The Unit Root Property and Optimality with a Continuum of States—Pure Exchange[1]

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ABSTRACT

We study general one good pure exchange stochastic overlapping generations economies with stationary Markovian uncertainty and a continuum of states for the Markov process. We show that a unit root property provides a complete characterization of the optimality properties of stationary equilibrium allocations when markets are sequentially complete and the welfare criterion is conditional Pareto optimality.

Keywords: Stochastic OLG, Continuum of states, Optimality, Unit root property
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1. Introduction

Competitive general equilibrium models with generational overlap exhibit the following curious but well known behaviour when the time horizon is infinite: the competitive mechanism need not lead to allocative efficiency, i.e. the first welfare theorem fails to hold. As a consequence, these models provide a natural vehicle for the analysis of a variety of macroeconomic questions all of which revolve around policy interventions that can induce Pareto improvements over equilibrium allocations in a dynamic environment. An easily stated and understood characterization of optimality is an essential tool in such a task with 'the net interest rate at a steady state of a deterministic economy is zero or positive' being a well known example. This paper shows that a corresponding simple condition, in the form of whether the value of a single endogenously determined variable is less than or equal to one, provides a characterization of optimality of stationary equilibria in quite a general stochastic version of the overlapping generations model.

A rich literature investigates the failure of the first welfare theorem in overlapping generations economies. Inspired by the seminal work of Cass (1972) on efficient growth, Balasko and Shell (1980) and Okuno and Zilcha (1980) obtained a general result characterizing those competitive equilibrium allocations which are also Pareto optimal allocations. Since then, in developments paralleling those in general equilibrium analysis, the overlapping generations model has been studied in successively more general frameworks with risk, limited market participation, etc. The analysis of the optimality of equilibrium allocations in stochastic environments has required appropriate specification of the markets that agents have access to as well as the criterion of optimality. The natural specification takes the demographic structure of the model seriously and assumes that agents cannot participate in markets to insure against the risky endowment received when young but can fully insure against risks faced when old, called sequentially complete markets in later literature, with a corresponding notion of welfare that treats agents born in different states as distinct entities; these two innovations had been introduced earlier by Muench (1977) under the names spot markets and conditional Pareto optimality.

This paper considers a pure exchange economy with risky endowments that are realized according to a stationary Markov process with a continuum of states, and provides a characterization of the optimality of stationary competitive allocations in sequentially complete markets using conditional Pareto optimality as the welfare criterion. It shows that the 'unit root property', which asks whether the dominant eigenvalue of an operator is less than or equal to one, is a necessary and sufficient condition for optimality. We highlight the fact that the condition identified is easily stated and is transparently interpretable. A key step in proving the result is to show that, associated with the dominant eigenvalue, there exists an eigenvector that is measurable, uniformly positive, as well as uniformly bounded. That 'positive' eigenvector plays an essential role in the analysis. The result is obtained in the case where the Markov transition function is described by a measurable density function that is uniformly positive and uniformly bounded.

Our approach is quite general since we allow the stationary equilibrium allocation to be any measurable function, while any stochastic process that obeys feasibility is allowed to be a potential candidate for a Pareto improvement. The operator we study identifies the prices of one period ahead Arrow securities induced at a stationary equilibrium; equivalently, the operator specifies the marginal rates of substitution between consumption when old and young in different states faced when old and young. We show that if the dominant eigenvalue exceeds one then the associated positive eigenvector can be used to construct an alternative stationary allocation that is a conditional Pareto improvement over the stationary equilibrium allocation. We then use a second order approximation approach and the positive eigenvector to show that any allocation that provides a conditional Pareto improvement over a stationary equilibrium allocation necessarily induces an inequality, with the weighted first moment of a bounded random variable on one side and the weighted sum of of its second moment at earlier dates on the other; that single inequality summarizes the value of additional resources required to fund the improvement. The fact that the positive eigenvector is uniformly positive and uniformly bounded is then used to argue that if the dominant eigenvalue of the operator is less than or equal to one then the inequality is necessarily violated at some finite date.

The key step in our argument builds on Birkhoff (1957 and 1962) which treats the eigenvalue-eigenvector problem for a class of linear operators on infinite dimensional spaces.[2] We show that when, as in our application, the kernel of the operator is uniformly positive and uniformly bounded, the eigenvector associated with the dominant eigenvalue also is uniformly positive and uniformly bounded. Continuity of the operator plays no role but, when assumed, delivers continuity of the eigenvector.[3]

Having stated what we do, we turn to discuss some aspects of our research question, namely, our notion of optimality, our use of a continuum state space, and our use of a model with one good.

The notion of optimality that we use to conduct the efficiency analysis is not the only one available although we believe that it is the most appropriate one in our context (see footnote 7 in Section 2.3). Weaker notions of optimality treat agents who are born in the same state but face distinct states when old as different entities. Such ex post notions of optimality run counter to the risk sharing opportunities provided by the markets that we assume exist thus rendering their use in making welfare judgements difficult to justify. The most obvious stronger notion does not distinguish between agents by the state of birth and therefore makes welfare judgements based on allocations that agents cannot access through the market. While such ex ante notions of optimality, which amount to

[2]The references to Birkhoff's results in the literature in economic theory that we are aware of are all to the finite dimensional case; footnote 11 in Section 3.1 lists some of them.

[3]The following could be another possible application: the results on the long-run risk-return relationship in Hansen and Scheinkman (2009) rest on the existence of a strictly positive eigenvector of the pricing kernel, which is an operator, and their Assumpitions 9.1-4 suffice to ensure existence. The recent work of Qin and Linetsky (2016) shows how Jentzsch's theorem, a result that provides the motivation and title for Birkhoff (1957), can be used to identify conditions that ensure the existence of a strictly positive eigenvector in models that are particular cases of the one in Hansen and Scheinkman.

providing insurance against the state in which an agent is born, have merit when they are used to evaluate the impact of policy interventions, their use is unlikely to produce useful indicators of the efficacy with which markets allocate resources.

Regarding our reasons for looking at a model with a continuum of states, we recall that Lucas (1972) invoked the overlapping generations model to study an important issue in macroeconomics. The context was one of risk and uncertainty and the model used had a continuum of states. Muench (1977) closely followed Lucas (1972), also in using a continuum state space. The move to analyzing finite state versions of the model appears to have been the result of the technical complexities of dealing with a continuum state space rather than any particular modelling preference grounded in economic analysis. An example is the particular issue we face in defining our welfare notion, an issue that arizes only because we have agents being born in one of a continuum of states at each date, and this is discussed and resolved prior to the introduction of Definition 5 in Section 2.3. Also, the fact that models of asset pricing in finance typically assume a continuum of states, regardless of whether they consider an overlapping generations structure or one with an infinitely lived agent, influences the choices made by researchers in the macro/finance literature and provides additional reasons for analyzing such models.

The type of stationary equilibrium that we consider is described by functions defined on the exogenous Markov state space. It is known from Spear (1985) that when there is more than one good at each date, such stationary equilibria do not exist in general. This is why we consider one good economies. For general pure exchange economies with uncertainty described by a tree with a uniformly bounded number of successor nodes to each node and many goods at each node, and with markets that are sequentially complete, Citanna and Siconolfi (2010) show the generic existence of stationary equilibria described by functions on an enlarged endogenous Markov state space. As noted below, the situation is similar when stochastic versions of the Diamond model are considered. Obtaining a 'unit root'-type characterization result for the optimality properties of stationary equilibrium allocations in such economies is the subject of ongoing research. However, the kernel of the operator is no longer uniformly positive, and need not be described by a density either; evidently, the more general environment requires a nontrivial extension of the analysis of the eigenvector problem presented here. That provides a more methodological reason for pursuing the research presented here.

To place our paper in context we relate it to two early contributions to the literature on optimality in stochastic overlapping generations economies: Manuelli (1990) considered a model very similar to the one treated in this paper, while Aiyagari and Peled (1991) looked at the case of a finite number of states but also allowed risky linear technologies. Both used a first order approximation approach. Manuelli provided a necessary and sufficient condition for optimality in the form of the existence of a function that satisfies a set of inequalities, one for each state.[4] Aiyagari and Peled (1991) identified the unit root

[4]Demange and Laroque (1999, 2000) and Barbie and Kaul (2015) note that some parts of the proof in Manuelli (1990) are unclear.

property by appealing to Perron's theorem on positive matrices. Importantly, Aiyagari and Peled (1991) restricted attention to the class of stationary allocations so that arbitrary feasible allocations were not admissible as possible Pareto improvements; later work by Demange and Laroque (1999), Chattopadhyay and Gottardi (1999), and Chattopadhyay (2001) showed that the unit root result holds even when nonstationary allocations are considered. The result in the present paper shows that the unit root property characterizes optimality in the much more general framework with a continuum of states.

This paper also relates to Chattopadhyay and Gottardi (1999) who consider general pure exchange economies with two period lifetimes, uncertainty described by a tree with a uniformly bounded number of successor nodes to each node, and many goods at each node, and with markets that are sequentially complete; they provide a characterization of those competitive allocations that are conditionally Pareto optimal. Their result is a generalized Cass criterion and they show how their result can be used to obtain the unit root property when attention is restricted to stationary equilibria with finite support. Barbie, Kaul, and Hagedorn (2007) extend the general characterization result in Chattopadhyay and Gottardi (1999) to one good economies with bounded neoclassical technologies. As noted earlier, and expanded upon below, stationary equilibria with finite support do not generally exist in such production economies.

Although we have confined our discussion to the case of exchange economies, there is an equally rich literature on optimality in stochastic overlapping generations models with production. Zilcha (1991) provides an early characterization result for a criterion of optimality that is weaker than conditional Pareto optimality.[5] Demange and Laroque (2000) extend their analysis of the pure exchange model with stochastic linear technologies and demographic shocks in Demange and Laroque (1999) to a neoclassical technology. In doing so the state space for capital becomes endogenous and, although they restrict attention to stationary allocations on a compact state space that are described by continuous functions that are uniformly positive, they are unable to obtain a characterization result. Barbie and Hillebrand (2017) extend the recursive approach (developed in Barbie and Kaul (2015) for pure exchange economies) to production economies to analyze stationary allocations that are continuous and uniformly positive on an endogenous state space and obtain conditions that are similar in spirit to the conditions in Manuelli (1990) but fall short of a characterization result.

The rest of the paper is organized as follows. Section 2 presents the model, namely, the notation relating to the probabilistic structure, agents and their characteristics, and aggregates, and the definitions of optimality, markets and agents' behaviour in markets, and stationary equilibrium. Section 3 introduces Birkhoff's work, presents the result that we need, and casts that result in terms of our model. Section 4 states our theorem and provides a number of remarks including a detailed technical discussion of the relevant literature. Many proofs are collected in Section 5.

[5]Similarly, Bloise and Calciano (2008) consider a pure exchange environment that is very similar to the one in Chattopadhyay and Gottardi (1999), and provide a result for a weaker notion of optimality.

2.1 Model: Temporal and Stochastic Structure

The economy evolves in discrete time with exogenous states drawn from a temporal state space, denoted S below. The probability of transition from one temporal state to another is described by a stationary Markov kernel, denoted Q below. We consider the case where the transition is induced by a family of density functions, denoted π below. We turn to a formal development.

$\mathcal{B}^n$ is the Borel σ-algebra on R^n. Λ^n denotes n-dimensional Lebesgue measure.

S is the temporal state space.

Assumption 1 (i): $S \in \mathcal{B}^n$. S is a compact set with $\Lambda^n(S) > 0$.

$\mathcal{S}$ is the restriction of $\mathcal{B}^n$ to S. Define Λ^S on $(S, \mathcal{S})$ by $\Lambda^S(F) := \frac{\Lambda^n(F)}{\Lambda^n(S)}$ for $F \in \mathcal{S}$.

$t = 1, 2, \cdots$.

One defines $\Omega := S^\infty$, with typical element ω which is a sequence. The projection function $s_t : \Omega \to S$ identifies the t-th coordinate of the sequence ω. We use the notation $\omega_t := s_t(\omega)$ and $\omega^t := (\omega_1, \omega_2, \cdots, \omega_t)$. $\mathcal{F}_t := \sigma(s_1, s_2, \cdots, s_t)$ is the σ-algebra generated by the collection of functions $(s_1, s_2, \cdots, s_t)$. $\mathcal{F}$ is a σ-algebra such that $\mathcal{F} = \sigma(\cup_{t=1}^\infty \mathcal{F}_t)$.

Definition 1: A *transition density function on S* is an element of
$\mathcal{D} := \{p : S^2 \to R_+ | p \text{ is } \mathcal{S}^2 - \text{measurable and for all } s \in S, \int_S p(s, s')\Lambda^S(ds') = 1\}$.

$p \in \mathcal{D}$ induces Q^p, a stationary Markov transition on the measurable space $(S, \mathcal{S})$. μ is a probability measure on $(S, \mathcal{S})$; it describes the probability with which the initial state is chosen. μ and Q^p induce $Q^{p,t}$, a probability measure on $(S^t, \mathcal{S}^t)$ for each $t = \{2, 3, \cdots\}$; $Q^{p,1} := \mu$. By Kolmogorov's Extension Theorem, there exists a unique probability measure P^p on $(\Omega, \mathcal{F})$ which is consistent with all the measures $Q^{p,t}$, where $t = \{1, 2, \cdots\}$:

$$\text{KET} \qquad \text{for} \quad F \in \mathcal{S}^t \qquad P^p(\{\omega | \omega^t \in F\}) = Q^{p,t}(F).$$

Assumption 1 (ii): $\pi \in \mathcal{D}$ is such that for all $(s, s') \in S^2$, $0 < \underline{\pi} \leq \pi(s, s') \leq \overline{\pi} < \infty$.

$(\Omega, \mathcal{F}, P^\pi)$ is our probability triple; $\{\mathcal{F}_t\}_{t=1}^\infty$ is the filtration that we will use.

For $F \in \mathcal{F}$, $I_F : \Omega \to \{0, 1\}$ is the indicator function, $I_F(\omega) = 1$ if and only if $\omega \in F$.

Given ω and t, define $A_t(\omega) := \{\tilde{\omega} \in \Omega | \tilde{\omega}^t = \omega^t\}$, i.e. those sample points that, up to date t, share the same history as ω. $A_t(\omega) \in \mathcal{F}_t$ and is a cylinder. Define

$\mathcal{A} := \{A \in \cup_{t \geq 1} \mathcal{F}_t | A = A_t(\omega) \text{ for some } \omega \in \Omega \text{ and some } t = 1, 2, \cdots\}$,

a distinguished set of cylinders. If $A \in \mathcal{A}$ then there exists a unique $t(A)$ such that for all $\omega \in A$, we have $A_{t(A)}(\omega) = A$ and $A_{t(A)+1}(\omega) \neq A$. So elements of $\mathcal{A}$ play the same role as date-events in an environment where the state space is finite.

That completes the description of the risky environment.

We introduce two sets of measurable functions. For $(X, \mathcal{X})$ a measurable space, define

$$B^{X,\mathcal{X}} := \{f : X \to R | f \text{ is } \mathcal{X}\text{-measurable and } \sup_{x \in X} |f(x)| < \infty\},$$

$$B_+^{X,\mathcal{X}} := B^{X,\mathcal{X}} \cap \{f : X \to R | f(x) \geq 0\}.$$

Elements of $B^{A,\mathcal{F}_{t(A)}}$ are constant functions and can be identified with their values.

