +\rangle$ is *not* a multiple of $|n, +\rangle$, then these states combine to form the representation Γ_+ , the flipping value being $+1$. Further, the state $|n, +^*\rangle$ is an eigenstate of the Hamiltonian with eigenvalue E_n^* . As the flipping number is real, linear combinations of $|n, +\rangle$ and $|n, +^*\rangle$ do exist which are eigenstates of \hat{A} —however, they are not eigenstates of \hat{H} . Consequently, the anti-unitary symmetry of the Hamiltonian makes itself felt (on a subspace with $(\mathcal{PT})^2 = +\hat{I}$) by either a single real eigenvalue or a pair of two complex conjugate eigenvalues.

If any of the two-dimensional representations Γ_* or Γ_{\pm} occurs and the associated eigenvalue happens to be real, the anti-unitary symmetry implies a twofold degeneracy

of the energy eigenvalue. Again, the symmetry provides no additional label, and simultaneous eigenstates of \hat{H} and \hat{A} can be constructed for Γ_+ only. These cases will be denoted by Γ_*^d or Γ_\pm^d .

It will be shown now that the properties of \mathcal{PT} -symmetric quantum systems are consistent with the representation theory of non-hermitean Hamiltonians possessing an anti-unitary symmetry. Upon identifying

$$\hat{A} = \mathcal{PT}, \quad (23)$$

one needs to check the value of $(\mathcal{PT})^2$ when applied to eigenstates of the Hamiltonian in order to decide which of the representations Γ_* , Γ_\pm , or γ_+ , is realized. Various explicit examples will be given now.

For parameters $Z < Z_0^c$, the eigenvalues of the \mathcal{PT} -symmetric square-well are real throughout, and the operators \hat{H} and \mathcal{PT} have common eigenstates. Thus, the relations (4) correspond to a multiple occurrence of the representation γ_+ , compatible with $(\mathcal{PT})^2 = +\hat{I}$.

For $Z > Z_0^c$, the energy eigenstate $\psi_0(x) \equiv \langle x|E_0, +\rangle$ in (7) satisfies $(\mathcal{PT})^2|E_0, +\rangle = +|E_0, +\rangle$. Therefore, the states $|E_0, +\rangle$ and $|E_0, +^*\rangle \equiv \mathcal{PT}|E_0, +\rangle$ carry a representation Γ_+ , and the presence of two complex energy eigenvalues, E_0 and E_0^* is justified. Eqs. (5) can be completed to read:

$$E \neq E^* : \quad \begin{aligned} \hat{H}|E_0, +\rangle &= E_0|E_0, +\rangle, & \mathcal{PT}|E_0, +\rangle &= +|E_0, +^*\rangle, \\ \hat{H}|E_0, +^*\rangle &= E_0^*|E_0, +^*\rangle, & \mathcal{PT}|E_0, +^*\rangle &= +|E_0, +\rangle. \end{aligned} \quad (24)$$

Consequently, \mathcal{PT} -symmetry is not broken but at $Z = Z_0^c$ the system switches between the representations Γ_+ and γ_+ , with a corresponding change of the energy spectrum.

The following examples are taken from a discrete family of non-hermitean operators [7],

$$\hat{H}_M = \hat{p}^2 - (\zeta \cosh 2x - iM)^2, \quad \zeta \in \mathbb{R}, \quad (25)$$

M taking positive integer values. Each operator \hat{H}_M is invariant under the combined action of \mathcal{PT} where \mathcal{P} is parity about the point $a = i\pi/2$: $x \rightarrow i\pi/2 - x$. Due to the reflection about a point off the real axis, the operators \mathcal{P} and \mathcal{T} do not commute as has been pointed out in [8]. However, this fact is not essential here since only the anti-unitary character of the symmetry \mathcal{PT} is essential.

