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TORSION HOMOLOGY GROWTH FOR NONCONGRUENCE SUBGROUPS OF BIANCHI GROUPS

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We carry out numerical experiments to investigate the growth of torsion in their first homology of noncongruence subgroups of Bianchi groups. The data we collect suggest that the torsion homology growth conjecture of Bergeron and Venkatesh for congruence subgroups may apply to the noncongruence case as well.

Keywords: Bianchi groups; growth of torsion homology; noncongruence subgroups.

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1. Introduction

In this paper we construct several explicit families of noncongruence subgroups of Bianchi groups and numerically test whether the asymptotic behaviour of torsion in their first homology groups agrees with the conjectural behaviour put forward by Bergeron and Venkatesh [4,5] in the case of congruence subgroups.

Let $\mathcal{O}_K$ denote the ring of integers of a number field K. It is a result of Serre (see [18]) that the group $\mathrm{PSL}_2(\mathcal{O}_K)$ has noncongruence subgroups if and only if $K = \mathbb{Q}$ or K is an imaginary quadratic field. In the latter case, we call the group $\mathrm{PSL}_2(\mathcal{O}_K)$ a Bianchi group. There is an extensive literature on the noncongruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Starting with the work of Atkin and Swinnerton-Dyer [2], the arithmetic of modular forms associated to these subgroups has received considerable attention and continues to do so. On the other hand, to the best of our knowledge, studies on noncongruence subgroups of the Bianchi groups are quite sparse in the literature (see [7,10]), and arithmetic of their associated modular forms is mostly unexplored (see [21]). We hope that our explicit families will help future studies.

In the recent years torsion in the homology of arithmetic groups has received much attention. A conjecture of Bergeron and Venkatesh [4,5] says that, in the case congruence subgroups of Bianchi groups, torsion in the homology should grow exponentially with respect to the volume (see Lê [14] for a similar conjecture for fundamental groups of hyperbolic 3-manifolds arising from knot complements). There is significant amount of numerical evidence related to this conjecture (e.g. [8,19,20]). These works deal with congruence type arithmetic groups and non-arithmetic groups. We consider the following extension of the conjecture of Bergeron and Venkatesh which also covers noncongruence type arithmetic subgroups.

Conjecture 1.1. *Let $\{\Gamma_n\}_n$ be a sequence of finite index subgroup of some fixed Bianchi group. Assume that $\{\Gamma_n\}_n$ is Benjamini-Schramm convergent. Then*

$$\lim_{n\to\infty} \frac{log|\left(\Gamma_n^{ab}\right)_{tor}|}{vol(\Gamma_n\backslash\mathbb{H})} = \frac{1}{6\pi}.$$

In Section 3, we construct five families of noncongruence subgroups. Two of them are obtained by fixing a noncongruence subgroup and intersecting it with a family of congruence subgroups. The other three families will be constructed, see Theorem 3.1, by looking at kernels of certain explicit surjective homomorphisms from the congruence subgroups $\Gamma_0(\mathfrak{p})$ to $\mathbb{F}_q$. Observe that the congruence closure index is bounded for the first two families. By increasing the prime q, we make sure that the congruence closure index is unbounded for the remaining three families. We discuss the notion of Benjamini-Schramm convergence in Section 6. In particular, we prove that all of the families that we construct are convergent in this sense, see Proposition 6.4. In Section 7, we proceed to numerically test Conjecture 1.1 on the families that we construct. The data we collect give support to Conjecture 1.1.

Let us finish this introduction by mentioning an interesting aspect of the torsion homology growth problem for noncongruence subgroups which does not arise in our experiments. As we mentioned earlier, the work of Bergeron and Venkatesh focuses on congruence arithmetic groups[a]. Noncongruence subgroups, just like congruence ones, are arithmetic as well. However as they are invisible to adelic methods, they do not fully enjoy the rich theory that surrounds congruence subgroups. A particular instance is the difference in the nature of Hecke action: it is known (see [21]) that, as in the case of $\mathrm{PSL}_2(\mathbb{Z})$ (see [3]), the action of Hecke operators on the cohomology of a noncongruence subgroup factors over the cohomology of its congruence closure. In the congruence case, the philosophy of Bergeron and Venkatesh is that the action of the Hecke operators prevents the free part of the integral cohomology from having any influence (via the so-called regulators) on the asymptotic growth of torsion. It is therefore natural to ask whether, due the defect in the action of the Hecke operators, the free part has any influence in the noncongruence case. The optimistic conjecture above says no. As we explain in Section 7, our experiments fail to shed any light on this issue.

[a]Although their arguments apply to cocompact lattices in general.

2. Bianchi Groups

Let $\mathbb{H} := \{(z, r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$ denote the 3-dimensional upper-half space. When $\mathbb{H}$ is equipped with the hyperbolic metric d induced from the line element ds defined by

$$ds = \frac{dx^2 + dy^2 + dr^2}{r^2},$$

with $z = x + iy$, then $\mathbb{H}$ becomes a model of hyperbolic 3-space. To make the computations easier, we can use quaternions to represent points in $\mathbb{H}$. Let $1, i, j, k$ be the standard $\mathbb{R}$-basis for the Hamilton's quaternions $\mathcal{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$. Then $\mathbb{H}$ can be seen as a subset of $\mathcal{H}$ via

$$(z, r) \mapsto z + rj.$$

Then the real Lie group $\mathrm{SL}_2(\mathbb{C})$ acts on a point $p = z + rj \in \mathbb{H}$ via the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p = (ap + b)(cp + d)^{-1}$$

where all the operations take place in $\mathcal{H}$. Observe that the center $\{\pm Id\}$ acts trivially so that the action descends to $\mathrm{PSL}_2(\mathbb{C})$. From now on, we shall adapt the standard abuse of the notation and denote elements of $\mathrm{PSL}_2(\mathbb{C})$ simply as matrices. The action can be extended to the boundary $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. If $(x : y)$ is an element of $\mathbb{P}^1(\mathbb{C})$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x : y) = (ax + by : cx + dy).$$