Let $s = 1, 2, \cdots$. For $f \in B^{\Omega,\mathcal{F}_{t+s}}$ and $p \in \mathcal{D}$, $E_{P^p}[f|\mathcal{F}_t](\omega)$ denotes the *conditional expectation of f given $\mathcal{F}_t$* where the integration is with respect to the measure P^p. We recall the defining property as well as two other properties of conditional expectations:

CE for $G \in \mathcal{F}_t$ $\int_G E_{P^p}[f|\mathcal{F}_t](\omega)P^p(d\omega) = \int_G f(\omega)P^p(d\omega)$,

CE (i) $E_{P^p}[E_{P^p}[f|\mathcal{F}_{t+s}]|\mathcal{F}_t](\omega) = E_{P^p}[f|\mathcal{F}_t](\omega)$ and

CE (ii) if $g \in B^{\Omega,\mathcal{F}_t}$ then $E_{P^p}[f \cdot g|\mathcal{F}_t](\omega) = g(\omega) \cdot E_{P^p}[f|\mathcal{F}_t](\omega)$.

When there is no scope for confusion, we write $E[f|\mathcal{F}_t](\omega)$ instead of $E_{P^\pi}[f|\mathcal{F}_t](\omega)$.

For $t \in \{1, 2, \cdots\}$ and $F \in \mathcal{F}_{t+1}$,

$$\text{CP} \qquad P^\pi(F|\omega^t) := E_{P^\pi}[I_F|\mathcal{F}_t] = \int_{\{s' \in s_{t+1}(A_t(\omega)\cap F)\}} \pi(s_t(\omega), s')\Lambda^S(ds');$$

it is the probability, conditional on the σ-algebra $\mathcal{F}_t$, of the occurrance of the event F.

2.2 Model: Agents

We turn to a description of the physical environment.

There is a single good at each $A \in \mathcal{A}$, i.e at date $t(A)$ and every $\omega \in A$.

Set $\mathcal{H} := \{1, \cdots, H\}$. H agents are born at each $A \in \mathcal{A}$ and live for two periods. They are denoted $(A, h) \in \mathcal{A} \times \mathcal{H}$. Also, H "initial old" agents denoted (A, h, o), where $A \in \mathcal{A}$ is such that $t(A) = 1$, enter the economy at $t = 1$ and live for only one period.

The consumption set of each agent is $X_{A,h} = B_+^{A,\mathcal{F}_{t(A)}} \times B_+^{A,\mathcal{F}_{t(A)+1}}$. For the initial old agent we have $X_{A,h,o} = B_+^{A,\mathcal{F}_{t(A)}}$.

We assume that the economy is stationary in that the agents' endowments and preferences depend on history only through the realization of the Markov state at the date at which the agent is born. Given $A \in \mathcal{A}$ and $\omega \in A$, the projection function allows us to identify $s_{t(A)}(\omega)$ as the Markov state of interest to us. Since to each $A \in \mathcal{A}$ one can assign a unique $t(A)$, we write s_A in place of $s_{t(A)}(\omega)$.

For endowments we use the notation $e(A, h)$ and require $e(A, h) \in X_{A,h}$; for the initial old we use $e_{2,1}(A, h, o) \in X_{A,h,o}$. Stationarity requires: for arbitrary pairs A and $\tilde{A}$, and $(\omega, \tilde{\omega}) \in A \times \tilde{A}$, if $s_A = s_{\tilde{A}}$ and $s_{t(A)+1}(\omega) = s_{t(\tilde{A})+1}(\tilde{\omega})$ then $e(\omega; A, h) = e(\tilde{\omega}; \tilde{A}, h)$. So consider $e : S \times \mathcal{H} \to R_+ \times B_+^{S,S}$ with the interpretation that, for $s' \in S$, $e(s, h)$ induces $(e_1(s, h), e_2(s'; s, h)) \in R_+^2$. Set $e(\omega; A, h) := (e_1(s_A, h), e_2(s_{t(A)+1}(\omega); s_A, h))$ where $\omega \in A$. In addition, we assume that, for some $\tilde{s} \in S$, $e_{2,1}(\omega; A, h, o) = e_2(s_A; \tilde{s}, h)$ for $\omega \in A$, for all $(A, h) \in \mathcal{A} \times \mathcal{H}$ where $t(A) = 1$. This ensures that the stochastic endowment of the initial old agents is compatible with the stationary structure of the economy.

Preferences have a subjective expected utility representation. Each agent has a state dependent Bernoulli function that depends on (i) the identity of the agent only through nature's choice at the date of the agent's birth and (ii) the state faced in the second period of life only through nature's choice when the agent is old. The function is denoted $u(\cdot; s_A, h) : R_+^2 \times S \to R$; for the initial old agent we use $u(z; s_A, h, o)$. The functions u are assumed to satisfy standard conditions formally specified in Assumption 2 below. For

(A, h) and $\omega \in A$, the agent's preferences over $(x_{1,t(A)}, x_{2,t(A)+1}) \in X_{A,h}$ are assumed to be representable in the form

$$V(x_{1,t(A)}, x_{2,t(A)+1}; A, h) := \int_A u(x_{1,t(A)}(\omega), x_{2,t(A)+1}(\omega), s_{t(A)+1}(\omega); s_A, h) P^\pi(d\omega|\omega^{t(A)}).$$

Assumption 2: (i) H is finite.

(ii) For all $(A, h) \in \mathcal{A} \times \mathcal{H}$ with $t(A) = 1$, $X_{A,h,o} = B_+^{A,\mathcal{F}_{t(A)}}$ and $u(\cdot; s_A, h, o) : R_+ \to R$ where $u(z; s_A, h, o)$ is strictly increasing in z for every $s_A \in S$.

(iii) For all $(A, h) \in \mathcal{A} \times \mathcal{H}$, $X_{A,h} = B_+^{A,\mathcal{F}_{t(A)}} \times B_+^{A,\mathcal{F}_{t(A)+1}}$, for $\omega \in A$, $e(\omega; A, h) = (e_1(s_A, h), e_2(s_{t(A)+1}(\omega); s_A, h))$ where $e : S \times \mathcal{H} \to R_+ \times B_+^{S,\mathcal{S}}$ is $\mathcal{S}-$measurable, and $u(\cdot; s_A, h) : R_+^2 \times S \to R$ where $u(y, z, s'; s_A, h)$ is

(a) $\mathcal{S}$-measurable in both s_A and s',

(b) for every $s' \in S$, it is twice continuously differentiable for $(y, z) \in R_{++}^2$ and differentiably strictly concave in (y, z), and strictly increasing in y and z, and

(c) for all $(x_{1,t(A)}, x_{2,t(A)+1}) \in X_{A,h}$, $V(x_{1,t(A)}, x_{2,t(A)+1}; A, h)$ is well defined.

(iv) For some $\tilde{s} \in S$, $e_{2,1}(\omega; A, h, o) = e_2(s_A; \tilde{s}, h)$ for $\omega \in A$, for all $(A, h) \in \mathcal{A} \times \mathcal{H}$ with $t(A) = 1$.

2.3 Model: Aggregates and Optimality

We introduce the notation that allows us to undertake welfare comparisons. The terms are all standard and adapted to the environment of the model.

Definition 2: An *allocation* $\mathfrak{x}$ is $\{x_{1,t,h}, x_{2,t,h}\}_{t=1}^\infty$, with $x_{i,t} \in B_+^{\Omega,\mathcal{F}_t}$ for $i = 1, 2$, for all $t = 1, 2, \cdots$, and for $h = 1, \cdots, H$, a pair of stochastic processes adapted to the filtration.

Definition 3: The *aggregate endowment* is the function $e : S \times S \to R_+$

$$e(s, s') := \sum_{h=1}^{H} \left[e_1(s', h) + e_2(s'; s, h)\right].$$

The next assumption imposes restrictions on the aggregate endowment. It amounts to requiring that a minimal (strictly positive) consumption level is always achieveable for the $2H$ agents who coexist; there is also a maximal level of consumption which is independent of the economy's history.

Assumption 3: The aggregate endowment is measurable, uniformly positive and uniformly bounded:

1. $e : S \times S \to R_+$ is $\mathcal{S} \times \mathcal{S}$-measurable,
2. there exist $\underline{e} > 0$ and $\overline{e} < \infty$ such that

$$\underline{e} \leq e(s_{t-1}(\omega), s_t(\omega)) \leq \overline{e} \quad \text{for all } t = 1, 2, \cdots, \text{ and for all } \omega.[6]$$

[6] Here and elsewhere we do not add the qualification "a.s."

Recall that $\tilde{s}$ was specified in Assumption 2 (iv).

Definition 4: A *feasible allocation* is an allocation that satisfies the restriction

$$\sum_{h=1}^{H} [x_{1,t,h}(\omega) + x_{2,t,h}(\omega)] \leq e(s_{t-1}(\omega), s_t(\omega)) \quad \text{for all } t = 1, 2, \cdots, \text{ and for all } \omega,$$

where $s_0(\omega) := \tilde{s}$ for all $\omega \in \Omega$.

Let (A, h)'s *consumption plan* be denoted $x(A, h) = (x_{1,t(A),h}, x_{2,t(A)+1,h}) \in X_{A,h}$. For the initial old agent (A, h, o) we use the notation $x_{2,1,h} \in B_+^{A,\mathcal{F}_{t(A)}}$. For $\omega \in A$, $x(A, h)$ induces $x(\omega; A, h) = (x_{1,t(A),h}(\omega), x_{2,t(A)+1,h}(\omega)) \in R_+^2$. Evidently, we can work interchangeably with an allocation or a set of consumption plans, one for each agent.

We use the Pareto criterion to make welfare judgements. We follow Muench (1977) who extends the Pareto criterion to environments in which agents are distinguished by the state in which they are born.[7] In our framework of Markovian uncertainty with a continuum of states, requiring the set of agents who are strictly improved to be a set of positive measure translates into the requirement that the allocation strictly improves a positive measure subset of the initial old agents; this seems to go against the grain of the overlapping generations model. Instead, 2a and 2b in Definition 5 capture the idea that a Pareto improving allocation must strictly improve all those agents who are born in some set of positive conditional measure; since an agent is a pair in $\mathcal{A} \times \mathcal{H}$, the formal requirement is that the conditional expected utility of at least one "date-event" contingent agent is strictly higher. As Remark 2 in Section 4 notes, since we restrict attention to equilibrium allocations that are stationary, if a CPO improvement as per Definition 5 exists then in fact agents in a set of positive measure can be improved.

Definition 5: A feasible allocation $\mathfrak{x} := \{x_{1,t,h}, x_{2,t,h}\}_{t=1}^{\infty}$ is *Conditionally Pareto Optimal (CPO)* if there does not exist another feasible allocation $\hat{\mathfrak{x}} := \{\hat{x}_{1,t,h}, \hat{x}_{2,t,h}\}_{t=1}^{\infty}$ such that

1a. for all $(A, h) \in \mathcal{A} \times \mathcal{H}$ with $t(A) = 1$,
$u(\hat{x}_{2,1,h}(\omega); s_A, h, o) \geq u(x_{2,1,h}(\omega); s_A, h, o)$ for $\omega \in A$,

1b. for all $(A, h) \in \mathcal{A} \times \mathcal{H}$,
$V(\hat{x}_{1,t(A),h}, \hat{x}_{2,t(A)+1,h}; A, h) \geq V(x_{1,t(A),h}, x_{2,t(A)+1,h}; A, h)$,

2a. either for some $(A', h') \in \mathcal{A} \times \mathcal{H}$ with $t(A') = 1$,
$u(\hat{x}_{2,1,h'}(\omega); s_{A'}, h', o) > u(x_{2,1,h'}(\omega); s_{A'}, h', o)$ for $\omega \in A'$,

2b. or for some $(A', h') \in \mathcal{A} \times \mathcal{H}$,
$V(\hat{x}_{1,t(A'),h'}, \hat{x}_{2,t(A')+1,h'}; A', h') > V(x_{1,t(A'),h'}, x_{2,t(A')+1,h'}; A', h')$.

[7]This is a well established approach to the assessment of welfare in models with generational overlap, e.g. see Peled (1982), Manuelli (1990), Aiyagari and Peled (1991), Demange and Laroque (1999), Chattopadhyay and Gottardi (1999), Demange and Laroque (2000), Chattopadhyay (2001), Barbie, Hagedorn, and Kaul (2007), and Barbie and Kaul (2015). Zilcha (1991) proposes a different definition.

2.4 Model: Demand and Intertemporal Prices

A complete set of Arrow securities is available at each $F \in \mathcal{A}$. So markets are sequentially complete.

Asset prices are given by $\{q_t\}_{t=1}^{\infty}$ where, for each t, $q_t : \Omega \to R_+$ is $\mathcal{F}_t$-measurable. The interpretation is that, for $F \in \mathcal{F}_{t+1}$, $\int_F q_{t+1}(\omega)P^{\pi}(d\omega|\omega^t)$ is the price paid at t in any (and every) state $\tilde{\omega} \in A_t(\omega)$ for one unit of the Arrow security which pays one unit of the good at $t+1$ if and only if F occurs.

We write a single budget constraint using the prices of the Arrow securities; the fact that markets are sequentially complete allows us to do this.

The optimization problem solved by (A, h) is to

$$\max_{(x_{1,t(A)},x_{2,t(A)+1})\in X_{A,h}} \int_A u(x_{1,t(A)}(\omega), x_{2,t(A)+1}(\omega), s_{t(A)+1}(\omega); s_A, h)P^{\pi}(d\omega|\omega^{t(A)})$$

$$\text{subject to} \qquad [x_{1,t(A)}(\omega) - e_1(s_A, h)]+$$

$$+\int_A q_{t(A)+1}(\omega)[x_{2,t(A)+1}(\omega) - e_2(s_{t(A)+1}(\omega); s_A, h)]P^{\pi}(d\omega|\omega^{t(A)}) \leq 0 \qquad \text{for} \quad \omega \in A.$$

Since the utility function is strictly increasing, the constraint holds with equality. At an interior solution $(\widehat{x}_{1,t(A),h}, \widehat{x}_{2,t(A)+1,h})$, the first order conditions imply that[8]

$$q_{t(A)+1}(\omega) = \frac{u_2(\widehat{x}_{1,t(A),h}(\omega), \widehat{x}_{2,t(A)+1,h}(\omega), s_{t(A)+1}(\omega); s_A, h)}{\int_A u_1(\widehat{x}_{1,t(A),h}(\omega), \widehat{x}_{2,t(A)+1,h}(\omega), s_{t(A)+1}(\omega); s_A, h)P^{\pi}(d\omega|\omega^{t(A)})}.$$

Definition 6: An allocation $\mathfrak{x}$ is *stationary* if for all $(A, \tilde{A}) \in \mathcal{A}\times\mathcal{A}$ and all $(\omega, \tilde{\omega}) \in A\times\tilde{A}$,

$$s_A = s_{\tilde{A}} \text{ and } s_{t(A)+1}(\omega) = s_{t(\tilde{A})+1}(\tilde{\omega}) \quad \Rightarrow \quad x(\omega; A, h) = x(\tilde{\omega}; \tilde{A}, h).$$

So consider a function $x : S \times \mathcal{H} \to R_+ \times B_+^{S,\mathcal{S}}$ with the interpretation that, for $s' \in S$, $x(s, h)$ induces $(x_1(s, h), x_2(s'; s, h)) \in R_+^2$. Set $x(\omega; A, h) := (x_1(s_A, h), x_2(s_{t(A)+1}(\omega); s_A, h))$, where $\omega \in A$, to generate the stationary allocation $\mathfrak{x}$ from the function x.

