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Game options with gradual exercise and cancellation under proportional transaction costs

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ABSTRACT
Game (Israeli) options in a multi-asset market model with proportional transaction costs are studied in the case when the buyer is allowed to exercise the option and the seller has the right to cancel the option gradually at a mixed (or randomised) stopping time, rather than instantly at an ordinary stopping time. Allowing gradual exercise and cancellation leads to increased flexibility in hedging, and hence tighter bounds on the option price as compared to the case of instantaneous exercise and cancellation. Algorithmic constructions for the bid and ask prices, and the associated superhedging strategies and optimal mixed stopping times for both exercise and cancellation are developed and illustrated. Probabilistic dual representations for bid and ask prices are also established.
Keywords: game options, randomised stopping, transaction costs, optimal stopping.

1. Introduction

A game (i.e. Israeli) option is a contract between an option buyer and seller, which allows the buyer the right to exercise the option, and the seller the right to cancel the option at any time up to expiry. The payoff associated with such a game option is due at the earliest of the exercise and cancellation times. If the option is cancelled before it is exercised, then the buyer also receives additional compensation from the seller. Game options were first introduced by Kifer [16] and have been studied in a frictionless setting in a number of papers; for a survey of this work see Kifer [17]. Game options have proved to be important not only in their own right, but also because they underpin the theory for other traded derivatives such as convertible bonds or callable options; see e.g. Kallsen and Kühn [15], Kühn and Kyprianou [22], Bielecki et al. [3], Wang and Jin [40], or Kwok [24].

Transaction costs were first considered in the context of game options by Kifer [18], who extended the results established for American options by Roux and Zastawniak [31] in the case of a market with a single risky security. Kifer's work [18] has recently been generalised by Roux [29] for game options in discrete multi-asset models with proportional transaction costs. Due to a negative result by Dolinsky [9] that the superreplication price of a game option in continuous time under proportional transaction costs is the initial value of a trivial buy-and-hold strategy, both Kifer [18] and Roux [29] study game options in discrete time. This approach is also adopted in the present paper.

Consistently with the wider literature on game options, the papers by Kifer [18] and Roux [29] take it for granted that the option can only be exercised or cancelled instantaneously and in full, in other words, at an ordinary stopping time. This means that pricing and hedging involves non-convex optimization problems for both the buyer and seller. In this case, Kifer [18] and Roux [29] showed that the bid and ask prices can be computed algorithmically, as can optimal strategies for both the buyer and the seller. Moreover, they established probabilistic dual representations for the bid and ask prices. In common with American options in this setting, the dual representations involve so-called mixed (or randomised) stopping times (used before in various contexts by Baxter and Chacon [1], Chalasani and Jha [6], Bouchard and Temam [4] and many others).

In the present paper we allow increased flexibility for both the buyer and seller by permitting both exercise and cancellation to take place gradually, i.e. at a mixed stopping time, rather than instantaneously at an ordinary stopping time. Such flexibility is available to investors who hold a portfolio of options and may wish to manage their exposure by exercising or cancelling some of these options at different times.

In the presence of proportional transaction costs, gradual exercise and cancellation is closely linked to the notion of deferred solvency; this has already been studied in the context of American options with gradual exercise by Roux and Zastawniak [32]. In the presence of a large bid-ask spread on the underlying assets, for example in the event of temporary illiquidity in the market, an agent may become insolvent in the traditional sense at some time instant t, but still able to return to solvency at a later time by trading in a self-financing way. Allowing such deferred solvency positions, rather than insisting on immediate solvency at all times, also leads to increased flexibility in constructing hedging strategies for both the seller and buyer of a game option.

In this setting, i.e. for game options with gradual exercise and cancellation under transaction costs and deferred solvency, we establish algorithmic constructions of the bid and ask prices and of optimal hedging strategies for both the seller and buyer of the option. In doing so, we extend the results of Kifer [18] and Roux [29], which apply to game options that allow only instantaneous exercise and cancellation with immediate solvency.

It turns out that, in the presence of proportional transaction costs, allowing deferred solvency along with gradual exercise and cancellation for game options leads to tighter bid-ask spreads as compared to the case of instantaneous exercise and cancellation. The reason for this is that the sets of buyer and seller superhedging strategies under instantaneous exercise and cancellation can be embedded in the corresponding sets for gradual exercise and cancellation. Taking the minimum (in the case of the seller) and the maximum (for the buyer) then leads to a lower (sometimes strictly lower) ask price and higher (sometimes strictly higher) bid price under gradual exercise and cancellation, hence a tighter (sometimes strictly tighter) bid-ask spread.

It is also important to note that allowing gradual exercise and cancellation turns pricing and hedging for the buyer and seller of a game option into convex optimization problems, massively enhancing the efficiency of the pricing and hedging algorithms as compared with the non-convex case studied by Kifer [18] and Roux [29]. Furthermore, convexity facilitates the use of duality methods, and could potentially allow extending the linear vector optimisation techniques which were developed by Löhne and Rudloff [25] for European options.

The methods and results presented in this paper build on a large body of work for European and American options under transaction costs, including papers by Merton [26], Dermody and Rockafellar [8], Boyle and Vorst [5], Bensaid, Lesne, Pagès and

Scheinkman [2], Edirisinghe, Naik and Uppal [10], Jouini and Kallal [12], Kusuoka [23], Koehl, Pham and Touzi [20, 21], Stettner [36, 37], Perrakis and Lefoll [28], Rutkowski [34], Touzi [39], Kabanov [13], Jouini [11], Palmer [27], Chalasani and Jha [6], Kabanov and Stricker [14], Kociński [19], Schachermayer [35], Bouchard and Temam [4], Tokarz and Zastawniak [38], Chen, Palmer and Sheu [7], Roux, Tokarz and Zastawniak [30], Roux and Zastawniak [31, 32, 33], Löhne and Rudloff [25].

The paper proceeds as follows. Section 2 introduces the multi-asset model with proportional transaction costs and summarizes the results from Roux and Zastawniak [32] that will be used in this paper. Game options with gradual exercise and cancellation are introduced in Section 3. Pricing algorithms for the seller and buyer, together with dual representations, are presented in Sections 4 and 5. The relative tightness of bid-ask spreads under gradual and instantaneous exercise and cancellation is explored in Section 6. Proofs of results in Sections 4–6 are deferred to Section 8. An example is provided in Section 7.

2. Preliminaries

2.1. Many-asset model with proportional transaction costs

We consider the discrete-time market model with many assets (conveniently thought of as currencies) and proportional transaction costs introduced by Kabanov [13] and studied by Kabanov and Stricker [14], Schachermayer [35] and others.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t=0}^T$. We assume that Ω is finite, $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For each $t = 0, \ldots, T$, by Ω_t we denote the collection of atoms of $\mathcal{F}_t$, called the *nodes* of the associated tree model.

The market model contains d assets or currencies. At each trading date $t = 0, 1, \ldots, T$ and for each $k, j = 1, \ldots, d$, one unit of asset k can be obtained by exchanging $\pi_t^{jk} > 0$ units of asset j. We assume that the exchange rates π_t^{jk} are $\mathcal{F}_t$-measurable and $\pi_t^{jj} = 1$ for all t and j, k.

For each $t = 0, \ldots, T$ let $\mathcal{L}_t := \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}_t)$ be the collection of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables. We can identify elements of $\mathcal{L}_t$ with $\mathbb{R}^d$-valued functions on Ω_t. Any $x \in \mathcal{L}_t$ can be thought of as a portfolio with positions $x^1, \ldots, x^d$ in the d assets. We say that a portfolio $x \in \mathcal{L}_t$ can be *exchanged* into a portfolio $y \in \mathcal{L}_t$ at time t whenever there are $\mathcal{F}_t$-measurable random variables $\beta^{jk} \geq 0$, $j, k = 1, \ldots, d$ such that for all $k = 1, \ldots, d$

$$y^k = x^k + \sum_{j=1}^d \beta^{jk} - \sum_{j=1}^d \beta^{kj} \pi_t^{kj},$$

where β^{jk} represents the number of units of asset k received as a result of exchanging some units of asset j.

The *solvency cone* $\mathcal{K}_t \subseteq \mathcal{L}_t$ is the set of portfolios that are *solvent* at time t, i.e. those portfolios at time t that can be exchanged into portfolios with non-negative positions in all d assets. It follows that $\mathcal{K}_t$ is the polyhedral convex cone generated by the canonical basis $e^1, \ldots, e^d$ of $\mathbb{R}^d$ and the vectors $\pi_t^{jk} e^j - e^k$ for $j, k = 1, \ldots, d$. We also refer to $\mathcal{K}_t$ as the *immediate solvency cone* to distinguish it from the so-called *deferred solvency cone* $\mathcal{Q}_t$ to be introduced later.

A *trading strategy* $y = (y_t)_{t=0}^{T+1}$ is a predictable $\mathbb{R}^d$-valued process with final value assumed to be $y_{T+1} = 0$ for notational convenience. For each $t > 0$ the portfolio $y_t \in \mathcal{L}_{t-1}$ is held from time $t-1$ to time t, and y_0 is the initial endowment. We denote by Φ the set of such trading strategies.

A trading strategy $y \in \Phi$ is said to be *self-financing* whenever

$$y_t - y_{t+1} \in \mathcal{K}_t \text{ for all } t = 0, \dots, T-1. \tag{1}$$

Note that no implicitly assumed self-financing condition is included in the definition of Φ.

A trading strategy $y \in \Phi$ is called an *arbitrage opportunity* if it is self-financing, $y_0 = 0$ and there is a portfolio $x \in \mathcal{L}_T \setminus \{0\}$ with $x^j \geq 0$ for each $j = 1, \dots, d$ and such that $y_T - x \in \mathcal{K}_T$. This notion of arbitrage was considered by Schachermayer [35]. The absence of arbitrage in this sense is formally different but equivalent to the weak no-arbitrage condition introduced by Kabanov and Stricker [14].

Theorem 2.1. [14, 35] *The model admits no arbitrage opportunity if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an $\mathbb{R}^d$-valued $\mathbb{Q}$-martingale $S = (S_t)_{t=0}^T$ such that*

$$S_t \in \mathcal{K}_t^* \setminus \{0\} \quad \textit{for all } t = 0, \dots, T, \tag{2}$$

where $\mathcal{K}_t^* := \{y \in \mathcal{L}_t \,|\, y \cdot x \geq 0 \textit{ for all } x \in \mathcal{K}_t\}$ is the polar of $-\mathcal{K}_t$.

We denote by $\mathcal{P}$ the set of pairs $(\mathbb{Q}, S)$ satisfying the conditions in Theorem 2.1, and by $\bar{\mathcal{P}}$ the set of pairs $(\mathbb{Q}, S)$ satisfying the conditions in Theorem 2.1 but with $\mathbb{Q}$ absolutely continuous with respect to (and not necessarily equivalent to) $\mathbb{P}$. We assume for the remainder of this paper that the model admits no arbitrage opportunities, i.e. $\mathcal{P} \neq \varnothing$.

Any portfolio $x \in \mathcal{K}_t$ is immediately solvent at time t, in the sense that it can be converted at time t into one with non-negative positions in all d assets. For American and game options under transaction costs, the following weaker type of solvency also proves useful. We denote by $\mathcal{Q}_t$ the collection of portfolios $x \in \mathcal{L}_t$ such that there is a sequence $y_s \in \mathcal{L}_{s-1}$ for $s = t+1, \dots, T+1$ satisfying the conditions

$$x - y_{t+1} \in \mathcal{K}_t, \quad y_s - y_{s+1} \in \mathcal{K}_s \text{ for all } s = t+1, \dots, T, \quad y_{T+1} = 0.$$

We call such a sequence $y_{t+1}, \dots, y_{T+1}$ a *liquidation strategy* starting from x at time t.

The portfolios in $\mathcal{Q}_t$ are those that can eventually (though possibly not immediately at time t) be converted by means of a sequence of self-financing transactions into a portfolio with zero positions in all assets. An equivalent way of constructing the deferred solvency cones is to put

$$\mathcal{Q}_T := \mathcal{K}_T$$

and then

$$Q_t := \mathcal{Q}_{t+1} \cap \mathcal{L}_t + \mathcal{K}_t \text{ for } t = T-1, \dots, 0$$

by backward induction. It turns out that $\mathcal{Q}_t$ is a convex polyhedral cone. We call it the *deferred solvency cone*. See [32] for more information on deferred solvency.

2.2. Mixed stopping times

A *mixed* (or *randomised*) *stopping time* is a non-negative adapted process $\phi = (\phi_t)_{t=0}^T$ such that

$$\sum_{t=0}^{T} \phi_t = 1.$$

The collection of mixed stopping times will be denoted by $\mathcal{X}$.

For any $\phi \in \mathcal{X}$ we put

$$\phi_t^* := \sum_{s=t}^{T} \phi_s \text{ for } t = 0, \ldots, T, \qquad \phi_{T+1}^* := 0. \tag{3}$$

Observe that ϕ^* is a predictable process since $\phi_0^* = 1$ is $\mathcal{F}_0$-measurable and $\phi_t^* = 1 - \sum_{s=0}^{t-1} \phi_s$ is $\mathcal{F}_{t-1}$-measurable for each $t = 1, \ldots, T$.

For example, in the case of a game option subject to gradual cancellation, ϕ_t could represent a fraction of the option that is cancelled at time t, whereas ϕ_t^* would be the part of the option that has not been cancelled before t.

For any adapted process X and for any $\phi \in \mathcal{X}$ we define the process X *evaluated at* ϕ as

$$X_\phi := \sum_{t=0}^{T} \phi_t X_t.$$

We also put

$$X_t^{\phi*} := \sum_{s=t}^{T} \phi_s X_s \text{ for } t = 0, \ldots, T, \qquad X_{T+1}^{\phi*} := 0.$$

The collection $\mathcal{T}$ of ordinary stopping times can be embedded in $\mathcal{X}$ by identifying every $\tau \in \mathcal{T}$ with the mixed stopping time $\chi^\tau \in \mathcal{X}$ defined as

$$\chi_t^\tau := \mathbf{1}_{\{t=\tau\}}$$

for each $t = 0, \ldots, T$. (Here $\mathbf{1}_A$ denotes the indicator function of an event $A \in \mathcal{F}$.)

2.3. American options with gradual exercise and cancellation

Here we collect the main notions and results concerning American options with gradual exercise and cancellation under proportional transaction costs; for full details, see [32]. These will be extended to game options, and will also be used as tools to establish some key results for this extension.

Consider an American option with adapted payoff process $Z = (Z_t)_{t=0}^T$, where $Z_t \in \mathcal{L}_t$ represents a portfolio of d assets for each $t = 0, \ldots, T$. If the buyer of the option is allowed to exercise it gradually according to a mixed stopping time $\psi \in \mathcal{X}$, then the sequence of portfolios $\psi_t Z_t$ is to be delivered by the option seller to the buyer at times $t = 0, \ldots, T$.