Let $K_d = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field where $d > 0$ is a square free integer and let $\mathcal{O}_d$ be its ring of integers. The groups $\mathrm{PSL}_2(\mathcal{O}_d)$ are called Bianchi groups. These groups are discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ and thus they act properly discontinuously on $\mathbb{H}$.

There is a distinguished class of finite index subgroups of $\mathrm{PSL}_2(\mathcal{O}_d)$ that we define now. Given an ideal $\mathfrak{a}$, the principal congruence subgroup of level $\mathfrak{a}$ is defined as the kernel of the natural surjection $\mathrm{PSL}_2(\mathcal{O}_d) \to \mathrm{PSL}_2(\mathcal{O}_d/\mathfrak{a})$, which is denoted by $\Gamma(\mathfrak{a})$. Any subgroup that contains a principal congruence subgroup is called a congruence subgroup. Besides the principal congruence subgroups, there are some important congruence subgroups such as

$$\Gamma_0(\mathfrak{a}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathcal{O}_d) \mid c \equiv 0 \pmod{\mathfrak{a}} \right\}.$$

For a congruence subgroup Γ of $\mathrm{PSL}_2(\mathcal{O}_d)$ the level of Γ is defined as the largest ideal $\mathfrak{a}_\Gamma$ of $\mathcal{O}_d$ such that $\Gamma(\mathfrak{a}_\Gamma) \leq \Gamma$. It can be easily seen that the level of $\Gamma_0(\mathfrak{a})$ and $\Gamma(\mathfrak{a})$ is $\mathfrak{a}$.

We shall extend the notion of level from congruence subgroups to all finite index subgroups. This can be done in either an algebraic way or, equivalently, in a geometric way. For a more leisurely discussion, see [11,16]. We start with the algebraic approach. Given a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_d$, we define the subgroup $M(\mathfrak{a})$ of unipotent elements of $\mathrm{PSL}_2(\mathcal{O}_d)$ as

$$M(\mathfrak{a}) := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathfrak{a} \right\}$$

and we denote the normal closure of $M(\mathfrak{a})$ in $\mathrm{PSL}_2(\mathcal{O}_d)$ by $\mathrm{Q}(\mathfrak{a})$, that is, the intersection of all normal subgroups of $\mathrm{PSL}_2(\mathcal{O}_d)$ that contain $gM(\mathfrak{a})g^{-1}$ for all $g \in \mathrm{PSL}_2(\mathcal{O}_d)$. It can be shown that there is a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_d$ such that $\mathrm{Q}(\mathfrak{a}) \leq \Gamma$ where Γ is a finite index subgroup of $\mathrm{PSL}_2(\mathcal{O}_d)$. The level of Γ, denoted $L(\Gamma)$, is defined as the maximal ideal $\mathfrak{a}_\Gamma$ with this property. So we can say that $L(\Gamma)$ is the largest ideal $\mathfrak{a}$ such that the intersection of all conjugate subgroups of Γ contains $M(\mathfrak{a})$.

Geometrically, the level of a subgroup Γ can be defined in terms of the cusp widths of the hyperbolic 3-fold associated to Γ. It is well-known that a Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$ acts on $\mathbb{P}^1(K_d)$ as a set of linear transformations with finitely many orbits, in fact, the number of orbits is equal to the class number of K_d. A cusp of Γ is a Γ-orbit in $\mathbb{P}^1(K_d)$. For example, if $\mathfrak{p}$ is a prime ideal of residue degree one, then $\Gamma^0(\mathfrak{p})$ has two cusps, namely $[0]$ and $[\infty]$. For each $g \in \mathrm{PSL}_2(\mathcal{O}_d)$, let $\mathfrak{c}_g$ denote the largest ideal $\mathfrak{a}$ with the property that

$$gM(\mathfrak{a})g^{-1} \leq \Gamma.$$

$\mathfrak{c}_g$ is called the cusp width of the cusp $[g \cdot \infty]$. It follows that

$$L(\Gamma) = \bigcap_{g \in \mathrm{PSL}_2(\mathcal{O}_d)} \mathfrak{c}_g.$$

The following theorem gives a characterization of congruence subgroups and is a key tool in identifying noncongruence subgroups.

Theorem 2.1 (Grunewald-Schwermer [11]). *The subgroup $\Gamma \leq \mathrm{PSL}_2(\mathcal{O}_d)$ is a congruence subgroup if and only if $\Gamma(\mathfrak{a}_\Gamma) \leq \Gamma$.*

The following proposition, which will be used later when computing levels, is given in [21] without a proof. We prove it here for the sake of completeness. Put

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix},$$

where $w = \sqrt{-d}$ if $d \not\equiv 3 \pmod 4$ and $w = \frac{1+\sqrt{-d}}{2}$ otherwise.