To reduce the notational burden, for $i = 1, 2$, we define

$$u_i^{x,s,h}(s') := u_i(x_1(s, h), x_2(s'; s, h), s'; s, h) \qquad (s, s') \in S^2.$$

Consider stationary asset prices given by the function $q : S \times S \to R_+$. So $q_{t(A)+1}(\omega) = q(s_A, s_{t(A)+1}(\omega))$. With stationary asset prices, at the interior stationary allocation $\widehat{\mathfrak{x}}$, the first order conditions take the form

$$q(s_A, s_{t(A)+1}(\omega)) = \frac{u_2^{\widehat{x},s_A,h}(s_{t(A)+1}(\omega))}{\int_S u_1^{\widehat{x},s_A,h}(s_{t(A)+1})\pi(s_A, s_{t(A)+1})\Lambda^S(ds_{t(A)+1})},$$

since, by CP of Section 2.1, $P^{\pi}(F|\omega^t) = \int_{\{s'\in s_{t+1}(A_t(\omega)\cap F)\}} \pi(s_t(\omega), s')\Lambda^S(ds')$ for $F \in \mathcal{F}_{t+1}$.

[8]We use u_i, $i \in \{1, 2\}$, to denote the first order partial derivatives of u with respect to its first two coordinates.

We have shown that $q^{\widehat{x}}$ satisfies

$$q^{\widehat{x}}(s,s') = \frac{u_2^{\widehat{x},s,h}(s')}{\int_S u_1^{\widehat{x},s,h}(s')\pi(s,s')\Lambda^S(ds')}.$$

2.5 Model: Stationary Equilibrium

We can now formally introduce the notion of a stationary equilibrium.

Definition 7: (x^*, q^*), where $x^* : S \times \mathcal{H} \to R_+ \times B_+^{S,\mathcal{S}}$, so that it induces $\mathfrak{x}^*$, an allocation that is stationary, and $q^* : S \times S \to R_+$, is a *stationary competitive equilibrium with sequentially complete markets* if the allocation induced is feasible and

1. for all $(A,h) \in \mathcal{A} \times \mathcal{H}$, for $\omega \in A$, $[x_1^*(s_A,h) - e_1(s_A,h)]+$

$$+\int_A q^*(s_A, s_{t(A)+1}(\omega))[x_2^*(s_{t(A)+1}(\omega); s_A, h) - e_2(s_{t(A)+1}(\omega); s_A, h)]P^\pi(d\omega|\omega^{t(A)}) \leq 0,$$

2. if $(x_{1,t(A)}, x_{2,t(A)+1}) \in X_{A,h}$ is such that

$$V(x_{1,t(A)}, x_{2,t(A)+1}; A, h) > \int_A u(x_1^*(s_A,h), x_2^*(s_{t(A)+1}(\omega); s_A, h), s_{t(A)+1}(\omega); s_A, h)P^\pi(d\omega|\omega^{t(A)})$$

then, for $\omega \in A$,

$$[x_{1,t(A)}(\omega) - e_1(s_A,h)] + \int_A q^*(s_A, s_{t(A)+1}(\omega))[x_{2,t(A)+1}(\omega) - e_2(s_{t(A)+1}(\omega); s_A, h)]P^\pi(d\omega|\omega^{t(A)}) > 0.$$

In order to be able to make any meaningful progress, we must work with an equilibrium process that has the minimal property of being measurable. Rather than verifying that our assumptions ensure that the stationary asset pricing function q^* is measurable, we directly assume this property.

Assumption 4: q^* is $\mathcal{S} \times \mathcal{S}$-measurable.

$q^* \cdot \pi : S \times S \to R_+$ defined as $q^* \cdot \pi(s,s') := q^*(s,s') \cdot \pi(s,s')$ is the *pricing kernel* associated with a stationary equilibrium.

3.1 Birkhoff's Generalization of the Perron-Frobenius Theorem

The result that we present in this section generalizes Perron's Theorem on the existence of a strictly positive eigenvector of a strictly positive matrix to a class of linear operators; we build on Birkhoff (1957, 1962), henceforth B '57 and B '62, respectively.[9] The key idea is to introduce the notion of a distance in which the linear operator is a strict contraction on a set which is complete.[10] We are unaware of a standard treatment that develops in

[9] B '57 treats the problem in the framework of vector lattices, while B '62 generalizes even further.

[10] Eveson (1995) provides a nice entry to the literature.

detail the specific result we need; therefore, we have opted to introduce the concepts and the results with references to B '57 and B '62.[11]

Recall that for $(X, \mathcal{X})$ a measurable space,

$$B^{X,\mathcal{X}} = \{f : X \to R | f \text{ is } \mathcal{X}\text{-measurable and } \sup_{x \in X} |f(x)| < \infty\}.$$

We follow B '57 (Example 3 on page 221) and define C^M for $M \geq 1$ by

$$C^M := B^{X,\mathcal{X}} \cap \{f : X \to R \,|\, 0 < \sup f(x) \leq M \inf f(x)\}.$$

C^M is a convex cone. Let 0 be the function that is identically zero, and define $C_0^M := C^M \cup \{0\}$. C_0^M induces the *partly ordered vector space* $(B^{X,\mathcal{X}}, C_0^M)$ where the partial order, $\geqq$, is defined by the rule: $f \geqq g$ if and only if $(f - g) \in C_0^M$.[12]

In B '62, the function $\theta : C_0^M \times C_0^M \to [0, \infty]$ is defined as

$$\theta(f,g) := \begin{cases} \infty & \text{if} \quad \{\alpha \in R | \alpha f \geqq g\} = \emptyset \quad \text{or} \quad \{\beta \in R | \beta g \geqq f\} = \emptyset, \\ \ln(\alpha_{f,g}\beta_{f,g}) & \text{if} \quad \alpha_{f,g} < \infty \quad \text{where} \quad \alpha_{f,g} := \inf\{\alpha \in R | \alpha f \geqq g\} \\ & \text{and} \quad \beta_{f,g} < \infty \quad \text{where} \quad \beta_{f,g} := \inf\{\beta \in R | \beta g \geqq f\}. \end{cases}$$

We have $g \leqq \alpha_{f,g} f \leqq \alpha_{f,g}\beta_{f,g} g$ and $f \leqq \beta_{f,g} g \leqq \alpha_{f,g}\beta_{f,g} f$. $\theta(\cdot)$ is Hilbert's *projective quasi-metric* on C_0^M;[13] it is not a metric since $\theta(f, g) = 0$ if and only if f and g are on the same ray of C^M. Lemma 8 in B '62 shows that $(C^M, \theta(\cdot))$ is a quasi-metric space in which the equivalence classes of vectors that are a zero distance apart are the rays of C^M. Furthermore, subsets of C^M with the property that $\theta(f, g) < \infty$ form *connected components.*

So $\theta(\cdot)$ is a metric on the set of rays in the same connected component of C^M.

The following key property is simply noted as a fact in B '57 (5. Applications, page 222). We provide a proof based on B '57, B '62, and Zabreiko et al (1972).

Lemma 1: *Each connected component of C^M is complete in $\theta(\cdot)$.*

The proof of Lemma 1 appears in Section 5.1.

Birkhoff considers a bounded linear operator that leaves invariant the cone which induces the partly ordered vector space, and introduces the following condition: the image of the cone under the operator has "finite diameter" in the quasi-metric $\theta(\cdot)$. He shows that such a bounded linear operator is uniformly contracting in the quasi-metric $\theta(\cdot)$, and hence there is a unique ray in the cone that is invariant.

[11]There are references to B '57 in the literature in economic theory but those applications are to the products of finite dimensional matrices. Kohlberg and Pratt (1982), which is largely restricted to the finite dimensional case, provides both a presentation of Birkhoff's approach and the result, as well as a discussion of applications together with references. Montrucchio (1998) uses a related result due to Thompson (1963) to study the differentiability of the policy function in dynamic programming problems.

[12]B '57 and B '62 consider pairs (X, C) where X is a vector space and C is a positive cone of elements.

[13]B '57 uses $\theta(f, g; C)$ and "*projective metric* associated with C" for the case of the pair (X, C).

Proposition 1: *Let $p : X \times X \to R$ be $\mathcal{X} \times \mathcal{X}-$measurable, and let it satisfy*

$$0 < \underline{p} = \inf p(x, y) \leq \sup p(x, y) = \kappa \underline{p} \quad \textit{for} \quad \kappa > 0,$$

and let $(X, \mathcal{X}, \mu)$ be a measure space with $0 < \mu(X) < \infty$. Then there exists $f^* \in C^M$, unique up to the ray it lies on, for $M \geq \kappa$ such that, for some $\phi > 0$,

$$\int_X p(x, y) f^*(y) \mu(dy) = \phi \cdot f^*(x) \quad \forall\, x \in X.$$

Proof: Consider the linear operator $P : B^{X,\mathcal{X}} \to B^{X,\mathcal{X}}$ defined by

$$[fP](x) = \int_X p(x, y) f(y) \mu(dy).$$

Clearly, P is bounded. For $f \in C^M$, set $I_f := \int_X f(y)\mu(dy)$. We have $I_f > 0$.

We show that P maps the space C^M into itself. Since $I_f > 0$ for $f \in C^M$, we have

$$\inf\,[fP](x) \geq \underline{p} I_f > 0 \qquad \text{and} \qquad \sup\,[fP](x) \leq \kappa \underline{p} I_f.$$

For $M \geq \kappa$ we have

$$M \inf\,[fP](x) \geq M \underline{p} I_f \geq \kappa \underline{p} I_f$$

so that

$$M \inf\,[fP](x) \geq \sup\,[fP](x) \geq \inf\,[fP](x) > 0,$$

and it follows that

$$[fP] \in C^M.$$

Consider $e \in B_+^{X,\mathcal{X}}$ such that $e(x) = 1$ for all $x \in X$; $e \in C^M$. For $f \in C^M$ we have

$$(\underline{p} I_f) e \leqq fP \leqq (\kappa \underline{p} I_f) e$$

so that, since $\kappa \leq M$,

$$(\underline{p} I_f) e \leqq fP \leqq (\kappa \underline{p} I_f) e \leqq (M \underline{p} I_f) e,$$

and, by linearity,

$$e \leqq \frac{f}{\underline{p} I_f} P \leqq M e,$$

and so $\alpha_{e, \frac{f}{\underline{p} I_f} P} \leq M$ and $\beta_{e, \frac{f}{\underline{p} I_f} P} \leq 1$. It follows that $\theta(e, \frac{f}{\underline{p} I_f} P) \leq \ln M$.

Since $\theta(f, g) = 0$ if and only if f and g are on the same ray of C^M, we have $\theta(e, fP) = \theta(e, \frac{f}{\underline{p} I_f} P)$. Therefore we have $\theta(e, fP) \leq \ln M$. By the triangle inequality

$$\theta(fP, gP) \leq \theta(e, fP) + \theta(e, gP) \leq 2 \ln M.$$

This shows that, in terms of the projective quasi-metric, the distance between pairs of images is uniformly bounded. By Lemma 1 in Section 4 in B '57 (or by the Corollary to

Theorem 3 on pg 48 in B '62) the operator P is a strict contraction in $\theta(\cdot)$. By Lemma 1 above, each connected component of rays of C^M is complete in $\theta(\cdot)$. From Theorem 1 in B '57 it follows that there is a unique characteristic ray $f^* \in C^M$ such that, for some $\phi > 0$, $[f^*P] = \phi f^*$. The result follows when we observe that

$$[f^*P] = \phi f^* \qquad \Leftrightarrow \qquad \int_X p(x,y) f^*(y)\mu(dy) = \phi \cdot f^*(x) \quad \forall\, x \in X. \qquad \blacksquare$$

Remark 1: B '57 also allows us to obtain a result where the eigenvector is also continuous. Let X be a topological space. Instead of the space of bounded measurable functions, one considers the space of continuous and bounded functions with the sup norm, together with the cone of continuous functions in C^M. The result follows by strengthening the hypotheses on the function p to continuity, and observing that Lemma 1 continues to hold for the smaller cone of functions.

3.2 Stationary Equilibrium Allocations and Positive Eigenvectors

Proposition 1 allows us to show that to any stationary equilibrium allocation one can associate a strictly positive eigenvector and a corresponding eigenvalue that must be a positive real number. Evidently, we need to ensure that the pricing kernel associated with a stationary equilibrium satisfies uniform upper and lower bounds.

Definition 8: A stationary allocation x is uniformly interior if there exists $\varepsilon > 0$ such that for all $(A,h) \in \mathcal{A} \times \mathcal{H}$, $\quad x_1(s_A,h) \geq \varepsilon, \quad$ and $\quad x_2(s'; s_A, h) \geq \varepsilon \quad \forall\, s' \in S.$

Lemma 2: *Let (x^*, q^*) be a stationary competitive equilibrium with sequentially complete markets in which the equilibrium allocation is uniformly interior. Under Assumptions 1-4, there exists $\gamma^* \in B^{S,S}$ and $\phi > 0$ such that*

$$\int_S q^*(s,s') \cdot \pi(s,s')\gamma^*(s')\Lambda^S(ds') = \phi \cdot \gamma^*(s) \quad \forall\, s \in S$$

and $\underline{\gamma}^* > 0$ and $\overline{\gamma}^* < \infty$, where $\underline{\gamma}^* := \inf_S \gamma^*(s)$ and $\overline{\gamma}^* := \sup_S \gamma^*(s)$.

The proof of Lemma 2 appears in Section 5.2.

4. The Unit Root Result: a Characterization of Optimality

We turn to the main result of the paper. We provide a characterization of the optimality properties of stationary competitive equilibria when markets are sequentially complete in terms of the magnitude of the eigenvalue identified in Lemma 2.

It is well known that *curvature inequalities* which provide quadratic approximations to an agent's utility play essential roles in the characterization results on optimality of competitive paths in OLG economies. Since fleshing them out is notationally laborious, we relegate the detailed development to Section 5.3 (see Lemma C). Here we state two independent *curvature assumptions* which require the existence of uniform lower and upper bounds on the ratio of the quadratic form obtained from the matrix of second derivatives of the Bernoulli function u to the value of first derivative of u with respect to its first coordinate. The notation is indicative and Section 5.3 has the details.