The seller needs to hedge against all mixed stopping times $\psi \in \mathcal{X}$ that can be chosen by the buyer. Because the seller can react to the buyer's choice of ψ, the hedging strategy may depend on ψ. We are going to write z_t^ψ for the time t position in the strategy.

On each trading date t, the seller needs to deliver the payoff $\psi_t Z_t$ and to rebalance the strategy from z_t^ψ to z_{t+1}^ψ without injecting any additional wealth, and can only use knowledge of ψ and the market up to and including time t. This leads to the following conditions.

Definition 2.2. Let $Z = (Z_t)_{t=0}^T$ be an adapted process. For an American option with payoff process Z and gradual exercise, a *seller's superhedging strategy* is a mapping $z : \mathcal{X} \to \Phi$ that satisfies the *rebalancing* condition

$$\forall \psi \in \mathcal{X}\ \forall t = 0, \dots, T : \ z_t^\psi - \psi_t Z_t - z_{t+1}^\psi \in \mathcal{K}_t \tag{4}$$

and the *non-anticipation* condition

$$\forall \psi, \psi' \in \mathcal{X}\ \forall t = 0, \dots, T : \ \textstyle\bigcap_{s=0}^{t-1} \{\psi_s = \psi'_s\} \subseteq \{z_t^\psi = z_t^{\psi'}\}. \tag{5}$$

The family of such strategies will be denoted by $\Psi^{\mathrm{a}}(Z)$.

Definition 2.3. The *seller's* (or *ask*) *price* in currency $j = 1, \dots, d$ of an American option with payoff process Z and gradual exercise is defined as

$$p_j^{\mathrm{a}}(Z) := \inf \left\{ x \in \mathbb{R} \,|\, \exists z \in \Psi^{\mathrm{a}}(Z) : x e^j = z_0 \right\}.$$

The following representation of the seller's price was obtained in [32]. In this representation, for any $\psi \in \mathcal{X}$, we denote by $\bar{\mathcal{P}}_j^{\mathrm{d}}(\psi)$ the collection of pairs $(\mathbb{Q}, S)$ such that $\mathbb{Q}$ is a probability measure absolutely continuous with respect to $\mathbb{P}$ and S is an $\mathbb{R}^d$-valued adapted process such that

$$S_t^j = 1, \qquad S_t \in \mathcal{Q}_t^* \setminus \{0\} \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}(S_{t+1}^{\psi *} | \mathcal{F}_t) \in \mathcal{Q}_t^* \quad \text{for all } t = 0, \dots, T,$$

where $\mathcal{Q}_t^* := \{y \in \mathcal{L}_t \,|\, y \cdot x \geq 0 \text{ for all } x \in \mathcal{Q}_t\}$ is the polar of $-\mathcal{Q}_t$.

Theorem 2.4. [32, Theorem 4.2] *The seller's price in currency $j = 1, \dots, d$ of an American option with payoff process Z and gradual exercise can be expressed as*

$$p_j^{\mathrm{a}}(Z) = \max_{\psi \in \mathcal{X}} \max_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}_j^{\mathrm{d}}(\psi)} \mathbb{E}_{\mathbb{Q}}((Z \cdot S)_\psi).$$

3. Game options with gradual exercise and cancellation

A game option as introduced in [16] and studied in [29] is a contract between an option buyer and seller, which gives the buyer the right to exercise the option at any stopping time $\tau \in \mathcal{T}$, and also gives the seller the right to cancel the option at any stopping time $\sigma \in \mathcal{T}$. There are two adapted processes $Y = (Y_t)_{t=0}^T$ and $X = (X_t)_{t=0}^T$ which determine, respectively, the payoffs due when exercising and cancelling the option. In the presence of transaction costs Y and X are $\mathbb{R}^d$-valued (i.e. portfolio-valued) processes; see [18] in the case when $d = 2$ and [29] for any $d \geq 2$. The seller has to deliver the portfolio Y_τ to the buyer at time τ when $\sigma \geq \tau$ or the portfolio X_σ at time σ when $\sigma < \tau$. That is, the option will be terminated at time $\sigma \wedge \tau$ due to exercise or cancellation, and the portfolio

$$Q_{\sigma,\tau} := \mathbf{1}_{\{\sigma \geq \tau\}} Y_\tau + \mathbf{1}_{\{\sigma < \tau\}} X_\sigma \tag{6}$$

will be changing hands at that time. Observe that exercising the option takes priority over cancellation when $\sigma = \tau$. Additionally, it is assumed that

$$X_t - Y_t \in \mathcal{K}_t \quad \text{for each } t = 0, \dots, T. \tag{7}$$

The difference $X_t - Y_t$ can be regarded as a penalty payable by the seller on top of the payoff Y_t when cancelling the option. We shall refer to an option of this kind as a *game* (or *Israeli*) *option with instant exercise and cancellation*, to distinguish it from one with gradual exercise and cancellation as described below.

In the present work we allow both the buyer and seller the freedom to exercise or, respectively, to cancel the option gradually according to mixed stopping times. If the buyer chooses a mixed stopping time $\psi \in \mathcal{X}$ as the exercise time and the seller selects a mixed stopping time $\phi \in \mathcal{X}$ to be the cancellation time, then on each trading date $t = 0, \dots, T$ the buyer will first be exercising a fraction ψ_t / ψ_t^* of the current position in the option, and then the seller will be cancelling a fraction ϕ_t / ϕ_t^* of the remaining position in the option, where ψ_t^* and ϕ_t^* are given by (3). Once again, exercising takes priority over cancellation.

In these circumstances, starting with an initial position of $\psi_0^* \phi_0^* = 1$ option at time 0, we are going to show by induction that $\psi_t^* \phi_t^*$ of the option will neither be exercised nor cancelled before time t, for each $t = 0, \dots, T$. It means that ψ_t / ψ_t^* of the current position $\psi_t^* \phi_t^*$, that is, $(\psi_t / \psi_t^*)\, \psi_t^* \phi_t^* = \psi_t \phi_t^*$ of the option will be exercised at t, given that the buyer has priority to exercise. The remaining position in the option will then be $\psi_t^* \phi_t^* - \psi_t \phi_t^* = (\psi_t^* - \psi_t)\, \phi_t^* = \psi_{t+1}^* \phi_t^*$, hence $(\phi_t / \phi_t^*)\, \psi_{t+1}^* \phi_t^* = \psi_{t+1}^* \phi_t$ of the option will be cancelled by the seller at t. Altogether, $\psi_t \phi_t^* + \psi_{t+1}^* \phi_t$ of the option will be terminated at t due to exercise or cancellation, leaving

$$\begin{aligned} \psi_t^* \phi_t^* - \psi_t \phi_t^* - \psi_{t+1}^* \phi_t &= \left(\psi_t + \psi_{t+1}^*\right) \left(\phi_t + \phi_{t+1}^*\right) - \psi_t \left(\phi_t + \phi_{t+1}^*\right) - \psi_{t+1}^* \phi_t \\ &= \psi_{t+1}^* \phi_{t+1}^* \end{aligned}$$

of the option neither exercised nor cancelled before or at t, to be carried forward to time $t+1$. This completes the induction.

Remark 1. The *minimum* $\psi \wedge \phi$ of mixed stopping times $\psi, \phi \in \mathcal{X}$ can be defined as

$$(\psi \wedge \phi)_t := \psi_t \phi_t^* + \psi_{t+1}^* \phi_t$$

for each $t = 0, \dots, T$; see [18]. The above argument shows that a game option with gradual exercise and cancellation will be terminated according to the mixed stopping time $\psi \wedge \phi$.

On each trading date $t = 0, \dots, T$, since $\psi_t \phi_t^*$ of the option is to be exercised and $\psi_{t+1}^* \phi_t$ of the option is to be cancelled, the seller will be delivering to the buyer the portfolio

$$G_t^{\phi,\psi} := \psi_t \phi_t^* Y_t + \psi_{t+1}^* \phi_t X_t, \tag{8}$$

where $Y = (Y_t)_{t=0}^T$ and $X = (X_t)_{t=0}^T$ are the exercise and cancellation processes characterising the game option, that is, $\mathbb{R}^d$-valued adapted processes that satisfy (7). Clearly, $G^{\phi,\psi} = (G_t^{\phi,\psi})_{t=0}^T$ is an $\mathbb{R}^d$-valued adapted process, which we shall be referring to as the *payoff process* for the game option.

Definition 3.1. A *game* (or *Israeli*) *option* (Y, X) *with gradual exercise and cancellation* is a derivative security that can be exercised according to a mixed stopping time $\psi \in \mathcal{X}$ chosen by the buyer or cancelled according to a mixed stopping time $\phi \in \mathcal{X}$ chosen by the seller, giving the buyer the right to receive and obliging the seller to deliver the portfolio $G_t^{\phi,\psi} = \psi_t \phi_t^* Y_t + \psi_{t+1}^* \phi_t X_t$ on each trading date $t = 0, \ldots, T$.

Remark 2. In contrast to the above payoff process $G_t^{\phi,\psi}$, Kifer [18] refers to the random variable

$$Q_{\phi,\psi} := \sum_{s=0}^{T} \sum_{t=0}^{T} \phi_s \psi_t Q_{s,t}$$

as the 'payoff' of a game option with exercise and cancellation according to mixed stopping times $\phi, \psi \in \mathcal{X}$, without specifying the time instant when this portfolio should be changing hands. However, the payoff of such an option should not be a single random variable but in fact an adapted process representing the flow of portfolios to be delivered on each trading date $t = 0, \ldots, T$. We observe that

$$Q_{\phi,\psi} = \sum_{t=0}^{T} G_t^{\phi,\psi},$$

i.e. $Q_{\phi,\psi}$ happens to be the total of all the $G_t^{\phi,\psi}$ for $t = 0, \ldots, T$.

In the present paper $Q_{\phi,\psi}$ will prove useful in a different role. Namely, identifying the mixed stopping time $\chi^t \in \mathcal{X}$ with a deterministic time t, we are going to use Q_{ϕ,χ^t} for $t = 0, \ldots, T$ as the payoff process of an American option with gradual exercise and invoke the results of [32] to establish a probabilistic representation of the seller's price for a game option under gradual exercise and cancellation; see Lemma 4.8 and Theorem 4.9. Similarly, in the buyer's case, we are going to use an American option with gradual exercise and payoff process $-Q_{\chi^t,\psi}$ for $t = 0, \ldots, T$; see Lemma 5.8 and Theorem 5.9.

4. Seller's price and superhedging strategies

The seller of a game option (Y, X) with gradual exercise and cancellation needs to hedge against any mixed stopping time $\psi \in \mathcal{X}$ chosen by the buyer to exercise the option. The seller can do this by following a trading strategy $u^\psi = (u_t^\psi)_{t=0}^T \in \Phi$, which may depend on ψ. Since u_t^ψ denotes a portfolio held over time step t, that is, between times $t-1$ and t, it follows that u_t^ψ may depend on the values $\psi_0, \ldots, \psi_{t-1}$ known to the seller at time $t-1$, when this portfolio is to be created, but not on the yet unknown (to the seller) values $\psi_t, \dots, \psi_T$. This is the reason for the non-anticipation condition (10) in Definition 4.1.

In addition to choosing the trading strategy $u^\psi \in \Phi$, the seller can select a mixed stopping time $\phi \in \mathcal{X}$ to cancel the option, and must be able to deliver the portfolio $G_t^{\phi,\psi}$ on each date $t = 0, \dots, T$ without injecting any additional wealth into the strategy. This justifies the rebalancing condition (9).

Definition 4.1. For a game option (Y, X) with gradual exercise and cancellation, a *seller's superhedging strategy* is a pair (ϕ, u), where $\phi \in \mathcal{X}$ and $u : \mathcal{X} \to \Phi$, that satisfies the *rebalancing* condition

$$\forall \psi \in \mathcal{X}\ \forall t = 0, \dots, T: \ u_t^\psi - G_t^{\phi,\psi} - u_{t+1}^\psi \in \mathcal{K}_t \tag{9}$$

and the *non-anticipation* condition

$$\forall \psi, \psi' \in \mathcal{X}\ \forall t = 0, \dots, T: \ \textstyle\bigcap_{s=0}^{t-1} \{\psi_s = \psi'_s\} \subseteq \{u_t^\psi = u_t^{\psi'}\}. \tag{10}$$

The family of such strategies will be denoted by $\Phi^{\mathrm{a}}(Y, X)$.

The least expensive (in a particular currency j) seller's superhedging strategy gives rise to the seller's price of the option.

Definition 4.2. The *seller's* (or *ask*) *price* in currency $j = 1, \dots, d$ of a game option (Y, X) with gradual exercise and cancellation is defined as

$$\pi_j^{\mathrm{a}}(Y, X) := \inf \left\{ x \in \mathbb{R} \,|\, \exists (\phi, u) \in \Phi^{\mathrm{a}}(Y, X) : xe^j = u_0 \right\}.$$

When comparing Definition 4.1 above with Definition 3.2 of [29], one can see that the set of seller's superhedging strategies under instantaneous exercise and cancellation can be embedded in the set $\Phi^{\mathrm{a}}(Y, X)$. Therefore the ask price under gradual exercise and cancellation must be lower under gradual exercise and cancellation, as will be shown in detail in Section 6. The example in Section 7 will demonstrate that it can be strictly lower.

4.1. Seller's pricing algorithm

The following is an iterative construction of the set of initial endowments that allow superhedging the seller's position in a game option with gradual exercise and cancellation.

Construction 4.3. *Construct adapted sequences* $\mathcal{Y}_t^{\mathrm{a}}, \mathcal{X}_t^{\mathrm{a}}, \mathcal{V}_t^{\mathrm{a}}, \mathcal{W}_t^{\mathrm{a}}, \mathcal{Z}_t^{\mathrm{a}}$ *for* $t = 0, \dots, T$ *as follows. First, put*

$$\mathcal{Y}_t^{\mathrm{a}} := Y_t + \mathcal{Q}_t, \quad \mathcal{X}_t^{\mathrm{a}} := X_t + \mathcal{Q}_t$$

for all $t = 0, \dots, T$ *and*

$$\mathcal{W}_T^{\mathrm{a}} := \mathcal{V}_T^{\mathrm{a}} := \mathcal{L}_T, \quad \mathcal{Z}_T^{\mathrm{a}} := \mathcal{Y}_T^{\mathrm{a}}.$$

Then, for $t = T-1, \dots, 0$ define by backward induction

$$\begin{aligned} \mathcal{W}_t^{\mathrm{a}} &:= \mathcal{Z}_{t+1}^{\mathrm{a}} \cap \mathcal{L}_t, \\ \mathcal{V}_t^{\mathrm{a}} &:= \mathcal{W}_t^{\mathrm{a}} + \mathcal{Q}_t, \\ \mathcal{Z}_t^{\mathrm{a}} &:= \operatorname{conv}\{\mathcal{V}_t^{\mathrm{a}}, \mathcal{X}_t^{\mathrm{a}}\} \cap \mathcal{Y}_t^{\mathrm{a}}, \end{aligned}$$

where $\operatorname{conv}\{\mathcal{V}_t^{\mathrm{a}}, \mathcal{X}_t^{\mathrm{a}}\}$ *is the convex hull of* $\mathcal{V}_t^{\mathrm{a}}$ *and* $\mathcal{X}_t^{\mathrm{a}}$*.*

By a similar argument as in the proof of Proposition 5.1 in [32], it follows that $\mathcal{Z}_t^{\mathrm{a}}$ are polyhedral convex sets for all t. We shall see that $\mathcal{Z}_0^{\mathrm{a}}$ is the set of initial endowments that allow the seller to superhedge their position in the game option (Y, X) with gradual exercise and cancellation. Once $\mathcal{Z}_0^{\mathrm{a}}$ has been constructed, the following result can be used to obtain the seller's price of the option.