For $M = 2$, two complex conjugate eigenvalues E_+ and $E_- = E_+^*$ of \hat{H}_2 exist, with associated eigenstates

$$\psi_+(x) = \Psi(x) \cosh x \equiv \langle x|E_+, -\rangle, \quad \psi_-(x) = \Psi(x) \sinh x \equiv \langle x|E_+, -^*\rangle, \quad (26)$$

and a \mathcal{PT} -invariant function $\Psi(x) = \exp[(i/2)\zeta \cosh 2x]$. These states are a flipping pair with flipping value i ,

$$\mathcal{PT}\psi_+(x) = -i\psi_-(x), \quad \mathcal{PT}\psi_-(x) = i\psi_+(x), \quad (27)$$

and the twofold application of \mathcal{PT} gives (-1) . Hence, the representation Γ_- is realized. Similarly, for $M = 4$, four eigenstates form two flipping pairs, i.e. two representations Γ_- , each being associated with a pair of complex conjugate eigenvalues.

For $M = 3$, three different real eigenvalues of the Hamiltonian \hat{H}_3 have been obtained analytically if $\zeta^2 < 1/4$. The corresponding eigenfunctions are given by

$$\psi(x) = \Psi(x) \sinh 2x, \quad \psi_{\pm}(x) = \Psi(x)(A \cosh 2x \pm iB), \quad (28)$$

with real coefficients A and B . Under the action of \mathcal{PT} , the state $\psi(x)$ is mapped to itself, while $\psi_{\pm}(x)$ each acquire an additional minus sign. Therefore, the states $\psi(x) \equiv \langle x|E, +\rangle$ and $i\psi_{\pm}(x) \equiv \langle x|E_{\pm}\rangle$ are simultaneous eigenstates of \hat{H} and \mathcal{PT} with eigenvalues $+1$. The part of Hilbert space spanned by these three states transforms according to three copies of the representation γ_+ . If $\zeta = 1/2$, the eigenvalues E_{\pm} turn degenerate, and the eigenstates given in (28) merge, $i\psi_+(x) = i\psi_-(x) \equiv \varphi(x)$. However, a second, independent \mathcal{PT} -invariant solution of Schrödinger's equation can be found,

$$\phi(x) = \Psi(x) \int_{x_0}^x dy \frac{e^{-i\varphi(y)/2}}{\varphi^2(y)}. \quad (29)$$

The solutions $\{\varphi, \phi\}$ transform according to $\gamma_+ \otimes \gamma_+ \equiv \Gamma_+^d$. So far, the representation Γ_* has apparently not been realized in \mathcal{PT} -symmetric quantum systems—a possible explanation is the constraint $\mathcal{T}^2 = \pm 1$ for time reversal [9].

In summary, the representation theory of anti-unitary symmetries of non-hermitean ‘Hamiltonians’ has been developed on the basis of Wigner’s normal form of anti-unitary operators. Typically, energy eigenvalues come in complex conjugate pairs, and the associated eigenstates of the Hamiltonian span a two-dimensional space carrying one of the two-dimensional representations Γ_* , or Γ_{\pm} . Furthermore, a single real eigenvalue may occur, related to a one-dimensional representation γ_+ . In this case a single \hat{A} -invariant energy eigenstate exists while there are no simultaneous eigenstates of the Hamiltonian and the symmetry operator in the two-dimensional \hat{A} -invariant subspaces. Instead, flipping pairs of states can be identified. Generally, the symmetry does not imply the existence of degenerate eigenvalues—only if the Hamiltonian happens to have a real eigenvalue, a two-dimensional degenerate subspace may exist occasionally. These results naturally explain the properties of eigenstates and eigenvalues of \mathcal{PT} -symmetric quantum systems. In particular, it is not necessary to invoke the concept of a *spontaneously broken* \mathcal{PT} -symmetry. Contrary to a unitary or hermitean symmetry, the presence of an anti-unitary symmetry does not imply the existence of a set of simultaneous eigenstates of \hat{H} and \mathcal{PT} —simply because an anti-linear operator is not guaranteed to have a complete set of eigenstates. Finally, the present approach provides a new perspective on the suggested modification of the scalar product in Hilbert space [10] which will be presented elsewhere [11] in detail.

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