Assumption C (ii): $\underline{\rho}^{\tilde{\mathfrak{r}}} := \frac{1}{2}\frac{\inf_{\omega\in\Omega}\underline{D}_2^{\tilde{\mathfrak{r}}}(\omega)}{\sup_{\omega\in\Omega}\overline{D}_1^{\tilde{\mathfrak{r}}}(\omega)} > 0,$ **C (iii):** $\overline{\rho}^{\tilde{\mathfrak{r}}} := \frac{1}{2}\frac{\sup_{\omega\in\Omega}\overline{D}_2^{\tilde{\mathfrak{r}}}(\omega)}{\inf_{\omega\in\Omega}\underline{D}_1^{\tilde{\mathfrak{r}}}(\omega)} < \infty.$

Remark C in Section 5.3 notes that if the Bernoulli function is independent of the state when young and when old, then Assumptions 2-3 together with the requirement that the equilibrium allocation is uniformly interior, ensure that Assumptions C (ii) and C (iii) are satisfied (the same is true if the state space has a finite number of elements).

The interior stationary equilibrium is said to satisfy the *unit root property* if the eigenvalue ϕ is less than or equal to one.

Theorem 1: *Let (x^*, q^*) be a stationary competitive equilibrium with sequentially complete markets in which the equilibrium allocation is uniformly interior. Under Assumptions 1-4 and C (ii) and C (iii), the equilibrium allocation is CPO if and only if it satisfies the unit root property.*

Proof: Lemma C in Section 5.3 applies since Assumption 3 and uniform interiority of the equilibrium allocation imply that Assumption C (i), also in Section 5.3, holds.

First we show that there exists a stationary improvement when ϕ exceeds one.[14]

We apply Lemma C (i) in Section 5.3 which provides sufficient conditions, including a *curvature inequality*, under which there is a local perturbation of the equilibrium consumption plan that gives strictly higher utility than the equilibrium consumption plan. We now show that when ϕ exceeds one, the inequality in Lemma C (i) holds.

From Lemma 2, the pair (ϕ, γ^*) is such that

$$\int_S q^*(s,s')\cdot\pi(s,s')\gamma^*(s')\Lambda^S(ds') = \phi\cdot\gamma^*(s) \quad \forall\, s\in S$$

$$\Leftrightarrow \quad \int_S \frac{u_2^{x^*,s,h}(s')}{\int_S u_1^{x^*,s,h}(s')\pi(s,s')ds'}\cdot\pi(s,s')\gamma^*(s')\Lambda^S(ds') - \gamma^*(s) \geq (\phi-1)\underline{\gamma}^* > 0 \quad \forall\, s\in S.$$

Recalling CP of Section 2.1, and identifying $\gamma^*(s)$ as the transfer received in state s by some (arbitrary) agent type h when old and also the transfer made in state s by the same type h when young, we see that Lemma C (i) indeed applies with $\overline{\delta} = (\phi-1)\underline{\gamma}^*$.

By choosing $\epsilon > 0$ to be sufficiently small and by reducing the consumption of an agent born in state s by $\epsilon\cdot\gamma^*(s)$ and increasing it by $\epsilon\cdot\gamma^*(s')$ in each state when old, one generates a strict Pareto improvement. Aggregate feasibility is obviously maintained; since the equilibrium allocation is uniformly interior, by choosing ϵ to be sufficiently small we can ensure that perturbed consumption when young is strictly positive. As the

[14] A first order condition argument appears in Aiyagari and Peled (1991), Demange and Laroque (1999), and Chattopadhyay (2001). The approach here is based on a second order approximation similar to the one in Chattopadhyay and Gottardi (1999).

transfers proposed are from the young to the old, the initial old are improved in every state. That completes the proof of Theorem 1 in one direction.

Now we proceed to an argument by contradiction to show that if the unit root property holds then there can be no improvements, stationary or otherwise.

Suppose that there exists an allocation $\hat{x}$ that CPO dominates x^*. Define

$$\Delta_{1,t(A),h}(\omega) := \hat{x}_{1,t(A),h}(\omega) - x_1^*(s_A, h), \quad \text{for } \omega \in A \quad \text{where } (A,h) \in \mathcal{A} \times \mathcal{H},$$

$$\Delta_{2,t(A)+1,h}(\omega) := \hat{x}_{2,t(A)+1,h}(\omega) - x_2^*(s_{t(A)+1}(\omega); s_A, h), \quad \text{for } \omega \in A \quad \text{where } (A,h) \in \mathcal{A} \times \mathcal{H},$$

where the notation allows us to treat the case in which the allocation $\hat{x}$ is not stationary.

We apply Lemma C (ii) in Section 5.3 which identifies a *curvature inequality* that must hold if a local perturbation of a consumption plan gives higher utility than the equilibrium consumption plan. Lemma C (ii) requires the perturbed plan to be interior; since preferences are convex, this can be achieved by taking a convex combination of the equilibrium plan (which is uniformly interior) and the perturbed plan. We assume that this has been done.

By Lemma C (ii) we have, for all agents $(A,h) \in \mathcal{A} \times \mathcal{H}$,

$$\int_{\omega \in A} q^* \left(s_{t(A)}(\omega), s_{t(A)+1}(\omega)\right) \Delta_{2,t(A)+1,h}(\omega) P^\pi \left(d\omega | \omega^{t(A)}\right) \geq -\Delta_{1,t(A),h}(\omega) + \underline{\rho} \left[\Delta_{1,t(A),h}(\omega)\right]^2, \tag{1}$$

where $\underline{\rho} := \underline{\rho}^{\mathfrak{r}^*}$ and $\mathfrak{r}^*$ is the stationary allocation induced by x^*.

Define

$$\bar{\Delta}_{t(A)}(\omega) := -\frac{1}{H} \sum_{h \in \mathcal{H}} \Delta_{1,t(A),h}(\omega), \quad \text{for } \omega \in A \quad \text{where } A \in \mathcal{A},$$

the negative of the change in the average consumption by the young born at A.

Feasibility of the alternative allocation implies that

$$\frac{1}{H} \sum_{h \in \mathcal{H}} \Delta_{2,t(A)+1,h}(\omega) - \bar{\Delta}_{t(A)+1}(\omega) \leq 0, \quad \text{for } \omega \in A.$$

It follows that if the first term is positive then so is the second. Clearly, the fact that the initial old must also be improved implies that $\bar{\Delta}_1(\omega) \geq 0$ for all $\omega \in \Omega$.

By averaging the inequality in (1) across the set of agents born at A, using Jensen's inequality applied to a quadratic function, and using the feasibility condition, we obtain:

$$\int_{\omega \in A} q^* \left(s_{t(A)}(\omega), s_{t(A)+1}(\omega)\right) \bar{\Delta}_{t(A)+1}(\omega) P^\pi \left(d\omega | \omega^{t(A)}\right) \geq \bar{\Delta}_{t(A)}(\omega) + \underline{\rho} \left[\bar{\Delta}_{t(A)}(\omega)\right]^2. \tag{2}$$

Next, we use Lemma 2 to rewrite (2). We use the pair (ϕ, γ^*) identified in Lemma 2 to induce $\upsilon : S \times S \to R_+$ by setting $\upsilon(s,s') := \frac{q^*(s,s') \cdot \gamma^*(s')}{\phi \gamma^*(s)}$. It is easy to check that υ is $\mathcal{S} \times \mathcal{S}$-measurable, uniformly positive and uniformly bounded, and $\int_{s' \in \mathcal{S}} \upsilon(s,s') \cdot \pi(s,s') \Lambda^S(ds') = 1$ for all $s \in \mathcal{S}$; in particular, $\upsilon \cdot \pi$ is a transition density function. We have

$$\int_{\omega \in A} q^* \left(s_{t(A)}(\omega), s_{t(A)+1}(\omega)\right) \bar{\Delta}_{t(A)+1}(\omega) P^\pi \left(d\omega | \omega^{t(A)}\right) =$$

$$= \phi \cdot \gamma^* \left(s_{t(A)}(\omega)\right) \int_{\omega \in A} \frac{q^*\left(s_{t(A)}(\omega), s_{t(A)+1}(\omega)\right) \cdot \gamma^*\left(s_{t(A)+1}(\omega)\right)}{\phi \gamma^*\left(s_{t(A)}(\omega)\right)} \cdot \frac{\bar{\Delta}_{t(A)+1}(\omega)}{\gamma^*\left(s_{t(A)+1}(\omega)\right)} P^\pi\left(d\omega | \omega^{t(A)}\right)$$

$$= \phi \cdot \gamma^* \left(s_{t(A)}(\omega)\right) \int_{\omega \in A} \upsilon\left(s_{t(A)}(\omega), s_{t(A)+1}(\omega)\right) \cdot \frac{\bar{\Delta}_{t(A)+1}(\omega)}{\gamma^*\left(s_{t(A)+1}(\omega)\right)} P^\pi\left(d\omega | \omega^{t(A)}\right)$$

$$= \phi \cdot \gamma^* \left(s_{t(A)}(\omega)\right) E\left[\upsilon\left(s_{t(A)}, s_{t(A)+1}\right) \cdot \frac{\bar{\Delta}_{t(A)+1}}{\gamma^*\left(s_{t(A)+1}\right)} \middle| \mathcal{F}_{t(A)}\right](\omega).$$

Since $A_t(\omega) \in \mathcal{A}$ is well defined for given ω and t, (2) can now be rewritten as

$$\phi \cdot E\left[\upsilon(s_t, s_{t+1}) \cdot \frac{\bar{\Delta}_{t+1}}{\gamma^*(s_{t+1})} \middle| \mathcal{F}_t\right](\omega) \geq \frac{\bar{\Delta}_t(\omega)}{\gamma^*(s_t(\omega))} + \underline{\rho} \frac{[\bar{\Delta}_t(\omega)]^2}{\gamma^*(s_t(\omega))}. \tag{3}$$

The existence of an improving allocation implies that (3) must hold.

As noted earlier, $\bar{\Delta}_1(\omega) \geq 0$ for all $\omega \in \Omega$. This fact together with an application of (3) shows that there must be some date $\bar{t}$ such that the change in average consumption by the young is zero for all agents born at dates $t < \bar{t}$, while some of those that are old at $\bar{t}$ are improved; so $\bar{\Delta}_{\bar{t}}(\omega) > 0$ for $\omega \in E$ where $P^\pi(E|\omega^{\bar{t}-1}) > 0$ and $E \in \mathcal{F}_{\bar{t}}$, i.e. a set of positive conditional measure. So the right hand side of (3) is positive on E for $t = \bar{t}$.

Inductively define $\Upsilon_{t+1}(\omega) := \Upsilon_t(\omega) \cdot \upsilon(s_t(\omega), s_{t+1}(\omega))$ with $\Upsilon_1(\omega) = 1$ for all $\omega \in \Omega$.

By iterating forwards the stochastic difference inequality (3) we obtain

Lemma 3: *Under A.1 and A.2, for $T = 1, 2, \cdots$ and $\omega \in E$,*

$$\phi^T \cdot E\left[\Upsilon_{\bar{t}+T} \cdot \frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})} \middle| \mathcal{F}_{\bar{t}}\right](\omega) \geq \Upsilon_{\bar{t}}(\omega) \cdot \frac{\bar{\Delta}_{\bar{t}}(\omega)}{\gamma^*(s_{\bar{t}}(\omega))} + \underline{\rho} \left\{ \sum_{\tau=0}^{T-1} \phi^\tau \cdot E\left[\Upsilon_{\bar{t}+\tau} \cdot \frac{[\bar{\Delta}_{\bar{t}+\tau}]^2}{\gamma^*(s_{\bar{t}+\tau})} \middle| \mathcal{F}_{\bar{t}}\right](\omega) \right\} > 0.$$

Next we note two implications of the fact that $\upsilon \cdot \pi$ is a transition density function; the implication in (b) follows from Jensen's Inequality.

Lemma 4: *For any $\bar{t} \geq 1$, any $T = 1, 2, \cdots$, and any $f \in B^{\Omega, \mathcal{F}_{\bar{t}+T}}$,*

(*a*) $E\left[\Upsilon_{\bar{t}+T} \cdot f | \mathcal{F}_{\bar{t}}\right](\omega) = \Upsilon_{\bar{t}}(\omega) \cdot E_{P^{\upsilon \cdot \pi}}\left[f | \mathcal{F}_{\bar{t}}\right](\omega)$,

(*b*) $E\left[\Upsilon_{\bar{t}+T} \cdot f | \mathcal{F}_{\bar{t}}\right](\omega) \geq a > 0 \quad \Rightarrow \quad \Upsilon_{\bar{t}}(\omega) \cdot E\left[\Upsilon_{\bar{t}+T} \cdot [f]^2 | \mathcal{F}_{\bar{t}}\right](\omega) \geq a^2.$

The proofs of Lemma 3 and Lemma 4 appear in Section 5.2.

By Assumption 3, the endowment is uniformly bounded and, by Lemma 2, $\underline{\gamma}^* > 0$, so feasibility implies that $\frac{\bar{\Delta}_t(\omega)}{\gamma^*(s_t(\omega))}$ is always bounded. When we ignore all but the first term on the right hand side, the expression in the statement of Lemma 3 becomes

$$E\left[\Upsilon_{\bar{t}+T} \cdot \frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})} \middle| \mathcal{F}_{\bar{t}}\right](\omega) \geq \frac{\Upsilon_{\bar{t}}(\omega)}{\phi^T} \cdot \frac{\bar{\Delta}_{\bar{t}}(\omega)}{\gamma^*(s_{\bar{t}}(\omega))} \quad \text{for all } T = 1, 2, \cdots,$$

so an application of Lemma 4 (b) shows that

$$\Upsilon_{\bar{t}}(\omega) \cdot E\left[\Upsilon_{\bar{t}+T} \cdot \frac{[\bar{\Delta}_{\bar{t}+T}]^2}{[\gamma^*(s_{\bar{t}+T})]^2}\middle|\mathcal{F}_{\bar{t}}\right](\omega) \geq \frac{[\Upsilon_{\bar{t}}(\omega)]^2}{\phi^{2T}} \cdot \frac{[\bar{\Delta}_{\bar{t}}(\omega)]^2}{[\gamma^*(s_{\bar{t}}(\omega))]^2} \quad \text{for all } T = 1, 2, \cdots,$$

$$\Rightarrow \qquad \phi^T \cdot E\left[\Upsilon_{\bar{t}+T} \cdot \frac{[\bar{\Delta}_{\bar{t}+T}]^2}{\gamma^*(s_{\bar{t}+T})}\middle|\mathcal{F}_{\bar{t}}\right](\omega) \geq \frac{\underline{\gamma}^*}{\overline{\gamma}^*} \cdot \frac{\Upsilon_{\bar{t}}(\omega)}{\phi^T} \cdot \frac{[\bar{\Delta}_{\bar{t}}(\omega)]^2}{\gamma^*(s_{\bar{t}}(\omega))} \quad \text{for all } T = 1, 2, \cdots,$$

where, by Lemma 2, $\underline{\gamma}^* > 0$ and $\overline{\gamma}^* < \infty$. It follows that the right hand side of the inequality in Lemma 3 is bounded below by

$$\Upsilon_{\bar{t}}(\omega) \cdot \frac{\bar{\Delta}_{\bar{t}}(\omega)}{\gamma^*(s_{\bar{t}}(\omega))}\left\{1 + \underline{\rho} \cdot \frac{\underline{\gamma}^*}{\overline{\gamma}^*} \cdot \bar{\Delta}_{\bar{t}}(\omega) \sum_{\tau=0}^{T} \frac{1}{\phi^\tau}\right\} > 0,$$

an expression that diverges if and only if $\phi \leq 1$. By Lemma 4 (a), the left hand side of the inequality in Lemma 3 is $\Upsilon_{\bar{t}}(\omega) \cdot E_{P^{v \cdot \pi}}\left[\frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})}|\mathcal{F}_{\bar{t}}\right](\omega)$, a constant multiplied by the conditional expected value of a bounded variable, multiplied by the growth factor ϕ^T. Clearly, if $\phi \leq 1$, the left hand side is uniformly bounded above and so there exists a T for which the inequality in Lemma 3 is violated. This yields the desired contradiction confirming that no such set $E \in \mathcal{F}_{\bar{t}}$ with $P^\pi(E|\omega^{\bar{t}-1}) > 0$ can exist when $\phi \leq 1$. ■

Remark 2: Evidently, the proof of Theorem 1 is simple in that it does not rely on the existence of random variables defined through limits.