Theorem 4.4. *The seller's price in currency $j = 1, \dots, d$ of a game option (Y, X) with gradual exercise and cancellation can be expressed as*

$$\pi_j^{\mathrm{a}}(Y, X) = \min\left\{x \in \mathbb{R} \,|\, xe^j \in \mathcal{Z}_0^{\mathrm{a}}\right\}.$$

To prove this theorem, we introduce an auxiliary family $\Lambda^{\mathrm{a}}(Y, X)$, the elements of which can be thought of as the strategies superhedging the seller's position in a game option with gradual cancellation, instant (rather than gradual) exercise and deferred (rather than immediate) solvency. The theorem and the following propositions are proved in the Appendix, Section 8.

Definition 4.5. We define $\Lambda^{\mathrm{a}}(Y, X)$ as the family consisting of all pairs (ϕ, z), where $\phi \in \mathcal{X}$ and $z \in \Phi$, that satisfy the conditions

$$\begin{aligned} z_t - \phi_t X_t - z_{t+1} \in \mathcal{Q}_t \quad &\text{for all } t = 0, \dots, T-1, \\ z_t - \phi_t^* Y_t \in \mathcal{Q}_t \quad &\text{for all } t = 0, \dots, T. \end{aligned}$$

According to the next proposition, $\mathcal{Z}_0^{\mathrm{a}}$ coincides with the set of initial endowments for the strategies in $\Lambda^{\mathrm{a}}(Y, X)$.

Proposition 4.6.

$$\mathcal{Z}_0^{\mathrm{a}} = \left\{z_0 \in \mathbb{R}^d \,|\, (\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)\right\}.$$

We also claim that the set of initial endowments for the strategies in $\Lambda^{\mathrm{a}}(Y, X)$ coincides with that for the strategies in $\Phi^{\mathrm{a}}(Y, X)$.

Proposition 4.7.

$$\left\{z_0 \in \mathbb{R}^d \,|\, (\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)\right\} = \left\{u_0 \in \mathbb{R}^d \,|\, (\phi, u) \in \Phi^{\mathrm{a}}(Y, X)\right\}.$$

It follows from Propositions 4.6 and 4.7 that $\mathcal{Z}_0^{\mathrm{a}}$ is the family of initial endowments for all strategies superhedging the seller's position in a game option with gradual exercise and cancellation. This is what's needed to prove Theorem 4.4, which links the seller's price $\pi_j^{\mathrm{a}}(Y, X)$ with $\mathcal{Z}_0^{\mathrm{a}}$. Full details can be found in the Appendix, Section 8.

4.2. Seller's price representation

In this section we obtain a dual representation of the seller's price for game options with gradual exercise and cancellation. This relies on a similar result established in [32] for American options with gradual exercise; see Theorem 2.4.

Observe that, by Definition 4.2,

$$\begin{aligned}\pi^{\mathrm{a}}_j(Y,X) &= \inf\left\{x\in\mathbb{R}\,|\,\exists(\phi,u)\in\Phi^{\mathrm{a}}(Y,X): xe^j=u_0\right\}\\ &= \inf_{\phi\in\chi}\inf\left\{x\in\mathbb{R}\,|\,\exists u:(\phi,u)\in\Phi^{\mathrm{a}}(Y,X), xe^j=u_0\right\}.\end{aligned}$$

Hence, as a consequence of Lemma 4.8 below, together with Definition 2.3, we have

$$\begin{aligned}\pi^{\mathrm{a}}_j(Y,X) &= \inf_{\phi\in\chi}\inf\left\{x\in\mathbb{R}\,|\,\exists z\in\Psi^{\mathrm{a}}(Q_{\phi,\cdot}): xe^j=z_0\right\}\\ &= \inf_{\phi\in\chi} p^{\mathrm{a}}_j(Q_{\phi,\cdot}),\end{aligned}$$

where $Q_{\phi,\cdot}=(Q_{\phi,t})_{t=0}^T$ with

$$Q_{\phi,t}:=Q_{\phi,\chi^t}\quad\text{for } t=0,\ldots,T$$

is the payoff process for an American option with gradual exercise, and where $p^{\mathrm{a}}_j(Q_{\phi,\cdot})$ is the seller's price of such an American option.

Lemma 4.8. For any $\phi\in\mathcal{X}$

$$\{u_0\,|\,(\phi,u)\in\Phi^{\mathrm{a}}(Y,X)\}=\{z_0\,|\,z\in\Psi^{\mathrm{a}}(Q_{\phi,\cdot})\},$$

where $Q_{\phi,\cdot}=(Q_{\phi,t})_{t=0}^T$ is the payoff process of an American option.

The lemma is proved in the Appendix, Section 8. It turns out that the infimum over $\phi\in\chi$ in

$$\pi^{\mathrm{a}}_j(Y,X)=\inf_{\phi\in\chi} p^{\mathrm{a}}_j(Q_{\phi,\cdot})$$

is, in fact, a minimum. Moreover, $p^{\mathrm{a}}_j(Q_{\phi,\cdot})$ can be represented as in Theorem 2.4. This leads to the following representation.

Theorem 4.9. *The seller's price in currency $j=1,\ldots,d$ of a game option (Y,X) with gradual exercise and cancellation can be represented as*

$$\pi^{\mathrm{a}}_j(Y,X)=\min_{\phi\in\mathcal{X}}\max_{\psi\in\mathcal{X}}\max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}^{\mathrm{d}}_j(\psi)}\mathbb{E}_{\mathbb{Q}}((Q_{\phi,\cdot}\cdot S)_\psi).$$

The details of the proof can be found, once again, in the Appendix, Section 8.

5. Buyer's price and superhedging strategies

The buyer of a game option (Y,X) will be able to select a mixed stopping time $\psi\in\mathcal{X}$ to exercise the option, and can follow a trading strategy $u^\phi=(u^\phi_t)_{t=0}^T\in\Phi$, which may depend on the cancellation time $\phi \in \mathcal{X}$ chosen by the seller. On each date $t = 0, \ldots, T$ the buyer will be taking delivery of the portfolio $G_t^{\phi,\psi}$ and can rebalance the current position u_t^ϕ in the strategy into u_{t+1}^ϕ in a self-financing way, i.e. without injecting any additional wealth. The portfolio u_t^ϕ created by the buyer at time $t-1$ may depend on the seller's cancellation strategy $\phi_0, \ldots, \phi_{t-1}$ up to and including time $t-1$, but not on the values $\phi_t, \ldots, \phi_T$, as these will not yet be known to the buyer at time $t-1$. These considerations lead to the following definition.

Definition 5.1. For a game option (Y, X) with gradual exercise and cancellation, a *buyer's superhedging strategy* is a pair (ψ, u), where $\psi \in \mathcal{X}$ and $u : \mathcal{X} \to \Phi$, that satisfies the *rebalancing* condition

$$\forall \phi \in \mathcal{X}\ \forall t = 0, \ldots, T:\ u_t^\phi + G_t^{\phi,\psi} - u_{t+1}^\phi \in \mathcal{K}_t \tag{11}$$

and the *non-anticipation* condition

$$\forall \phi, \phi' \in \mathcal{X}\ \forall t = 0, \ldots, T:\ \textstyle\bigcap_{s=0}^{t-1} \{\phi_s = \phi'_s\} \subseteq \{u_t^\phi = u_t^{\phi'}\}. \tag{12}$$

The family of such strategies will be denoted by $\Phi^{\mathrm{b}}(Y, X)$.

The buyer's price of the game option in currency j can be understood as the largest amount in that currency which can be raised against a long position in the option used as surety. The precise definition is as follows.

Definition 5.2. The *buyer's* (or *bid*) *price* in currency $j = 1, \ldots, d$ of a game option (Y, X) under gradual exercise and cancellation is defined as

$$\pi_j^{\mathrm{b}}(Y, X) := \sup \left\{ -x \in \mathbb{R} \,|\, \exists (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) : x e^j = u_0 \right\}.$$

Comparing Definition 5.1 above with Definition 3.4 of [29], one can observe that the set of buyer's superhedging strategies under instantaneous exercise and cancellation can be embedded in the set $\Phi^{\mathrm{b}}(Y, X)$, and thus the bid price is higher (sometimes strictly higher; see the example in Section 7) under gradual exercise and cancellation. This will be shown in Section 6, and the example in Section 7 will demonstrate that it can sometimes be strictly higher.

5.1. Buyer's pricing algorithm

As is well known, there is a symmetry between the buyer's and seller's superhedging and pricing problems for a European option. The symmetry consists, essentially, in reversing the sign of the payoff while also reversing the roles of buyer and seller. Hence, solving the seller's problem also yields a solution to the buyer's problem, and *vice versa*. However, for an American option this symmetry is broken, and one needs to solve the buyer's and seller's problems separately; see for example [32] or [33].

On first sight, it might appear that the symmetry between the buyer and seller might be restored in the case of a game option. However, in fact, this is not so when the buyer has priority to exercise the option before the seller can cancel it. Reversing their roles would give priority to the seller. Combined with condition (7), this breaks the symmetry, and so a specific solution to the buyer's problem is needed. This is facilitated by the following construction.

Construction 5.3. Construct adapted sequences $\mathcal{Y}^{\mathrm{b}}_t, \mathcal{X}^{\mathrm{b}}_t, \mathcal{V}^{\mathrm{b}}_t, \mathcal{W}^{\mathrm{b}}_t, \mathcal{Z}^{\mathrm{b}}_t$ for $t = 0, \dots, T$ as follows. First, put

$$\mathcal{Y}^{\mathrm{b}}_t := -Y_t + \mathcal{Q}_t, \quad \mathcal{X}^{\mathrm{b}}_t := -X_t + \mathcal{Q}_t$$

for all $t = 0, \dots, T$ and

$$\mathcal{W}^{\mathrm{b}}_T := \mathcal{V}^{\mathrm{b}}_T := \mathcal{L}_T, \quad \mathcal{Z}^{\mathrm{b}}_T := \mathcal{Y}^{\mathrm{b}}_T.$$

Then, for $t = T-1, \dots, 0$ define by backward induction

$$\begin{aligned} \mathcal{W}^{\mathrm{b}}_t &:= \mathcal{Z}^{\mathrm{b}}_{t+1} \cap \mathcal{L}_t, \\ \mathcal{V}^{\mathrm{b}}_t &:= \mathcal{W}^{\mathrm{b}}_t + \mathcal{Q}_t, \\ \mathcal{Z}^{\mathrm{b}}_t &:= \operatorname{conv}\{\mathcal{V}^{\mathrm{b}}_t \cap \mathcal{X}^{\mathrm{b}}_t, \mathcal{Y}^{\mathrm{b}}_t\}. \end{aligned}$$

As compared to the seller's Construction 4.3, apart from swapping the payoff processes Y, X for $-X, -Y$, which would have been enough had there been a simple symmetry between the buyer and seller, the operations of intersection and convex hull are taken in the reverse order in the last line of this construction.

The proofs of the results below concerning the buyer's case resemble those for the seller, but certain details follow a diverse pattern to account for the differences between the seller's and buyer's pricing constructions.

Just as in the seller's case, the same argument as in the proof of Proposition 5.1 in [32] shows that the $\mathcal{Z}^{\mathrm{b}}_t$ are polyhedral convex sets. Moreover, we shall see that $\mathcal{Z}^{\mathrm{b}}_0$ plays a similar role for the buyer as $\mathcal{Z}^{\mathrm{a}}_0$ does for the seller, namely it is the set of all initial endowments allowing the option buyer to superhedge their position. This leads to the following result.

Theorem 5.4. *The buyer's price in currency $j = 1, \dots, d$ of a game option (Y, X) with gradual exercise and cancellation can be expressed as*

$$\pi^{\mathrm{b}}_j(Y, X) = \max\left\{-x \in \mathbb{R} \,|\, xe^j \in \mathcal{Z}^{\mathrm{b}}_0\right\}.$$

To prove this theorem we need the following family $\Lambda^{\mathrm{b}}(Y, X)$, the elements of which can be seen as strategies superhedging the buyer's position in a game option with instant (rather than gradual) cancellation, gradual exercise and deferred (rather than immediate) solvency.

Definition 5.5. We define $\Lambda^{\mathrm{b}}(Y, X)$ as the family consisting of all pairs (ψ, z), where $\psi \in \mathcal{X}$ and $z \in \Phi$, that satisfy the conditions

$$\begin{aligned} z_t + \psi_t Y_t - z_{t+1} \in \mathcal{Q}_t &\quad \text{for all } t = 0, \dots, T-1, \\ z_t + \psi_t Y + \psi^*_{t+1} X_t \in \mathcal{Q}_t &\quad \text{for all } t = 0, \dots, T. \end{aligned}$$

The next two results are similar to Propositions 4.6 and 4.7. First, $\mathcal{Z}^{\mathrm{b}}_0$ is shown to be equal to the set of initial endowments for the strategies in $\Lambda^{\mathrm{b}}(Y, X)$.

Proposition 5.6.

$$\mathcal{Z}_0^{\mathrm{b}} = \left\{ z_0 \in \mathbb{R}^d \,|\, (\psi, z) \in \Lambda^{\mathrm{b}}(Y, X) \right\}.$$

The set of initial endowments for the strategies in $\Lambda^{\mathrm{b}}(Y, X)$ is then shown to coincide with that for the strategies in $\Phi^{\mathrm{b}}(Y, X)$.

Proposition 5.7.

$$\left\{ z_0 \in \mathbb{R}^d \,|\, (\psi, z) \in \Lambda^{\mathrm{b}}(Y, X) \right\} = \left\{ u_0 \in \mathbb{R}^d \,|\, (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) \right\}.$$

The proofs of these two propositions are in the Appendix, Section 8. Once these results have been established, proving that $\mathcal{Z}_0^{\mathrm{b}}$ is the set of initial endowments for the strategies in $\Phi^{\mathrm{b}}(Y, X)$, Theorem 5.4 follows; for details, see the proof in the Appendix, Section 8.

