Reconstructing a pure state of a spin s through three Stern-Gerlach measurements

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Consider a spin s prepared in a *pure* state. It is shown that, generically, the moduli of the $(2s+1)$ spin components along three directions in space determine the state unambigously. These probabilities are accessible experimentally by means of a standard Stern-Gerlach apparatus. To reconstruct a pure state is therefore possible on the basis of $3(2s+1)$ measured intensities.

The reconstruction of a particle density-operator is possible in principle through repeated measurements on an ensemble of identically prepared systems [1,2]. Quantum states of vibrating molecules [3], of trapped ions [4], as well as the state of atoms in motion [5] have been reconstructed successfully in the laboratory. Similarly, quantum optical experiments [6] have been performed.

For a spin of length s, this question arises for states in a Hilbert space of finite dimension. There is an explicit expression for the density matrix ρ in terms of the moduli of spin components along $(4s+1)$ appropriate directions in space [7]. This number can be reduced to $(2s+1)$ upon adopting a different approach [8]. A standard Stern-Gerlach apparatus with variable orientation in space provides the corresponding probabilities in an experiment. Alternatively, a Wigner function defined on the discrete phase space associated with a finite-dimensional Hilbert space allows one to reconstruct quantum states [9]. This method has been adapted in [10] in order to determine a quantized electromagnetic mode of a cavity. Every proposed method of state reconstruction is bound to reflect on the link between the outcomes of a finite number of measurements obtained in an actual experiment and the mathematical probabilities which refer to *infinite* ensembles (see [11], for example).

Suppose now that the spin state to be reconstructed is known to be prepared in a *pure* state which is determined by less parameters than a mixed one. How to exploit this additional knowledge in the most efficient way? Reconstruction of pure states has been turned into a question as early as 1933 for a *particle* by Pauli [12] who did not provide an answer. One solution of the spin version of the problem [13] makes use of a *Feynman filter*. This is an advanced version of a Stern-Gerlach apparatus which is assumed to reveal the relative phases of the expansion coefficients of a pure spin state. Another approach relates expectation values of spin multipoles with the parameters which define the quantum state [14].

As shown in this letter, the pure state of a spin s is determined unambigously if the *intensities* of the spin components are measured along *three* axes. Compared to the $(2s+1)$ axes required for a mixed state [8], the experimental effort to perform state reconstruction is thus reduced considerably for large spins. Further, this result is satisfactory from a mathematical point of view since it generalizes an earlier result: the intensities along two *infinitesimally close* axes spanning a plane define a unique pure state when complemented by the expectation value of a spin component "out of plane" [15]. Effectively, this means to measure $(2s+1)$ probabilities along a third direction.

The states of a spin of magnitude s live in a Hilbert space $\mathbf{H}^s$ of complex dimension $(2s+1)$, which carries an irreducible representation of the group $SU(2)$. The components of the spin operator $\vec{\boldsymbol{S}} \equiv \hbar \vec{\boldsymbol{s}}$ with standard commutation relations $[\boldsymbol{s}_x, \boldsymbol{s}_y] = i\boldsymbol{s}_z, \ldots$ generate rotations about the corresponding axes. The standard basis of the space $\mathbf{H}^s$ is given by the eigenvectors of the z component of the spin, denoted by $|s, \mu_z\rangle, -s \leq \mu_z \leq s$. The transformation under the anti-unitary time reversal operator T fixes their phases, $T|s,\mu_z\rangle = (-1)^{s-\mu_z}|s, -\mu_z\rangle$. When expanded in the z basis ($\mu_k \equiv \mu_z$),

$$|\psi\rangle = \sum_{\mu_k=-s}^{s} \psi_{\mu_k} |s, \mu_k\rangle \,, \qquad k = x, y, z \,, \tag{1}$$

a pure state is seen to be determined by $(2s+1)$ complex coefficients $\psi_{\mu_z} \equiv \langle s, \mu_z|\psi\rangle$. If normalized, rays $|\psi\rangle$ depend on $4s$ real parameters. Two other bases of the space $\mathbf{H}^s$ are used in Eq. (1): the sets $\{|s,\mu_x\rangle\}$ and $\{|s,\mu_y\rangle\}$ with $-s \leq \mu_x, \mu_y \leq s$, made up from the eigenvectors of the spin components $\boldsymbol{s}_x$ and $\boldsymbol{s}_y$, respectively. Rotations about appropriate axes by an angle $\pi/2$ map them to the z basis:

$$|s,\mu_z\rangle = e^{-i\pi \boldsymbol{s}_y/2}|s,\mu_x\rangle = e^{i\pi \boldsymbol{s}_x/2}|s,\mu_y\rangle \,. \tag{2}$$

