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Article

On the Basel Liquidity Formula for Elliptical Distributions

Janine Balter ^{1,†} and Alexander J. McNeil ^{2,*}  0000-0002-6137-2890

¹ Deutsche Bundesbank; janinba@web.de

² The York Management School, University of York; alexander.mcneil@york.ac.uk

* Correspondence: The York Management School, Freboys Lane, York YO10 5GD, UK; Tel.: +44-1904-325307

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Abstract: A justification of the Basel liquidity formula for risk capital in the trading book is given under the assumption that market risk-factor changes form a Gaussian white noise process over 10-day time steps and changes to P&L are linear in the risk-factor changes. A generalization of the formula is derived under the more general assumption that risk-factor changes are multivariate elliptical. It is shown that the Basel formula tends to be conservative when the elliptical distributions are from the heavier-tailed generalized hyperbolic family. As a by-product of the analysis a Fourier approach to calculating expected shortfall for general symmetric loss distributions is developed.

Keywords: Basel Accords; liquidity risk; risk measures; expected shortfall; elliptical distributions; generalized hyperbolic distributions.

1. Introduction

As a result of the fundamental review of the trading book (FRTB) ([Basel Committee on Banking Supervision 2013](#)) a new minimum capital standard for the trading book has emerged ([Basel Committee on Banking Supervision 2016](#)). Under this standard, banks are now required to calculate a liquidity-adjusted expected shortfall risk measure on a daily basis. This calculation is carried out at both the level of the whole trading book and the level of individual desks using an aggregation formula that is based on the concepts of liquidity horizons and square-root-of-time scaling.

The Basel liquidity formula uses the language of risk factors. These are the fundamental quantities such as asset prices, index values, interest rates and exchange rates that are required to value the various positions in the trading book at any point in time. In addition to prices and rates, the set of risk factors contains a number of market-observable parameters including implied volatilities, which are used as inputs to model-based formulas for the valuation of derivative securities such as options.

Every risk factor is assigned to a unique liquidity bucket j associated with a liquidity horizon LH_j which may be 10, 20, 40, 60 or 120 days. These horizons are conservative estimates of the amount of time that would be required to execute trades that would eliminate the portfolio's sensitivity to changes in these risk factors during a period of market illiquidity. For example, risk factors describing the price risk of large-cap equities are assigned to the bucket with the shortest horizon of 10 days; risk factors describing volatility risk for derivatives involving large-cap equities are given a risk horizon of 20 days; risk factors for structured credit instruments (e.g. CDOs) have the longest liquidity horizon of 120 days.

The liquidity formula requires that a series of expected shortfall charges are calculated with respect to 'shocks' to certain risk factors while other risk factors are held constant. The shocks are estimates of

the more extreme joint changes in risk-factor values that could occur over a fixed horizon of T days. For most banks, what this means in practice is that historical risk-factor changes for the selected risk factors over the horizon T are applied to the positions to obtain a so-called P&L or profit-and-loss distribution. While this P&L distribution can be obtained by full revaluation of the positions in the portfolio, most banks use a simpler approach in which they consider only the P&L resulting from first-order (delta) and possibly second-order (gamma) sensitivities to the risk-factor changes. Having obtained the distribution, the effect of the shock is computed by applying the expected shortfall risk measure.

To make the calculation explicit, we give the formula and notation as published on page 52 of the revised capital standard ([Basel Committee on Banking Supervision 2016](#)).

- let $T = LH_1$ denote the so-called base liquidity horizon of 10 days.
- Let $ES_T(P)$ denote the expected shortfall at horizon T and a 97.5% confidence level for a portfolio P with respect to shocks to all risk factors to which the positions in the portfolio are exposed.
- Let $ES_T(P, j)$ denote the expected shortfall at horizon T and a 97.5% confidence level for a portfolio P with respect to shocks to the risk factors which have a liquidity horizon of length LH_j or greater, with all other risk factors held fixed.

The liquidity-adjusted expected shortfall is

$$ES = \sqrt{(ES_T(P))^2 + \sum_{j \geq 2} \left(ES_T(P, j) \sqrt{\frac{LH_j - LH_{j-1}}{T}} \right)^2}. \quad (1)$$

The bank computes the expected shortfall charges $ES_T(P)$ and $ES_T(P, j)$ and evaluates the right-hand side of (1). The resulting number ES is then an important determinant of the bank's overall capital charge for trading activities. (There are a number of further adjustments and add-on charges that we will not go in to.)

The formula is very mysterious at first glance but some rough intuition can be gained by observing that the squared capital charge ES^2 is given by a sum of terms $ES_T(P, j)^2(LH_j - LH_{j-1})/T$ for $j = 1, \dots, 5$, where $ES_T(P)^2$ corresponds to $j = 1$. The square root of each of these terms can be thought of as measuring the risk contribution arising from position liquidations between LH_{j-1} and LH_j . The scaling factors $\sqrt{(LH_j - LH_{j-1})/T}$ take into account that $ES_T(P, j)$ is an expected shortfall charge calculated over the interval $[0, T]$. They are an example of square-root-of-time scaling which is widely used in finance to translate certain measures of risk (e.g. volatility, value-at-risk and expected shortfall) calculated on shorter time intervals to longer time intervals.

The first objective of this paper is to provide a principles-based derivation of this formula that relates it to the concept of expected shortfall as a risk measure applied to a loss distribution. Most practitioners know that an assumption of normality underlies the formula but exact details are not available in the main regulatory documents in the public domain. We make it precise that the formula can be justified by assuming that risk-factor changes over time steps equal to the base liquidity horizon form a multivariate Gaussian white noise with mean zero and portfolio losses are all attributable to first-order (delta) sensitivities to the risk-factor changes.

The second and major objective of the paper is to extend the formula under the more general assumption that risk-factor changes have a multivariate elliptical distribution. This allows us to consider some particular cases with heavy tails and tail dependencies that might be considered more realistic models for market risk-factor changes.