5.2. Buyer's price representation

In this section we obtain a representation of the buyer's price of a game option with gradual exercise, by exploiting a link with the price of an American option with gradual exercise and payoff process $-Q_{\cdot,\psi} = (-Q_{t,\psi})_{t=0}^T$ defined for any $\psi \in \chi$, where

$$Q_{t,\psi} := Q_{\chi^t,\psi} \quad \text{for } t = 0, \ldots, T.$$

Such a link is furnished by the next lemma.

Lemma 5.8. For any $\psi \in \mathcal{X}$

$$\left\{ u_0 \,|\, (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) \right\} = \left\{ z_0 \,|\, z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi}) \right\},$$

where $-Q_{\cdot,\psi} = (-Q_{t,\psi})_{t=0}^T$ is the payoff process of an American option.

With the aid of this lemma, in a similar manner as in Section 4.2, we can establish the following representation of the buyer's price.

Theorem 5.9. *The buyer's price in currency $j = 1, \ldots, d$ of a game option (Y, X) with gradual exercise and cancellation can be represented as*

$$\pi_j^{\mathrm{b}}(Y, X) = \max_{\psi \in \mathcal{X}} \min_{\phi \in \mathcal{X}} \min_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}_j^{\mathrm{d}}(\phi)} \mathbb{E}_{\mathbb{Q}}((Q_{\cdot,\psi} \cdot S)_\phi).$$

The proofs of Lemma 5.8 and Theorem 5.9 can be found in the Appendix, Section 8.

6. Comparison with instantaneous exercise and cancellation

We define the ask (seller's) and bid (buyer's) prices under instantaneous exercise and cancellation (see Definitions 3.3 and 3.5 in [29]) within the notational conventions of the present paper as

$$\hat{\pi}^{\mathrm{a}}_j(Y,X) := \inf\{x \in \mathbb{R} \,|\, \exists\,(\sigma,y) \in \mathcal{T}\times\Phi : y_0 = xe^j \text{ and } (\sigma,y) \text{ hedges } (Y,X) \text{ for the seller}\},$$
$$\hat{\pi}^{\mathrm{b}}_j(Y,X) := \sup\{-x \in \mathbb{R} \,|\, \exists\,(\sigma,y) \in \mathcal{T}\times\Phi : y_0 = xe^j \text{ and } (\sigma,y) \text{ hedges } (Y,X) \text{ for the buyer}\}.$$

Here, a pair (σ,y) is said to hedge (Y,X) for the seller under instantaneous exercise and cancellation if y is self-financing and

$$y_{\sigma\wedge\tau} - Q_{\sigma,\tau} \in \mathcal{K}_{\sigma\wedge\tau} \text{ for all } \tau \in \mathcal{T}. \tag{13}$$

A pair (σ,y) is said to hedge (Y,X) for the buyer under instantaneous exercise and cancellation if y is self-financing and

$$y_{\tau\wedge\sigma} + Q_{\tau,\sigma} \in \mathcal{K}_{\tau\wedge\sigma} \text{ for all } \tau \in \mathcal{T}. \tag{14}$$

The following result shows that bid-ask spreads under gradual exercise and cancellation are generally tigher than under instantaneous exercise and cancellation.

Theorem 6.1.

$$\pi^{\mathrm{a}}_j(Y,X) \le \hat{\pi}^{\mathrm{a}}_j(Y,X), \tag{15}$$
$$\pi^{\mathrm{b}}_j(Y,X) \ge \hat{\pi}^{\mathrm{b}}_j(Y,X). \tag{16}$$

The proof of Theorem 6.1 appears in the Appendix, Section 8. The example in Section 7 shows that bid-ask spreads under gradual exercise and cancellation can sometimes be strictly tighter than under instantaneous exercise and cancellation.

7. Example

A game option (Y,X) in a binary two-step two-currency model is presented in Figure 1. The model is recombinant; the option payoff is path-independent and has no cancellation penalties at time 2. The model has transaction costs only at the node u at time 1.

Constructions 3.1 and 3.4 of [29] give the bid-ask spread of the game option (Y,X) with instant exercise and cancellation in terms of currency 2 to be $[3.2,5]$. We will show below that the bid-ask spread of (Y,X) with gradual exercise and cancellation is

$$[\pi^{\mathrm{b}}_2(Y,X), \pi^{\mathrm{a}}_2(Y,X)] = [\tfrac{11}{3}, \tfrac{14}{3}] \approx [3.6667, 4.6667] \subset [3.2,5].$$

(Indeed the bid and ask prices of (Y,X) can be read off the vertical axes in Figures 3 and 5 below.) Thus gradual exercise and cancellation leads to a smaller bid-ask spread in this example.

Let us use Construction 4.3 to find the set $\mathcal{Z}^{\mathrm{a}}_0$ of initial endowments that allow the seller to superhedge (Y,X) with gradual exercise and cancellation. At time $t=2$ we

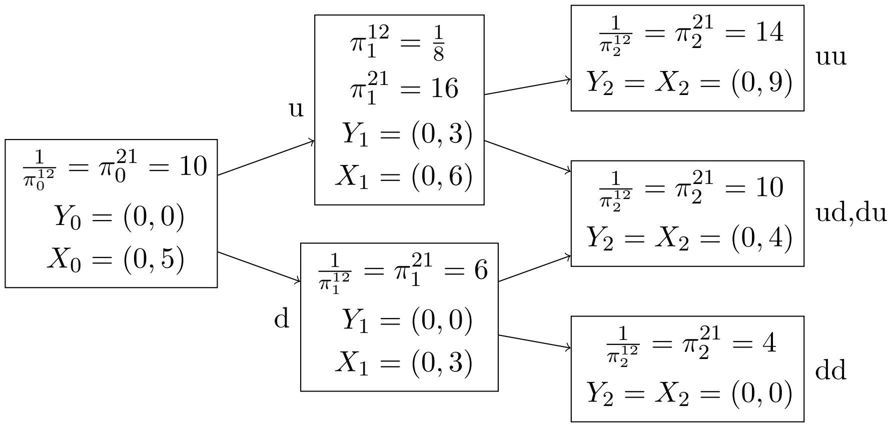

Figure 1. Game option in binary two-step two-currency model

have

$$\begin{aligned}
\mathcal{Z}_2^{\mathrm{auu}} &= \{(x^1, x^2) \in \mathbb{R}^2 : 14x^1 + x^2 \geq 9\}, \\
\mathcal{Z}_2^{\mathrm{aud}} = \mathcal{Z}_2^{\mathrm{adu}} &= \{(x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq 4\}, \\
\mathcal{Z}_2^{\mathrm{add}} &= \{(x^1, x^2) \in \mathbb{R}^2 : 4x^1 + x^2 \geq 0\}.
\end{aligned}$$

Figure 2 illustrates the construction at time $t = 1$ at the node u, which results in

$$\mathcal{Z}_1^{\mathrm{au}} = \{(x^1, x^2) \in \mathbb{R}^2 : 14x^1 + x^2 \geq 6, \tfrac{58}{5}x^1 + x^2 \geq 6, 10x^1 + x^2 \geq 4\}.$$

Similar considerations at the node d give

$$\mathcal{Z}_1^{\mathrm{ad}} = \{(x^1, x^2) \in \mathbb{R}^2 : 6x^1 + x^2 \geq \tfrac{4}{3}\}.$$

The construction at time $t = 0$ gives

$$\mathcal{Z}_0^{\mathrm{a}} = \{(x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq \tfrac{14}{3}\},$$

as illustrated in Figure 3.

A superhedging strategy for the seller starting from the initial endowment $(0, \pi_2^{\mathrm{a}}(Y, X))$ can be constructed by following similar lines as in the proof of Proposition 4.6 to assemble $(\phi, z^{\mathrm{a}}) \in \Lambda^{\mathrm{a}}(Y, X)$, and then converting it into a superhedging strategy $(\phi, u^{\mathrm{a}}) \in \Phi^{\mathrm{a}}(Y, X)$ using the arguments in the proof of Proposition 4.7. We illustrate the first part of the process here for the scenario uu. Define first $z_0^{\mathrm{a}} := \left(0, \frac{14}{3}\right)$; it is clear from Figure 3 that $z_0^{\mathrm{a}} \notin \mathcal{X}_0^{\mathrm{a}}$, leading to $\phi_0 := 0$. Choosing $z_1^{\mathrm{a}} := \left(\frac{5}{6}, -\frac{11}{3}\right) \in \mathcal{W}_0^{\mathrm{a}}$ then gives that

$$z_0^{\mathrm{a}} - \phi_0 X_0 - z_1^{\mathrm{a}} = z_0^{\mathrm{a}} - z_1^{\mathrm{a}} \in \mathcal{Q}_0.$$

Figure 2 shows that $z_1^{\mathrm{a}} \in \mathcal{Z}_1^{\mathrm{a}} \subset \mathcal{Y}_1^{\mathrm{a}}$. It also shows that defining $\phi_1^{\mathrm{u}} := \frac{2}{3}$ and $z_2^{\mathrm{au}} := \left(\frac{5}{4}, -\frac{17}{2}\right)$ leads to

$$z_1^{\mathrm{a}} = \phi_1^{\mathrm{u}}(0, 6) + (1 - \phi_1^{\mathrm{u}}) z_2^{\mathrm{au}},$$

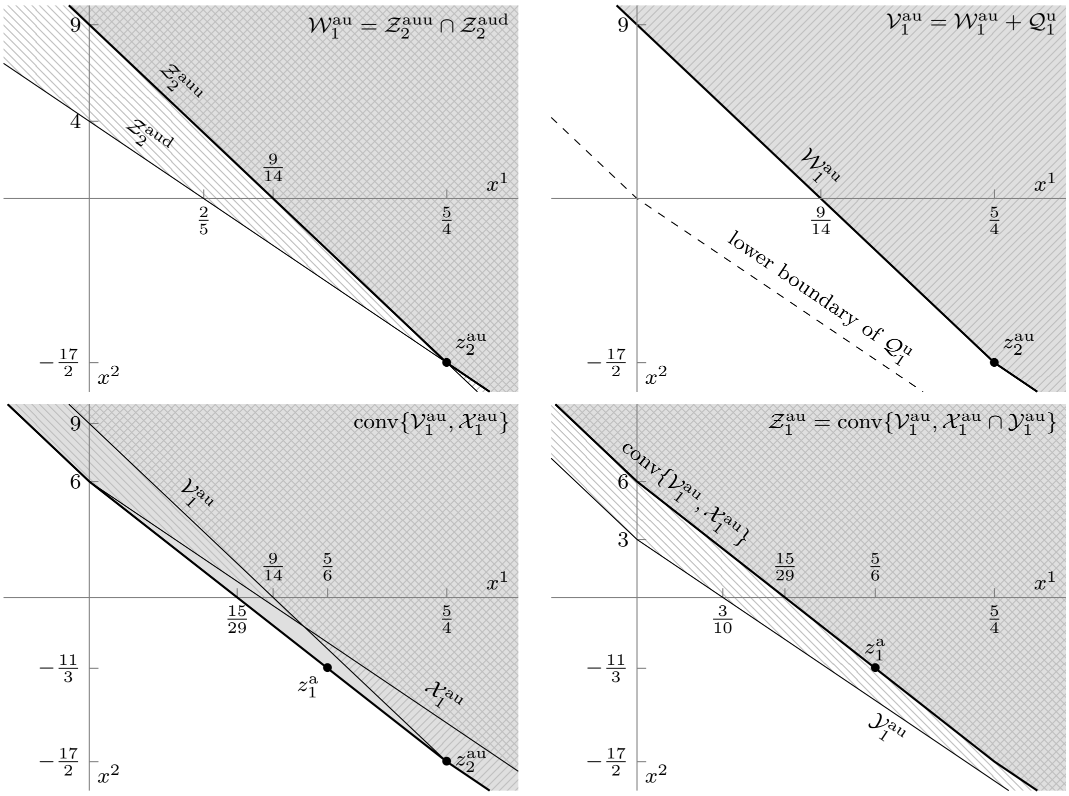

Figure 2. $\mathcal{W}_1^{\mathrm{au}}$, $\mathcal{V}_1^{\mathrm{au}}$, $\mathrm{conv}\{\mathcal{V}_1^{\mathrm{au}}, \mathcal{X}_1^{\mathrm{au}}\}$, $\mathcal{Z}_1^{\mathrm{au}}$, z_1^{a}, z_2^{au}

with $(0,6) \in \mathcal{X}_1^{\mathrm{au}}$ and $z_2^{\mathrm{au}} \in \mathcal{V}_1^{\mathrm{au}} = \mathcal{W}_1^{\mathrm{au}} \subset \mathcal{Z}_2^{\mathrm{auu}} = \mathcal{Y}_2^{\mathrm{auu}}$. Thus this strategy corresponds to cancellation of $\frac{1}{3}$ of the option at time 1 at the node u and the remaining $\phi_1^{\mathrm{uu}} := \frac{1}{3}$ at time 2.