A measurement of the intensities $\{|\langle s,\mu_z|\psi\rangle|^2\}$ does not fix a single state $|\psi\rangle$ since the phases of the coefficients ψ_{μ_z} remain undetermined. However:
a spin state $|\psi\rangle \in \mathbf{H}^s$ is determined unambiguously if $3(2s+1)$ probabilities

$$p(\mu_k) = |\psi_{\mu_k}|^2 \,, \quad k = x, y, z \,, \tag{3}$$

are measured with a Stern-Gerlach apparatus along three *axes not in a plane. For some exceptional states of measure zero in Hilbert space $\mathbf{H}^s$, the probabilities $p(\mu_k)$ might be compatible with a finite number of states.*

For simplicity, the proof is carried out for orthogonal axes, the generalization being straightforward. Measuring with respect to *two* axes provides $2(2s+1)$ intensities which are usually compatible with a huge number of isolated states, in agreement with the result of [15]: the parameters fullfil nonlinear relations which may have multiple solutions. Enumerating the ensemble of possible "partner" states is complicated, so a distinctive third measurement is included from the very beginning.

It is useful to rephrase the statement at stake differently. According to (3) a state $|\widetilde{\psi}\rangle$ gives rise to the *same* intensities as does $|\psi\rangle$ if its coefficients $\widetilde{\psi}_{\mu_k} = \langle s, \mu_k|\widetilde{\psi}\rangle$ differ from ψ_{μ_k} by phase factors only. Using (1) one writes thus

$$\sum_{\mu_k=-s}^{s} \psi_{\mu_k} e^{i\chi_k(\mu_k)}|s,\mu_k\rangle = \exp[i\chi_k(\boldsymbol{s}_k)]|\psi\rangle \,, \qquad (4)$$

with three polynomials $\chi_k(\mu)$ of order $2s$ in μ at most. From now on, the index k is understood to take the values x, y, and z throughout. The coefficients in (4) thus define three states $|\psi_k\rangle = W_k^s|\psi\rangle$, where $W_k^s = \exp\left[i\chi_k(\boldsymbol{s}_k)\right]$ is a unitary operator diagonal in the k basis. Consequently, a state $|\widetilde{\psi}\rangle$ compatible with (3) exists if and only if there are nontrivial unitary operators W_k^s such that

$$W_x^s|\psi\rangle = W_y^s|\psi\rangle = W_z^s|\psi\rangle \equiv |\widetilde{\psi}\rangle \,. \qquad (5)$$

It will turn out that this relation is satisfied only if the operators W_k^s are multiples of the identity, implying that $|\widetilde{\psi}\rangle$ and $|\psi\rangle$ represent the *same* ray in Hilbert space.

Before turning to the proof, the intensities $p(\mu_k)$ in (3) are represented in a more compact way. Define three functions $m_k(\alpha)$ of a complex variable $\alpha \in \boldsymbol{C}$ by

$$m_k(\alpha) = \langle\psi|U_k^s(\alpha)|\psi\rangle \equiv \sum_{\mu_k=-s}^{s} e^{i\mu_k\alpha} p(\mu_k) \,, \qquad (6)$$

where the operator $U_k^s(\alpha) = \exp(i\alpha\boldsymbol{s}_k)$ rotates a state $|\psi\rangle$ about the k axis if $\alpha \in \boldsymbol{R}$. Eq. (6) is inverted easily using the orthogonality of the functions $\exp[-i\mu_k\alpha]$ on the interval $0 \leq \alpha < 2\pi$.

The proof showing that the data (3) are sufficient for state reconstruction is divided into five steps. ($\imath$) A 2^{2s} dimensional "parent" space $\mathcal{H}^s$ is introduced which contains the Hilbert space $\mathbf{H}^s$ of the spin s as a subspace. ($\imath\imath$) To each state $|\psi\rangle \in \mathbf{H}^s$ an equivalence class of product states $\{|\Psi\rangle \in \mathcal{H}^s\}$ is associated. ($\imath\imath\imath$) A natural definition of *generic* states emerges for *product* states in $\mathcal{H}^s$ and, *a fortiori*, in $\mathbf{H}^s$. ($\imath v$) An appropriate set of expectation values of the parent states $|\Psi\rangle$ fixes them uniquely. (v) Finally, it is shown that all states $|\widetilde{\psi}\rangle$ satisfying (5) have parents in the *same* equivalence class as the original $|\psi\rangle$. Consequently, the (generic) state $|\psi\rangle$ is the only one giving rise to the intensities (3).