Many results in quantitative risk management (QRM) continue to hold when multivariate normal assumptions are generalized to multivariate elliptical assumptions. In particular, when losses are linear in a set of underlying elliptically-distributed risk factors, aggregation of risk measures across different business lines, desks or risk factors can generally be based on a common formulaic approach, regardless of the exact choice of elliptical distribution; see Chapter 8 of [McNeil et al. \(2015\)](#). The

76 difference in the current paper is that aggregation takes place, not only across risk factors, but also
 77 across time and therefore a ‘central limit effect’ takes place. This means that (1) does not hold in the
 78 general elliptical case.

79 We derive a generalization of (1) that applies to all elliptical distributions. Using this generalization
 80 we consider, in particular, a number of the heavier-tailed distributions in the symmetric generalized
 81 hyperbolic family (a sub-family of the elliptical distributions). We infer that, for these distributions,
 82 the use of the standard aggregation rule in (1) would lead to a conservative capital charge in the sense
 83 that the resulting amount of capital ES is larger than is actually necessary to achieve the level of risk
 84 targeted by FRTB (expected shortfall at a 97.5% confidence level).

85 As a by-product of our analyses we also demonstrate a new approach to calculating VaR and
 86 expected shortfall for symmetric distributions with a known characteristic function. This approach is
 87 particularly useful in cases where we take convolutions of elliptically distributed random vectors and
 88 often lose the ability to write simple closed-form expressions for their probability densities.

89 We present all ideas in terms of the standard probabilistic approach to risk measures. Losses
 90 (or P&L variables) are represented by random variables L . Expected shortfall (ES_α) and value-at-risk
 91 (VaR_α) at level α are risk measures applied to L . If F_L denotes the distribution function of L and F_L^\leftarrow
 92 its generalized inverse, they are given by $VaR_\alpha(L) = F_L^\leftarrow(\alpha)$ and $ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 F_L^\leftarrow(u) du$. If F_L is
 93 continuous then the formula $ES_\alpha(L) = \mathbb{E}(L \mid L \geq VaR_\alpha(L))$ also holds.

94 2. Justifying and extending the Basel liquidity formula

95 Let (X_t) be a d -dimensional time series of risk-factor changes for all relevant risk factors and
 96 assume that these are all defined in terms of simple differences or log-differences. We interpret X_{t+1} as
 97 the vector of risk-factor changes over the time step $[t, t + 1]$. In practice this time step will be equal to
 98 the base liquidity horizon of 10 days.

For $h \in \mathbb{N}$, the risk factor changes over the time step $[t, t + h]$ are given additively by

$$X_{[t,t+h]} := \sum_{j=1}^h X_{t+j}. \quad (2)$$

99 Without loss of generality let the risk calculation be made at time $t = 0$. We make the following
 100 assumptions.

- 101 **Assumption 1.** (i) The risk-factor changes (X_t) form a stationary white noise process (a serially
 102 uncorrelated process) with mean zero and covariance matrix Σ .
 103 (ii) Each risk factor may be assigned to a unique liquidity bucket B_k defined by a liquidity horizon $h_k \in \mathbb{N}$,
 104 $k = 1, \dots, n$.
 105 (iii) In the event of a portfolio liquidation action the loss (or profit) attributable to risk factors in bucket B_k is
 106 given by $\mathbf{b}'_k X_{[0,h_k]}$ where \mathbf{b}_k is a weight vector with zeros in any position that corresponds to a risk factor
 107 that is not in B_k .

108 Assumption 1(iii) contains the linearity assumption and adopts the pessimistic view that the full
 109 liquidity horizon h_k is required to remove the portfolio’s sensitivity to all the risk factors in liquidity
 110 bucket B_k .

Under these assumptions we compute the portfolio loss L over the maximum time horizon h_n ,
 which is the time required to remove the portfolio’s sensitivity to all risk factors. It follows from
 Assumption 1(ii) and (iii) that

$$L = \sum_{k=1}^n \mathbf{b}'_k X_{[0,h_k]} = \sum_{k=1}^n \sum_{j=1}^k \mathbf{b}'_k X_{[h_{j-1},h_j]} = \sum_{k=1}^n \sum_{j=k}^n \mathbf{b}'_j X_{[h_{k-1},h_k]} = \sum_{k=1}^n \boldsymbol{\beta}'_k X_{[h_{k-1},h_k]} \quad (3)$$

111 where $\beta_k = \sum_{j=k}^n \mathbf{b}_j$ and $h_0 = 0$. The vector β_k contains the weights for all risk factors in the union of
 112 liquidity buckets $B_k \cup \dots \cup B_n$.

Let us write $L_k := \beta_k' \mathbf{X}_{[h_{k-1}, h_k]}$ for $k = 1, \dots, n$ for the summands in the final expression in (3).
 These are uncorrelated by Assumption 1(i) and we may easily calculate that

$$\text{var}(L) = \sum_{k=1}^n \text{var}(L_k) = \sum_{k=1}^n \beta_k' \text{cov}(\mathbf{X}_{[h_{k-1}, h_k]}) \beta_k = \sum_{k=1}^n (h_k - h_{k-1}) \beta_k' \Sigma \beta_k \quad (4)$$

113 where the final step follows because (2) implies that $\mathbf{X}_{[h_{k-1}, h_k]} = \sum_{j=1}^{h_k - h_{k-1}} \mathbf{X}_{h_{k-1} + j}$.

We now introduce random variables

$$L^{(k)} = \beta_k' \mathbf{X}_{[0, h_1]} \quad (5)$$

for $k = 1, \dots, n$. These represent losses attributable to all risk factors in the union of liquidity buckets
 $B_k \cup \dots \cup B_n$ over the liquidity horizon h_1 . Note that the L_k and $L^{(k)}$ variables differ (unless $k = 1$).
 Since $\text{var}(L^{(k)}) = h_1 \beta_k' \Sigma \beta_k$, we obtain from (4) the formula

$$\text{sd}(L) = \sqrt{\sum_{k=1}^n \left(\sqrt{\frac{h_k - h_{k-1}}{h_1}} \text{sd}(L^{(k)}) \right)^2}. \quad (6)$$

114 It may be noted that the presence of positive correlation between the variables L_k in (4), caused by
 115 serial correlation in the underlying risk-factor changes $\mathbf{X}_{[h_{k-1}, h_k]}$, would tend to lead to the left-hand
 116 side of (6) being larger than the right-hand side. Negative correlation would lead to it being smaller.