The set of superhedging strategies for the buyer of (Y, X) can be computed by following Construction 5.3. At time $t = 2$,

$$\begin{aligned}
\mathcal{Z}_2^{\mathrm{buu}} &= \{(x^1, x^2) \in \mathbb{R}^2 : 14x^1 + x^2 \geq -9\}, \\
\mathcal{Z}_2^{\mathrm{bud}} &= \mathcal{Z}_2^{\mathrm{bdu}} = \{(x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq -4\}, \\
\mathcal{Z}_2^{\mathrm{bdd}} &= \{(x^1, x^2) \in \mathbb{R}^2 : 4x^1 + x^2 \geq 0\}.
\end{aligned}$$

The construction at time $t = 1$ at the node u gives

$$\mathcal{Z}_1^{\mathrm{bu}} = \{(x^1, x^2) \in \mathbb{R}^2 : 14x^1 + x^2 \geq -6, 10x^1 + x^2 \geq -4\},$$

as illustrated in Figure 4. Similar considerations at the node d result in

$$\mathcal{Z}_1^{\mathrm{bd}} = \{(x^1, x^2) \in \mathbb{R}^2 : 6x^1 + x^2 \geq -\tfrac{4}{3}\}.$$

Figure 5 demonstrates the construction at time $t = 0$, which leads to

$$\mathcal{Z}_0^{\mathrm{a}} = \{(x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq -\tfrac{11}{3}\}.$$

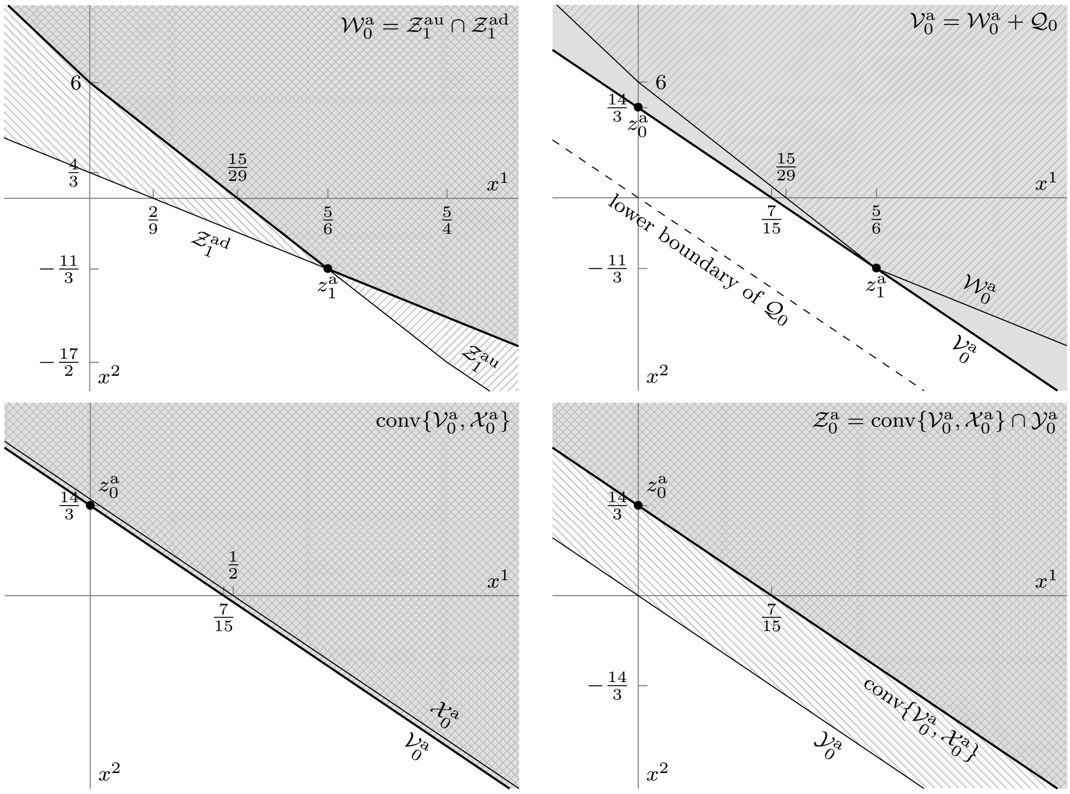

Figure 3. $\mathcal{W}_0^{\mathrm{a}}$, $\mathcal{V}_0^{\mathrm{a}}$, $\mathrm{conv}\{\mathcal{V}_0^{\mathrm{a}}, \mathcal{X}_0^{\mathrm{a}}\}$, $\mathcal{Z}_0^{\mathrm{a}}$, z_0^{a}, z_1^{a}

Similarly to the seller's case, the construction of a superhedging strategy for the buyer starting from the initial endowment $\big(0, -\pi_2^{\mathrm{b}}(Y,X)\big)$ involves two steps, namely assembling $(\psi, z^{\mathrm{b}}) \in \Lambda^{\mathrm{b}}(Y,X)$ using the construction in the proof of Proposition 5.6, and then converting it into a superhedging strategy $(\psi, u^{\mathrm{b}}) \in \Phi^{\mathrm{b}}(Y,X)$ following the lines in the proof of Proposition 5.7. Let us consider again the first step for the scenario uu. Define $z_0^{\mathrm{b}} := \big(0, -\frac{11}{3}\big)$; then Figure 5 shows that $z_0^{\mathrm{b}} \in \mathcal{X}_0^{\mathrm{b}}$ but $z_0^{\mathrm{b}} \notin \mathcal{Y}_0^{\mathrm{b}}$, which leads to $\psi_0 := 0$. Choosing $z_1^{\mathrm{b}} := \big(-\frac{7}{12}, \frac{13}{6}\big) \in \mathcal{W}_0^{\mathrm{b}}$ ensures that

$$z_0^{\mathrm{b}} - \psi_0 Y_0 - z_1^{\mathrm{b}} = z_0^{\mathrm{b}} - z_1^{\mathrm{b}} \in \mathcal{Q}_0.$$

Figure 4 shows that $z_1^{\mathrm{b}} \in \mathcal{X}_1^{\mathrm{bu}}$; however $z_1^{\mathrm{b}} \notin \mathcal{Y}_1^{\mathrm{bu}}$ again leads to the choice $\psi_1^{\mathrm{u}} := 0$. Moreover $z_1^{\mathrm{b}} \in \mathcal{W}_1^{\mathrm{bu}}$, so choosing $z_2^{\mathrm{bu}} := z_1^{\mathrm{b}}$ gives

$$z_1^{\mathrm{b}} - \psi_1^{\mathrm{u}} Y_1^{\mathrm{u}} - z_2^{\mathrm{bu}} = 0 \in \mathcal{Q}_1^{\mathrm{u}}.$$

Note finally that $z_2^{\mathrm{bu}} \in \mathcal{Z}_2^{\mathrm{buu}} = \mathcal{Y}_2^{\mathrm{buu}}$, which leads to $\psi_2^{\mathrm{uu}} := 1$. Thus this strategy corresponds to exercising the entire option at time 2 on the node uu.

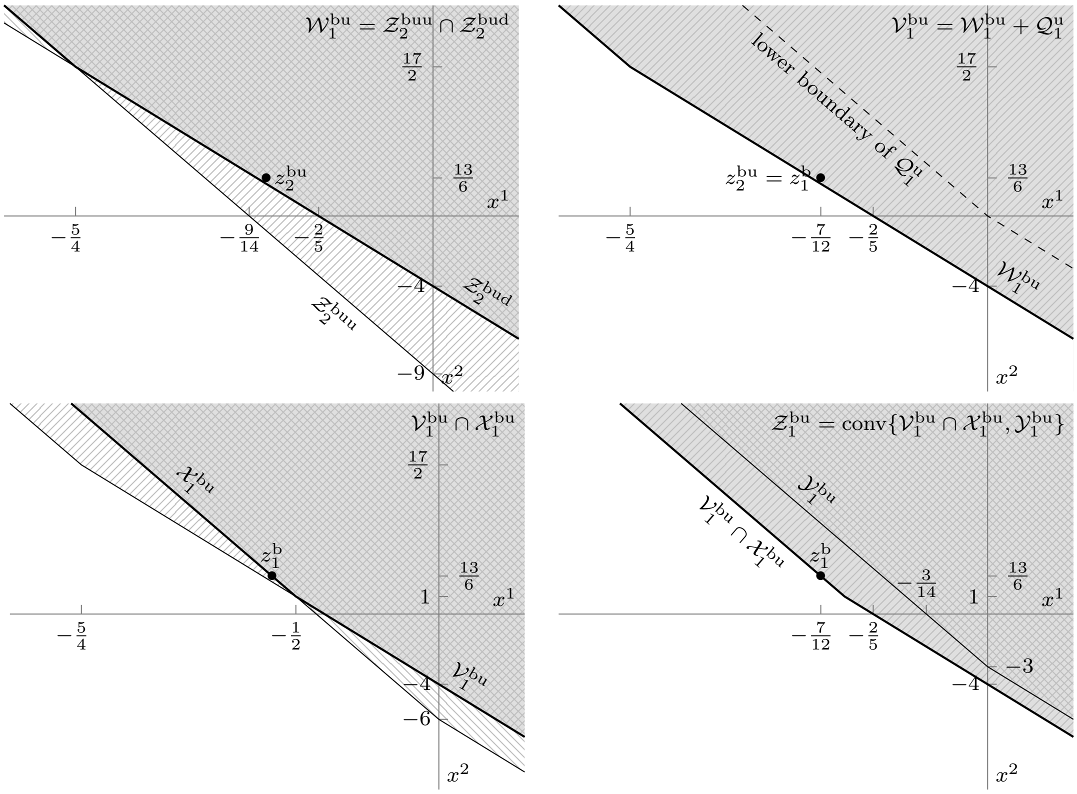

Figure 4. $\mathcal{W}_1^{\rm bu}$, $\mathcal{V}_1^{\rm bu}$, $\mathcal{V}_1^{\rm bu} \cap \mathcal{X}_1^{\rm bu}$, $\mathcal{Z}_1^{\rm bu}$, $z_1^{\rm b}$, $z_2^{\rm bu}$

8. Appendix: proofs

Proof of Theorem 4.4: By Definition 4.2,

$$\pi_j^{\rm a}(Y,X) = \inf\left\{x \in \mathbb{R} \,|\, \exists(\phi,u) \in \Phi^{\rm a}(Y,X) : xe^j = u_0\right\}.$$

Hence, according to Proposition 4.7,

$$\pi_j^{\rm a}(Y,X) = \inf\left\{x \in \mathbb{R} \,|\, \exists(\phi,u) \in \Lambda^{\rm a}(Y,X) : xe^j = u_0\right\}$$

and, by Proposition 4.6,

$$\pi_j^{\rm a}(Y,X) = \inf\left\{x \in \mathbb{R} \,|\, xe^j \in \mathcal{Z}_0^{\rm a}\right\}.$$

Because $\mathcal{Z}_0^{\rm a}$ is a polyhedral set, it is closed. It follows that $\left\{x \in \mathbb{R} \,|\, xe^j \in \mathcal{Z}_0^{\rm a}\right\}$ is closed. It is non-empty and bounded below because $xe^j \in \mathcal{Z}_0^{\rm a}$ for any $x \in \mathbb{R}$ large enough, and $xe^j \notin \mathcal{Z}_0^{\rm a}$ for any $x \in \mathbb{R}$ small enough. It follows that the infimum is in fact a minimum. □

Proof of Proposition 4.6: Let $a \in \mathcal{Z}_0^{\rm a}$. We construct a mixed stopping time $\phi \in \mathcal{X}$ together with the corresponding process ϕ^* and a strategy $z \in \Phi$ by induction. First we put $\phi_0^* := 1$ and $z_0 := a$. Clearly, $z_0 \in \phi_0^* \mathcal{Z}_0^{\rm a}$. Now suppose that for some $t = 0, \ldots, T-1$

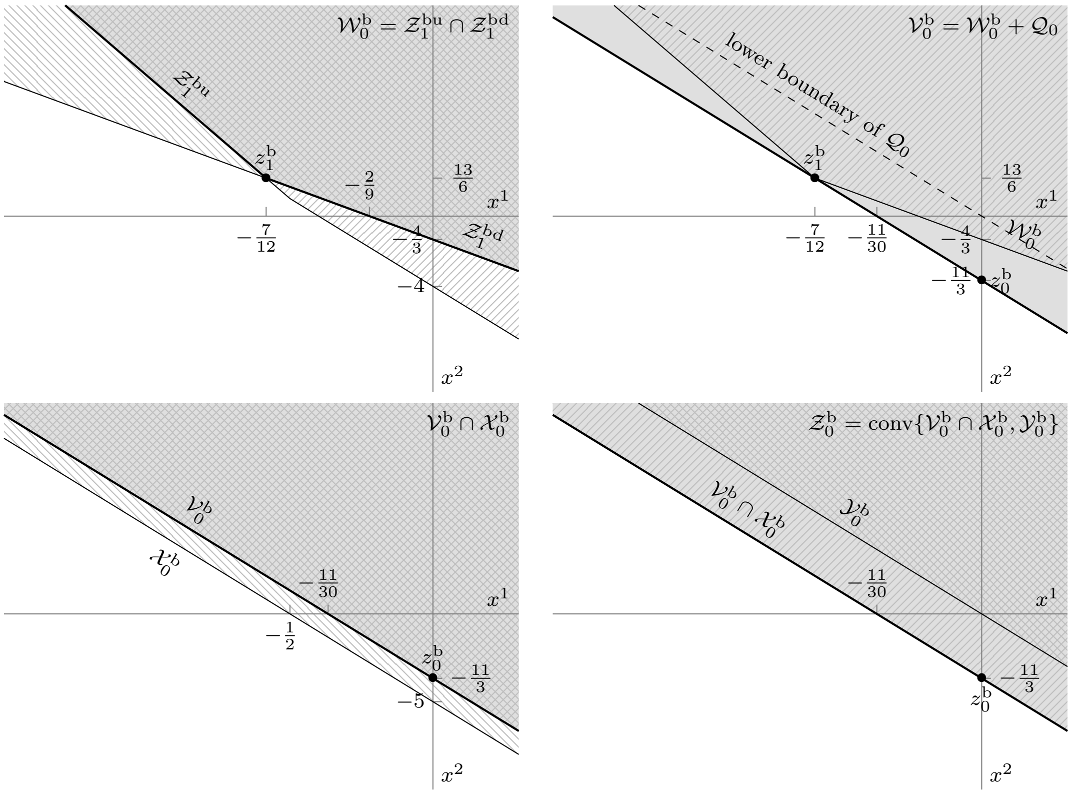

Figure 5. $\mathcal{W}_0^{\mathrm{b}}$, $\mathcal{V}_0^{\mathrm{b}}$, $\mathcal{V}_0^{\mathrm{b}} \cap \mathcal{X}_0^{\mathrm{b}}$, $\mathcal{Z}_0^{\mathrm{b}}$, z_0^{b}, z_1^{b}

we have constructed z_t and ϕ_t^* such that $z_t \in \phi_t^* \mathcal{Z}_t^{\mathrm{a}}$. It follows that $z_t \in \phi_t^* \mathcal{Y}_t^{\mathrm{a}}$, hence

$$z_t - \phi_t^* Y_t \in \mathcal{Q}_t.$$

It also follows that $z_t \in \phi_t^* \operatorname{conv}\{\mathcal{V}_t^{\mathrm{a}}, \mathcal{X}_t^{\mathrm{a}}\}$, hence there exist $\lambda_t \in [0,1]$, $v_t \in \mathcal{V}_t^{\mathrm{a}}$ and $x_t \in \mathcal{X}_t^{\mathrm{a}}$ such that $z_t = \phi_t^* \left((1-\lambda_t)v_t + \lambda_t x_t\right)$. We put $\phi_t := \phi_t^* \lambda_t$, and then $\phi_{t+1}^* := \phi_t^* - \phi_t = \phi_t^*(1-\lambda_t)$, so $z_t = \phi_{t+1}^* v_t + \phi_t x_t$. Since $x_t \in \mathcal{X}_t^{\mathrm{a}}$ and $v_t \in \mathcal{V}_t^{\mathrm{a}}$, it follows that $x_t - X_t \in \mathcal{Q}_t$ and there is $z_{t+1} \in \phi_{t+1}^* \mathcal{W}_t^{\mathrm{a}}$ such that $\phi_{t+1}^* v_t - z_{t+1} \in \mathcal{Q}_t$. As a result,

$$\begin{aligned} z_t - \phi_t X_t - z_{t+1} &= \phi_{t+1}^* v_t + \phi_t x_t - \phi_t X_t - z_{t+1} \\ &= \phi_{t+1}^* v_t - z_{t+1} + \phi_t \left(x_t - X_t\right) \in \mathcal{Q}_t. \end{aligned}$$

Since $\mathcal{W}_t^{\mathrm{a}} \subseteq \mathcal{Z}_{t+1}^{\mathrm{a}}$, it also follows that $z_{t+1} \in \phi_{t+1}^* \mathcal{Z}_{t+1}^{\mathrm{a}}$. Finally, given that $z_T \in \phi_T^* \mathcal{Z}_T^{\mathrm{a}}$, we get $z_T \in \phi_T^* \mathcal{Y}_T^{\mathrm{a}}$, so $z_T - \phi_T^* Y_T \in \mathcal{Q}_T$, and we put $\phi_T := \phi_T^*$, $\phi_{T+1}^* := 0$ and $z_{T+1} := 0$. We have constructed $(\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)$ such that $a = z_0$.