($\imath$) The 2^{2s} dimensional "parent" space $\mathcal{H}^s$ of $\mathbf{H}^s$ is obtained from tensoring $2s$ copies of the Hilbert space $\boldsymbol{C}^2$ of a spin $1/2$:

$$\mathcal{H}^s = \bigotimes_{r=1}^{2s} \boldsymbol{C}_r^2 \,. \qquad (7)$$

A basis of $\boldsymbol{C}^2$ is given by the eigenstates $|\sigma\rangle \equiv |s = 1/2, \mu_3 = \sigma/2\rangle, \sigma = \pm 1$, of the third component of the spin $1/2$: $\boldsymbol{\sigma}_3|\sigma\rangle = \sigma|\sigma\rangle$. This choice induces a basis of $\mathcal{H}^s$ formed by all product states

$$|\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} |\sigma_r\rangle \,. \qquad (8)$$

The parent space $\mathcal{H}^s$ decomposes into a subspace $\mathcal{H}^s_{\mathrm{sym}}$ and its complement,

$$\mathcal{H}^s = \mathcal{H}^s_{\mathrm{sym}} \oplus \left(\mathcal{H}^s_{\mathrm{sym}}\right)^{\perp} \,, \qquad (9)$$

where $\mathcal{H}^s_{\mathrm{sym}}$ is spanned by the $(2s+1)$ states obtained from completely symmetrizing those in (8):

$$\begin{aligned}|s,\mu_3\rangle &= \mathcal{S}_{2s}|\{\sigma_r\}\rangle \\ &\equiv N^s_{\mu_3} \sum_{\{\sigma_r\}} \delta(\sigma_1 + \cdots + \sigma_{2s} - 2\mu_3)|\{\sigma_r\}\rangle \,, \end{aligned} \qquad (10)$$

where $-s \leq \mu_3 \leq s$, using a symmetrizer of $2s$ objects, $\mathcal{S}_{2s}$, and the normalization factor $N^s_{\mu_3} = ((s-\mu_3)!(s+\mu_3)!/(2s)!)^{1/2}$. The space $\mathcal{H}^s_{\mathrm{sym}}$ is important here because it carries a $(2s+1)$ dimensional irreducible representation of the group of rotations, $SU(2)$, obtained upon reducing the product representation [16]

$$\mathcal{U}\,|\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} \sum_{\sigma_r'=\pm 1} |\sigma_r'\rangle\langle\sigma_r'|u_r|\sigma_r\rangle \,, \qquad (11)$$

where u_r is the r-th copy of a rotation $u \in SU(2)$ of the fundamental representation acting on $\boldsymbol{C}^2$, and $\mathcal{U}$ is an operator defined on $\mathcal{H}^s$. Since Hilbert spaces of the same dimension are isomorphic, $\mathcal{H}^s_{\mathrm{sym}}$ and $\mathbf{H}^s$ will be identified from now on.

($\imath\imath$) There is a one-to-one relation between states $|\psi\rangle \in \mathcal{H}^s_{\mathrm{sym}}$ and equivalence classes of *product* states $|\Psi\rangle \in \mathcal{H}^s$:

$$|\Psi\rangle \equiv |\{\Psi^r\}\rangle = \bigotimes_{r=1}^{2s} \left(\sum_{\sigma_r} \Psi^r_{\sigma_r}|\sigma_r\rangle \right) \,. \qquad (12)$$

The equivalence relation $\sim$ is defined as follows: the projection of a state $|\Psi\rangle$ in (12) onto a basis state $|s,\mu_3\rangle \in \mathcal{H}^s_{\mathrm{sym}}$ must equal the corresponding expansion coefficient of $|\psi\rangle$ in the z basis, i.e. ,

$$\langle s,\mu_3|\Psi\rangle = N_\psi\langle s,\mu_z|\psi\rangle \,, \quad -s \leq \mu_3 = \mu_z \leq s \,, \qquad (13)$$

and the factor $N_\psi > 0$ may depend on the state $|\psi\rangle$ under consideration but *not* on the index μ_z. Thus, $|\Psi\rangle \sim |\Psi'\rangle$

means that for a fixed $|\psi\rangle$, the Eqs. (13) hold for both product states, $|\Psi\rangle \sim |\Psi'\rangle$. The association of spin states $|\psi\rangle$ with product or "parent" states $|\Psi\rangle$ is essential for the following.