Proposition 2.2. *Let I and J be two proper nonzero ideals of O_d such that $I \subseteq J$ and $m, n \in \mathbb{Z}$. We have*

(1) $M(I) \leq Q(I) \leq \Gamma(I)$.
(2) $M(I) \leq M(J)$, $Q(I) \leq Q(J)$, and $\Gamma(I) \leq \Gamma(J)$.
(3) If $w = \frac{1+\sqrt{-d}}{2}$, that is if $d \equiv 3 (mod\ 4)$, then

$$M(\langle m + nw \rangle) = \langle a^m u^n, a^{-n(d+1)/4} u^{m+n} \rangle.$$

(4) If $w = \sqrt{-d}$*, that is if* $d \not\equiv 3 (mod\ 4)$*, then*

$$M(\langle m + nw \rangle) = \langle a^m u^n, a^{-nd} u^m \rangle.$$

Proof. (1) and (2) is clear. We will give the proof of (3). (4) can be done similarly.

Since $a^m u^n$ and, $a^{-n(d+1)/4} u^{m+n}$ is an element of $M(\langle m + nw \rangle)$, we have

$$\langle a^m u^n, a^{-n(d+1)/4} u^{m+n} \rangle \leq M(\langle m + nw \rangle).$$

For the other side, let $X \in M(\langle m + nw \rangle)$. Then X is of the form

$$\begin{pmatrix} 1 & (m+nw)(x+yw) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & mx + w(my+nx) + nyw^2 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & mx + w(my+nx) + ny(w - n(d+1)/4) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & (mx - ny(d+1)/4) + w(nx + (m+n)y) \\ 0 & 1 \end{pmatrix}.$$

Thus X can be written in the form

$$(a^m)^x (u^n)^x (a^{-n(d+1)/4})^y (u^{m+n})^y \in \langle a^m u^n, a^{-n(d+1)/4} u^{m+n} \rangle.$$

Hence the result follows. □

Let H be a subgroup of $\mathrm{PSL}_2(\mathcal{O}_d)$. We define the congruence closure of H as the smallest congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_d)$ containing H. Accordingly, we define the congruence closure index (c.c.i.) of H as the index of H in its congruence closure.

3. Construction noncongruence subgroups of Bianchi groups

3.1. *Bounded congruence closure index*

It is a simple observation that the intersection of a congruence subgroup and a noncongruence subgroup is again a noncongruence subgroup. This leads to a straightforward way of constructing families of noncongruence subgroups with bounded congruence closure index. Indeed, fix a noncongruence subgroup H and a sequence $\{\Gamma_n\}_n$ of congruence subgroups and consider the family $\{\Gamma_n \cap H\}_n$. The congruence closure of $\Gamma_n \cap H$ is inside Γ_n, thus the c.c.i. is bounded by $[\mathrm{PSL}_2(\mathcal{O}_d) : H]$. Noncongruence subgroups of small index can be easily found as we will discuss below.

3.2. *Increasing congruence closure index*

We will now obtain families of noncongruence subgroups of $\mathrm{PSL}_2(\mathcal{O}_d)$ for $d = 2, 7, 11$ in which the congruence closure index is increasing. More precisely, we will consider the congruence family $\{\Gamma_0(\mathfrak{p}_n)\}_n$ with prime levels $\mathfrak{p}_n$ and consider special normal subgroups with increasing prime index such that each subgroup is noncongruence.

We will use the following theorem (compare with [13, Prop. 18]) to construct these special normal subgroups.

Theorem 3.1. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_d$ of residue degree one over the rational prime p and $q \geq 5$ a rational prime such that q does not divide $p(p-1)$ and the discriminant of $K_d/\mathbb{Q}$. Let H be a normal subgroup of $\Gamma_0(\mathfrak{p})$ with index q. If the level of H is $\mathfrak{p}\langle q\rangle$, then H is a noncongruence subgroup.*

Proof. Assume that H is a congruence subgroup. By Theorem 2.1, H contains the principal congruence subgroup $\Gamma(\mathfrak{p}\langle q\rangle)$. As $H/\Gamma(\mathfrak{p}\langle q\rangle)$ is normal in $\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$ with the same index q, the simple cyclic group $C_q \simeq \Gamma_0(\mathfrak{p})/H$ is a composition factor of $\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$. We will obtain a contradiction by showing that this cannot be the case.

We have the normal series

$$\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \supset \Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \supset \langle I\rangle.$$

Therefore the composition factors of $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$ are the union of those of

$$\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$$

and of

$$\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)\Big/\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \cong \Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p}).$$

Now, the order of $\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p})$ is $p(p-1)/2$. By assumption we have $q \nmid p(p-1)$ and so the composition factors of $\Gamma_0(\mathfrak{p})/\Gamma(\mathfrak{p})$ can not be of order q.

