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A SIMPLE PROOF OF THE FORMULA FOR THE BETTI NUMBERS OF THE QUASIHOMOGENEOUS HILBERT SCHEMES.

ALEXANDR BURYAK, BORIS LVOVICH FEIGIN, AND HIRAKU NAKAJIMA

ABSTRACT. In a recent paper the first two authors proved that the generating series of the Poincare polynomials of the quasihomogeneous Hilbert schemes of points in the plane has a simple decomposition in an infinite product. In this paper we give a very short geometrical proof of that formula.

1. INTRODUCTION

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in the plane \mathbb{C}^2 parametrizes ideals $I \subset \mathbb{C}[x, y]$ of colength n : $\dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n$. It is a nonsingular, irreducible, quasiprojective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [7, 13] for an introduction.

The cohomology groups of $(\mathbb{C}^2)^{[n]}$ were computed in [5], and the ring structure in the cohomology was determined independently in the papers [10] and [14].

There is a $(\mathbb{C}^*)^2$ -action on $(\mathbb{C}^2)^{[n]}$ that plays a central role in this subject. The algebraic torus $(\mathbb{C}^*)^2$ acts on \mathbb{C}^2 by scaling the coordinates, $(t_1, t_2) \cdot (x, y) = (t_1x, t_2y)$. This action lifts to the $(\mathbb{C}^*)^2$ -action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

For arbitrary non-negative integers α and β , such that $\alpha + \beta \geq 1$, let $T_{\alpha, \beta} = \{(t^\alpha, t^\beta) \in (\mathbb{C}^*)^2 \mid t \in \mathbb{C}^*\}$ be a one-dimensional subtorus of $(\mathbb{C}^*)^2$. If α and β are non-zero, then the fixed point set $((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}}$ is called the quasihomogeneous Hilbert scheme of points on the plane \mathbb{C}^2 .

The quasihomogeneous Hilbert scheme $((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}}$ is compact and in general has many irreducible components. They were described in [6]. In the case $\alpha = 1$ the Poincare polynomials of the irreducible components were computed in [3].

The Poincare polynomial of a manifold X is defined by $P_q(X) = \sum_{i \geq 0} \dim H_i(X; \mathbb{Q}) q^{\frac{i}{2}}$. In [4] the first two authors proved the following theorem.

Theorem 1.1. *Suppose α and β are positive coprime integers, then*

$$\sum_{n \geq 0} P_q \left(((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta}} \right) t^n = \prod_{\substack{i \geq 1 \\ (\alpha + \beta) \nmid i}} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha + \beta)i}}.$$

In this paper we give another proof of this theorem. In [4] the large part of the proof consists of non-trivial combinatorial computations with Young diagrams. Our new proof is more geometrical and is much shorter. In fact, we prove a slightly more general statement.

Let Γ_m be the finite subgroup of $(\mathbb{C}^*)^2$ defined by

$$\Gamma_m = \left\{ (\zeta^j, \zeta^{-j}) \in (\mathbb{C}^*)^2 \mid \zeta = \exp \left(\frac{2\pi i}{m} \right), j = 0, 1, \dots, m - 1 \right\}.$$

For a manifold X let $H_*^{BM}(X; \mathbb{Q})$ denote the Borel-Moore homology group of X with rational coefficients. Let $P_q^{BM}(X) = \sum_{i \geq 0} \dim H_i^{BM}(X; \mathbb{Q}) q^{\frac{i}{2}}$.

We prove the following theorem.

Theorem 1.2. *Let α and β be any two non-negative integers, such that $\alpha + \beta \geq 1$. Then we have*

$$(1) \quad \sum_{n \geq 0} P_q^{BM} \left(((\mathbb{C}^2)^{[n]})^{T_{\alpha, \beta} \times \Gamma_{\alpha + \beta}} \right) t^n = \prod_{\substack{i \geq 1 \\ (\alpha + \beta) \nmid i}} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha + \beta)i}}.$$

Here we use Borel-Moore homology, because the variety $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}}$ is in general not compact, if $\alpha = 0$.

If α and β are coprime, then $\Gamma_{\alpha+\beta} \subset T_{\alpha,\beta}$. Hence, Theorem 1.1 follows from Theorem 1.2.

Our proof of Theorem 1.2 consists of two steps. First, we prove that the left-hand side of (1) depends only on the sum $\alpha + \beta$. We use an argument with an equivariant symplectic form that is very similar to the one that was applied by the third author in [12] (proof of Proposition 5.7). After that the case $\alpha = 0$ can be done using a notion of a power structure over the Grothendieck ring of quasiprojective varieties.