In order to determine the class of states satisfying Eq. (13) for a prescribed vector $|\psi\rangle$ (with definite phase), multiply by the factor $1/N^s_\mu$, by powers $(-z)^{\mu+s}$ and sum all terms. The right-hand-side then defines an analytic function

$$f_R(z) = N_\psi \sum_{\mu=-s}^{s} \frac{(-z)^{\mu+s}}{N^s_\mu} \psi_\mu \propto \prod_{r=1}^{2s} (z_r - z) \,, \qquad (14)$$

specified by the location of its $2s$ zeroes z_r in the complex plane. The left-hand-side yields a second analytic function of z,

$$f_L(z) = \sum_{\mu=-s}^{s} \sum_{\{\sigma_r\}} (-z)^{\mu+s} \delta(\sigma_1 + \cdots + \sigma_{2s} - 2\mu) \Psi^1_{\sigma_1} \ldots \Psi^{2s}_{\sigma_{2s}}$$

$$\equiv \prod_{r=1}^{2s} (\Psi^r_- - z\,\Psi^r_+) \,, \qquad \Psi^r_\pm \equiv \Psi^r_{\pm 1} \,. \qquad (15)$$

The $(2s+1)$ equations (13) are satisfied if $f_L(z)$ and $f_R(z)$ coincide. Being two polynomials of degree $2s$, this requires them to have identical zeroes,

$$\frac{\Psi^r_-}{\Psi^r_+} = z_r \,, \qquad r = 1, \ldots, 2s \,; \qquad (16)$$

in addition, $f_L(0) = f_R(0)$ must hold. Due to the normalization $\langle \Psi^r | \Psi^r \rangle = |\Psi^r_+|^2 + |\Psi^r_-|^2 = 1$, one can write

$$\begin{pmatrix} \Psi^r_+ \\ \Psi^r_- \end{pmatrix} = \frac{e^{i\kappa_r}}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix} , \quad \kappa_r \in [0, 2\pi) \,. \qquad (17)$$

Thus, there are $2s$ undetermined phase factors $e^{i\kappa_r}$ with a product equal to 1 (remember that $|\psi\rangle$ denotes a *vector*). However, the overall ambiguity is even larger: when comparing the zeroes of the functions $f_L(z)$ and $f_R(z)$, there is no rule which would indicate what order to choose when writing down the product state $|\{\Psi^r\}\rangle$. In other words, the equivalence class of states defined by (13) consists of all states with coefficients (17) distributed in any order over the $2s$ spinors in (12). All these states are parents of the same $|\psi\rangle$ since they satisfy Eq. (13).

A given product state $|\Psi\rangle$ with components

$$\langle \{\sigma_r\} | \Psi \rangle = \Psi_{\{\sigma_r\}} = \prod_{r=1}^{2s} \Psi^r_{\sigma_r} \,, \qquad (18)$$

has a unique "daughter" $|\psi\rangle$ to be read off directly. Upon parametrizing each factor $|\Psi^r\rangle$ by a complex number z_r,

$$\begin{pmatrix} \Psi^r_+ \\ \Psi^r_- \end{pmatrix} = \frac{1}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix} , \qquad (19)$$

one sees that the ensemble $\{z_r\} \equiv (z_1, \ldots, z_{2s})$ (*no* order implied) defines the daughter $|\psi\rangle$ completely while a maximum of $(2s)!$ different parent states $|\Psi\rangle$ is associated with a given set $\{z_r\}$.

(*iii*) Suppose that three ensembles of $2s$ real numbers each, $\{x_r\}$, $\{y_r\}$, and $\{|z_r|\} \equiv (|z_1|, \ldots, |z_{2s}|)$ with $z_r = x_r + iy_r$ are given in disorder. If one is able to construct the disordered ensemble of $2s$ complex numbers $\{z_r = x_r + iy_r\}$ upon using the $2s$ conditions $|z_r|^2 = x_r^2 + y_r^2$, the equivalence class with representant $|\Psi\rangle$ is called *generic*. In other words, it must be possible to combine unambiguously real and imaginary parts into complex numbers z_r. In this spirit, a daughter $|\psi\rangle \in \mathcal{H}^s_{\text{sym}}$ will be called *generic* if it has generic parents $|\Psi\rangle$. The procedure does not work if equalities such as $x_r = \pm y_{r'}, r \neq r'$ exist; hence *exceptional* states have measure zero.