Conversely, we take any $(\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)$, and want to show that $z_0 \in \mathcal{Z}_0^{\mathrm{a}}$. More generally, we will show by backward induction that for each $t = 0, \ldots, T$

$$z_t \in \left\{ \begin{array}{ll} \phi_t^* \mathcal{Z}_t^{\mathrm{a}} & \text{on } \{\phi_t^* > 0\} \\ \mathcal{Q}_t & \text{on } \{\phi_t^* = 0\} \end{array} \right. . \tag{17}$$

Since $z_T - \phi_T^* Y_T = z_T - \phi_T Y_T \in \mathcal{Q}_T$, it follows that

$$z_T \in \phi_T^* Y_T + \mathcal{Q}_T = \begin{cases} \phi_T^* \mathcal{Y}_T^{\mathrm{a}} & \text{on } \{\phi_T^* > 0\} \\ \mathcal{Q}_T & \text{on } \{\phi_T^* = 0\} \end{cases} = \begin{cases} \phi_T^* \mathcal{Z}_T^{\mathrm{a}} & \text{on } \{\phi_T^* > 0\} \\ \mathcal{Q}_T & \text{on } \{\phi_T^* = 0\} \end{cases} .$$

Next, suppose that (17) holds for some $t = 1, \ldots, T$. Since z is predictable, it follows that $z_t \in \mathcal{L}_{t-1}$, so

$$z_t \in \begin{cases} \phi_t^* (\mathcal{Z}_t^{\mathrm{a}} \cap \mathcal{L}_{t-1}) & \text{on } \{\phi_t^* > 0\} \\ \mathcal{Q}_t \cap \mathcal{L}_{t-1} & \text{on } \{\phi_t^* = 0\} \end{cases} \subseteq \begin{cases} \phi_t^* \mathcal{W}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* = 0\} \end{cases} .$$

Hence, using

$$z_{t-1} - \phi_{t-1} X_{t-1} - z_t \in \mathcal{Q}_{t-1},$$

we obtain

$$\begin{aligned} z_{t-1} - \phi_{t-1} X_{t-1} &\in z_t + \mathcal{Q}_{t-1} \\ &\subseteq \begin{cases} \phi_t^* \mathcal{W}_{t-1}^{\mathrm{a}} + \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* > 0\} \\ \mathcal{Q}_{t-1} + \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* = 0\} \end{cases} \\ &= \begin{cases} \phi_t^* \mathcal{V}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* = 0\} \end{cases} . \end{aligned}$$

It follows that

$$\begin{aligned} z_{t-1} &\in \begin{cases} \phi_{t-1} X_{t-1} + \phi_t^* \mathcal{V}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_t^* > 0\} \\ \phi_{t-1} X_{t-1} + \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* = 0\} \end{cases} \\ &= \begin{cases} \phi_t^* \mathcal{V}_{t-1}^{\mathrm{a}} + \phi_{t-1} \mathcal{X}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_t^* > 0\} \\ \phi_{t-1} \mathcal{X}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_t^* = 0, \phi_{t-1} > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_t^* = 0, \phi_{t-1} = 0\} \end{cases} \\ &= \begin{cases} \phi_t^* \mathcal{V}_{t-1}^{\mathrm{a}} + \phi_{t-1} \mathcal{X}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_{t-1}^* = 0\} \end{cases} \\ &\subseteq \begin{cases} \phi_t^* \operatorname{conv} \left\{ \mathcal{V}_{t-1}^{\mathrm{a}}, \mathcal{X}_{t-1}^{\mathrm{a}} \right\} & \text{on } \{\phi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_{t-1}^* = 0\} \end{cases} . \end{aligned}$$

Moreover, sice $z_{t-1} - \phi_{t-1}^* Y_{t-1} \in \mathcal{Q}_{t-1}$, it follows that

$$z_{t-1} \in \phi_{t-1}^* Y_{t-1} + \mathcal{Q}_{t-1} = \begin{cases} \phi_t^* \mathcal{Y}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_{t-1}^* = 0\} \end{cases} .$$

As a result,

$$\begin{aligned} z_{t-1} &\in \begin{cases} \phi_t^* \operatorname{conv} \left\{ \mathcal{V}_{t-1}^{\mathrm{a}}, \mathcal{X}_{t-1}^{\mathrm{a}} \right\} \cap \phi_t^* \mathcal{Y}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_{t-1}^* = 0\} \end{cases} \\ &= \begin{cases} \phi_t^* \mathcal{Z}_{t-1}^{\mathrm{a}} & \text{on } \{\phi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\phi_{t-1}^* = 0\} \end{cases} , \end{aligned}$$

which completes the proof. □

Proof of Proposition 4.7: Suppose that $(\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)$. Then, for each $t = 0, \dots, T-1$

$$z_t - \phi_t X_t - z_{t+1} \in \mathcal{Q}_t,$$

so there is a liquidation strategy $y^t_{t+1}, \dots, y^t_{T+1}$ starting from $z_t - \phi_t Y_t - z_{t+1}$ at time t. We also put $y^T_{T+1} := 0$ for notational convenience. Moreover,

$$z_t - \phi^*_t Y_t \in \mathcal{Q}_t \quad \text{for each } t = 0, \dots, T,$$

so there is a liquidation strategy $x^t_{t+1}, \dots, x^t_{T+1}$ starting from $z_t - \phi^*_t Y_t$ at time t. For each $\psi \in \mathcal{X}$ we put

$$\begin{aligned} u^\psi_0 &:= z_0, \\ u^\psi_t &:= \psi^*_t z_t + \sum_{s=0}^{t-1} \psi^*_{s+1} y^s_t + \sum_{s=0}^{t-1} \psi_s x^s_t \quad \text{for } t = 1, \dots, T+1. \end{aligned}$$

This defines $u : \mathcal{X} \to \Phi$, which satisfies the non-anticipation condition (10). Moreover, for each $\psi \in \mathcal{X}$ and for each $t = 0, \dots, T$,

$$\begin{aligned} & u^\psi_t - G^{\phi,\psi}_t - u^\psi_{t+1} \\ &= \psi^*_t z_t + \sum_{s=0}^{t-1} \psi^*_{s+1} y^s_t + \sum_{s=0}^{t-1} \psi_s x^s_t - \psi_t \phi^*_t Y_t - \psi^*_{t+1} \phi_t X_t \\ &\quad - \psi^*_{t+1} z_{t+1} - \sum_{s=0}^{t} \psi^*_{s+1} y^s_{t+1} - \sum_{s=0}^{t} \psi_s x^s_{t+1} \\ &= \psi^*_{t+1} \left(z_t - \phi_t X_t - z_{t+1} - y^t_{t+1} \right) + \psi_t \left(z_t - \phi^*_t Y_t - x^t_{t+1} \right) \\ &\quad + \sum_{s=0}^{t-1} \psi^*_{s+1} \left(y^s_t - y^s_{t+1} \right) + \sum_{s=0}^{t-1} \psi_s \left(x^s_t - x^s_{t+1} \right) \\ &\in \psi^*_{t+1} \mathcal{K}_t + \psi_t \mathcal{K}_t + \sum_{s=0}^{t-1} \psi^*_{s+1} \mathcal{K}_t + \sum_{s=0}^{t-1} \psi_s \mathcal{K}_t \subseteq \mathcal{K}_t. \end{aligned}$$

This means that $(\phi, u) \in \Phi^{\mathrm{a}}(Y, X)$, with $z_0 = u_0$.

Conversely, suppose that $(\phi, u) \in \Phi^{\mathrm{a}}(Y, X)$. Then we put

$$z := u^{\chi^T}.$$

It follows that for each $t = 0, \dots, T-1$

$$z_t - \phi_t X_t - z_{t+1} = u^{\chi^T}_t + G^{\phi,\chi^T}_t - u^{\chi^T}_{t+1} \in \mathcal{K}_t \subseteq \mathcal{Q}_t$$

since

$$G^{\phi,\chi^T}_t = \chi^T_t \phi^*_t Y_t + \chi^{T*}_{t+1} \phi_t X_t = \phi_t X_t.$$

Next, take any $t = 0, \dots, T$. Then $\chi^T_s = \chi^t_s = 0$ for each $s = 0, \dots, t-1$, and because u satisfies the non-anticipation condition (10), we have $z_t = u^{\chi^T}_t = u^{\chi^t}_t$. Since $\chi^t_t = 1$, $\chi^{t*}_{t+1} = 0$ and

$$G^{\phi,\chi^t}_t = \chi^t_t \phi^*_t Y_t + \chi^{t*}_{t+1} \phi_t X_t = \phi^*_t Y_t,$$

it means that

$$z_t - \phi^*_t Y_t - u^{\chi^t}_{t+1} = u^{\chi^t}_t + G^{\phi,\chi^t}_t - u^{\chi^t}_{t+1} \in \mathcal{K}_t \subseteq \mathcal{Q}_t. \tag{18}$$

Moreover, for each $s = t+1, \dots, T$ we have $\chi^t_s = \chi^{t*}_{s+1} = 0$ and

$$G^{\phi,\chi^t}_s = \chi^t_s \phi^*_s Y_s + \chi^{t*}_{s+1} \phi_t X_t = 0,$$

hence

$$u^{\chi^t}_s - u^{\chi^t}_{s+1} = u^{\chi^t}_s + G^{\phi,\chi^t}_s - u^{\chi^t}_{s+1} \in \mathcal{K}_s \subseteq \mathcal{Q}_s.$$

We can verify by backward induction that $u^{\chi^t}_{s+1} \in \mathcal{Q}_s$ for each $s = t, \dots, T$. Clearly, $u^{\chi^t}_{T+1} = 0 \in \mathcal{Q}_T$. Now suppose that $u^{\chi^t}_{s+1} \in \mathcal{Q}_s$ for some $s = t+1, \dots, T$. It follows that $u^{\chi^t}_s = (u^{\chi^t}_s - u^{\chi^t}_{s+1}) + u^{\chi^t}_{s+1} \in \mathcal{Q}_s + \mathcal{Q}_s = \mathcal{Q}_s$. By predictability, $u^{\chi^t}_s \in \mathcal{L}_{s-1}$, so we can conclude that $u^{\chi^t}_s \in \mathcal{Q}_s \cap \mathcal{L}_{s-1} \subseteq \mathcal{Q}_{s-1}$, which completes the backward induction argument. In particular, we have shown that $u^{\chi^t}_{t+1} \in \mathcal{Q}_t$. Together with (18), this shows that

$$z_t - \phi^*_t Y_t \in \mathcal{Q}_t$$

for each $t = 0, \dots, T$. As a result, $(\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)$ with $z_0 = u_0$, which completes the proof. □

Proof of Lemma 4.8: Take any $u \in \Phi$ such that $(\phi, u) \in \Phi^{\mathrm{a}}(Y, X)$. Observe that

$$\begin{aligned} Q_{\phi,t} = Q_{\phi,\chi^t} &= \sum_{s=0}^{T} \sum_{u=0}^{T} \phi_s \chi^t_u Q_{s,u} = \sum_{s=0}^{T} \phi_s \mathbf{1}_{\{s \geq t\}} Y_t + \sum_{s=0}^{T} \phi_s \mathbf{1}_{\{s < t\}} X_s \\ &= \phi^*_t Y_t + \sum_{s=0}^{t-1} \phi_s X_s \end{aligned}$$

and define $z : \mathcal{X} \to \Phi$ such that

$$z^\psi_t := u^\psi_t + \psi^*_t \sum_{s=0}^{t-1} \phi_s X_s \tag{19}$$

for any $\psi \in \mathcal{X}$ and any $t = 0, \dots, T+1$. Then z satisfies the non-anticipation condition (5), $z_0 = u_0$, and it also satisfies the rebalancing condition (4) for an American option with payoff process $Q_{\phi,\cdot}$ since for any $\psi \in \mathcal{X}$ and any $t = 0, \dots, T$

$$\begin{aligned}
&z_t^\psi - \psi_t Q_{\phi,t} - z_{t+1}^\psi \\
&= \left(u_t^\psi + \psi_t^* \sum_{s=0}^{t-1} \phi_s X_s\right) - \psi_t \left(\phi_t^* Y_t + \sum_{s=0}^{t-1} \phi_s X_s\right) - \left(u_{t+1}^\psi + \psi_{t+1}^* \sum_{s=0}^{t} \phi_s X_s\right) \\
&= u_t^\psi - \psi_t \phi_t^* Y_t - \psi_{t+1}^* \phi_t X_t - u_{t+1}^\psi \\
&= u_t^\psi - G_t^{\phi,\psi} - u_{t+1}^\psi \in \mathcal{K}_t.
\end{aligned}$$

Conversely, take any $z \in \Psi^{\mathrm{a}}(Q_{\phi,\cdot})$ and define $u : \mathcal{X} \to \Phi$ such that

$$u_t^\psi := z_t^\psi - \psi_t^* \sum_{s=0}^{t-1} \phi_s X_s$$

for any $\psi \in \mathcal{X}$ and any $t = 0, \dots, T+1$. Then u satisfies the non-anticipation condition (10), $u_0 = z_0$, and

$$\begin{aligned}
&u_t^\psi - G_t^{\phi,\psi} - u_{t+1}^\psi \\
&= \left(z_t^\psi - \psi_t^* \sum_{s=0}^{t-1} \phi_s X_s\right) - \left(\psi_t \phi_t^* Y_t + \psi_{t+1}^* \phi_t X_t\right) - \left(z_{t+1}^\psi - \psi_{t+1}^* \sum_{s=0}^{t} \phi_s X_s\right) \\
&= z_t^\psi - \psi_t \left(\phi_t^* Y_t + \sum_{s=0}^{t-1} \phi_s X_s\right) - z_{t+1}^\psi \\
&= z_t^\psi - \psi_t Q_{\phi,t} - z_{t+1}^\psi \in \mathcal{K}_t
\end{aligned}$$

for any $\psi \in \mathcal{X}$ and any $t = 0, \dots, T$, so the rebalancing condition (9) holds. The lemma follows because (19) defines a one-to-one map between strategies $z \in \Psi^{\mathrm{a}}(Q_{\phi,\cdot})$ and strategies u such that $(\phi, u) \in \Phi^{\mathrm{a}}(Y, X)$ with $u_0 = z_0$. $\square$