117 2.1. The Gaussian case

Suppose that (\mathbf{X}_t) is a Gaussian process; in this case (\mathbf{X}_t) is actually a strict white noise (a process
 of independent and identically distributed vectors). It follows that $L_k \sim N(0, (h_k - h_{k-1}) \beta_k' \Sigma \beta_k)$ and
 the L_k are independent for all k . Thus, by the convolution property for independent normals,

$$L \sim N \left(0, \sum_{k=1}^n (h_k - h_{k-1}) \beta_k' \Sigma \beta_k \right). \quad (7)$$

118 Moreover, we clearly have $L^{(k)} \sim N(0, h_1 \beta_k' \Sigma \beta_k)$.

For any mean-zero normal random variable V it is easy to show that $\text{ES}_\alpha(V) = c_\alpha \text{sd}(V)$ where
 $c_\alpha = \phi(\Phi^{-1}(\alpha)) / (1 - \alpha)$, ϕ denotes the density of the standard normal distribution and $\Phi^{-1}(\alpha)$
 denotes the α -quantile of the standard normal distribution function Φ (see McNeil et al. 2015, Chapter
 2). It follows from (6) that

$$\text{ES}_\alpha(L) = \sqrt{\sum_{k=1}^n \left(\sqrt{\frac{h_k - h_{k-1}}{h_1}} \text{ES}_\alpha(L^{(k)}) \right)^2} \quad (8)$$

119 which is the proposed standard formula for the trading book (1) rewritten in our notation.

120 2.2. An extension to the formula for elliptical distributions

121 In this section we assume a centred elliptical distribution for the risk-factor changes, which
 122 subsumes the multivariate normal distribution as a special case. The class of elliptical distributions
 123 contains a number of particular distributions which are popular models for financial returns including
 124 the multivariate Student t and the symmetric generalized hyperbolic distributions. There is much
 125 empirical evidence that 10-day and even monthly risk-factor returns of different types are heavier
 126 tailed than Gaussian; see, for example, Section 6.2.4 of McNeil et al. (2015). Distributions in the

127 symmetric generalized hyperbolic family provide a superior fit, although they do not address the issue
128 of asymmetry which is a feature of certain risk-factor returns such as equity returns.

129 In simple terms, elliptical distributions are affine transformations of spherical distributions and
130 spherical distributions can be thought of as distributions that are invariant under rotations. More
131 formally, a random vector \mathbf{Y}_t is said to have a d -dimensional spherical distribution with characteristic
132 generator ψ , written $\mathbf{Y}_t \sim S_d(\psi)$, if its characteristic function satisfies $\phi(\mathbf{s}) = E(e^{i\mathbf{s}'\mathbf{Y}_t}) = \psi(\mathbf{s}'\mathbf{s})$ for a
133 function of a scalar variable ψ . The covariance matrix of \mathbf{Y}_t is a scalar multiple of the d -dimensional
134 identity matrix I_d satisfying $\text{cov}(\mathbf{Y}_t) = \text{var}(Y)I_d$ where $Y \sim S_1(\psi)$ denotes any component of the
135 vector \mathbf{Y}_t .

136 A random vector \mathbf{X}_t is said to have a d -dimensional elliptical distribution with location vector $\boldsymbol{\mu}$,
137 positive-definite dispersion matrix Ω and characteristic generator function ψ , written $\mathbf{X}_t \sim E_d(\boldsymbol{\mu}, \Omega, \psi)$
138 if $\mathbf{X}_t = \boldsymbol{\mu} + A\mathbf{Y}_t$ for some matrix $A \in \mathbb{R}^{d \times d}$ satisfying $\Omega = AA'$ and some spherically distributed
139 random vector $\mathbf{Y}_t \sim S_d(\psi)$. It follows that the covariance matrix of \mathbf{X}_t is given by $\Sigma = \text{var}(Y)\Omega$, which
140 shows that the covariance matrix is in general a scalar multiple of Ω . See Fang et al. (1990) and McNeil
141 et al. (2015) for further details of these distributions.

142 In addition to Assumption 1 we assume that the following holds in this section.

143 **Assumption 2.** (i) The process (\mathbf{X}_t) is a multivariate strict white noise (an iid process).
144 (ii) For every t , $\mathbf{X}_t \sim E_d(\mathbf{0}, \Omega, \psi)$ where Ω is a positive-definite matrix.

145 Assumption 2(i) may seem strong but in practice we assume that (\mathbf{X}_t) is a process of 10-day
146 returns so that the iid assumption, while unlikely to be true, is less problematic than for daily financial
147 returns. The assumption is required in order to analyse convolutions of elliptically distributed random
148 vectors with different characteristic generators.

We need three key properties of an elliptical distribution for our calculation. Let $\mathbf{X} \sim E_d(\mathbf{0}, \Omega, \psi)$
and $\tilde{\mathbf{X}} \sim E_d(\mathbf{0}, \Omega, \tilde{\psi})$ be independent elliptically-distributed variables with the same dispersion matrix
 Ω and possibly different characteristic generators ψ and $\tilde{\psi}$.

$$\boldsymbol{\beta}'\mathbf{X} \sim E_1(0, \boldsymbol{\beta}'\Omega\boldsymbol{\beta}, \psi) \quad \text{for } \boldsymbol{\beta} \in \mathbb{R}^d \text{ and } \boldsymbol{\beta} \neq \mathbf{0}. \quad (9)$$

$$\mathbf{X} \sim E_d(\mathbf{0}, c\Omega, \psi(s/c)) \quad \text{for any } c > 0. \quad (10)$$

$$\mathbf{X} + \tilde{\mathbf{X}} \sim E_d(\mathbf{0}, \Omega, \psi^*) \quad \text{where } \psi^*(s) = \psi(s)\tilde{\psi}(s). \quad (11)$$

149 We will use (9) and (11) to find the characteristic functions of elliptical random vectors under linear
150 combinations and convolutions respectively. The property in (10) shows that we have some discretion
151 in how we represent the characteristic generator of an elliptical random variable in terms of its
152 characteristic generator and its scaling.