In [4], as a corollary of Theorem 1.1, there was derived a combinatorial identity. In the same way Theorem 1.2 leads to a more general combinatorial identity. Denote by \mathcal{Y} the set of all Young diagrams. The number of boxes in a Young diagram Y is denoted by $|Y|$. For a box $s \in Y$ we define the numbers $l_Y(s)$ and $a_Y(s)$, as it is shown on Fig. 1.

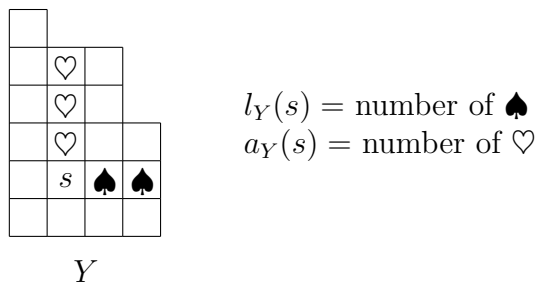


FIGURE 1. Definition of the numbers $a_Y(s)$ and $l_Y(s)$

For a Young diagram Y define the number $h_{\alpha,\beta}(Y)$ by

$$h_{\alpha,\beta}(Y) = \left\{ s \in Y \mid \begin{array}{l} \alpha l_Y(s) = \beta(a_Y(s)+1) \\ (\alpha+\beta) | l_Y(s) + a_Y(s) + 1 \end{array} \right\}.$$

The following corollary is a generalization of Theorem 1.2 from [4].

Corollary 1.3. *Let α and β be arbitrary non-negative integers, such that $\alpha + \beta \geq 1$. Then we have*

$$(2) \quad \sum_{Y \in \mathcal{Y}} q^{h_{\alpha,\beta}(Y)} t^{|Y|} = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-qt^{(\alpha+\beta)i}}.$$

Proof. The proof is similar to the proof of Theorem 1.2 in [4]. We apply the results from [1, 2], in order to construct a cell decomposition of the variety $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}}$, and show that the left-hand side of (1) is equal to the left-hand side of (2). \square

We thank Ole Warnaar for suggesting this more general combinatorial identity after the paper [4] was published in arXiv.

1.1. *Organization of the paper.* In Section 2 we recall the definition of the Grothendieck ring of complex quasiprojective varieties and the properties of the natural power structure over it. Section 3 contains the proof of Theorem 1.2.

2. POWER STRUCTURE OVER THE GROTHENDIECK RING $K_0(\nu_{\mathbb{C}})$

In this section we review the definition of the Grothendieck ring of complex quasiprojective varieties and the power structure over it.

2.1. *Grothendieck ring.* The Grothendieck ring $K_0(\nu_{\mathbb{C}})$ of complex quasiprojective varieties is the abelian group generated by the classes $[X]$ of all complex quasiprojective varieties X modulo the relations:

- (1) if varieties X and Y are isomorphic, then $[X] = [Y]$;
- (2) if Y is a Zariski closed subvariety of X , then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\nu_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The class $[\mathbb{A}_{\mathbb{C}}^1] \in K_0(\nu_{\mathbb{C}})$ of the complex affine line is denoted by \mathbb{L} .

We will need the following property of the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. There is a natural homomorphism $\theta: \mathbb{Z}[z] \rightarrow K_0(\nu_{\mathbb{C}})$, defined by $z \mapsto \mathbb{L}$. This homomorphism is injective (see e.g. [11]).

2.2. *Power structure.* In [8] there was defined a notion of a power structure over a ring and there was described a natural power structure over the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. This means that for a series $A(t) = 1 + a_1 t + a_2 t^2 + \dots \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$ and for an element $m \in K_0(\nu_{\mathbb{C}})$ one defines a series $(A(t))^m \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$, so that all the usual properties of the exponential function hold.