It remains to show that C_q cannot be a composition factor of $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$. Observe that for any ideal $\mathfrak{n}$, the group $\Gamma(\mathfrak{n})$ is isomorphic to the group

$$S\Gamma(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_d) \mid a-1 \equiv b \equiv c \equiv d-1 \equiv 0 \pmod{\mathfrak{n}} \right\},$$

its counterpart inside $\mathrm{SL}_2(\mathcal{O}_d)$. Indeed, if $\overline{\Gamma}(\mathfrak{n})$ is the preimage of $\Gamma(\mathfrak{n})$ inside $\mathrm{SL}_2(\mathcal{O}_d)$ under the standard map $\pi : \mathrm{SL}_2(\mathcal{O}_d) \to \mathrm{PSL}_2(\mathcal{O}_d)$, then

$$\Gamma(\mathfrak{n}) = \pi\left(\overline{\Gamma}(\mathfrak{n})\right) = \overline{\Gamma}(\mathfrak{n})/\{\pm Id\} \simeq S\Gamma(\mathfrak{n}).$$

The quotient $S\Gamma(\mathfrak{p})/S\Gamma(\mathfrak{p}\langle q\rangle)$ is equal to the kernel of the natural surjection from $\mathrm{SL}_2(\mathcal{O}_d/\mathfrak{p}\langle q\rangle)$ onto $\mathrm{SL}_2(\mathcal{O}_d/\mathfrak{p})$, which in turn is isomorphic to $\mathrm{SL}_2(\mathcal{O}_d/\langle q\rangle)$ since $q \notin \mathfrak{p}$. We obtain that

$$\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \simeq S\Gamma(\mathfrak{p})/S\Gamma(\mathfrak{p}\langle q\rangle) \simeq \mathrm{SL}_2(\mathcal{O}_d/\langle q\rangle).$$

Case 1: Assume that q is inert in $\mathcal{O}_d$. Then we have

$$\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \cong \mathrm{SL}_2(\mathbb{F}_{q^2}).$$

Since $\mathrm{PSL}_2(\mathbb{F}_{q^2})$ is simple for any prime q, a composition series for $\mathrm{SL}_2(\mathbb{F}_{q^2})$ is

$$\{Id\} \lhd \{\pm Id\} \lhd \mathrm{SL}_2(\mathbb{F}_{q^2}).$$

Thus the composition factors for $\mathrm{SL}_2(\mathbb{F}_{q^2})$ are C_2 and $\mathrm{PSL}_2(\mathbb{F}_{q^2})$. So for $q \geq 3$, C_q cannot be a composition factor for $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle)$.

Case 2: Assume that q splits in $\mathcal{O}_d$. Now we have

$$\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}\langle q\rangle) \cong \mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q).$$

If C_q is a composition factor for $\mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q)$ then, by the Jordan-Hölder Theorem, it must be also a composition factor for $\mathrm{SL}_2(\mathbb{F}_q)$. Since $\mathrm{PSL}_2(\mathbb{F}_q)$ is simple for $q \geq 5$, a composition series for $\mathrm{SL}_2(\mathbb{F}_q)$ is

$$\{Id\} \triangleright \{\pm Id\} \triangleright \mathrm{SL}_2(\mathbb{F}_q).$$

Therefore the composition factors for $\mathrm{SL}_2(\mathbb{F}_q)$ are C_2 and $\mathrm{PSL}_2(\mathbb{F}_q)$. Hence for $q \geq 5$, C_q cannot be a composition factor for $\mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q)$. □

Remark 3.2. The reviewer kindly informed us that Theorem 3.1 is essentially known: it follows immediately from the fact that any congruence subgroup containing the derived subgroup $[\Gamma_0(\mathfrak{p}), \Gamma_0(\mathfrak{p})]$ must contain the subgroup $\Gamma_0(\mathfrak{p}^3)$. This fact in turn follows from "super-strong approximation".

4. Explicit families with fixed congruence closure index

In this section we will consider two noncongruence subgroups of small index given in [11]. We construct families of noncongruence subgroups by intersecting these two groups with families of congruence subgroups. The resulting families will have fixed congruence closure index.

4.1. *Family 1.*

A presentation of $\mathrm{PSL}_2(\mathcal{O}_7)$ can be given as

$$\mathrm{PSL}_2(\mathcal{O}_7) = \langle\, a, b, u \mid (ba)^3 = b^2 = aua^{-1}u^{-1} = (bau^{-1}bu)^2 \,\rangle.$$

There is an index 3 normal subgroup H of $\mathrm{PSL}_2(\mathcal{O}_7)$ generated by the elements

$$a, b, ubu^{-1}, u^3, u^{-1}bu.$$

We see that $M(3\mathcal{O}_7) \leq Q(3\mathcal{O}_7) \leq H$. The ideal $3\mathcal{O}_7$ is a prime ideal so the maximal ideal $\mathfrak{a}$ where $Q(\mathfrak{a}) \leq H$ is $3\mathcal{O}_7$ and hence the level of H has to be $3\mathcal{O}_7$ by Proposition 2.2. If H is a congruence subgroup, that is $\Gamma(3\mathcal{O}_7)$ is contained in H, then $\mathrm{PSL}_2(\mathcal{O}_7/3\mathcal{O}_7) = \mathrm{PSL}_2(\mathbb{F}_9)$ must contain a normal subgroup of index 3 which is not the case. Hence H is a noncongruence subgroup with index 3 (see [11, Page 213]).

Let $S = \{\mathfrak{p}_n\}_n$ be a sequence of prime ideals in $\mathcal{O}_7$ with prime norm, one prime ideal chosen over every rational prime splitting in $\mathbb{Q}(\sqrt{-7})$. Let $\Gamma_n = \Gamma^0(\mathfrak{p}_n) \cap H$. Obviously each Γ_n is an index 3 subgroup of $\Gamma^0(\mathfrak{p}_n)$ and is noncongruence.