Theorem 1. Under Assumptions 1 and 2 the loss L in (3) is a univariate spherical random variable $L \sim S_1(\psi_L)$
with characteristic generator

$$\psi_L(s) = \prod_{k=1}^n \psi_k(s\boldsymbol{\beta}'_k\Omega\boldsymbol{\beta}_k), \quad (12)$$

153 where $\psi_k = \psi^{h_k - h_{k-1}}$ is the $(h_k - h_{k-1})$ -fold product of ψ for $k = 1, \dots, n$.

For $\alpha > 0.5$ the expected shortfall of L is related to the expected shortfall of the variables $L^{(k)}$ in (5) by

$$\text{ES}_\alpha(L) = \frac{c_{\alpha, \psi_L}}{c_{\alpha, \psi_1}} \sqrt{\sum_{k=1}^n \left(\sqrt{\frac{h_k - h_{k-1}}{h_1}} \text{ES}_\alpha(L^{(k)}) \right)^2}. \quad (13)$$

154 where c_{α, ψ_L} represents the ratio of expected shortfall to standard deviation for L and c_{α, ψ_1} is the equivalent ratio
155 for a univariate spherical variable $Z \sim S_1(\psi_1)$.

Proof. We need to derive the distributions of

$$L_k = \beta'_k \mathbf{X}_{[h_{k-1}, h_k]}, \quad L = \sum_{k=1}^n L_k \quad \text{and} \quad L^{(k)} = \beta'_k \mathbf{X}_{[0, h_1]}. \quad (14)$$

First note that if $\mathbf{X}_t \sim E_d(\mathbf{0}, \Omega, \psi)$ then it follows from (2) and (11) that $\mathbf{X}_{[h_{k-1}, h_k]} \sim E_d(\mathbf{0}, \Omega, \psi_k)$ where $\psi_k = \psi^{h_k - h_{k-1}}$. Using (9) we have that

$$L_k \sim E_1(0, \beta'_k \Omega \beta_k, \psi_k) \quad \text{and} \quad L^{(k)} \sim E_1(0, \beta'_k \Omega \beta_k, \psi_1).$$

Using (10) we write the former as $L_k \sim E_1(0, 1, \psi_k(s\beta'_k \Omega \beta_k))$ or $L_k \sim S_1(\psi_k(s\beta'_k \Omega \beta_k))$ and then use the convolution property (11) to conclude that $L \sim S_1(\psi_L)$ where ψ_L is given in (12).

Now $\text{ES}_\alpha(L^{(k)}) = \sqrt{\beta'_k \Omega \beta_k} \text{ES}_\alpha(Z)$ and $\text{sd}(L^{(k)}) = \sqrt{\beta'_k \Omega \beta_k} \text{sd}(Z)$ where $Z \sim S_1(\psi_1)$. Hence it follows that $\text{ES}_\alpha(L^{(k)}) = c_{\alpha, \psi_1} \text{sd}(L^{(k)})$ for all k and

$$\begin{aligned} \text{ES}_\alpha(L) &= c_{\alpha, \psi_L} \text{sd}(L) = c_{\alpha, \psi_L} \sqrt{\sum_{k=1}^n \left(\sqrt{\frac{h_k - h_{k-1}}{h_1}} \text{sd}(L^{(k)}) \right)^2} \\ &= c_{\alpha, \psi_L} \sqrt{\sum_{k=1}^n \left(\sqrt{\frac{h_k - h_{k-1}}{h_1}} \frac{\text{ES}_\alpha(L^{(k)})}{c_{\alpha, \psi_1}} \right)^2} \end{aligned}$$

which yields (13). \square

It may be easily verified that when $\psi(s) = \exp(-s/2)$ (the Gaussian case), the characteristic function $\phi(s) = \psi_L(s^2)$ implied by (12) is the characteristic function of the normal distribution in (7). In this case the constants c_{α, ψ_L} and c_{α, ψ_1} are identical.

When the risk factors have a heavier-tailed distribution than normal we expect that $c_{\alpha, \psi_L} \leq c_{\alpha, \psi_1}$. This is because the aggregation across time periods that takes place in the definition of L should lead to a central limit effect whereby L is closer to Gaussian than the $L^{(k)}$ variables. In this case we expect that the standard Basel liquidity formula should give an upper bound on $\text{ES}_\alpha(L)$.

3. Calculating the scaling ratio in practice

We turn to the problem of calculating the ratio $r_\alpha := c_{\alpha, \psi_L} / c_{\alpha, \psi_1}$ when the underlying risk factors have an elliptical distribution with generator ψ . To compute c_{α, ψ_1} we calculate the ratio $\text{ES}_\alpha(Z) / \text{sd}(Z)$ for a univariate spherical random variable Z with characteristic generator $\psi_1 = \psi^{h_1}$. To compute c_{α, ψ_L} we calculate the ratio $\text{ES}_\alpha(L) / \text{sd}(L)$ for a univariate spherical variable L with characteristic generator given by (12).

The problem of calculating expected shortfall for linear portfolios of elliptically distributed risk factors is tackled in Kamdem (2005) and Dobrev et al. (2017). The proposed method relies on knowing the so-called density generator of the elliptical distribution. In our application the taking of convolutions means that the density generator required to calculate $\text{ES}_\alpha(Z)$ and $\text{ES}_\alpha(L)$ may not be available in a simple closed forms for the underlying distributions of \mathbf{X}_t which interest us.

However, the characteristic generator is always available in our application. In the following section we give results that can be used to compute expected shortfall directly from the characteristic function of a spherical random variable.

3.1. Calculating expected shortfall by Fourier inversion

A univariate spherical random variable $Y \sim S_1(\psi)$ is symmetric about the origin with a real-valued even characteristic function given by $\phi_Y(s) := \psi(s^2)$. We give a general result that applies to univariate random variables that are symmetric about the origin.