Also, when $\phi > 1$ we perturb the consumption plan of every agent, including the initial old, of type h where $h \in \hat{\mathcal{H}} \subset \mathcal{H}$ and $\hat{\mathcal{H}} \neq \emptyset$. So a subset of the set of all agents of "size at least $1/H$" is strictly improved. Conversely, the existence of any CPO improvement implies that $\phi > 1$. We can conclude that the stationarity of both the model and the equilibrium allocation ensure that the characterization result for CPO optimality remains unchanged even if the requirement to be an improvement is tightened from "some agent" to "a set of agents of positive measure."

Remark 3: By Remark 1, Proposition 1 generalizes to the case in which p is a continuous and bounded function. Assume that q^* is continuous, and strengthen Assumption 1 (ii) so that π is continuous. It follows that Lemma 2 generalizes to the case in which $q^* \cdot \pi$ is continuous and bounded, with the conclusion that γ^* is uniformly positive, uniformly bounded, and continuous. Although Theorem 1 remains unchanged, now if $\phi > 1$ there exists an improving transfer which is a continuous function; conversely, if a CPO improvement exists then $\phi > 1$ and there must exist a stationary and continuous improvement.

Remark 4: In the finite state case, an argument that uses only first order terms easily shows that if there exists a stationary allocation that is a CPO improvement, then $\phi > 1$ (see, e.g. Chattopadhyay (2001)). We now show that an analogous result can be proved with a continuum of states if $q^* \cdot \pi$ is known to be continuous.

By Remark 3, Lemma 2 generalizes to that case with the conclusion that γ^* is uniformly positive, uniformly bounded, and continuous.

Suppose there is a stationary improvement, i.e., a function ξ where $\xi(s)$ is the transfer received in state s by an agent when old, where ξ is measurable, bounded, and upper semi-continuous. Since the initial old must also be improved, ξ must be nonnegative. Also,

$$\int_S u_2^{x^*,s,h}(s') \cdot \pi(s,s')\xi(s')\Lambda^S(ds') > \xi(s) \int_S u_1^{x^*,s,h}(s')\pi(s,s')\Lambda^S(ds') \quad \forall\, s \in S,$$

since every agent is improved (here we require a strict inequality—in this way we can ignore third and higher order terms while the strict inequality takes into account the second order term since it has a negative effect). Equivalently,

$$\int_S q^*(s,s') \cdot \pi(s,s')\xi(s')\Lambda^S(ds') > \xi(s) \quad \forall\, s \in S$$

$$\Leftrightarrow \qquad \phi \int_S \frac{q^*(s,s') \cdot \pi(s,s')\gamma^*(s')}{\phi\gamma^*(s)} \frac{\xi(s')}{\gamma^*(s')}\Lambda^S(ds') > \frac{\xi(s)}{\gamma^*(s)} \quad \forall\, s \in S$$

where we use the fact, stated in Lemma 2, that $\underline{\gamma}^* := \inf_S \gamma^*(s) > 0$.

Let $\bar{s} \in \mathcal{S}$ denote a state in which

$$\frac{\xi(\bar{s})}{\gamma^*(\bar{s})} \geq \frac{\xi(s)}{\gamma^*(s)} \quad \text{for all } s \in \mathcal{S}.$$

Continuity of γ^* and compactness of S, together with the facts that $\underline{\gamma}^* := \inf_S \gamma^*(s) > 0$ and that ξ is upper semi-continuous, ensure that $\bar{s}$ exists.

So the existence of a stationary improvement ξ that is upper semi-continuous implies that

$$\phi \int_S \frac{q^*(\bar{s},s') \cdot \pi(\bar{s},s')\gamma^*(s')}{\phi\gamma^*(\bar{s})} \frac{\xi(s')}{\gamma^*(s')}\Lambda^S(ds') > \frac{\xi(\bar{s})}{\gamma^*(\bar{s})} \qquad \text{and} \qquad \frac{\xi(\bar{s})}{\gamma^*(\bar{s})} \geq \frac{\xi(s)}{\gamma^*(s)} \quad \forall\, s \in S.$$

Since

$$\int_S q^*(s,s') \cdot \pi(s,s')\gamma^*(s')\Lambda^S(ds') = \phi \cdot \gamma^*(s) \quad \forall\, s \in S$$

it follows that

$$\frac{\xi(\bar{s})}{\gamma^*(\bar{s})} \geq \int_S \frac{q^*(\bar{s},s') \cdot \pi(\bar{s},s')\gamma^*(s')}{\phi\gamma^*(\bar{s})} \frac{\xi(s')}{\gamma^*(s')}\Lambda^S(ds').$$

But then, the existence of such a ξ that is upper semi-continuous implies that $\phi > 1$.

Notice that measurability of ξ does not suffice to guarantee the existence of the state $\bar{s}$. Yet, the proof of Theorem 1 is able to handle the case where an improving allocation has no regularity properties beyond being a stochastic process.

Remark 5: We are now in a position to relate our result to the literature. That an eigenvalue completely characterizes optimality of stationary allocations can be seen as follows: restate Lemma 2 in the form

$$\int_S q^*(s,s') \cdot \pi(s,s')\gamma^*(s')\Lambda^S(ds') - \gamma^*(s) = (\phi - 1)\gamma^*(s) \quad \forall\, s \in S, \qquad (*)$$

where γ^* is measurable, uniformly positive, and uniformly bounded. Evidently, $\phi > 1$ if and only if the left hand side of $(*)$ is uniformly positive. By Lemma C (i), a rescaled γ^* generates a CPO improvement if the left hand side of $(*)$ is uniformly positive; conversely, using Lemma C (ii), the existence of a CPO improvement can be shown to be incompatible with $\phi \leq 1$, i.e. then $\phi > 1$. By the latter, if $\phi \leq 1$ then the allocation is CPO.

The model in Manuelli (1990) is identical to ours except that he works with a general Markov transition function on a compact subset of a metric space.[15] The result in Manuelli is: for a stationary allocation to be CPO it is necessary and sufficient that there is a measurable function ξ that is uniformly positive and uniformly bounded such that

$$\int_S q^*(s,s') \cdot \pi(s,s')\xi(s')\Lambda^S(ds') - \xi(s) \leq 0 \quad \forall\, s \in S. \tag{M}$$

With Lemma 2 in hand, Manuelli's condition can be written in terms of $(*)$ as $\phi \leq 1$; in the absence of Lemma 2 one cannot make that connection. This observation underlines the difficulty in using a condition in terms of the existence of a function that satisfies a set of inequalities since it is less informative, particularly when it comes to the negation of the condition. Also, $(*)$ is a tighter condition when $\phi \neq 1$ and this difference can be traced to the fact that our proof uses a second order approximation which is more informative than the first order approach followed by Manuelli.

Demange and Laroque (2000) consider a model similar to ours with production, a finite number of exogenous states and a continuum of endogenous states for capital. Whether Theorem 1 can be extended to their framework is an open question. They consider strictly positive continuous stationary allocations on a compact state space. They show that ξ that is uniformly positive and continuous generates a CPO improvement if the left hand side of $(*)$ is strictly positive; instead, if (M) holds then the allocation is CPO. As they recognize, their result is not a characterization. They go further by studying the spectral value but are unable to close the gap completely.

Barbie and Kaul (2015) consider a model that is almost identical to ours except that their state space S is a compact Polish space. For stationary allocations that are continuous and Markov transition functions that satisfy a strong form of continuity, they show that: ξ that is strictly positive and continuous generates a CPO improvement if the left hand side of $(*)$ is strictly positive; instead, if there exists a CPO improvement then there exists a ξ that is strictly positive and continuous for which the left hand side of $(*)$ is strictly positive (with a weaker assumption on the Markov transition functions, a weaker version of the latter result is obtained as ξ is measurable but might fail to be continuous). This agrees exactly with the result noted in Remark 3 and corresponds to $\phi > 1$ in terms of our result. Of course, $(*)$ delivers a better bound, Theorem 1 applies even when $q^* \cdot \pi$ fails to be continuous, and a necessary and sufficient condition in terms of a single number is manifestly simpler.

[15] He assumes that the transition satisfies the Feller condition but it is not clear that this is used in the result on optimality. We have many agent types but that makes no difference to the result.

5.1 Proof of Lemma 1

Lemma 1: *Each connected component of C^M is complete in $\theta(\cdot)$.*
Proof: Let $\{f_n\}$ be a sequence in a connected component of C^M. Assume that $\{f_n\}$ is Cauchy with respect to $\theta(\cdot)$, i.e. for every $\epsilon > 0$ there exists $N(\epsilon)$ such that, for all $m, n \geq N(\epsilon)$, $\theta(f_m, f_n) = \ln(\alpha_{m,n}\beta_{m,n}) < \epsilon$; equivalently, for all $m, n \geq N(\epsilon)$, $\alpha_{m,n}\beta_{m,n} < e^\epsilon$. Choose a subsequence $\{f_{n(k)}\}$ such that $\theta(f_{n(k)}, f_{n(k+1)}) < -\ln\left(1 - \frac{1}{a^k}\right)$, where $a > 1$.

From the definition of $\theta(\cdot)$ we know that

$$f_{n(k)} \leqq \beta_{n(k),n(k+1)} f_{n(k+1)} \leqq \alpha_{n(k),n(k+1)}\beta_{n(k),n(k+1)} f_{n(k)}.$$

Furthermore, $\alpha_{n(k),n(k+1)}\beta_{n(k),n(k+1)} < \left(1 - \frac{1}{a^k}\right)^{-1}$ since $\theta(f_{n(k)}, f_{n(k+1)}) < -\ln\left(1 - \frac{1}{a^k}\right)$. So

$$f_{n(k)} \leqq \beta_{n(k),n(k+1)} f_{n(k+1)} \leqq \left(1 - \frac{1}{a^k}\right)^{-1} f_{n(k)}.$$

Set

$$h_1 := \left(1 - \frac{1}{a}\right) f_{n(1)} \qquad \text{and} \qquad \text{for } k \geq 2, \quad h_k := \left(\prod_{i=1}^{k-1} \left[\left(1 - \frac{1}{a^i}\right) \beta_{n(i),n(i+1)}\right]\right) f_{n(k)}.$$

Clearly, for each i, h_i lies on the same ray as $f_{n(i)}$, so $\theta(f_{n(i)}, h_i) = 0$. In addition, it is easy to check that the sequence $\{h_k\}$ is such that $\left(1 - \frac{1}{a^k}\right) h_k \leqq h_{k+1} \leqq h_k$. Therefore, by iterating on the two inequalities one obtains, for all $k > 1$, $\prod_{i=1}^{k-1}\left(1 - \frac{1}{a^i}\right) h_1 \leqq h_k$ and $h_k \leqq h_1$, while rewriting the first inequality leads to $h_k - h_{k+1} \leqq \frac{1}{a^k} h_k$.

We shall show that (i) $\delta h_1 \leqq h_k \leqq h_1$, where $\delta := \left[e^{a/(a-1)^2}\right]^{-1}$, and (ii) for $k > j \geq 1$, $0 \leqq h_j - h_k \leqq \delta_j h_1$ where $\delta_j := \frac{a^{1-j}}{a-1} \geq \frac{1}{a^j}\frac{a}{a-1}\left[1 - \frac{1}{a^{k-j}}\right]$.

We have

$$\sum_{i=j}^{k-1} \frac{1}{a^i} = \frac{1}{a^j}\left(\sum_{i=0}^{k-1-j} \frac{1}{a^i}\right) = \frac{1}{a^j}\frac{a}{a-1}\left[1 - \frac{1}{a^{k-j}}\right].$$

Since $1 + x < e^x$ for $x > 0$, from the above we also have

$$\prod_{i=j}^{k-1}\left(1 + \frac{a}{a-1}\frac{1}{a^i}\right) = e^{\sum_{i=j}^{k-1} \ln\left(1 + \frac{a}{a-1}\frac{1}{a^i}\right)} < e^{\sum_{i=j}^{k-1} \frac{a}{a-1}\frac{1}{a^i}} = e^{\frac{a}{a-1}\frac{1}{a^j}\frac{a}{a-1}\left[1 - \frac{1}{a^{k-j}}\right]};$$

in particular, $\prod_{i=1}^{k-1}\left(1 + \frac{a}{a-1}\frac{1}{a^i}\right) < e^{\frac{a}{a-1}\frac{1}{a}\frac{a}{a-1}\left[1 - \frac{1}{a^{k-1}}\right]} < e^{\frac{a}{(a-1)^2}}$.

Since $a > 1$ we have $\left(1 - \frac{1}{a^i}\right)\left(1 + \frac{a}{a-1}\frac{1}{a^i}\right) \geq 1$, and therefore, as in the proof of Theorem 4 on page 220 in Knopp (1990), we may conclude that

$$\prod_{i=1}^{k-1}\left(1 - \frac{1}{a^i}\right) \geq \left[\prod_{i=1}^{k-1}\left(1 + \frac{a}{(a-1)a^i}\right)\right]^{-1} > \left[e^{\frac{a}{(a-1)^2}}\right]^{-1}.$$

That completes the proof of (i). The proof of (ii) follows from (i) and the fact that

$$h_j - h_k = \sum_{i=j}^{k-1}(h_i - h_{i+1}) \leqq \sum_{i=j}^{k-1} \frac{1}{a^i} h_i.$$

By (ii), there exists $z_{j,k} \in C_0^M$ such that $h_j - h_k + z_{j,k} = \delta_j h_1$, so that, for $k > j \geq 1$,

$$\| h_j - h_k \|_\infty \leq \| h_j - h_k \|_\infty + \| z_{j,k} \|_\infty = \| h_j - h_k + z_{j,k} \|_\infty = \delta_j \| h_1 \|_\infty,$$

where the first equality follows from the fact that $z_{j,k} \in C_0^M$ and that, for $f \in B^{X,\mathcal{X}}$, $\| f \|_\infty := \sup_{x \in X}|f(x)|$. It follows that $\{h_k\}$ is a Cauchy sequence in $(C_0^M, \| \cdot \|_\infty)$. Since $(B^{X,\mathcal{X}}, \| \cdot \|_\infty)$ is a Banach space, and C_0^M is a closed subset of $(B^{X,\mathcal{X}}, \| \cdot \|_\infty)$, $(C_0^M, \| \cdot \|_\infty)$ is complete. So there is $h \in C_0^M$ such that $\| h_k - h \|_\infty \to 0$ as $k \to \infty$.