4.2. *Family 2.*

The group $\mathrm{PSL}_2(\mathcal{O}_{11})$ has a presentation

$$\mathrm{PSL}_2(\mathcal{O}_{11}) = \langle\, a, b, u \mid (ba)^3 = b^2 = aua^{-1}u^{-1} = (bau^{-1}bu)^3 \,\rangle.$$

The abelianization of $\mathrm{PSL}_2(\mathcal{O}_{11})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}$. Hence there is a surjective homomorphism $\phi : \mathrm{PSL}_2(\mathcal{O}_{11})^{ab} \to \mathbb{Z}/2\mathbb{Z}$. Let H be the kernel of ϕ. Then H is an index 2 subgroup generated by the elements

$$a, b, u^{-2}, ubu^{-1}.$$

Here $u \notin H$, and this implies that the level of H is $2\mathcal{O}_{11}$ by Proposition 2.2. If $\Gamma(2\mathcal{O}_{11})$ were contained in H, then $\mathrm{PSL}_2(\mathcal{O}_{11}/2\mathcal{O}_{11})$ would have a subgroup of index 2, where $\mathcal{O}_{11}/2\mathcal{O}_{11}$ is a field, which is not the case. Therefore H is a noncongruence subgroup (see [11, Page 213]).

Let $S = \{\mathfrak{p}_n\}_n$ be a sequence of prime ideals in $\mathcal{O}_{11}$ with prime norm, one prime ideal chosen over every rational prime splitting in $\mathbb{Q}(\sqrt{-11})$. Let $\Gamma_n = \Gamma^0(\mathfrak{p}_n) \cap H$. Then it is clear that Γ_n is an index 2 subgroup of $\Gamma^0(\mathfrak{p}_n)$ and is noncongruence.

5. Explicit families with increasing congruence closure index

In this section we will consider three families of noncongruence subgroups. These subgroups will be constructed via Thm. 3.1 using explicit surjective homomorphisms from the congruence subgroups $\Gamma_0(\mathfrak{p})$ to finite fields $\mathbb{F}_q$. These explicit homomorphisms arise simply from the "boundary cohomology". We thank the referee for pointing out this fact which greatly simplified our description of the homomorphisms.

5.1. *Families 3,4,5*

Let $d = 2, 7, 11$ and let $w = \frac{1+\sqrt{-d}}{2}$ if $d = 7, 11$ and $w = \sqrt{-d}$ if $d = 2$. For $d = 7, 11$, we have already seen presentations for $\mathrm{PSL}_2(\mathcal{O}_d)$. For $d = 2$, a well-known presentation is

$$\mathrm{PSL}_2(\mathcal{O}_2) = \langle\, a, b, u \mid (ab)^3 = b^2 = aua^{-1}u^{-1} = (bu^{-1}bu)^2 \,\rangle.$$

In all three cases, it is clear that the map $\phi : \mathrm{PSL}_2(\mathcal{O}_d) \to \mathbb{Z}$ sending a, b to 0 and u to 1 is a homomorphism.

Let $S = \{\mathfrak{p}_n\}_n$ be an infinite sequence of prime ideals in $\mathcal{O}_d$ with prime norm, one prime ideal chosen over every rational prime splitting in $\mathcal{O}_d$. For $\mathfrak{p} \in S$, the elements $\{Id, ba^i\}$ for $0 \le i \le p-1$ form a system of coset representatives for $\Gamma_0(\mathfrak{p})$ in $\mathrm{PSL}_2(\mathcal{O}_d)$ (here p is the rational prime that $\mathfrak{p}$ is over). Identify $\mathcal{O}_d/\mathfrak{p} \simeq \mathbb{F}_p = \{0, 1, \ldots, p-1\}$ and let t denote the image of w in $\mathcal{O}_d/\mathfrak{p}$. Using the Reidemeister-Schreier algorithm (see [12]), we obtain the following generators for $\Gamma_0(\mathfrak{p})$

$$a, u, ba^t ub, ba^p b, ba^k ba^{-k_*} b$$

where $1 \le k, k_* \le p-1$ and $kk_* \equiv -1 \mod p$.

We restrict ϕ to the subgroup $\Gamma_0(\mathfrak{p})$ to obtain a homomorphism $\phi(p) : \Gamma_0(\mathfrak{p}) \to \mathbb{Z}$ sending the generators u, ba^tub to 1 and all the rest to 0. For a positive integer $m \geq 2$, we can compose ϕ with the reduction map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ and obtain the homomorphism $\phi(p,m) : \Gamma_0(\mathfrak{p}) \to \mathbb{Z}/m\mathbb{Z}$.

Let q be a prime and let $H(p,q)$ be the kernel of $\phi(p,q)$. Let us compute its level. Since $H(p,q)$ is a normal subgroup, we can determine the number of cusps of $H(p,q)$ by looking at whether the cusp ∞ splits into cusps in $H(p,q)$ or not. For this purpose, we shall need a set of coset representatives for $H(p,q)$ in $\Gamma_0(\mathfrak{p})$; we use $\{Id, u, u^2, \ldots, u^{q-1}\}$. Since for every $k \in \mathbb{N}$, u^k stabilizes ∞, we have only two cusps, 0 and ∞. As $\Gamma_0(\mathfrak{p})_\infty = \langle a, u \rangle$ and $\Gamma_0(\mathfrak{p})_0 = \langle ba^pb, ba^tub \rangle$, we have $H(p,q)_\infty = \langle a, u^q \rangle$ and $H(p,q)_0 = \langle ba^pb, (ba^tub)^q \rangle$. Then the level of $H(p,q)$ is $\mathfrak{p}\langle q \rangle$, see [16].

If we choose $q \geq 5$ a rational prime which is not ramified in $\mathcal{O}_d$ and is such that $q \nmid p(p-1)$, then by Theorem 3.1 we see that $H(p,q)$ is a noncongruence subgroup.