For a series $A(t) = (1 - t)^{-1}$ and a quasiprojective variety M the series $A(t)^{[M]}$ coincides with the motivic zeta function $\zeta_{[M]}(t)$ introduced by M. Kapranov ([9]):

$$(1 - t)^{-[M]} = 1 + \sum_{i \geq 1} [S^i M] t^i,$$

where $S^i M$ is the i -th symmetric product of the variety M . There is the following generalization of this formula. Suppose that M_1, M_2, \dots and N are quasiprojective varieties. Then we have

$$(3) \quad \left(1 + \sum_{i \geq 1} [M_i] t^i \right)^{[N]} = 1 + \sum_{n \geq 1} X_n t^n, \quad \text{where}$$

$$X_n = \sum_{\sum_{i \geq 1} i d_i = n} \left[\left((N^{\sum d_i} \setminus \Delta) \times \left(\prod M_i^{d_i} \right) \right) / \prod S_{d_i} \right].$$

Here Δ is the "large diagonal" in $N^{\sum d_i}$, which consists of $(\sum d_i)$ points of N with at least two coinciding ones. The permutation group S_{d_i} acts by permuting corresponding d_i factors in $\prod N^{d_i}$ and $\prod M_i^{d_i}$ simultaneously.

We also need the following property of the power structure over $K_0(\nu_{\mathbb{C}})$. For any $i \geq 1$ and $j \geq 0$ we have

$$(4) \quad (1 - \mathbb{L}^j t^i)^{-\mathbb{L}} = (1 - \mathbb{L}^{j+1} t^i)^{-1}.$$

It can be derived from several statements from [8] as follows. Let $a_i, i \geq 1$, and m be from the Grothendieck ring $K_0(\nu_{\mathbb{C}})$ and $A(t) = 1 + \sum_{i \geq 1} a_i t^i$. Then for any $s \geq 0$ we have

$$(5) \quad A(\mathbb{L}^s t)^m = (A(t)^m)|_{t \rightarrow \mathbb{L}^s t},$$

$$(6) \quad (1 - t)^{-\mathbb{L}^s m} = (1 - t)^{-m}|_{t \rightarrow \mathbb{L}^s t}.$$

Formula (5) follows from Statement 2 in [8] and equation (6) follows from Statement 3 in [8]. Also for any $s \geq 1$ we have (see [8])

$$(7) \quad A(t^s)^m = (A(t)^m)|_{t \rightarrow t^s}.$$

Obviously, formula (4) follows from (5), (6) and (7).

3. PROOF OF THEOREM 1.2

Using the $(\mathbb{C}^*)^2$ -action on $(\mathbb{C}^2)^{[n]}$ and the results from [1, 2] one can easily construct a cell decomposition of $((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$. Thus, Theorem 1.2 is equivalent to the following formula

$$(8) \quad \sum_{n \geq 0} \left[((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha,\beta} \times \Gamma_{\alpha+\beta}} \right] t^n = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{(\alpha+\beta)i}}.$$

It is clear that equation (8) is a corollary of the following two lemmas.

Lemma 3.1. *For any $\alpha, \beta \geq 0$, such that $\alpha + \beta \geq 1$, we have*

$$\left[((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha,\beta} \times \Gamma_{\alpha+\beta}} \right] = \left[((\mathbb{C}^2)^{[n]})^{T_{0,\alpha+\beta} \times \Gamma_{\alpha+\beta}} \right].$$

Lemma 3.2. *For any $m \geq 1$ we have*

$$\sum_{n \geq 0} \left[((\mathbb{C}^2)^{[n]})^{T_{0,m} \times \Gamma_m} \right] t^n = \prod_{\substack{i \geq 1 \\ m \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{mi}}.$$

Proof of Lemma 3.1. Let $((\mathbb{C}^2)^{[n]})^{\Gamma_{\alpha+\beta}} = \coprod_i ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$ be the decomposition in the irreducible components. It is sufficient to prove that

$$\left[((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right] = \mathbb{L}^{\frac{d_i}{2}} \left[\left(((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right)^{T_{\alpha,\beta}} \right],$$

where $d_i = \dim ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$. The subvarieties $((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$ are quiver varieties of affine type $\tilde{A}_{\alpha+\beta-1}$. We prove the above equality by using the idea in [12, Proposition 5.7].