Theorem 2. Let Y be symmetrically distributed about the origin with an integrable characteristic function $\phi_Y(s)$. Let $-\infty < a < b < \infty$. Then the following formulas hold:

$$f_Y(y) = \frac{1}{\pi} \int_0^\infty \cos(sy) \phi_Y(s) ds, \quad (15)$$

$$F_Y(y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin(sy)}{s} \phi_Y(s) ds, \quad (16)$$

$$E(YI_{\{a \leq Y \leq b\}}) = \frac{1}{\pi} \int_0^\infty \frac{bs \sin(bs) + \cos(bs) - as \sin(as) - \cos(as)}{s^2} \phi_Y(s) ds. \quad (17)$$

Proof. The characteristic function $\phi_Y(s)$ of a random variables that is symmetric about the origin is real-valued and even. If ϕ_Y is integrable then the density exists and the standard Fourier inversion formula for the characteristic formula yields

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-isy} \phi_Y(s) ds = \frac{1}{\pi} \int_0^\infty \cos(sy) \phi_Y(s) ds.$$

The formula (16) for the distribution function is obtained from a well-known representation of the distribution by [Gil-Pelaez \(1951\)](#). To derive (17) we observe that

$$\begin{aligned} \int_a^b y f_Y(y) dy &= \frac{1}{\pi} \int_a^b \int_0^\infty y \cos(sy) \phi_Y(s) ds dy \\ &= \frac{1}{\pi} \int_0^\infty \left(\int_a^b y \cos(sy) dy \right) \phi_Y(s) ds \end{aligned}$$

by Fubini's Theorem since $|y \cos(sy) \phi_Y(s)| \leq |y| |\phi_Y(s)|$ and the latter is integrable on $[a, b] \times [0, \infty)$. The inner integral can be solved by parts to obtain

$$\int_a^b y \cos(sy) dy = \frac{bs \sin(bs) + \cos(bs) - as \sin(as) - \cos(as)}{s^2}$$

184 and (17) follows. \square

These formulas permit the accurate evaluation of $\text{VaR}_\alpha(Y)$ and expected shortfall using one-dimensional integration. Calculation of $\text{VaR}_\alpha(Y)$ for $\alpha > 0.5$ is accomplished by numerical root finding using (16). If $\mathbb{E}|Y| < \infty$ for the distribution in question, then expected shortfall is defined and it can be calculated by setting $a = \text{VaR}_\alpha(Y)$ and computing the limit

$$\text{ES}_\alpha(Y) = \lim_{b \rightarrow \infty} \frac{1}{\pi(1-\alpha)} \int_0^\infty \frac{bs \sin(bs) + \cos(bs) - as \sin(as) - \cos(as)}{s^2} \phi_Y(s) ds. \quad (18)$$

185 Our experiments confirm that calculating the integral in (18) for increasing b does result in stable
186 limiting values for $\text{ES}_\alpha(Y)$ which agree to a high level of accuracy with theoretical values for
187 well-known distributions such as Student t .

188 3.2. The case of generalized hyperbolic distributions

189 We will apply Theorem 2 to the family of symmetric generalized hyperbolic (GH) distributions.
190 This is a very popular family for modelling financial returns and there are many useful sources for the
191 properties of these distributions including [Barndorff-Nielsen \(1978\)](#), [Barndorff-Nielsen and Blæsild \(1981\)](#),
192 [Eberlein \(2010\)](#) and [McNeil et al. \(2015\)](#). While some special cases of the GH family are known to
193 be invariant under convolutions ([Podgórski and Wallin 2016](#)) the complicated aggregation of variables
194 with different scaling that we undertake means that, even for these cases, we generally need to use (18)
195 to compute expected shortfall for the aggregate loss L .

Let $\mathbf{Y} = (Y_1, \dots, Y_d)'$ have the stochastic representation $\mathbf{Y} = \sqrt{W}\mathbf{V}$ where $\mathbf{V} = (V_1, \dots, V_d)'$ is a vector of independent standard normal variables and W is an independent positive random variable with a so-called generalized inverse Gaussian (GIG) distribution $W \sim N^-(\lambda, \chi, \kappa)$; see formula (A1) in the Appendix for the density of this distribution. The vector \mathbf{Y} has a spherical distribution $\mathbf{Y} \sim S_d(\psi)$, and any component Y has a univariate spherical distribution $Y \sim S_1(\psi)$, for a characteristic generator ψ that depends on the particular choice of the parameters λ , χ and κ . An elliptical model of the kind described in Assumption 2(ii) is obtained by taking $\mathbf{X} = A\mathbf{Y}$ for $A \in \mathbb{R}^{d \times d}$ and satisfies $\mathbf{X} \sim E_d(\mathbf{0}, \Omega, \psi)$ where $\Omega = AA'$. \mathbf{X} is said to have a d -dimensional symmetric generalized hyperbolic (GH) distribution.

To carry out our calculations it suffices to consider the single component Y . The variance of Y satisfies $\text{var}(Y) = \mathbb{E}(W)$ and an explicit formula for the case where $\chi > 0$ and $\kappa > 0$ is given in (A3). A formula for the characteristic function ϕ_Y is given in (A4) and the characteristic generator of the elliptical family can be inferred from the identity $\psi(s^2) = \phi_Y(s)$.

We consider four special one-parameter cases of this distribution resulting from particular choices of the parameters λ , χ and κ of the GIG distribution:

1. The student t distribution with degree of freedom ν . This corresponds to the case where $\kappa = 0$, $\lambda = -\nu/2$ and $\chi = \nu$ or where W has an inverse gamma distribution $W \sim \text{IG}(\nu/2, \nu/2)$. In this case $\text{var}(Y) = \nu/(\nu - 2)$, provided $\nu > 2$, and the characteristic function is given by (A5) in the Appendix.
2. The variance gamma (VG) distribution. This corresponds to the case where $\chi = 0$ or where W has a gamma distribution $W \sim \text{Ga}(\lambda, \kappa/2)$. Without loss of generality we set the scaling parameter $\kappa = 2$ so that $\text{var}(Y) = \lambda$. The corresponding characteristic function is given by (A6).
3. The normal-inverse-Gaussian (NIG) distribution. This corresponds to the case where $\lambda = -1/2$. The distribution can be reparameterized in terms of $\theta = \sqrt{\chi\kappa}$ and χ ; the latter parameter can be treated as a scaling parameter and set to one. The variance is then $\text{var}(Y) = \theta^{-1}$ and the characteristic function is given by (A7).
4. The hyperbolic (Hyp) distribution. This corresponds to the case where $\lambda = 1$. The distribution can be reparameterized in exactly the same way as the NIG distribution. The variance is $\text{var}(Y) = \theta^{-1}K_2(\theta)/K_1(\theta)$ and the characteristic function is given by (A8).