6. Torsion homology growth

In this section, we will report on our experiments on the growth of torsion in the homology of noncongruence subgroups of Bianchi groups.

The following is a special case of a conjecture of Bergeron and Venkatesh [4] [5, Conjecture 6.1.] which treats general *congruence* arithmetic groups.

Conjecture 6.1. *Let* $\{\Gamma_n\}_n$ *be a sequence of congruence subgroups of a fixed Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_d)$*. Then*

$$\lim_{n \to \infty} \frac{log|\left(\Gamma_n^{ab}\right)_{tor}|}{vol(\Gamma_n \backslash \mathbb{H})} = \frac{1}{6\pi}.$$

We wish to test the analogue of this conjecture for noncongruence subgroups of Bianchi groups. An important ingredient to formulate the analogue is the *BS-convergence* notion of [1], named after Benjamini and Schramm. Recall that for a hyperbolic manifold M and positive real number R, the R-thin part $M_{<R}$ of M is the part of M in which the local injectivity radius is less than R. A sequence $\{M_n\}_n$ of hyperbolic 3-manifolds of finite volume *BS-converges* to the universal cover $\mathbb{H}$ if, informally speaking, "the injectivity radius of M_n goes to infinity at almost every point", that is, for every $R > 0$,

$$\lim_{n \to \infty} \frac{vol((M_n)_{<R})}{vol(M_n)} = 0.$$

For convenience, we will say that a sequence $\{\Gamma_n\}_n$ of cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is BS-convergent if the sequence $\{\Gamma_n \backslash \mathbb{H}\}_n$ of associated hyperbolic 3-manifolds BS-converge to $\mathbb{H}$. In [17, Theorem A], Raimbault proves that the for any sequence $\{\Gamma_n\}_n$ of congruence subgroups of Bianchi groups (not necessarily inside a fixed Bianchi group) is BS-convergent. Thus the following conjecture extends the scope of Conjecture 6.1 to include noncongruence subgroups as well.

Conjecture 6.2. *Let* $\{\Gamma_n\}_n$ *be a sequence of finite index subgroups of a fixed*

Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$*. Assume that* $\{\Gamma_n\}_n$ *is BS-convergent. Then*

$$\lim_{n\to\infty} \frac{log|\left(\Gamma_n^{ab}\right)_{tor}|}{vol(\Gamma_n\backslash\mathbb{H})} = \frac{1}{6\pi}.$$

Remark 6.3. As the notion of BS-convergence is more flexible, one can relax the above by not requiring the Bianchi group to be fixed.

We will test this conjecture on our families of noncongruence subgroups.

6.1. *BS-convergence*

In this section, we prove that the five families that we have constructed above are BS-convergent. For this end, we will use the notion of trace convergence of Farber, see [9,15] which equals BS-convergence when the sequence members Γ_n are all inside a fixed Bianchi group.

Say we have a sequence of finite-index subgroups $\{\Gamma_n\}_n$ of a cofinite discrete subgroup G of $\mathrm{PSL}_2(\mathbb{C})$. The permutation action of G on the right cosets of Γ_n gives rise to an action of G on the $\mathbb{C}$-vector space $\mathbb{C}[\Gamma_n\backslash G]$ with basis $\Gamma_n\backslash G$. For $g \in G$, let $\mathrm{tr}(g,\Gamma_n)$ be the trace of the operator given by the action of g on $\mathbb{C}[\Gamma_n\backslash G]$.

We say that $\{\Gamma_n\}_n$ is trace convergent if

$$\frac{\mathrm{tr}(g,\Gamma_n)}{[G:\Gamma_n]} \to 0$$

as $n \to \infty$ for every nontrivial $g \in G$. It is well-known that a sequence $\{\Gamma_n\}_n$ as above is BS-convergent if and only if it is trace convergent, see [15]. Note that BS-convergence does not need the necessity to be inside a fixed G.

Proposition 6.4. *Let* $\{H_n\}_n$ *be a trace convergent sequence of finite index subgroups of a fixed Bianchi group* G*. For every* n*, let* K_n *be a finite index subgroup of* H_n*. Then the sequence* $\{K_n\}_n$ *is trace convergent as well.*

Proof. Let $k_n := [H_n : K_n]$. We will show that for any $g \in G$,

$$tr(g, K_n) \leq k_n \cdot tr(g, H_n). \tag{6.1}$$

Assuming this, let us see how the claim follows. Noting that $[G : K_n] = k_n \cdot [G : H_n]$, we have

$$\frac{tr(g,K_n)}{[G:K_n]} \leq \frac{k_n \cdot tr(g,H_n)}{[G:K_n]} = \frac{tr(g,H_n)}{[G:H_n]}$$

If $g \neq 1$, then $\frac{tr(g,H_n)}{[G:H_n]} \to 0$ as $n \to \infty$ by hypothesis, implying $\frac{tr(g,K_n)}{[G:K_n]} \to 0$ as $n \to \infty$ as desired.