(*iv*) It is shown now that the expectation values of rotations $\mathcal{U}_k(\alpha)$ about the axes x, y, and z, fix generic product states $|\Psi\rangle = |\{\Psi^r\}\rangle$ up to a permutation of the factors $|\Psi^r\rangle$ and an overall phase factor. A generic $|\Psi\rangle \in \mathcal{H}^s$ leads to three expectation values

$$M_k(\alpha) = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \equiv \prod_{r=1}^{2s} \langle \Psi^r | u^r_k(\alpha) | \Psi^r \rangle \,, \qquad (20)$$

where $u_k(\alpha) = \mathbf{1}\cos(\alpha/2) + \boldsymbol{\sigma}_k \sin(\alpha/2)$ represents a rotation about axis k in $\boldsymbol{C}^2$. Using the parametrization of Eq. (19), the functions $M_k(\alpha)$ defined in (20) read explicitly

$$M_x(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2ix_r \sin(\alpha/2)}{1+|z_r|^2} \,, \qquad (21a)$$

$$M_y(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2y_r \sin(\alpha/2)}{1+|z_r|^2} \,, \qquad (21b)$$

$$M_z(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + i(1-|z_r|^2) \sin(\alpha/2)}{1+|z_r|^2} \,, \qquad (21c)$$

where again $z_r = x_r + iy_r$. Denote by $|\widetilde{\Psi}\rangle \equiv |\{\widetilde{\Psi}^r\}\rangle$ another product state with expectations $\widetilde{M}_k(\alpha)$:

$$\widetilde{M}_k(\alpha) = \langle \widetilde{\Psi} | \mathcal{U}_k(\alpha) | \widetilde{\Psi} \rangle \equiv \prod_{r=1}^{2s} \langle \widetilde{\Psi}^r | u^r_k(\alpha) | \widetilde{\Psi}^r \rangle \,. \qquad (22)$$

Upon describing the state $|\widetilde{\Psi}\rangle$ by the sequence $\{\tilde{z}_r\}$, the three functions $\widetilde{M}_k(\alpha)$ are given by Eqs. 21) after replacing each z_r by $\tilde{z}_r$. It is shown now that the conditions

$$\langle \widetilde{\Psi} | \mathcal{U}_k(\alpha) | \widetilde{\Psi} \rangle = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \,, \qquad k = x, y, z \,. \qquad (23)$$

necessitate $|\widetilde{\Psi}\rangle \sim |\Psi\rangle$. Being analytic in the complex α plane, the functions $M_k(\alpha)$ and $\widetilde{M}_k(\alpha)$ are equal if they have same zeroes. The equality $\widetilde{M}_z(\alpha) = M_z(\alpha)$ requires $|\tilde{z}_r| = |z_r|$. The condition $\widetilde{M}_x(\alpha) = M_x(\alpha)$ in turn implies $\tilde{x}_r = x_r$; finally, $\tilde{y}_r = y_r$ follows from $\widetilde{M}_y(\alpha) = M_y(\alpha)$. However, this procedure determines the ensembles $\{x_r\}, \{y_r\}$, and $\{|z_r|\}$ *without* any order of its members. Nevertheless, one can reconstruct the ensemble $\{z_r\}$ (*no* order implied) according to $(\imath\imath\imath)$ if $|\Psi\rangle$ is *generic* providing thus a *unique* equivalence class. For exceptional states, the $2s$ complex numbers cannot be reconstructed unambigously since they might allow for parents contained in different equivalence classes.

(v) The results $(\imath)$ to $(\imath v)$ imply that the probabilities $p(\mu_k)$ for three directions x, y, and z as given in Eq. (3) determine a generic state $|\psi\rangle$ unambigously. According to Eq. (5), a state $|\widetilde{\psi}\rangle$ gives rise to the same probabilities as does $|\psi\rangle$ if one has

$$|\psi_x\rangle = |\psi_y\rangle = |\psi_z\rangle = |\widetilde{\psi}\rangle \, . \tag{24}$$

For parent states $|\Psi_k\rangle$ of $|\psi_k\rangle$ this relation says that

$$|\Psi_x\rangle \sim |\Psi_y\rangle \sim |\Psi_z\rangle \sim |\widetilde{\Psi}\rangle \, . \tag{25}$$