3.3. Summary of the steps in the calculation

We return to the problem of calculating the scaling ratios $r_\alpha = c_{\alpha, \psi_L} / c_{\alpha, \psi_1}$ in (13) when the underlying risk-factor returns have symmetric distributions in the multivariate generalized hyperbolic family.

We recall the basic components that are required for the calculation: $Y \sim S_1(\psi)$ is spherically distributed with known standard deviation $\text{sd}(Y)$ and known characteristic function $\phi_Y(s) = \psi(s^2)$; $Z \sim S_1(\psi_1)$ where $\psi_1 = \psi^{h_1}$; $L \sim S_1(\psi_L)$ where ψ_L is given in (12). The steps are:

1. Calculate $\text{ES}_\alpha(Z)$ using (18) and $\phi_Z(s) = \phi_Y^{h_1}(s)$.
2. Calculate $\text{sd}(Z) = \sqrt{h_1} \text{sd}(Y)$.
3. Hence calculate $c_{\alpha, \psi_1} = \text{ES}_\alpha(Z) / \text{sd}(Z)$.
4. Calculate $\text{ES}_\alpha(L)$ using (18) and the fact that

$$\phi_L(s) = \prod_{k=1}^n \phi_Y^{h_k - h_{k-1}} \left(s \sqrt{\beta_k' \Omega \beta_k} \right).$$

5. Calculate $\text{sd}(L)$ using the formula

$$\text{sd}(L) = \text{sd}(Y) \sqrt{\sum_{k=1}^n (h_k - h_{k-1}) \beta_k' \Omega \beta_k}.$$

6. Hence calculate $c_{\alpha, \psi_L} = \text{ES}_\alpha(L) / \text{sd}(L)$.

236 7. Hence calculate the ratio $r_\alpha = c_{\alpha,\psi_L}/c_{\alpha,\psi_1}$.

237 4. Results

238 In the analyses of this section we make explicit choices of parametric distributions for the
239 risk-factor changes in order to study the possible extent of risk overestimation that results from
240 using the standard Basel liquidity formula. It is important to stress that most banks do not estimate
241 parametric models for X_t in practice.

242 The vast majority of banks employ resampling techniques known as historical simulation. This
243 means, effectively, that they estimate their models non-parametrically. It is certainly possible to fit
244 multivariate hyperbolic distributions to data, even in high dimensions, using variants of the EM
245 algorithm (McNeil et al. 2015; Protassov 2004), but we are not aware of banks that do this.

246 In the absence of data on the risk factors that affect a particular bank, we choose plausible values
247 for the parameters of the generalized hyperbolic distributions by fitting univariate models to the broad
248 market returns of the S&P500 index. We also choose illustrative values for the elements of the matrix
249 Ω , since this matrix is not explicitly estimated by banks using the historical simulation method.

250 The values of the vectors β_k depend on the sensitivities of the trading book positions to the risk
251 factors. These would be known to a bank in practice. In the absence of data, we again make simple
252 stylized choices.

253 4.1. Design of experiments

254 In order to calibrate our distributions, we use 2132 observations of adjusted daily closing prices for
255 the S&P500 index, from 17.7.2007 to 31.12.2015, which have been converted to two-weekly log-returns
256 (conforming approximately to 10 trading days, the base liquidity horizon required under FRTB).

257 We fit the various distributions discussed in Section 3.2 to the 10-day return data using the R
258 package `ghyp`. Table 1 gives the estimated shape parameters for the distributions of interest; scale
259 parameters are not required in our analysis. Note that we also confirm that the calculations for the
260 Gaussian case yield a ratio of 1, as a check on our implementation.

Table 1. Distribution parameters used in the calculation experiments. These have been derived by fitting these distributions to two-weekly log-returns of the S&P500 index over the period from 17.07.2007 to 31.12.2015.

Distribution Parameters	λ	θ	Remarks
t	-1.46		$\nu = -2\lambda$
NIG	-0.5	0.49	λ fixed
Hyp	1	0.11	λ fixed
VG	0.95		$\kappa = 2$

261 We carry out two experiments:

- 262 • In the first, we consider two risk factors, one in B_1 with a liquidity horizon of 10 days ($h_1 = 1$)
263 and the other in B_2 with a liquidity horizon of 20 days ($h_2 = 2$). The dispersion matrix Ω is either
264 taken to be the identity $\Omega = I_2$ (no correlation) or a correlation matrix with correlation $\rho = 0.5$.
- 265 • The second experiment follows in the same fashion but we assume there are 5 risk factors with
266 liquidity horizons 10, 20, 40, 60 and 120 days ($h_1 = 1, h_2 = 2, h_3 = 4, h_4 = 6, h_5 = 12$). We
267 consider both the case where $\Omega = I_5$ and the case where Ω is an equicorrelation matrix with
268 element $\rho = 0.5$.

269 We present values of $c_{\alpha,\psi_1}, c_{\alpha,\psi_L}$ as well as the scaling ratio r_α for various confidence levels α . The
270 case of two risk factors is reported in Table 2 and the case of five risk factors is reported in Table 3.

271 4.2. Results

272 In both tables it is clear that the scaling ratios are less than one for all non-Gaussian cases meaning
 273 that the Basel liquidity formula is indeed conservative when the risk factors have a multivariate
 274 elliptical distribution from one of the four generalized hyperbolic sub-families considered in Section 3.2
 275 and Table 1.

276 The second experiment with five liquidity buckets leads in general to smaller values for the
 277 scaling ratios than the first experiment with two buckets. Thus the degree of conservatism of the
 278 formula increases with the number of liquidity buckets. This is in line with the increase in the central
 279 limit effect as we aggregate over more time periods.