Let us now prove (6.1). Since the action of g on $\mathbb{C}[H_n\backslash G]$ arises from the permutation action of g on the coset space $H_n\backslash G$, the trace of G equals the number of cosets of H_n in G fixed by g. Similarly, the trace of g on $\mathbb{C}[K_n\backslash G]$ equals the number of cosets of K_n in G fixed by g. Let $K_n g'$ be a coset in $K_n\backslash G$ fixed by g, that is,

$$K_n g' g = K_n g'.$$

Since $K_n g'$ is contained only in the coset $H_n g'$ in $H_n\backslash G$, we have to have

$$H_n g' g = H_n g'.$$

Since every coset of H_n in G splits into k_n cosets of K_n in G, it follows that

$$\mathrm{tr}(g, K_n) \leq k_n \cdot \mathrm{tr}(g, H_n).$$ □

It follows immediately from the above proposition that all our families are BS-convergent since the sequence of congruence subgroups $\{\Gamma_0(\mathfrak{p}_n)\}_n$ is BS-convergent, and thus trace convergent, by Raimbault's above mentioned result [17, Theorem A].

7. Numerical experiments

In this section, we report on the torsion homology growth experiments we conducted using the families above. We will group the families, under light of Remark 6.3, into two groups according to the nature of the congruence closure index.

7.1. *Experiment 1: fixed congruence closure index.*

Here we work with Family 1 and Family 2. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_d$ of degree 1. The index of $\Gamma^0(\mathfrak{p})$ in $\mathrm{PSL}_2(\mathcal{O}_d)$ is $p+1$. The elements a^i, $0 \leq i \leq p-1$, and b form a system of coset representatives for $\Gamma^0(\mathfrak{p})$ in $\mathrm{PSL}_2(\mathcal{O}_d)$. From these matrices we obtain the following generating system for $\Gamma^0(\mathfrak{p})$ by using the Reidemeister-Schreier algorithm:

$$a^p, bab, ua^{-t}, bub, a^i ba^j \tag{7.1}$$

where $ij \equiv 1 \pmod{p}$ and $i, j \in \mathbb{F}_p^*$ and $t \in \mathbb{F}_p$ such that $w = t = -\frac{x}{y}$ in $\mathbb{F}_p$ for $p = (x+yw)\overline{(x+yw)}$. We automated this in Magma [6] and computed the ratios

$$\frac{log|\left(\Gamma_n^{ab}\right)_{\mathrm{tor}}|}{vol(\Gamma_n\backslash\mathbb{H})} \tag{7.2}$$

for all $\mathfrak{p}_n \in S$ with norm less than or equal to 15000 and 18000 for $d = 7, 11$, respectively. There are 1895 such ideals in total. We plotted the ratios (7.2) against the index of Γ_n in $\mathrm{PSL}_2(\mathcal{O}_d)$ in Fig. 1. There is a clear convergence to $\frac{1}{6\pi}$.

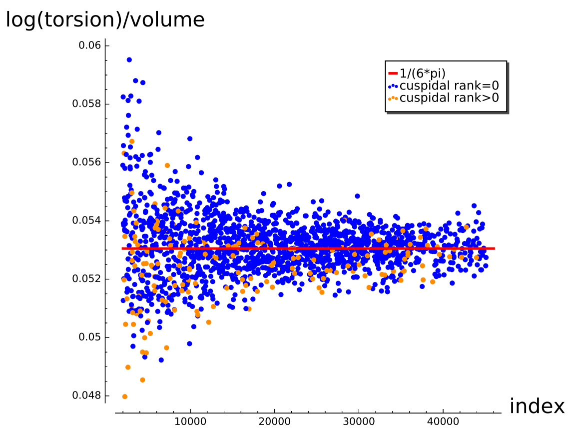

Fig. 1. The growth of torsion for the Γ_n's.

It is also interesting to compare the behaviour of the rank of the free part of the abelianization (this is related to the so-called "regulator", see [4]). It is easy to compute that Γ_n also has only two cusps using the fact that $\Gamma^0(\mathfrak{p}_n)$ with $\mathfrak{p}_n \in S$ has two cusps, namely $[0]$ and $[\infty]$. Existence of two cusps imply that the ranks of the abelianizations of Γ_n and of $\Gamma^0(\mathfrak{p}_n)$ are at least 2.

We computed the ranks of the abelianizations of Γ_n and of $\Gamma^0(\mathfrak{p}_n)$; within the range of our computations, they always have the same rank. We will call the rank of Γ_n^{ab} minus cusp contribution as cuspidal rank. As was observed in [19], the cuspidal rank is typically 0, see also [8,20]. We tabulated the ranks in Table 1 where the first column represents the cuspidal rank and the second column corresponds to the number of primes in S in which we observed this cuspidal rank frequency. In Fig. 2, we plot the cuspidal rank against the index.

Table 1. Distribution of cuspidal rank of Γ_n^{ab}.

cuspidal rank	frequency
0	1700/1895
1	148/1895
2	41/1895
3	4/1895
5	1/1895
6	1/1895

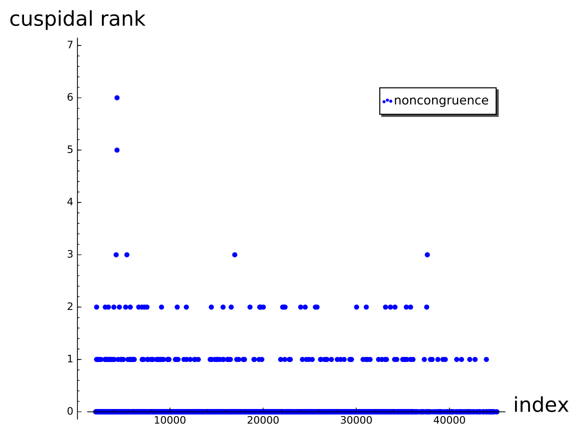

Fig. 2. Cuspidal rank of Γ_n^{ab} against the index.