This implies that the mean values $\langle \Psi_k | \mathcal{U}_x(\alpha) | \Psi_k \rangle$ of the operator $\mathcal{U}_x(\alpha) = \otimes_r \exp[i\alpha\boldsymbol{\sigma}_x/2]$ are equal for $k = x, y, z$: as products they are invariant under a permutation of their factors. This also holds for expectation values of the operators $\mathcal{U}_y(\alpha)$ and $\mathcal{U}_z(\alpha)$. Write the parent states $|\Psi_k\rangle$ in the form $\mathcal{W}_k|\Psi\rangle$ with operators $\mathcal{W}_k(\{\alpha_{k,r}\}) = \otimes_r \exp[i\alpha_{k,r}\boldsymbol{\sigma}_k/2]$ defined on the parent space $\mathcal{H}^s$ such that they have W_k^s as component acting in $\mathcal{H}^s_{\text{sym}}$. Contrary to the rotations $\mathcal{U}_k(\alpha)$ which depend linearly on the generators $\boldsymbol{s}_k$, the operators W_k^s are *non-linear* functions $\chi(\boldsymbol{s}_k)$ of them, Eq. (4). Therefore, the operators $\mathcal{W}_k(\{\alpha_{k,r}\})$ depend on a set of $2s$ *different* angles $\{\alpha_{k,r}\}$. Using (25) one concludes

$$\begin{aligned}\langle \widetilde{\Psi} | \mathcal{U}_k | \widetilde{\Psi} \rangle &= \langle \Psi_k | \mathcal{U}_k | \Psi_k \rangle \\ &= \langle \Psi | \mathcal{W}_k^\dagger \mathcal{U}_k \mathcal{W}_k | \Psi \rangle = \langle \Psi | \mathcal{U}_k | \Psi \rangle \, . \end{aligned} \tag{26}$$

The third equality follows because $\mathcal{W}_k$ and $\mathcal{U}_k$ do commute, both being functions of $\boldsymbol{s}_k$ only. Eq. (26) comes down to saying that the functions $M_k(\alpha)$ and $\widetilde{M}_k(\alpha)$ coincide for all k and α. One concludes thus with $(\imath v)$ that the state $|\widetilde{\Psi}\rangle$, a parent of $|\widetilde{\psi}\rangle$, is necessarily a member of the *same* equivalence class as the parent $|\Psi\rangle$ of $|\psi\rangle$. In other words, the application of the operators $\mathcal{W}_k$ on a parent $|\Psi\rangle$ does not map it into another equivalence class. In the generic case, there is thus no state different from $|\psi\rangle$ with the same data (3) what was to be shown.

The reasoning $(\imath)$ to (v) remains valid if one measures the intensities along directions characterized by unit vectors $\mathbf{n}_\zeta, \mathbf{n}_\eta$, and $\mathbf{n}_\xi$ instead of three orthogonal axes. These vectors must be linearly independent, that is, they have to span a *volume* in space: $\mathbf{n}_\zeta \cdot \mathbf{n}_\eta \times \mathbf{n}_\xi \neq 0$.

As a matter of fact, it is not excluded that the set of data (3) be also sufficient to determine exceptional states unambiguously. Suppose that the numbers $\{z_r\}$ are associated with a parent state $|\Psi\rangle$ and $\{z'_r\}$ with another one, $|\Psi'\rangle$, where both sets of complex numbers are obtained from the ensembles $\{x_r\}$ and $\{y_r\}$ through $|z_r|^2 = x_r^2 + y_r^2$. This does not necessarily imply the existence of an independent $|\psi'\rangle \neq |\psi\rangle$ since it is the basic conditions $|\psi'_{\mu_k}| = |\psi_{\mu_k}|$ which must be satisfied. Explicit calculations for low values of spin s show that this happens only if $\psi'_{\mu_k} = \psi^*_{\mu_k}$, resulting in $\langle\psi|(\boldsymbol{s}_y)^{2n+1}|\psi\rangle \equiv 0$ for all integers n. In any case one expects every non-genericity to vanish if the spatial directions involved are slightly modified.

To sum up, state reconstruction is possible if based on the $3(2s+1)$ moduli of the spin components with respect to three directions in space not all in the same plane. Compared to a constructive method using $(2s+1)^2$ real numbers, the non-constructive method presented here requires that considerably less parameters be determined experimentally, namely $3(2s+1)$.

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