280 Introducing correlation leads to an increase in the constants c_{α, ψ_L} and hence an increase in the
 281 scaling ratio. In other words, the weaker the correlation, the more conservative the liquidity formula.
 282 To understand why this is the case, note that the constants c_{α, ψ_L} depend on the characteristic generator
 283 ψ_L in (12) and hence on the set of values $\{\beta'_k \Omega \beta_k, k = 1, \dots, n\}$. By considering formula (4) we can
 284 think of these as the relative weights attached to each of the n liquidity buckets. When $\rho = 0$ these
 285 weights are (5, 4, 3, 2, 1) but when $\rho = 0.5$ they are (15, 10, 6, 3, 1). The intuition is that, in the second
 286 case, the first few liquidity buckets dominate more in the convolution calculation and the central limit
 287 effect is mitigated.

288 Considering the different generalized hyperbolic special cases we see that the ratios are usually
 289 largest for the t distribution followed by the other three distributions; the exact ordering depends on
 290 the confidence level α used in the calculation. In other words, use of the Basel liquidity formula is least
 291 conservative in the case of t and more conservative for the other distributions.

292 When we look at the confidence level of $\alpha = 0.975$ which is the level used in the new
 293 capital standard (Basel Committee on Banking Supervision 2016) the normal inverse Gaussian (NIG)
 294 distribution leads to the highest level of conservatism. This distribution is often a plausible model in
 295 market risk applications. The ratio in the case where $n = 5$ and $\rho = 0$ is 0.837 which means that the
 296 Basel liquidity formula would tend to overstate capital by around 19.4%.

Table 2. Constants c_{α, ψ_1} , c_{α, ψ_L} and ratios r_α in the experiment with 2 risk factors.

Model	α Quantity ρ	0.95		0.975		0.99	
		0	0.5	0	0.5	0	0.5
Gauss	c_{α, ψ_1}	2.063	2.063	2.338	2.338	2.665	2.665
	c_{α, ψ_L}	2.063	2.063	2.338	2.338	2.665	2.665
	r_α	1.000	1.000	1.000	1.000	1.000	1.000
t	c_{α, ψ_1}	2.223	2.223	2.906	2.906	4.065	4.065
	c_{α, ψ_L}	2.212	2.169	2.831	2.671	3.868	3.486
	r_α	0.995	0.975	0.974	0.919	0.952	0.858
VG	c_{α, ψ_1}	2.345	2.345	2.841	2.841	3.509	3.509
	c_{α, ψ_L}	2.247	2.132	2.670	2.468	3.225	2.891
	r_α	0.958	0.909	0.940	0.869	0.919	0.824
Hyp	c_{α, ψ_1}	2.330	2.330	2.816	2.816	3.459	3.459
	c_{α, ψ_L}	2.237	2.128	2.653	2.459	3.194	2.877
	r_α	0.960	0.913	0.942	0.873	0.923	0.832
NIG	c_{α, ψ_1}	2.374	2.374	2.976	2.976	3.832	3.832
	c_{α, ψ_L}	2.296	2.167	2.801	2.544	3.502	3.042
	r_α	0.967	0.913	0.941	0.855	0.914	0.794

297 It would be appealing to link the values of r_α and the resulting levels of conservatism of the
 298 Basel formula to some parameter that describes the heavy-tailedness of the distributions under
 299 consideration, such as their tail index or kurtosis. However the only distribution in Tables 2 and 3
 300 which has a regularly varying tail, and thus a finite tail index, is the t distribution. Although the
 301 VG, hyperbolic and NIG distributions have infinite tail indices, they tend to give smaller ratios r_α .

Table 3. Constants c_{α, ψ_1} , c_{α, ψ_L} and ratios r_α in the experiment with 5 risk factors.

Model	α Quantity ρ	0.95		0.975		0.99	
		0	0.5	0	0.5	0	0.5
Gauss	c_{α, ψ_1}	2.063	2.063	2.338	2.338	2.665	2.665
	c_{α, ψ_L}	2.063	2.063	2.338	2.338	2.665	2.665
	r_α	1.000	1.000	1.000	1.000	1.000	1.000
t	c_{α, ψ_1}	2.223	2.223	2.906	2.906	4.065	4.065
	c_{α, ψ_L}	2.160	2.169	2.637	2.671	3.402	3.486
	r_α	0.972	0.975	0.908	0.919	0.837	0.858
VG	c_{α, ψ_1}	2.345	2.345	2.841	2.841	3.509	3.509
	c_{α, ψ_L}	2.112	2.132	2.429	2.468	2.824	2.891
	r_α	0.901	0.909	0.855	0.869	0.805	0.824
Hyp	c_{α, ψ_1}	2.330	2.330	2.816	2.816	3.459	3.459
	c_{α, ψ_L}	2.108	2.128	2.423	2.459	2.814	2.877
	r_α	0.905	0.913	0.860	0.873	0.813	0.832
NIG	c_{α, ψ_1}	2.374	2.374	2.976	2.976	3.832	3.832
	c_{α, ψ_L}	2.142	2.167	2.492	2.544	2.942	3.042
	r_α	0.902	0.913	0.837	0.855	0.768	0.794

302 Moreover the t distribution has infinite kurtosis while the other distributions have finite kurtosis. It
 303 would seem that there is a more complex story behind the precise ordering of the r_α values. However,
 304 the results are sufficient to show that a range of differing heavy-tailed distributions all lead to ratios
 305 less than one.

306 5. Conclusion

307 We have presented evidence that the Basel liquidity formula tends to lead to conservative capital
 308 charges when financial risk factors come from heavier-tailed elliptical distributions.

309 The Basel formula is clearly a heavily stylized formula and makes a number of crude assumptions.
 310 We have concentrated on the effect of changing the underlying distribution of the risk factors when
 311 portfolio sensitivities are linear. However, there are other important effects we have not considered
 312 which will have an influence on the ability of the formula to capture risk. In particular, the true effect of
 313 risk-factor changes on portfolio risk is likely to be highly non-linear over the kind of time horizons we
 314 consider. Moreover, as we have already noted, positive serial correlation between losses over different
 315 sub-intervals $[h_{k-1}, h_k]$ of the overall liquidity horizon $[0, h_n]$ will tend to lead to a tendency towards
 316 underestimation which may counteract the central limit effect.

317 We note that our assumption that risk-factor changes are elliptically distributed implies that
 318 their marginal distributions are symmetric. This is clearly a limiting assumption and it would be of
 319 interest to see if the liquidity formula could be further generalized to classes of distribution that admit
 320 skewness, such as the full generalized hyperbolic family.