Since $0 \leqq h_j - h_k \leqq \delta_j h_1$ for $k > j \geq 1$, and $\| h_k - h \|_\infty \to 0$ as $k \to \infty$, we have

$$0 \leqq h_j - h \leqq \delta_j h_1 \qquad \Leftrightarrow \qquad 0 \leqq h_j - h \leqq \delta_j (\delta)^{-1} h_k,$$

where we use (i) to get from the first to the second pair of inequalities. We can now use the fact that $\| h_k - h \|_\infty \to 0$ as $k \to \infty$ to obtain

$$h \leqq h_j \leqq \left(1 + \delta_j (\delta)^{-1}\right) h \qquad \Leftrightarrow \qquad \theta(h_j, h) = \ln\left(1 + \delta_j(\delta)^{-1}\right)$$

and so, recalling that $\delta_j := \frac{a^{1-j}}{a-1}$, we may conclude that $\theta(h_j, h) \to 0$ as $j \to \infty$. As

$$\theta(f_m, h) \leq \theta(f_m, f_{n(k)}) + \theta(f_{n(k)}, h_k) + \theta(h_k, h),$$

where $\theta(f_{n(k)}, h_k) = 0$ since, by construction, the two vectors lie on the same ray, and $\theta(f_m, f_{n(k)}) \to 0$ since $\{f_m\}$ is a Cauchy sequence with respect to $\theta(\cdot)$, we have $\theta(f_m, h) \to 0$ as $m \to \infty$. Since $\theta(f_m, h) < \infty$, we can conclude that h is an element of the connected component that contains $\{f_m\}$. We have shown that every sequence in a connected component of C^M which is Cauchy with respect to $\theta(\cdot)$ has a subsequence with a limit in the same connected component; this implies that every such Cauchy sequence has a limit in the same connected component. That completes the proof. ∎

5.2 Proofs of Lemma 2, 3, and 4

Lemma 2: *Let (x^*, q^*) be a stationary competitive equilibrium with sequentially complete markets in which the equilibrium allocation is uniformly interior. Under Assumptions 1-4, there exists $\gamma^* \in B^{S,\mathcal{S}}$ and $\phi > 0$ such that*

$$\int_S q^*(s, s') \cdot \pi(s, s') \gamma^*(s') \Lambda^S(ds') = \phi \cdot \gamma^*(s) \quad \forall\, s \in S$$

and $\underline{\gamma}^* > 0$ and $\overline{\gamma}^* < \infty$, where $\underline{\gamma}^* := \inf_S \gamma^*(s)$ and $\overline{\gamma}^* := \sup_S \gamma^*(s)$.

Proof: Since the aggregate endowment is uniformly bounded and $H < \infty$, x^* must be uniformly bounded. By assumption x^* is uniformly interior. With our assumption that the Bernoulli utility functions u are all strictly increasing and continuously differentiable, the function q^* is uniformly positive and uniformly bounded, i.e., there exist $\underline{q}$ and $\overline{q}$ such that

$$0 < \underline{q} = \inf_{(s,s')\in S\times S} q^*(s,s') \leq \sup_{(s,s')\in S\times S} q^*(s,s') = \overline{q}.$$

Now consider the function $q^* \cdot \pi$. It is $\mathcal{S}\times\mathcal{S}-$measurable and by what we have just seen and Assumption 1, there exist $\underline{q}$ and $\underline{\pi}$ such that, for some $\kappa > 0$,

$$0 < \underline{q}\underline{\pi} \leq \inf_{(s,s')\in S\times S} q^*(s,s')\cdot\pi(s,s') \leq \sup_{(s,s')\in S\times S} q^*(s,s')\cdot\pi(s,s') \leq \kappa\cdot\underline{q}\underline{\pi}.$$

It follows from Proposition 1 in Section 3 that there exists an eigenvector

$$\gamma^* \in B^{S,\mathcal{S}} \cap \{f : S \to R \,|\, 0 < \sup f(x) \leq M \inf f(x)\},$$

where $M \geq \kappa$, so γ^* is measurable, uniformly positive and bounded, such that, for some $\phi > 0$, an eigenvalue,

$$\int_S q^*(s,s')\cdot\pi(s,s')\gamma^*(s')\Lambda^S(ds') = \phi\cdot\gamma^*(s) \quad \forall\, s\in S.$$

Define $\underline{\gamma}^* := \inf_S \gamma^*(s)$ and $\overline{\gamma}^* := \sup_S \gamma^*(s)$. Evidently, $\underline{\gamma}^* > 0$ and $\overline{\gamma}^* < \infty$. ∎

Lemma 3: *Under A.1 and A.2, for $T = 1, 2, \cdots$ and $\omega \in E$*

$$\phi^T\cdot E\left[\Upsilon_{\bar{t}+T}\cdot\frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})}\middle|\mathcal{F}_{\bar{t}}\right](\omega) \geq \Upsilon_{\bar{t}}(\omega)\cdot\frac{\bar{\Delta}_{\bar{t}}(\omega)}{\gamma^*(s_{\bar{t}}(\omega))} + \underline{\rho}\left\{\sum_{\tau=0}^{T-1}\phi^\tau\cdot E\left[\Upsilon_{\bar{t}+\tau}\cdot\frac{[\bar{\Delta}_{\bar{t}+\tau}]^2}{\gamma^*(s_{\bar{t}+\tau})}\middle|\mathcal{F}_{\bar{t}}\right](\omega)\right\} > 0.$$

Proof: When we defined υ we noted that it is uniformly positive and uniformly bounded. By CE (i) and CE (ii) of Section 2.1, we have

$$\phi^T\cdot E\left[\Upsilon_{\bar{t}+T}\cdot\frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})}\middle|\mathcal{F}_{\bar{t}}\right](\omega) =$$

$$= \Upsilon_{\bar{t}}(\omega)\cdot\phi\cdot E\left[\upsilon(s_{\bar{t}},s_{\bar{t}+1})\cdots\left\{\phi\cdot E\left[\upsilon(s_{\bar{t}+T-1},s_{\bar{t}+T})\cdot\frac{\bar{\Delta}_{\bar{t}+T}}{\gamma^*(s_{\bar{t}+T})}\middle|\mathcal{F}_{\bar{t}+T-1}\right]\right\}\cdots\middle|\mathcal{F}_{\bar{t}}\right](\omega),$$

so that for $\omega \in E$, upon using (3), we have

$$\geq \Upsilon_{\bar{t}}(\omega)\cdot\phi\cdot E\left[\upsilon(s_{\bar{t}},s_{\bar{t}+1})\cdots\phi\cdot E\left[\upsilon(s_{\bar{t}+T-2},s_{\bar{t}+T-1})\left\{\frac{\bar{\Delta}_{\bar{t}+T-1}}{\gamma^*(s_{\bar{t}+T-1})}+\underline{\rho}\frac{[\bar{\Delta}_{\bar{t}+T-1}]^2}{\gamma^*(s_{\bar{t}+T-1})}\right\}\middle|\mathcal{F}_{\bar{t}+T-2}\right]\cdots\middle|\mathcal{F}_{\bar{t}}\right](\omega)$$

$$= \Upsilon_{\bar{t}}(\omega)\cdot\phi\cdot E\left[\upsilon(s_{\bar{t}},s_{\bar{t}+1})\cdots\left\{\phi\cdot E\left[\upsilon(s_{\bar{t}+T-2},s_{\bar{t}+T-1})\frac{\bar{\Delta}_{\bar{t}+T-1}}{\gamma^*(s_{\bar{t}+T-1})}\middle|\mathcal{F}_{\bar{t}+T-2}\right]\right\}\cdots\middle|\mathcal{F}_{\bar{t}}\right](\omega)+$$

$$+\Upsilon_{\bar{t}}(\omega)\cdot\phi\cdot E\left[\upsilon(s_{\bar{t}},s_{\bar{t}+1})\cdots\phi\cdot E\left[\upsilon(s_{\bar{t}+T-2},s_{\bar{t}+T-1})\underline{\rho}\frac{[\bar{\Delta}_{\bar{t}+T-1}]^2}{\gamma^*(s_{\bar{t}+T-1})}\middle|\mathcal{F}_{\bar{t}+T-2}\right]\cdots\middle|\mathcal{F}_{\bar{t}}\right](\omega)$$

so that, upon using (3) again in the first term and using CE (ii) in the second term, we obtain

$$\geq \Upsilon_{\bar{t}}(\omega) \cdot \phi \cdot E\left[\upsilon(s_{\bar{t}}, s_{\bar{t}+1}) \cdots \phi \cdot E\left[\upsilon(s_{\bar{t}+T-3}, s_{\bar{t}+T-2}) \left\{ \frac{\bar{\Delta}_{\bar{t}+T-2}}{\gamma^*(s_{\bar{t}+T-2})} + \underline{\rho} \frac{[\bar{\Delta}_{\bar{t}+T-2}]^2}{\gamma^*(s_{\bar{t}+T-2})} \right\} \middle| \mathcal{F}_{\bar{t}+T-3} \right] \cdots \middle| \mathcal{F}_{\bar{t}} \right](\omega) +$$

$$+\underline{\rho} \cdot \phi^{T-1} \cdot E\left[\Upsilon_{\bar{t}+T-1} \frac{[\bar{\Delta}_{\bar{t}+T-1}]^2}{\gamma^*(s_{\bar{t}+T-1})} \middle| \mathcal{F}_{\bar{t}} \right](\omega)$$

$$= \Upsilon_{\bar{t}}(\omega) \cdot \phi \cdot E\left[\upsilon(s_{\bar{t}}, s_{\bar{t}+1}) \cdots \left\{ \phi \cdot E\left[\upsilon(s_{\bar{t}+T-3}, s_{\bar{t}+T-2}) \frac{\bar{\Delta}_{\bar{t}+T-2}}{\gamma^*(s_{\bar{t}+T-2})} \middle| \mathcal{F}_{\bar{t}+T-3} \right] \right\} \cdots \middle| \mathcal{F}_{\bar{t}} \right](\omega) +$$

$$+\Upsilon_{\bar{t}}(\omega) \cdot \phi \cdot E\left[\upsilon(s_{\bar{t}}, s_{\bar{t}+1}) \cdots \phi \cdot E\left[\upsilon(s_{\bar{t}+T-3}, s_{\bar{t}+T-2}) \underline{\rho} \frac{[\bar{\Delta}_{\bar{t}+T-2}]^2}{\gamma^*(s_{\bar{t}+T-2})} \middle| \mathcal{F}_{\bar{t}+T-3} \right] \cdots \middle| \mathcal{F}_{\bar{t}} \right](\omega)$$

$$+\underline{\rho} \cdot \phi^{T-1} \cdot E\left[\Upsilon_{\bar{t}+T-1} \frac{[\bar{\Delta}_{\bar{t}+T-1}]^2}{\gamma^*(s_{\bar{t}+T-1})} \middle| \mathcal{F}_{\bar{t}} \right](\omega)$$

$$= \Upsilon_{\bar{t}}(\omega) \cdot \phi \cdot E\left[\upsilon(s_{\bar{t}}, s_{\bar{t}+1}) \cdots \left\{ \phi \cdot E\left[\upsilon(s_{\bar{t}+T-3}, s_{\bar{t}+T-2}) \frac{\bar{\Delta}_{\bar{t}+T-2}}{\gamma^*(s_{\bar{t}+T-2})} \middle| \mathcal{F}_{\bar{t}+T-3} \right] \right\} \cdots \middle| \mathcal{F}_{\bar{t}} \right](\omega) +$$

$$+\underline{\rho} \cdot \phi^{T-2} \cdot E\left[\Upsilon_{\bar{t}+T-2} \frac{[\bar{\Delta}_{\bar{t}+T-2}]^2}{\gamma^*(s_{\bar{t}+T-2})} \middle| \mathcal{F}_{\bar{t}} \right](\omega) + \underline{\rho} \cdot \phi^{T-1} \cdot E\left[\Upsilon_{\bar{t}+T-1} \frac{[\bar{\Delta}_{\bar{t}+T-1}]^2}{\gamma^*(s_{\bar{t}+T-1})} \middle| \mathcal{F}_{\bar{t}} \right](\omega)$$

$$\cdots$$

$$\geq \Upsilon_{\bar{t}}(\omega) \cdot \frac{\bar{\Delta}_{\bar{t}}(\omega)}{\gamma^*(s_{\bar{t}}(\omega))} + \underline{\rho} \left\{ \sum_{\tau=0}^{T-1} \phi^{\tau} \cdot E\left[\Upsilon_{\bar{t}+\tau} \frac{[\bar{\Delta}_{\bar{t}+\tau}]^2}{\gamma^*(s_{\bar{t}+\tau})} \middle| \mathcal{F}_{\bar{t}} \right](\omega) \right\}.$$

■

Lemma 4: *For any* $\bar{t} \geq 1$*, any* $T = 1, 2, \cdots$*, and any* $f \in B^{\Omega, \mathcal{F}_{\bar{t}+T}}$*,*

(*a*) $\quad E\left[\Upsilon_{\bar{t}+T} \cdot f | \mathcal{F}_{\bar{t}}\right](\omega) = \Upsilon_{\bar{t}}(\omega) \cdot E_{P^{\upsilon \cdot \pi}}\left[f | \mathcal{F}_{\bar{t}}\right](\omega),$

(*b*) $\quad E\left[\Upsilon_{\bar{t}+T} \cdot f | \mathcal{F}_{\bar{t}}\right](\omega) \geq a > 0 \quad \Rightarrow \quad \Upsilon_{\bar{t}}(\omega) \cdot E\left[\Upsilon_{\bar{t}+T} \cdot [f]^2 | \mathcal{F}_{\bar{t}}\right](\omega) \geq a^2.$

Proof: Let $t \in \{1, 2, \cdots\}$ and consider any set B^t such that $B^t := B_1 \times \cdots \times B_t$ where, for $s \in \{1, \cdots, t\}$, $B_s \subset S$; $B^t \in \mathcal{S}^t$ and is a measurable rectangle. Any transition density $p \in \mathcal{D}$, together with the measure μ on $(S, \mathcal{S})$, induces the measure $Q^{p,t}$ on $(S^t, \mathcal{S}^t)$ where

$$\text{(i)} \qquad Q^{p,t}\Big(B^t\Big) = \int_{B_1} \cdots \int_{B_t} Q^p(s_{t-1}, ds_t) \cdots Q^p(s_1, ds_2) \cdot \mu(ds_1)$$

$$= \int_{B_1} \left[\int_{B_2} \cdots \left[\int_{B_t} p(s_{t-1}, s_t) \Lambda^S(ds_t) \right] \cdots p(s_1, s_2) \Lambda^S(ds_2) \right] \cdot \mu(ds_1),$$

where the last equality follows since the stationary transition Q^p is induced by $p \in \mathcal{D}$, and, by KET in Section 2.1,

$$\text{(ii)} \qquad P^p\Big(B^t \times \prod_{\tau>t} S\Big) = Q^{p,t}\Big(B^t\Big).$$

Since $\Upsilon_t(\omega) := \upsilon(s_1(\omega), s_2(\omega)) \cdots \upsilon(s_{t-1}(\omega), s_t(\omega))$, and $\pi \in \mathcal{D}$, we have

$$\text{(iii)} \qquad \int_{B^t \times \prod_{\tau>t} S} \Upsilon_t(\omega) P^\pi(d\omega) := \int_{B^t \times \prod_{\tau>t} S} \upsilon(s_1(\omega), s_2(\omega)) \cdots \upsilon(s_{t-1}(\omega), s_t(\omega)) P^\pi(d\omega).$$