Proof of Theorem 4.9: According to Definition 4.2,

$$\pi_j^{\mathrm{a}}(Y, X) = \inf \left\{x \in \mathbb{R} \,|\, \exists (\phi, u) \in \Phi^{\mathrm{a}}(Y, X) : x e^j = u_0\right\}. \tag{20}$$

By Theorem 4.4, $\pi_j^{\mathrm{a}}(Y, X) e^j \in \mathcal{Z}_0^{\mathrm{a}}$. Hence, by Proposition 4.6, there is a $(\phi, z) \in \Lambda^{\mathrm{a}}(Y, X)$ such that $\pi_j^{\mathrm{a}}(Y, X) e^j = z_0$. It follows by Proposition 4.7 that there is a $(\phi, u) \in \Phi^{\mathrm{a}}(Y, X)$ such that $\pi_j^{\mathrm{a}}(Y, X) e^j = u_0$, so the infimum in (20) is in fact a minimum,

$$\pi_j^{\mathrm{a}}(Y, X) = \min \left\{x \in \mathbb{R} \,|\, \exists (\phi, u) \in \Phi^{\mathrm{a}}(Y, X) : x e^j = u_0\right\}.$$

As a result, according to Lemma 4.8 and Definition 2.3,

$$\begin{aligned}
\pi_j^{\mathrm{a}}(Y, X) &= \min \left\{x \in \mathbb{R} \,|\, \exists \phi \in \chi \exists z \in \Psi^{\mathrm{a}}(Q_{\phi,\cdot}) : x e^j = z_0\right\} \\
&= \min_{\phi \in \chi} \inf \left\{x \in \mathbb{R} \,|\, \exists z \in \Psi^{\mathrm{a}}(Q_{\phi,\cdot}) : x e^j = z_0\right\} \\
&= \min_{\phi \in \chi} p_j^{\mathrm{a}}(Q_{\phi,\cdot}),
\end{aligned}$$

where $p^{\mathrm{a}}_j(Q_{\phi,\cdot})$ is the seller's price of an American option with gradual exercise and payoff process $Q_{\phi,\cdot}$, which can be expressed as

$$p^{\mathrm{a}}_j(Q_{\phi,\cdot}) = \max_{\psi\in\mathcal{X}} \max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}^{\mathrm{d}}_j(\psi)} \mathbb{E}_{\mathbb{Q}}((Q_{\phi,\cdot}\cdot S)_\psi)$$

by Theorem 2.4. It follows that

$$\pi^{\mathrm{a}}_j(Y,X) = \min_{\phi\in\mathcal{X}} \max_{\psi\in\mathcal{X}} \max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}^{\mathrm{d}}_j(\psi)} \mathbb{E}_{\mathbb{Q}}((Q_{\phi,\cdot}\cdot S)_\psi),$$

completing the proof. □

Proof of Theorem 5.4: Using Definition 5.2 and Propositions 5.6 and 5.7, we obtain

$$\begin{aligned}\pi^{\mathrm{b}}_j(Y,X) &= \sup\left\{-x\in\mathbb{R}\,|\,\exists(\psi,u)\in\Phi^{\mathrm{b}}(Y,X): xe^j = u_0\right\}\\ &= \sup\left\{-x\in\mathbb{R}\,|\,\exists(\psi,u)\in\Lambda^{\mathrm{b}}(Y,X): xe^j = u_0\right\}\\ &= \sup\left\{-x\in\mathbb{R}\,|\,xe^j\in\mathcal{Z}^{\mathrm{b}}_0\right\}.\end{aligned}$$

Being a polyhedral set, $\mathcal{Z}^{\mathrm{b}}_0$ is closed, hence $\left\{-x\in\mathbb{R}\,|\,xe^j\in\mathcal{Z}^{\mathrm{b}}_0\right\}$ is closed. Moreover, $\left\{-x\in\mathbb{R}\,|\,xe^j\in\mathcal{Z}^{\mathrm{b}}_0\right\}$ is non-empty and bounded above because $xe^j\in\mathcal{Z}^{\mathrm{b}}_0$ for any $x\in\mathbb{R}$ large enough and $xe^j\notin\mathcal{Z}^{\mathrm{b}}_0$ for any $x\in\mathbb{R}$ small enough, so the supremum is in fact a maximum. □

Proof of Proposition 5.6: Let $a\in\mathcal{Z}^{\mathrm{b}}_0$. We construct a mixed stopping time $\psi\in\mathcal{X}$ and a strategy $z\in\Phi$ by induction. First we put $\psi^*_0 := 1$ and $z_0 := a$. Clearly, $z_0\in\psi^*_0\mathcal{Z}^{\mathrm{b}}_0$. Next, suppose that $z_t\in\psi^*_t\mathcal{Z}^{\mathrm{b}}_t$ for some $t=0,\dots,T-1$. Then $z_t\in\psi^*_t\operatorname{conv}\{\mathcal{V}^{\mathrm{b}}_t\cap\mathcal{X}^{\mathrm{b}}_t,\mathcal{Y}^{\mathrm{b}}_t\}$, so there exist $\lambda_t\in[0,1]$, $v_t\in\mathcal{V}^{\mathrm{b}}_t\cap\mathcal{X}^{\mathrm{b}}_t$ and $y_t\in\mathcal{Y}^{\mathrm{b}}_t$ such that $z_t=\psi^*_t((1-\lambda_t)v_t+\lambda_t y_t)$. We put $\psi_t:=\psi^*_t\lambda_t$, and then $\psi^*_{t+1}:=\psi^*_t-\psi_t=\psi^*_t(1-\lambda_t)$, so $z_t=\psi^*_{t+1}v_t+\psi_t y_t$. Because $v_t\in\mathcal{X}^{\mathrm{b}}_t$ and $y_t\in\mathcal{Y}^{\mathrm{b}}_t$, we have $v_t+X_t\in\mathcal{Q}_t$ and $y_t+Y_t\in\mathcal{Q}_t$. It follows that

$$\begin{aligned}z_t+\psi_tY_t+\psi^*_{t+1}X_t &= \psi^*_{t+1}v_t+\psi_ty_t+\psi_tY_t+\psi^*_{t+1}X_t\\ &= \psi^*_{t+1}(v_t+X_t)+\psi_t(y_t+Y_t)\\ &\in \psi^*_{t+1}\mathcal{Q}_t+\psi_t\mathcal{Q}_t\subseteq\mathcal{Q}_t.\end{aligned}$$

Since $v_t\in\mathcal{V}^{\mathrm{b}}_t$, there is a $z_{t+1}\in\psi^*_{t+1}\mathcal{W}^{\mathrm{b}}_t$ such that $\psi^*_{t+1}v_t-z_{t+1}\in\mathcal{Q}_t$. It follows that

$$\begin{aligned}z_t+\psi_tY_t-z_{t+1} &= \psi^*_{t+1}v_t+\psi_ty_t+\psi_tY_t-z_{t+1}\\ &= (\psi^*_{t+1}v_t-z_{t+1})+\psi_t(y_t+Y_t)\in\mathcal{Q}_t.\end{aligned}$$

Since $\mathcal{W}^{\mathrm{b}}_t\subseteq\mathcal{Z}^{\mathrm{b}}_{t+1}$, it also follows that $z_{t+1}\in\psi^*_{t+1}\mathcal{Z}^{\mathrm{b}}_{t+1}$. Finally, given that $z_T\in\psi^*_T\mathcal{Z}^{\mathrm{b}}_T=\psi^*_T\mathcal{Y}^{\mathrm{b}}_T$, we get $z_T+\psi_TY_T=z_T+\psi^*_TY_T\in\mathcal{Q}_T$. Putting and $\psi_T:=\psi^*_T$, $\psi^*_{T+1}:=0$ and $z_{T+1}:=0$, we obtain

$$z_T+\psi_TY_T+\psi^*_{T+1}X_T\in\mathcal{Q}_T.$$

We have constructed $(\psi,z)\in\Lambda^{\mathrm{b}}(Y,X)$ such that $a=z_0$.

Conversely, we take any $(\psi, z) \in \Lambda^{\mathrm{b}}(Y, X)$, and want to show that $z_0 \in \mathcal{Z}_0^{\mathrm{b}}$. This a consequence of the following fact, which will be proved by backward induction: for each $t = 0, \ldots, T$

$$z_t \in \left\{ \begin{array}{ll} \psi_t^* \mathcal{Z}_t^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_t & \text{on } \{\psi_t^* = 0\} \end{array} \right. . \tag{21}$$

We start the proof with $t = T$. Since $z_T + \psi_T^* Y_T = z_T + \psi_T Y_T + \psi_{T+1}^* X_T \in \mathcal{Q}_T$, it follows that indeed

$$z_T \in -\psi_T^* Y_T + \mathcal{Q}_T = \left\{ \begin{array}{ll} \psi_T^* \mathcal{Y}_T^{\mathrm{b}} & \text{on } \{\psi_T^* > 0\} \\ \mathcal{Q}_T & \text{on } \{\psi_T^* = 0\} \end{array} \right. = \left\{ \begin{array}{ll} \psi_T^* \mathcal{Z}_T^{\mathrm{b}} & \text{on } \{\psi_T^* > 0\} \\ \mathcal{Q}_T & \text{on } \{\psi_T^* = 0\} \end{array} \right. .$$

Next, suppose that (21) holds for some $t = 1, \ldots, T$. Since z is predictable, it follows that $z_t \in \mathcal{L}_{t-1}$, so

$$z_t \in \left\{ \begin{array}{ll} \psi_t^* (\mathcal{Z}_t^{\mathrm{b}} \cap \mathcal{L}_{t-1}) & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_t \cap \mathcal{L}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. \subseteq \left\{ \begin{array}{ll} \psi_t^* \mathcal{W}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. .$$

Hence, using

$$z_{t-1} + \psi_{t-1} Y_{t-1} - z_t \in \mathcal{Q}_{t-1},$$

we obtain

$$\begin{aligned} z_{t-1} + \psi_{t-1} Y_{t-1} &\in z_t + \mathcal{Q}_{t-1} \\ &\subseteq \left\{ \begin{array}{ll} \psi_t^* \mathcal{W}_{t-1}^{\mathrm{b}} + \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} + \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. \\ &= \left\{ \begin{array}{ll} \psi_t^* \mathcal{V}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. . \end{aligned}$$

Moreover, since

$$z_{t-1} + \psi_{t-1} Y_{t-1} + \psi_t^* X_{t-1} \in \mathcal{Q}_{t-1},$$

we obtain

$$z_{t-1} + \psi_{t-1} Y_{t-1} \in \mathcal{Q}_{t-1} - \psi_t^* X_{t-1} = \left\{ \begin{array}{ll} \psi_t^* \mathcal{X}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. .$$

It follows that

$$z_{t-1} + \psi_{t-1} Y_{t-1} \in \left\{ \begin{array}{ll} \psi_t^* (\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}) & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_t^* = 0\} \end{array} \right. ,$$

and so

$$
\begin{aligned}
z_{t-1} &\in \begin{cases} \psi_t^*(\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}) + \psi_{t-1}Y_{t-1} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} + \psi_{t-1}Y_{t-1} & \text{on } \{\psi_t^* = 0\} \end{cases} \\
&\subseteq \begin{cases} \psi_t^*(\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}) + \psi_{t-1}\mathcal{Y}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \mathcal{Q}_{t-1} + \psi_{t-1}\mathcal{Y}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* = 0\} \end{cases} \\
&= \begin{cases} \psi_t^*(\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}) + \psi_{t-1}\mathcal{Y}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* > 0\} \\ \psi_{t-1}\mathcal{Y}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_t^* = 0, \psi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_{t-1}^* = 0\} \end{cases} \\
&= \begin{cases} \psi_t^*(\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}) + \psi_{t-1}\mathcal{Y}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_{t-1}^* = 0\} \end{cases} \\
&\subseteq \begin{cases} \psi_{t-1}^* \operatorname{conv}\left\{\mathcal{V}_{t-1}^{\mathrm{b}} \cap \mathcal{X}_{t-1}^{\mathrm{b}}, \mathcal{Y}_{t-1}^{\mathrm{b}}\right\} & \text{on } \{\psi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_{t-1}^* = 0\} \end{cases} \\
&= \begin{cases} \psi_{t-1}^* \mathcal{Z}_{t-1}^{\mathrm{b}} & \text{on } \{\psi_{t-1}^* > 0\} \\ \mathcal{Q}_{t-1} & \text{on } \{\psi_{t-1}^* = 0\} \end{cases},
\end{aligned}
$$

which completes the proof. □

Proof of Proposition 5.7: Suppose that $(\psi, z) \in \Lambda^{\mathrm{b}}(Y, X)$. Then, for each $t = 0, \ldots, T-1$,

$$z_t + \psi_t Y_t - z_{t+1} \in \mathcal{Q}_t,$$

so there is a liquidation strategy $y_{t+1}^t, \ldots, y_{T+1}^t$ starting from $z_t + \psi_t Y_t - z_{t+1}$ at time t. We also put $y_{T+1}^T := 0$ for notational convenience. Moreover,

$$z_t + \psi_t Y_t + \psi_{t+1}^* X_t \in \mathcal{Q}_t \quad \text{for each } t = 0, \ldots, T,$$

so there is a liquidation strategy $x_{t+1}^t, \ldots, x_{T+1}^t$ starting from $z_t + \psi_t Y_t + \psi_{t+1}^* X_t$ at time t. For each $\phi \in \mathcal{X}$ we put

$$
\begin{aligned}
u_0^\phi &:= z_0, \\
u_t^\phi &:= \phi_t^* z_t + \sum_{s=0}^{t-1} \phi_{s+1}^* y_t^s + \sum_{s=0}^{t-1} \phi_s x_t^s \quad \text{for } t = 1, \ldots, T+1.
\end{aligned}
$$

This defines $u : \mathcal{X} \to \Phi$, which satisfies the non-anticipation condition (12). Moreover, for each $\psi \in \mathcal{X}$ and for each $t = 0, \dots, T$

$$\begin{aligned}
&u_t^\phi + G_t^{\phi,\psi} - u_{t+1}^\phi \\
&= \phi_t^* z_t + \sum_{s=0}^{t-1} \phi_{s+1}^* y_t^s + \sum_{s=0}^{t-1} \phi_s x_t^s + \psi_t \phi_t^* Y_t + \psi_{t+1}^* \phi_t X_t \\
&\quad - \phi_{t+1}^* z_{t+1} - \sum_{s=0}^{t} \phi_{s+1}^* y_{t+1}^s - \sum_{s=0}^{t} \phi_s x_{t+1}^s \\
&= \phi_{t+1}^* \left(z_t + \psi_t Y_t - z_{t+1} - y_{t+1}^t \right) + \phi_t \left(z_t + \psi_t Y_t + \psi_{t+1}^* X_t - x_{t+1}^t \right) \\
&\quad + \sum_{s=0}^{t-1} \phi_{s+1}^* \left(y_t^s - y_{t+1}^s \right) + \sum_{s=0}^{t-1} \phi_s \left(x_t^s - x_{t+1}^s \right) \\
&\in \phi_{t+1}^* \mathcal{K}_t + \phi_t \mathcal{K}_t + \sum_{s=0}^{t-1} \phi_{s+1}^* \mathcal{K}_t + \sum_{s=0}^{t-1} \phi_s \mathcal{K}_t \subseteq \mathcal{K}_t.
\end{aligned}$$

This means that $(\psi, u) \in \Phi^{\mathrm{b}}(Y, X)$, with $z_0 = u_0$.