321 In writing about inherent conservatism in the liquidity formula we are well aware that there are
 322 many further layers of conservatism built into the new system of risk charges for the trading book,
 323 such as the requirement to calibrate the model to stress periods and the requirement to adjust the
 324 calculation to understate the possible diversification effects across risk factors. These other features
 325 may have greater impact than the issue we address.

326 Nonetheless it is important to be clear about the workings of the formula and the extent to which
 327 it may be interpreted as a principles-based approach to the measurement of market risk. Our study
 328 should be understood as a contribution to the clarification of this issue.

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331 Appendix

A standardized univariate generalized hyperbolic random variable Y has the stochastic representation $Y = \sqrt{W}V$ where V is a standard normal variable and W is an independent positive random variable with a generalized-inverse-Gaussian (GIG) distribution. The density of the latter is

$$f_W(w) = \frac{\chi^{-\lambda}(\sqrt{\chi\kappa})^\lambda}{2K_\lambda(\sqrt{\chi\kappa})} w^{\lambda-1} \exp(-\frac{1}{2}(\chi w^{-1} + \kappa w)), \quad \begin{cases} \chi > 0, \kappa \geq 0 & \text{if } \lambda < 0 \\ \chi > 0, \kappa > 0 & \text{if } \lambda = 0 \\ \chi \geq 0, \kappa > 0 & \text{if } \lambda > 0 \end{cases} \quad (\text{A1})$$

where K_λ denotes a Bessel function of the third kind. The characteristic function of Y is given by

$$\phi_Y(s) = \mathbb{E} \left(\mathbb{E} \left(\exp(is\sqrt{W}V) \mid W \right) \right) = \mathbb{E} \left(\exp(-\frac{1}{2}s^2W) \right) = \int_0^\infty e^{-\frac{1}{2}s^2w} f_W(w) dw \quad (\text{A2})$$

332 and the variance by $\text{var}(Y) = \mathbb{E}(W)$.

We first consider the case where $\chi > 0$ and $\kappa > 0$. In this case the variance of Y is

$$\text{var}(Y) = \left(\frac{\chi}{\kappa} \right)^{1/2} \frac{K_{\lambda+1}(\sqrt{\chi\kappa})}{K_\lambda(\sqrt{\chi\kappa})} \quad (\text{A3})$$

and the characteristic function is

$$\begin{aligned} \phi_Y(s) &= \int_0^\infty e^{-\frac{1}{2}(\chi w^{-1} + (s^2 + \kappa)w)} \frac{\chi^{-\lambda} (\chi\kappa)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\kappa})} x^{\lambda-1} dw \\ &= \left(\frac{\kappa}{s^2 + \kappa} \right)^{\lambda/2} \frac{K_\lambda(\sqrt{\chi(s^2 + \kappa)})}{K_\lambda(\sqrt{\chi\kappa})}. \end{aligned} \quad (\text{A4})$$

We next consider the case of a Student t distribution which corresponds to $\kappa = 0$, $\lambda = -\nu/2$ and $\chi = \nu$. In this case W has an inverse gamma distribution $W \sim \text{IG}(\nu/2, \nu/2)$ and $\text{var}(Y) = \mathbb{E}(W) = \nu/(\nu - 2)$, provided $\nu > 2$. The characteristic function should be interpreted as the limit of (A4) as $\kappa \rightarrow 0$. Substituting the density of an inverse gamma distribution into (A2) yields

$$\begin{aligned} \phi_Y(s) &= \int_0^\infty e^{-\frac{1}{2}s^2w} \frac{(\frac{1}{2}\nu)^{\nu/2}}{\Gamma(\frac{1}{2}\nu)} w^{-\frac{\nu}{2}-1} e^{-\frac{1}{2}\nu w^{-1}} dw \\ &= \frac{(\nu s^2)^{\nu/4}}{2^{\nu/2-1} \Gamma(\frac{1}{2}\nu)} K_{\nu/2}(\sqrt{\nu s^2}). \end{aligned} \quad (\text{A5})$$

The special case of variance gamma (VG) corresponds to $\chi = 0$; without loss of generality we set the scaling parameter $\kappa = 2$. In this case W has a gamma distribution $W \sim \text{Ga}(\lambda, 1)$ and $\text{var}(Y) = \mathbb{E}(W) = \lambda$. The characteristic function in this case should be interpreted as the limit of (A4) as $\chi \rightarrow 0$. Substituting the density of a gamma distribution $W \sim \text{Ga}(\lambda, 1)$ for f_W in (A2) we obtain

$$\begin{aligned} \phi_Y(s) &= \int_0^\infty e^{-\frac{1}{2}s^2w} \frac{w^{\lambda-1} e^{-w}}{\Gamma(\lambda)} dw \\ &= \left(1 + \frac{1}{2}s^2 \right)^{-\lambda}. \end{aligned} \quad (\text{A6})$$

333 Two further special cases are the normal inverse Gaussian (NIG) and hyperbolic distributions. In
334 both cases we fix the parameter λ and reparameterize the GH distribution in terms of $\theta = \sqrt{\chi\kappa}$ and κ ;
335 the latter then appears only as a scaling parameter and can be set to one.

For the NIG distribution $\lambda = -1/2$ and $\text{var}(Y) = \theta^{-1}$. The identity $K_\lambda(x) = K_{-\lambda}(x)$ can be used to infer that

$$\phi_Y(s) = \left(\frac{\sqrt{\theta^2 + s^2}}{\theta} \right)^{1/2} \frac{K_{1/2}(\sqrt{\theta^2 + s^2})}{K_{1/2}(\theta)}. \quad (\text{A7})$$

For the hyperbolic (Hyp) distribution $\lambda = 1$ and $\text{var}(Y) = \theta^{-1}K_2(\theta)/K_1(\theta)$. The characteristic function is

$$\phi_Y(s) = \left(\frac{\theta}{\sqrt{\theta^2 + s^2}} \right) \frac{K_1(\sqrt{\theta^2 + s^2})}{K_1(\theta)}. \quad (\text{A8})$$

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