7.2. *Experiment 2: increasing congruence closure index.*

In this experiment, we work with the families for which the congruence closure index is increasing (see Section 5).

Using Magma, we computed the ratios

$$\frac{log|\left(H(p,q)^{ab}\right)_{\rm tor}|}{vol(H(p,q)\backslash\mathbb{H})} \tag{7.3}$$

for 103 different noncongruence subgroups $H(p,q)$ over the three fields with $d = 2, 7, 11$. We increased p and q simultaneously in order to make sure that the congruence closure index was increasing. The table below lists the tuples (p,q).

Table 2. The tuples (p, q).

$d = 2$
$(17, 11), (19, 13), (41, 17), (43, 19), (59, 23), (67, 29), (73, 31), (83, 37), (89, 41), (97, 43), (107, 47), (113, 53), (131, 59), (137, 61), (139, 67), (163, 71), (179, 73), (193, 79), (211, 83), (227, 89), (233, 97), (241, 101), (251, 103), (257, 107), (281, 109), (283, 113), (307, 127), (313, 131), (331, 137), (337, 139), (347, 149), (353, 151), (379, 157), (401, 163)$
$d = 7$
$(23, 13), (29, 11), (37, 17), (43, 19), (53, 23), (67, 29), (71, 31), (79, 37), (107, 41), (109, 43), (113, 47), (127, 53), (137, 59), (149, 61), (151, 67), (163, 71), (179, 73), (191, 79), (193, 83), (197, 89), (211, 97), (233, 101), (239, 103), (263, 107), (277, 109), (281, 113), (317, 127), (331, 131), (337, 137), (347, 139), (359, 149), (373, 151), (379, 157), (389, 163), (401, 167), (421, 173), (431, 179), (443, 181), (449, 191)$
$d = 11$
$(23, 13), (31, 11), (37, 17), (47, 19), (53, 23), (59, 31), (67, 29), (71, 37), (89, 41), (97, 43), (103, 47), (113, 53), (137, 59), (157, 61), (163, 67), (179, 71), (181, 73), (191, 79), (199, 83), (223, 89), (229, 97), (251, 101), (257, 103), (269, 107), (311, 109), (313, 113), (317, 127), (331, 131), (353, 137), (367, 139)$

We plotted the ratios (7.3) against the index of noncongruence subgroups $H(p, q)$'s in $\mathrm{PSL}_2(\mathcal{O}_d)$'s in Fig. 3. It seems to converge to $\frac{1}{6\pi}$.

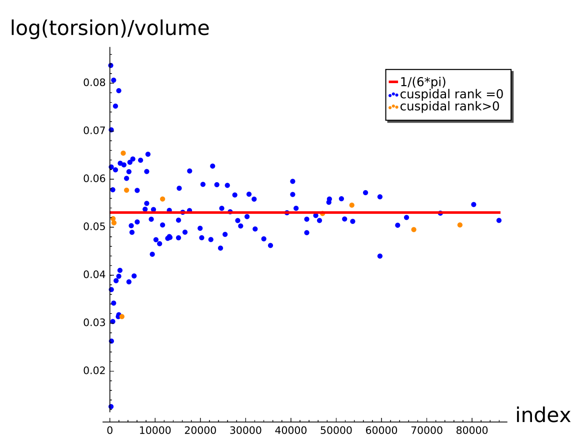

Fig. 3. The growth of torsion of $H(p, q)$'s against the index.

We see that $H(p, q)$ has only two cusps using the fact that $\Gamma_0(\mathfrak{p}_n)$ where $\mathfrak{p}_n$ is a prime ideal in $\mathcal{O}_d$ with prime norm has two cusps. Existence of two cusps imply that the ranks of the abelianizations of $H(p, q)$ and $\Gamma_0(\mathfrak{p}_n)$ are at least 2.

We computed the ranks of the abelianizations of $H(p,q)$ and of $\Gamma_0(\mathfrak{p}_n)$; within the range of our computations and we see that they always have the same rank. We tabulated the ranks in Table 3. In Fig. 4, we plot the cuspidal rank against the index.

Table 3. Distribution of cuspidal rank of $H(p,q)$.

cuspidal rank	frequency
0	93/103
1	7/103
2	3/103

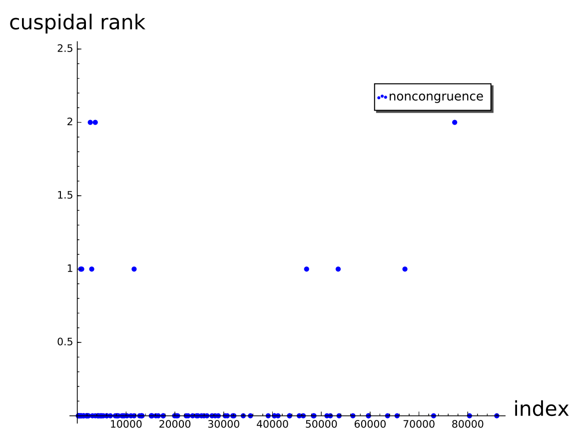

Fig. 4. Cuspidal rank of $H(p,q)$'s against the index.

Remark 7.1. In order to try to gain insight into the regulator growth issue that was discussed at the end of the Introduction, we computed the ranks of *all* $H(p,q)$ with $pq \leq 30000$ within the families $3, 4, 5$. Unfortunately, as in the examples above, the rank always equaled the rank of the congruence closure.

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