Conversely, suppose that $(\psi, u) \in \Phi^{\mathrm{b}}(Y, X)$. Then we put

$$z := u^{\chi^T}.$$

It follows that for each $t = 0, \dots, T-1$

$$z_t + \psi_t Y_t - z_{t+1} = u_t^{\chi^T} - G_t^{\chi^T,\psi} - u_{t+1}^{\chi^T} \in \mathcal{K}_t \subseteq \mathcal{Q}_t$$

since

$$G_t^{\chi^T,\psi} = \psi_t \chi_t^{T*} Y_t + \psi_{t+1}^* \chi_t^T X_t = \psi_t Y_t.$$

Next, take any $t = 0, \dots, T$. Then $\chi_s^T = \chi_s^t = 0$ for each $s = 0, \dots, t-1$, and because u satisfies the non-anticipation condition (12), we have $z_t = u_t^{\chi^T} = u_t^{\chi^t}$. Since $\chi_t^t = \chi_t^{t*} = 1$ and

$$G_t^{\chi^t,\psi} = \psi_t \chi_t^{t*} Y_t + \psi_{t+1}^* \chi_t^t X_t = \psi_t Y_t + \psi_{t+1}^* X_t,$$

it means that

$$z_t + \psi_t Y_t + \psi_{t+1}^* X_t - u_{t+1}^{\chi^t} = u_t^{\chi^t} + G_t^{\chi^t,\psi} - u_{t+1}^{\chi^t} \in \mathcal{K}_t \subseteq \mathcal{Q}_t. \tag{22}$$

Moreover, for each $s = t+1, \dots, T$ we have $\chi_s^t = \chi_s^{t*} = 0$ and

$$G_s^{\chi^t,s} = \psi_s \chi_s^{t*} Y_s + \psi_{s+1}^* \chi_s^t X_s = 0,$$

hence

$$u_s^{\chi^t} - u_{s+1}^{\chi^t} = u_s^{\chi^t} + G_s^{\chi^t,\psi} - u_{s+1}^{\chi^t} \in \mathcal{K}_s \subseteq \mathcal{Q}_s.$$

We can verify by backward induction that $u^{\chi^t}_{s+1} \in \mathcal{Q}_s$ for each $s = t, \dots, T$. Clearly, $u^{\chi^t}_{T+1} = 0 \in \mathcal{Q}_T$. Now suppose that $u^{\chi^t}_{s+1} \in \mathcal{Q}_s$ for some $s = t+1, \dots, T$. It follows that $u^{\chi^t}_s = (u^{\chi^t}_s - u^{\chi^t}_{s+1}) + u^{\chi^t}_{s+1} \in \mathcal{Q}_s + \mathcal{Q}_s = \mathcal{Q}_s$. By predictability, $u^{\chi^t}_s \in \mathcal{L}_{s-1}$, so we can conclude that $u^{\chi^t}_s \in \mathcal{Q}_s \cap \mathcal{L}_{s-1} \subseteq \mathcal{Q}_{s-1}$, which completes the backward induction argument. In particular, we have shown that $u^{\chi^t}_{t+1} \in \mathcal{Q}_t$. Together with (22), this shows that

$$z_t + \psi_t Y_t + \psi^*_{t+1} X_t \in \mathcal{Q}_t$$

for each $t = 0, \dots, T$. As a result, $(\psi, z) \in \Lambda^{\mathrm{b}}(Y, X)$ with $z_0 = u_0$, which completes the proof. □

Proof of Lemma 5.8: Observe that for any $\psi \in \mathcal{X}$

$$\begin{aligned} Q_{t,\psi} = Q_{\chi^t,\psi} &= \sum_{u=0}^{T} \sum_{s=0}^{T} \chi^t_u \psi_s Q_{u,s} = \sum_{s=0}^{T} \psi_s \left(\mathbf{1}_{\{t \geq s\}} Y_s + \mathbf{1}_{\{t < s\}} X_t \right) \\ &= \sum_{s=0}^{t} \psi_s Y_s + \psi^*_{t+1} X_t. \end{aligned}$$

Now take any $u \in \Phi$ such that $(\psi, u) \in \Phi^{\mathrm{b}}(Y, X)$ and define $z : \mathcal{X} \to \Phi$ such that

$$z^\phi_t := u^\phi_t - \phi^*_t \sum_{s=0}^{t-1} \psi_s Y_s \tag{23}$$

for any $\phi \in \mathcal{X}$ and any $t = 0, \dots, T+1$. Then z satisfies the non-anticipation condition (5), $z_0 = u_0$, and it also satisfies the rebalancing condition (4) for an American option with payoff process $Z = Q_{\cdot,\psi}$ since for any $t = 0, \dots, T$

$$\begin{aligned} & z^\phi_t + \phi_t Q_{t,\psi} - z^\phi_{t+1} \\ &= \left(u^\phi_t - \phi^*_t \sum_{s=0}^{t-1} \psi_s Y_s \right) + \phi_t \left(\sum_{s=0}^{t} \psi_s Y_s + \psi^*_{t+1} X_t \right) - \left(u^\phi_{t+1} - \phi^*_{t+1} \sum_{s=0}^{t} \psi_s Y_s \right) \\ &= u^\phi_t + \psi_t \phi^*_t Y_t + \psi^*_{t+1} \phi_t X_t - u^\phi_{t+1} \\ &= u^\phi_t + G^{\phi,\psi}_t - u^\phi_{t+1} \in \mathcal{K}_t. \end{aligned}$$

Conversely, take any $z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi})$ and define $u : \mathcal{X} \to \Phi$ such that

$$u^\phi_t := z^\phi_t + \phi^*_t \sum_{s=0}^{t-1} \psi_s Y_s$$

for any $\phi \in \mathcal{X}$ and any $t = 0, \dots, T+1$. Then u satisfies the non-anticipation condition (12), $u_0 = z_0$, and

$$\begin{aligned}
&u_t^\phi + G_t^{\phi,\psi} - u_{t+1}^\phi \\
&= \left(z_t^\phi + \phi_t^* \sum_{s=0}^{t-1} \psi_s Y_s\right) + \left(\psi_t \phi_t^* Y_t + \psi_{t+1}^* \phi_t X_t\right) - \left(z_{t+1}^\phi + \phi_{t+1}^* \sum_{s=0}^{t} \psi_s Y_s\right) \\
&= z_t^\phi + \phi_t \left(\sum_{s=0}^{t} \psi_s Y_s + \psi_{t+1}^* X_t\right) - z_{t+1}^\phi \\
&= z_t^\phi + \phi_t Q_{t,\psi} - z_{t+1}^\phi \in \mathcal{K}_t
\end{aligned}$$

for any $\phi \in \mathcal{X}$ and any $t = 0, \ldots, T$, that is, the rebalancing condition (11) holds. The lemma follows because (23) defines a one-to-one map between strategies $z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi})$ and strategies u such that $(\psi, u) \in \Phi^{\mathrm{b}}(Y, X)$ with $u_0 = z_0$. □

Proof of Theorem 5.9: By Definition 5.2,

$$\pi_j^{\mathrm{b}}(Y, X) = \sup\left\{-x \in \mathbb{R} \,|\, \exists (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) : xe^j = u_0\right\}. \tag{24}$$

According to Theorem 5.4, $-\pi_j^{\mathrm{b}}(Y, X)e^j \in \mathcal{Z}_0^{\mathrm{b}}$. Hence, by Proposition 5.6, there is a $(\psi, z) \in \Lambda^{\mathrm{b}}(Y, X)$ such that $-\pi_j^{\mathrm{b}}(Y, X)e^j = z_0$, and so, by Proposition 5.7, there is a $(\psi, u) \in \Phi^{\mathrm{b}}(Y, X)$ such that $-\pi_j^{\mathrm{b}}(Y, X)e^j = u_0$. It follows that the supremum in (24) is attained,

$$\begin{aligned}
\pi_j^{\mathrm{b}}(Y, X) &= \max\left\{-x \in \mathbb{R} \,|\, \exists (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) : xe^j = u_0\right\} \\
&= -\min\left\{x \in \mathbb{R} \,|\, \exists (\psi, u) \in \Phi^{\mathrm{b}}(Y, X) : xe^j = u_0\right\}.
\end{aligned}$$

Hence, according to Lemma 5.8 and Definition 2.3,

$$\begin{aligned}
\pi_j^{\mathrm{b}}(Y, X) &= -\min\left\{x \in \mathbb{R} \,|\, \exists \psi \in \mathcal{X} \exists z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi}) : xe^j = z_0\right\} \\
&= -\min_{\psi \in \mathcal{X}} \inf\left\{x \in \mathbb{R} \,|\, \exists z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi}) : xe^j = z_0\right\} \\
&= -\min_{\psi \in \mathcal{X}} p_j^{\mathrm{a}}(-Q_{\cdot,\psi}),
\end{aligned}$$

where

$$p_j^{\mathrm{a}}(-Q_{\cdot,\psi}) = \inf\left\{x \in \mathbb{R} \,|\, \exists z \in \Psi^{\mathrm{a}}(-Q_{\cdot,\psi}) : xe^j = z_0\right\}$$

is the seller's price of an American option under gradual exercise with payoff process $-Q_{\cdot,\psi}$. By Theorem 2.4,

$$p_j^{\mathrm{a}}(-Q_{\cdot,\psi}) = \max_{\phi \in \mathcal{X}} \max_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}_j^{\mathrm{d}}(\phi)} \mathbb{E}_{\mathbb{Q}}((-Q_{\cdot,\psi} \cdot S)_\phi),$$

so

$$
\begin{aligned}
\pi_j^{\mathrm{b}}(Y,X) &= -\min_{\psi\in\mathcal{X}} p_j^{\mathrm{a}}(-Q_{\cdot,\psi}) \\
&= -\min_{\psi\in\mathcal{X}}\max_{\phi\in\mathcal{X}}\max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}_j^{\mathrm{d}}(\phi)} \mathbb{E}_{\mathbb{Q}}((-Q_{\cdot,\psi}\cdot S)_{\phi}) \\
&= \max_{\psi\in\mathcal{X}}\min_{\phi\in\mathcal{X}}\min_{(\mathbb{Q},S)\in\bar{\mathcal{P}}_j^{\mathrm{d}}(\phi)} \mathbb{E}_{\mathbb{Q}}((Q_{\cdot,\psi}\cdot S)_{\phi}).
\end{aligned}
$$

Theorem 5.9 has been proved. □

Proof of Theorem 6.1: Comparing the definitions of $\pi_j^{\mathrm{a}}(Y,X)$ and $\hat{\pi}_j^{\mathrm{a}}(Y,X)$, as well as those of $\pi_j^{\mathrm{b}}(Y,X)$ and $\hat{\pi}_j^{\mathrm{b}}(Y,X)$, we can see that to prove (15) and (16) it suffices to show that for every pair $(\sigma,y)\in\mathcal{T}\times\Phi$ which hedges (Y,X) for the seller there is a pair $(\phi,u)\in\Phi^{\mathrm{a}}(Y,X)$ such that $y_0=u_0$, and that for every pair $(\sigma,y)\in\mathcal{T}\times\Phi$ which hedges (Y,X) for the buyer there is a pair $(\phi,u)\in\Phi^{\mathrm{b}}(Y,X)$ such that $y_0=u_0$. To this end, in either case we put

$$
\begin{aligned}
&\phi := \chi^{\sigma}, \\
&u_t^{\psi} := \psi_t^* y_t \mathbf{1}_{\{\sigma\geq t\}} \quad \text{for each } \psi\in\mathcal{X} \text{ and each } t=0,\ldots,T+1.
\end{aligned}
$$

Then, clearly, $y_0=u_0$. It remains to show that $(\phi,u)\in\Phi^{\mathrm{a}}(Y,X)$ if (σ,y) hedges for the seller, and $(\phi,u)\in\Phi^{\mathrm{b}}(Y,X)$ if (σ,y) hedges for the buyer. Since $\psi_t^*=1-\sum_{s=0}^{t-1}\psi_s$, we can see that u_t^{ψ} depends on ψ via $\psi_0,\ldots,\psi_{t-1}$, so it satisfies the non-anticipation condition (10) in Definition 4.1 or (12) in Definition 5.1. Finally, we need to verify the rebalancing condition (9) in the seller's case and (11) in the buyer's case. From (8), we obtain

$$
u_t^{\psi} - G_t^{\phi,\psi} - u_{t+1}^{\psi} = \begin{cases} \psi_t\,(y_t - Y_t) + \psi_{t+1}^*\,(y_t - y_{t+1}) & \text{on } \{t<\sigma\}, \\ \psi_t\,(y_t - Y_t) + \psi_{t+1}^*\,(y_t - X_t) & \text{on } \{t=\sigma\}, \\ 0 & \text{on } \{t>\sigma\} \end{cases}
$$

in the seller's case, and in the buyer's case

$$
u_t^{\psi} + G_t^{\psi,\phi} - u_{t+1}^{\psi} = \begin{cases} \psi_t\,(y_t + X_t) + \psi_{t+1}^*\,(y_t - y_{t+1}) & \text{on } \{t<\sigma\}, \\ \psi_t^*\,(y_t + Y_t) & \text{on } \{t=\sigma\}, \\ 0 & \text{on } \{t>\sigma\}. \end{cases}
$$

In both cases, the self-financing condition (1) gives that $y_t - y_{t+1}\in\mathcal{K}_t$ for each $t=0,\ldots,T-1$. Condition (13) for the seller together with (6) implies that $y_t - Y_t\in\mathcal{K}_t$ on $\{t\leq\sigma\}$, and also for $t<T$ that $y_t - X_t\in\mathcal{K}_t$ on $\{t=\sigma\}$. Because $\mathcal{K}_t$ is a convex cone and $\psi_{T+1}^*=0$, it follows that $u_t^{\psi}-G_t^{\phi,\psi}-u_{t+1}^{\psi}\in\mathcal{K}_t$, completing the proof of (9). Moreover, condition (14) for the buyer implies that $y_t+X_t\in\mathcal{K}_t$ on $\{t<\sigma\}$ and $y_t+Y_t\in\mathcal{K}_t$ on $\{t=\sigma\}$, hence it follows that $u_t^{\psi}+G_t^{\psi,\phi}-u_{t+1}^{\psi}\in\mathcal{K}_t$, completing the proof of (11). □

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