Recall that υ is uniformly positive, uniformly bounded, and $\int_{s' \in \mathcal{S}} \upsilon(s, s') \cdot \pi(s, s') \Lambda^S(ds') = 1$ for all $s \in \mathcal{S}$. It follows that $\upsilon \cdot \pi$ is a transition density function, $\upsilon \cdot \pi \in \mathcal{D}$. Now consider the probability measure $P^{\upsilon \cdot \pi}$. Since (i) and (ii) above hold for all $p \in \mathcal{D}$ we have

$$P^{\upsilon\cdot\pi}\Big(B^t \times \prod_{\tau>t} S\Big) = \int_{B_1} \left[\int_{B_2} \cdots \left[\int_{B_t} \upsilon(s_{t-1}, s_t) \cdot \pi(s_{t-1}, s_t) \Lambda^S(ds_t) \right] \cdots \upsilon(s_1, s_2) \cdot \pi(s_1, s_2) \Lambda^S(ds_2) \right] \cdot \mu(ds$$

$$= \int_{B_1} \left[\int_{B_2} \cdots \left[\int_{B_t} \upsilon(s_1, s_2) \cdots \upsilon(s_{t-1}, s_t) \cdot \pi(s_{t-1}, s_t) \Lambda^S(ds_t) \right] \cdots \pi(s_1, s_2) \Lambda^S(ds_2) \right] \cdot \mu(ds_1).$$

But then, using (iii), it follows that, for any $F \in \mathcal{F}_t$,

$$\int_F \Upsilon_t(\omega) P^\pi(d\omega) = \int_F P^{\upsilon\cdot\pi}(d\omega).$$

Since υ is uniformly positive and uniformly bounded, the definition of Υ_t lets us conclude that $\Upsilon_t \in B_+^{\Omega, \mathcal{F}_t}$. Evidently, for $F \in \mathcal{F}_t$,

$$P^\pi|_{\mathcal{F}_t}(F) = 0 \qquad \Rightarrow \qquad P^{\upsilon\cdot\pi}|_{\mathcal{F}_t}(F) = 0,$$

where the measures $P^\pi|_{\mathcal{F}_t}$ and $P^{\upsilon\cdot\pi}|_{\mathcal{F}_t}$ are the restrictions of the measures P^π and $P^{\upsilon\cdot\pi}$, respectively, to the measurable space $(\Omega, \mathcal{F}_t)$. It follows from the Radon-Nikodym Theorem that

$$\Upsilon_t = \frac{dP^{\upsilon\cdot\pi}|_{\mathcal{F}_t}}{dP^\pi|_{\mathcal{F}_t}}.$$

By Theorem 2 on page 79 in Gikhman and Skorokhod, for $F \in \mathcal{F}_t$ and any $f \in B^{\Omega, \mathcal{F}_t}$,

$$\int_F f(\omega) P^{\upsilon\cdot\pi}|_{\mathcal{F}_t}(d\omega) = \int_F f(\omega) \Upsilon_t(\omega) P^\pi|_{\mathcal{F}_t}(d\omega).$$

Now consider any $\bar{t} \geq 1$, any $T = 1, 2, \cdots$, and any $f \in B^{\Omega, \mathcal{F}_{\bar{t}+T}}$. For $G \in \mathcal{F}_{\bar{t}}$, we can use, in order, CE, the last result with $t = \bar{t} + T$, CE again, $\Upsilon_{\bar{t}+T} = \frac{dP^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}{dP^\pi|_{\mathcal{F}_{\bar{t}+T}}}$, CE once again, and CE (ii) together with $\mathcal{F}_{\bar{t}}$-measurability of $E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega)$, to write

$$\int_G E_{P^\pi|_{\mathcal{F}_{\bar{t}+T}}}\left[\Upsilon_{\bar{t}+T} \cdot f|\mathcal{F}_{\bar{t}}\right](\omega) P^\pi|_{\mathcal{F}_{\bar{t}+T}}(d\omega) = \int_G f(\omega) \Upsilon_{\bar{t}+T}(\omega) P^\pi|_{\mathcal{F}_{\bar{t}+T}}(d\omega) = \int_G f(\omega) P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}(d\omega)$$

$$= \int_G E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega)P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}(d\omega) = \int_G \Upsilon_{\bar{t}+T}(\omega)E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega)P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}(d\omega)$$

$$= \int_G E_{P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[\left(\Upsilon_{\bar{t}+T}E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right]\right)|\mathcal{F}_{\bar{t}}\right](\omega)P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}(d\omega)$$

$$= \int_G E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega)\cdot E_{P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[\Upsilon_{\bar{t}+T}|\mathcal{F}_{\bar{t}}\right](\omega)P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}(d\omega)$$

which suggests the following implication in terms of conditional expectations[16]

$$E_{P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[\Upsilon_{\bar{t}+T}\cdot f|\mathcal{F}_{\bar{t}}\right](\omega) = E_{P^{\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[\Upsilon_{\bar{t}+T}|\mathcal{F}_{\bar{t}}\right](\omega)\cdot E_{P^{\upsilon\cdot\pi}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega).$$

The last line may be rewritten as

$$E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T}\cdot f|\mathcal{F}_{\bar{t}}\right](\omega) = E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T}|\mathcal{F}_{\bar{t}}\right](\omega)\cdot E_{P^{\upsilon\cdot\pi}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega)$$

since, for any probability measure, in particular for the measure induced by any transition density function $p \in \mathcal{D}$, the fact that f is $\mathcal{F}_{\bar{t}+T}$-measurable implies that

$$E_{P^{p}|_{\mathcal{F}_{\bar{t}+T}}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega) = E_{P^{p}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega).$$

Therefore, to complete the proof of Lemma 4 (a) it suffices to show that

$$E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T}|\mathcal{F}_{\bar{t}}\right](\omega) = \Upsilon_{\bar{t}}(\omega).$$

By CE (i), it suffices to show that $E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T+1}|\mathcal{F}_{\bar{t}+T}\right](\omega) = \Upsilon_{\bar{t}+T}(\omega)$. Using the fact that $\Upsilon_t(\omega) := \upsilon(s_1(\omega), s_2(\omega))\cdots\upsilon(s_{t-1}(\omega), s_t(\omega))$ and CE (ii), we have

$$E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T+1}|\mathcal{F}_{\bar{t}+T}\right](\omega) = E_{P^{\pi}}\left[\Upsilon_{\bar{t}+T}\cdot\upsilon(s_{\bar{t}+T}, s_{\bar{t}+T+1})|\mathcal{F}_{\bar{t}+T}\right](\omega)$$

$$= \Upsilon_{\bar{t}+T}(\omega)\cdot E_{P^{\pi}}\left[\upsilon(s_{\bar{t}+T}, s_{\bar{t}+T+1})|\mathcal{F}_{\bar{t}+T}\right](\omega)$$

$$= \Upsilon_{\bar{t}+T}(\omega)\cdot\int_{\{s'\in s_{\bar{t}+T+1}(A_{\bar{t}+T}(\omega))\}}\upsilon(s_{\bar{t}+T}(\omega), s')\cdot\pi(s_{\bar{t}+T}(\omega), s')\Lambda^S(ds'),$$

where we use CP,

$$= \Upsilon_{\bar{t}+T}(\omega),$$

since $s_{\bar{t}+T+1}(A_{\bar{t}+T}(\omega)) = S$, and $\int_{s'\in\mathcal{S}}\upsilon(s,s')\cdot\pi(s,s')\Lambda^S(ds') = 1$ for all $s \in \mathcal{S}$.

That completes the proof of Lemma 4 (a).

We turn to the proof of Lemma 4 (b). By the conditional form of Jensen's Inequality,

$$E_{P^{\upsilon\cdot\pi}}\left[\Upsilon_{\bar{t}}\cdot f|\mathcal{F}_{\bar{t}}\right](\omega) \geq a > 0 \qquad\Rightarrow\qquad E_{P^{\upsilon\cdot\pi}}\left[\left[\Upsilon_{\bar{t}}\right]^2\cdot[f]^2|\mathcal{F}_{\bar{t}}\right](\omega) \geq a^2.$$

Upon using CE (ii) on the terms on the left hand side of each of the inequalities above we obtain

$$\Upsilon_{\bar{t}}(\omega)\cdot E_{P^{\upsilon\cdot\pi}}\left[f|\mathcal{F}_{\bar{t}}\right](\omega) \geq a > 0 \qquad\Rightarrow\qquad \left[\Upsilon_{\bar{t}}(\omega)\right]^2\cdot E_{P^{\upsilon\cdot\pi}}\left[[f]^2|\mathcal{F}_{\bar{t}}\right](\omega) \geq a^2,$$

so that by applying Lemma 3 (a) to the terms on the left hand side of each of the inequalities above we obtain

$$E\left[\Upsilon_{\bar{t}+T}\cdot f|\mathcal{F}_{\bar{t}}\right](\omega) \geq a > 0 \qquad\Rightarrow\qquad \Upsilon_{\bar{t}}(\omega)\cdot E\left[\Upsilon_{\bar{t}+T}\cdot[f]^2|\mathcal{F}_{\bar{t}}\right](\omega) \geq a^2$$

as required. ■

[16]For a general statement see, e.g. Lemma 4.29 in Kopp, Malczak, and Zastawniak (2013).

5.3 Statement and Proof of Lemma C

In this section we present a result in two parts: (i) we specify conditions on parameters under which if a consumption plan and a local perturbation of that plan jointly satisfy a certain inequality, then the perturbed plan induces higher utility than the unperturbed one, and (ii) we specify another set of conditions on parameters under which if a local perturbation of a consumption plan induces higher utility than the unperturbed plan then a certain inequality must hold. As we noted in Section 4, it is well known that such *curvature inequalities*, which provide quadratic approximations to an agent's utility, play essential roles in the characterization results on optimality of competitive paths in OLG economies; fleshing them out is notationally laborious and the conditions can be phrased in different ways. We present one consistent way of setting things up and note that we have sacrificed some generality in the interest of brevity.[17]

We start with some notation and preliminary definitions.

Recall that an agent (A,h)'s consumption plan is denoted $x(A,h) \in X_{A,h}$, with $x(A,h) = \left(x_{1,t(A),h}, x_{2,t(A)+1,h}\right)$. From here on, for an agent (A,h)'s consumption plan $x(A,h) \in X_{A,h}$, and $\omega \in A$, we use the notation $x_{A,h}(\omega) := \left(x_{1,t(A),h}\left(\omega\right), x_{2,t(A)+1,h}\left(\omega\right)\right)$.

To reduce the notational burden, we define

$$u^{x,A,h}\left(\omega\right) := u\left(x_{A,h}\left(\omega\right), s_{t(A)+1}\left(\omega\right); s_A, h\right).$$

Recall that u is a twice differentiable function. Let u_i, $i \in \{1,2\}$, denote the first order partial derivatives of u with respect to its first two coordinates, and let D^2u denote the matrix of partial derivatives of order two.

We extend the notation to first and second derivatives for $x_{A,h}\left(\omega\right) \in R^2_{++}$, e.g.

$$u_i^{x,A,h}\left(\omega\right) := u_i\left(x_{A,h}\left(\omega\right), s_{t(A)+1}\left(\omega\right); s_A, h\right) \quad \text{for } i \in \{1,2\}.$$

Although the economy depends on ω in a very general way, the existence of derivatives can be ensured if all plans $x(A,h) \in X_{A,h}$ are such that, for all $\omega \in A$, $x_{A,h}\left(\omega\right) \in R^2_{++}$.

So let $\tilde{\mathfrak{x}} := \{\tilde{x}_{1,t,h}, \tilde{x}_{2,t,h}\}_{t=1}^{\infty}$ be a feasible allocation which induces consumption plans $\tilde{x}(A,h)$ for all $(A,h) \in \mathcal{A} \times \mathcal{H}$ and for all initial old agents (A,h,o). Recall that, by Assumption 3, $\bar{e}$ is a uniform upper bound on the aggregate endowment. Define $X^{\bar{e}}_{A,h} := B^{A,\mathcal{F}_{t(A)}}_{\bar{e}} \times B^{A,\mathcal{F}_{t(A)+1}}_{\bar{e}}$ where $B^{A,\mathcal{F}_{t(A)+i}}_{\bar{e}} = \{f : A \to (0,\bar{e}]\} \cap B^{A,\mathcal{F}_{t(A)+i}}$ for $i \in \{0,1\}$.

Assumption C (i): $\tilde{\mathfrak{x}}$ is such that $\tilde{x}(A,h) \in X^{\bar{e}}_{A,h}$ for all $(A,h) \in \mathcal{A} \times \mathcal{H}$.

Consider $\hat{x}(A,h) \in X^{\bar{e}}_{A,h}$ and $\tilde{x}(A,h) \in X^{\bar{e}}_{A,h}$, and define $\Delta x(A,h) := \hat{x}(A,h) - \tilde{x}(A,h)$. It follows that, for $\omega \in A$, we have

$$\Delta x_{1,t(A),h}\left(\omega\right) := \hat{x}_{1,t(A),h}\left(\omega\right) - \tilde{x}_{1,t(A),h}\left(\omega\right),$$

[17] See Demange and Laroque (1999, 2000), Chattopadhyay and Gottardi (1999), Barbie, Hagedorn, and Kaul (2007), and Barbie and Kaul (2015); we draw on all of the above.

$$\Delta x_{2,t(A)+1,h}(\omega) := \hat{x}_{2,t(A)+1,h}(\omega) - \tilde{x}_{2,t(A)+1,h}(\omega),$$

$$\Delta x_{t(A),h}(\omega) := \left(\Delta x_{1,t(A),h}(\omega), \Delta x_{2,t(A)+1,h}(\omega)\right) \in R^2.$$

For $\epsilon \in [0,1]$, we extend the notation $u^{\tilde{x},A,h}(\omega)$ by defining

$$u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) := u\left(\tilde{x}_{A,h}(\omega) + \epsilon \cdot \Delta x_{t(A),h}(\omega), s_{t(A)+1}(\omega); s_A, h\right).$$

Using a second order Taylor expansion, for some $\tau \in [0,1]$ we must have

$$V(\tilde{x}(A,h) + \epsilon \cdot \Delta x(A,h); A,h) - V(\tilde{x}(A,h); A,h) = \int_A \left[u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) - u^{\tilde{x},A,h}(\omega)\right] P^\pi\left(d\omega|\omega^{t(A)}\right)$$

$$= \int_A \Big\{u_1^{\tilde{x},A,h}(\omega) \cdot \epsilon \cdot \Delta x_{1,t(A),h}(\omega) + u_2^{\tilde{x},A,h}(\omega) \cdot \epsilon \cdot \Delta x_{2,t(A)+1,h}(\omega)$$

$$+ \left[\frac{1}{2} \cdot (\epsilon)^2 \cdot \Delta x_{t(A),h}(\omega) \cdot D^2 u^{\tilde{x},\tau\cdot\epsilon,\Delta x,A,h}(\omega) \cdot \Delta x_{t(A),h}(\omega)\right]\Big\} P^\pi\left(d\omega|\omega^{t(A)}\right)$$

$$= \left\{\int_A u_1^{\tilde{x},A,h}(\omega) P^\pi\left(d\omega|\omega^{t(A)}\right)\right\} \cdot \epsilon \cdot \Bigg\{\int_A \frac{u_2^{\tilde{x},A,h}(\omega)}{\int_A u_1^{\tilde{x},A,h}(\omega') P^\pi(d\omega'|\omega^{t(A)})} \Delta x_{2,t(A)+1,h}(\omega) P^\pi\left(d\omega|\omega^{t(A)}\right)$$

$$+\Delta x_{1,t(A),h}(\omega) + \epsilon \cdot \int_A \frac{\frac{1}{2}\Delta x_{t(A),h}(\omega) \cdot D^2 u^{\tilde{x},\tau\cdot\epsilon,\Delta x,A,h}(\omega) \cdot \Delta x_{t(A),h}(\omega)}{\int_A u_1^{\tilde{x},A,h}(\omega') P^\pi(d\omega'|\omega^{t(A)})} P^\pi\left(d\omega|\omega^{t(A)}\right)\Bigg\} \qquad (*)$$

where we use the fact that u is strictly increasing and differentiable.

Bounds on derivatives can now be defined ($\|\cdot\|$ denotes the Euclidean norm in R^2):

$$\underline{D}_1^{\tilde{\mathfrak{x}}}(\omega) := \inf_{(A,h)\in\mathcal{A}\times\mathcal{H}} u_1^{\tilde{x},A,h}(\omega), \qquad \overline{D}_1^{\tilde{\mathfrak{x}}}(\omega) := \sup_{(A,h)\in\mathcal{A}\times\mathcal{H}} u_1^{\tilde{x},A,h}(\omega),$$

$$\underline{D}_2^{\tilde{\mathfrak{x}}}(\omega) := \inf_{(A,h)\in\mathcal{A}\times\mathcal{H}} \min_{\epsilon\in[0,1]} \min_{\{x\in R^2|\|x\|=1\}} (-1)x \cdot D^2 u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) \cdot x,$$

$$\overline{D}_2^{\tilde{\mathfrak{x}}}(\omega) := \sup_{(A,h)\in\mathcal{A}\times\mathcal{H}} \max_{\epsilon\in[0,1]} \max_{\{x\in R^2|\|x\|=1\}} (-1)x \cdot D^2 u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) \cdot x.$$

Given ω, $\underline{D}_1^{\tilde{\mathfrak{x}}}(\omega)$ and $\overline{D}_1^{\tilde{\mathfrak{x}}}(\omega)$ provide lower and upper bounds for $u_1^{\tilde{x},A,h}(\omega)$ that are uniform across agents (note that the initial old agents have no role here); similarly, $\underline{D}_2^{\tilde{\mathfrak{x}}}(\omega) \cdot \| \Delta x_{t(A),h}(\omega) \|^2$ and $\overline{D}_2^{\tilde{\mathfrak{x}}}(\omega) \cdot \| \Delta x_{t(A),h}(\omega) \|^2$ provide bounds for

$$(-1)\Delta x_{t(A),h}(\omega) \cdot D^2 u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) \cdot \Delta x_{t(A),h}(\omega).$$

We make two independent *curvature assumptions* which require the existence of certain uniform bounds. These are not implied by Assumption 2 as no continuity assumptions were made regarding the influence of ω on the economy.

Assumption C (ii): $\underline{\rho}^{\tilde{\mathfrak{x}}} := \frac{1}{2}\frac{\inf_{\omega\in\Omega}\underline{D}_2^{\tilde{\mathfrak{x}}}(\omega)}{\sup_{\omega\in\Omega}\overline{D}_1^{\tilde{\mathfrak{x}}}(\omega)} > 0,$ **C (iii):** $\overline{\rho}^{\tilde{\mathfrak{x}}} := \frac{1}{2}\frac{\sup_{\omega\in\Omega}\overline{D}_2^{\tilde{\mathfrak{x}}}(\omega)}{\inf_{\omega\in\Omega}\underline{D}_1^{\tilde{\mathfrak{x}}}(\omega)} < \infty.$

Remark C: If we consider an allocation that is uniformly interior as per Definition 8, and recall that by Assumption 3 the aggregate endowment is uniformly bounded, we can be certain that all consumption vectors lie in a compact subset of R^2_{++}; also, since (iii) (b) in Assumption 2 imposes twice continuous differentiability of the Bernoulli utility function of each agent, the first and second derivatives are uniformly bounded on compact subsets of R^2_{++}. It follows that if the Bernoulli function is independent of the state when young and when old then, under Assumptions 2-3, and the requirement that the allocation under consideration is uniformly interior, Assumptions C (ii) and C (iii) necessarily hold.

Note that $\left[\Delta x_{1,t(A),h}(\omega)\right]^2 \leq \| \Delta x_{t(A),h}(\omega) \|^2$ and $\Delta x_{1,t(A),h}(\omega)$ is $\mathcal{F}_{t(A)}$-measurable. Therefore, from the definitions, we have

$$\begin{aligned}
\underline{\rho}^{\tilde{\mathfrak{r}}} \left[\Delta x_{1,t(A),h}(\omega)\right]^2 &= \frac{1}{2}\frac{\inf_{\omega\in\Omega}\underline{D}_2^{\tilde{\mathfrak{r}}}(\omega)}{\sup_{\omega\in\Omega}\overline{D}_1^{\tilde{\mathfrak{r}}}(\omega)} \left[\Delta x_{1,t(A),h}(\omega)\right]^2 \\
&\leq -\int_A \frac{\frac{1}{2}\Delta x_{t(A),h}(\omega)\cdot D^2 u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega)\cdot \Delta x_{t(A),h}(\omega)}{\int_A u_1^{\tilde{x},A,h}(\omega') P^\pi(d\omega'|\omega^{t(A)})} P^\pi\left(d\omega|\omega^{t(A)}\right) \\
&\leq \frac{1}{2}\frac{\sup_{\omega\in\Omega}\overline{D}_2^{\tilde{\mathfrak{r}}}(\omega)}{\inf_{\omega\in\Omega}\underline{D}_1^{\tilde{\mathfrak{r}}}(\omega)}\int_A \| \Delta x_{t(A),h}(\omega) \|^2 P^\pi\left(d\omega|\omega^{t(A)}\right) \\
&= \overline{\rho}^{\tilde{\mathfrak{r}}}\int_A \| \Delta x_{t(A),h}(\omega) \|^2 P^\pi\left(d\omega|\omega^{t(A)}\right). \qquad (**)
\end{aligned}$$

With the preliminaries in place, we can move on to the results.

Observe that Assumption 3 and uniform interiority of the equilibrium allocation imply that Assumption C (i) holds.

Lemma C (i): *Under A.1-4 and A.C (i) and (ii), if $\hat{x}(A,h)\in X^{\overline{e}}_{A,h}$ and*

$$\int_A \frac{u_2^{\tilde{x},A,h}(\omega)}{\int_A u_1^{\tilde{x},A,h}(\omega') P^\pi(d\omega'|\omega^{t(A)})}\Delta x_{2,t(A)+1,h}(\omega) P^\pi\left(d\omega|\omega^{t(A)}\right) + \Delta x_{1,t(A),h}(\omega) \geq \overline{\delta} > 0,$$

then there exists an $\epsilon > 0$ such that

$$V\left(\tilde{x}(A,h) + \epsilon\cdot\Delta x(A,h); A,h\right) > V\left(\tilde{x}(A,h); A,h\right).$$

Proof: A.3 implies that $\| \Delta x_{t(A),h}(\omega) \| \leq \sqrt{2}\overline{e}$ uniformly. So the last term in $(**)$ is bounded above by $\overline{\rho}^{\tilde{\mathfrak{r}}}\cdot 2(\overline{e})^2$.

We can combine $(*)$ with the second inequality in $(**)$ to conclude that

$$\begin{aligned}
&\int_A \left[u^{\tilde{x},\epsilon,\Delta x,A,h}(\omega) - u^{\tilde{x},A,h}(\omega)\right] P^\pi\left(d\omega|\omega^{t(A)}\right) \\
&\qquad \geq \left\{\int_A u_1^{\tilde{x},A,h}(\omega) P^\pi\left(d\omega|\omega^{t(A)}\right)\right\}\cdot\epsilon\cdot\left\{\overline{\delta} - \epsilon\cdot\overline{\rho}^{\tilde{\mathfrak{r}}}\cdot 2(\overline{e})^2\right\}.
\end{aligned}$$

The first term on the right is positive since u is strictly increasing and differentiable. Hence, by choosing $\epsilon > 0$ to be sufficiently small, it is possible to ensure that the last term is positive thus completing the proof. ∎

Lemma C (ii): *Under A.1-4 and A.C (i) and (iii), if* $\hat{x}(A,h) \in X^{\overline{e}}_{A,h}$ *and*

$$V\left(\hat{x}\left(A,h\right);A,h\right) \geq V\left(\tilde{x}\left(A,h\right);A,h\right),$$

then

$$\int_A \frac{u_2^{\tilde{x},A,h}\left(\omega\right)}{\int_A u_1^{\tilde{x},A,h}\left(\omega'\right) P^{\pi}\left(d\omega'|\omega^{t(A)}\right)} \Delta x_{2,t(A)+1,h}\left(\omega\right) P^{\pi}\left(d\omega|\omega^{t(A)}\right) + \Delta x_{1,t(A),h}\left(\omega\right) - \underline{\rho}^{\tilde{\mathfrak{r}}}\left[\Delta x_{1,t(A),h}\left(\omega\right)\right]^2 \geq 0.$$

Proof: Under the stated hypothesis, for $\omega \in A$, we have

$$\int_A \left[u^{\tilde{x},1,\Delta x,A,h}\left(\omega\right) - u^{\tilde{x},A,h}\left(\omega\right)\right] P^{\pi}\left(d\omega|\omega^{t(A)}\right) \geq 0.$$

By combining $(*)$ with the first inequality in $(**)$, and using the fact that u is strictly increasing and differentiable, we can conclude that the required inequality holds. ∎

References

AIYAGARI, S. R. AND D. PELED (1991): "Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Monetary Equilibria in Stochastic OLG Models," *Journal of Economic Theory* 54, 69-83.

BALASKO, Y. AND K. SHELL (1980): "The Overlapping Generations Model: I. The Case of Pure Exchange without Money," *Journal of Economic Theory*, 23, 281-306.

BARBIE, M., M. HAGEDORN, AND A. KAUL (2007): "On the Interaction between Risk Sharing and Capital Accumulation in a Stochastic OLG Model with Production," *Journal of Economic Theory* 137, 568-579.

BARBIE, M. AND M. HILLEBRAND (2017): "Bubbly Markov Equilibria," Working Paper.

BARBIE, M. AND A. KAUL (2015): "Pareto Optimality and Existence of Monetary Equilibria in a Stochastic OLG Model: A Recursive Approach," Working Paper.

BIRKHOFF, G. (1957): "Extensions of Jentzsch's Theorem," *Transactions of the American Mathematical Society* 85, 219-227.

BIRKHOFF, G. (1962): "Uniformly Semi-primitive Multiplicative Processes," *Transactions of the American Mathematical Society* 104, 37-51.

BLOISE, G. AND F. CALCIANO (2008): "A Characterization of Inefficiency in Stochastic Overlapping Generations Economies," *Journal of Economic Theory* 143, 442-468.

CASS, D. (1972): "On Capital Overaccumulation in the Aggregative Neoclassical Model of Economic Growth: A Complete Characterization," *Journal of Economic Theory* 4, 200-223.

CHATTOPADHYAY, S. (2001): "The Unit Root Property and Optimality: A Simple Proof," *Journal of Mathematical Economics* 36, 151-159.

CHATTOPADHYAY, S. AND P. GOTTARDI (1999): "Stochastic OLG Models, Market Structure, and Optimality," *Journal of Economic Theory* 89, 21-67.

CITANNA, A. AND P. SICONOLFI (2010): “Recursive Equilibrium in Stochastic Overlapping-Generations Economies,” *Econometrica*, 78, 309-347.

DEMANGE, G. AND G. LAROQUE (1999): “Social Security and Demographic Shocks,” *Econometrica* 67, 527-542.

DEMANGE, G. AND G. LAROQUE (2000): “Social Security, Optimality, and Equilibria in a Stochastic Overlapping Generations Economy,” *Journal of Public Economic Theory* 2, 1-23.

EVESON, S. (1995): “Hilbert’s Projecive Metric and the Spectral Properties of Positive Linear Operators,” *Proceedings of the London Mathematical Society*, 70, 411-440.

GIKHMAN, I. I. AND A. V. SKOROKHOD (1996): “Introduction to the Theory of Random Processes,” Dover Publications, New York.

HANSEN, L. P. AND J. A. SCHEINKMAN (2009): “Long-Term Risk: An Operator Approach,” *Econometrica* 77, 177-234.

KOHLBERG, E. AND J. W. PRATT (1982): “The Contraction Mapping Approach to the Perron-Frobenius Theory: Why Hilbert’s Matric,” *Mathematics of Operations Research* 7, 198-210.

KOPP, E., J. MALCZAK, AND T. ZASTAWNIAK (2013): “Probability for Finance,” Cambridge University Press, Cambridge.

KNOPP, K. (1990): “Theory and Application of Infinite Series,” Dover Publications, New York.

LUCAS, R. E. (1972): “Expectations and the Neutrality of Money,” *Journal of Economic Theory*, 4, 103-124.

MANUELLI, R. (1990): “Existence and Optimality of Currency Equilibrium in Stochastic Overlapping Generations Models: the Pure Endowment Case,” *Journal of Economic Theory* 51, 268-294.

MONTRUCCHIO, L. (1998): “Thompson Metric, Contraction Property and Differentiability of Policy Functions,” *Journal of Economic Behavior and Organization* 33, 449-466.

MUENCH, T. J. (1977): “Optimality, the Interaction of Spot and Futures Markets, and the Non-neutrality of Money in the Lucas Model,” *Journal of Economic Theory* 15, 325-344.

OKUNO, M. AND I. ZILCHA (1980): “On the Efficiency of a Competitive Equilibrium in Infinite Horizon Monetary Economies,” *Review of Economic Studies* 42, 797-807.

QIN, L. AND V. LINETSKY (2016): “Positive Eigenfunctions of Markovian Pricing Operators: Hansen-Scheinkman Factorization, Ross Recovery, and Long-Term Pricing,” *Operations Research* 64, 99-117.

SPEAR, S. E. (1985): “Rational Expectations in the Overlapping Generations Model,” *Journal of Economic Theory*, 35, 251-275.

THOMPSON, A. C. (1963): “On Certain Contraction Mappings in a Partially Ordered Vector Space,” *Proceedings of the American Mathematical Society*, 14, 438-443.

ZABREIKO, P. P., M. A. KRASNOSEL’SKII, AND YU V. POKORNYI (1972): “On a Class of Positive Linear Operators,” *Functional Analysis Applications*, 5, 272-279.

ZILCHA, I. (1991): "Characterizing Efficiency in Stochastic Overlapping Generations Models.," *Journal of Economic Theory* 55, 1-16.