Let $\left(((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \right)^{T_{\alpha,\beta}} = \coprod_j ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$ be the decomposition in the irreducible components. Consider the \mathbb{C}^* -action on $(\mathbb{C}^2)^{[n]}$ induced by the homomorphism $\mathbb{C}^* \rightarrow (\mathbb{C}^*)^2, t \mapsto (t^\alpha, t^\beta)$. Define the sets $C_{i,j}$ by

$$C_{i,j} = \left\{ z \in ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} \mid \lim_{t \rightarrow 0, t \in \mathbb{C}^*} t \cdot z \in ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}} \right\}.$$

From [1, 2] it follows that the set $C_{i,j}$ is a locally trivial fiber bundle over $((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$ with an affine space as a fiber. Let us denote by $d_{i,j}$ the dimension of a fiber. For $p \in ((\mathbb{C}^2)^{[n]})_{i,j}^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$ the tangent space $T_p ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$ is a \mathbb{C}^* -module. Let

$$T_p ((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}} = \sum_{m \in \mathbb{Z}} H(m)$$

be the weight decomposition. It is clear that $d_{i,j} = \dim \left(\bigoplus_{m \geq 1} H(m) \right)$.

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ has the canonical symplectic form ω that is induced from the symplectic form $dx \wedge dy$ on \mathbb{C}^2 (see e.g. [13]). The form ω has weight $-\alpha - \beta$ with respect to the \mathbb{C}^* -action on $(\mathbb{C}^2)^{[n]}$. The restriction $\omega|_{((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}}$ is the canonical symplectic form on the quiver variety (see [12]). Therefore, the spaces $\bigoplus_{m \leq 0} H(m)$ and $\bigoplus_{m \geq \alpha+\beta} H(m)$ are dual with respect to this form. Obviously, the $(\alpha + \beta)$ -th root of unity $\alpha+\beta\sqrt[1]{1}$ acts trivially on $((\mathbb{C}^2)^{[n]})_i^{\Gamma_{\alpha+\beta}}$, thus, $H(m) = 0$, if $(\alpha + \beta) \nmid m$. We get $\bigoplus_{m \geq \alpha+\beta} H(m) = \bigoplus_{m \geq 1} H(m)$ and $d_{i,j} = \dim \left(\bigoplus_{m \geq 1} H(m) \right) = \frac{d_i}{2}$. This completes the proof of the lemma. \square

Proof of Lemma 3.2. Obviously, we have $((\mathbb{C}^2)^{[n]})^{T_{0,m}} = ((\mathbb{C}^2)^{[n]})^{T_{0,1}}$. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 1$, and a point $x_0 \in \mathbb{C}$ define the ideal $I_{\lambda, x_0} \subset \mathbb{C}[x, y]$ by

$$I_{\lambda, x_0} = (y^{\lambda_1}, (x - x_0)y^{\lambda_2}, \dots, (x - x_0)^{l-1}y^{\lambda_l}, (x - x_0)^l).$$

In [13] it is proved that each element $I \in ((\mathbb{C}^2)^{[n]})^{T_{0,1}}$ can be uniquely expressed as

$$I = I_{\lambda^1, x_1} \cap \dots \cap I_{\lambda^k, x_k}$$

for some distinct points $x_1, \dots, x_k \in \mathbb{C}$ and for some partitions $\lambda^1, \dots, \lambda^k$ satisfying $\sum_{i=1}^k |\lambda^i| = n$.

Denote by \mathbb{C}_x the x -axis in the plane \mathbb{C}^2 . Consider the map $\pi_n: ((\mathbb{C}^2)^{[n]})^{T_{0,1}} \rightarrow S^n \mathbb{C}_x$ defined by

$$\pi_n (I_{\lambda^1, x_1} \cap \dots \cap I_{\lambda^k, x_k}) = \sum_{i=1}^k |\lambda^i| [x_i].$$

Suppose Z is an open subset of \mathbb{C}_x . From (3) it follows that

$$\sum_{n \geq 0} [\pi_n^{-1}(S^n Z)] t^n = \left(\prod_{i \geq 1} \frac{1}{1-t^i} \right)^{[Z]}.$$

The Γ_m -action on $\mathbb{C}_x \setminus \{0\}$ is free and $(\mathbb{C}_x \setminus \{0\})/\Gamma_m \cong \mathbb{C}_x \setminus \{0\}$, therefore,

$$(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m} \cong \begin{cases} \emptyset, & \text{if } m \nmid n, \\ \pi_l^{-1}(S^l(\mathbb{C}_x \setminus \{0\})), & \text{if } n = ml. \end{cases}$$

We obtain

$$\sum_{n \geq 0} [(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m}] t^n = \left(\prod_{i \geq 1} \frac{1}{1-t^{mi}} \right)^{\mathbb{L}-1}.$$

Therefore, we get

$$\begin{aligned} \sum_{n \geq 0} [((\mathbb{C}^2)^{[n]})^{T_{0,1} \times \Gamma_m}] t^n &= \left(\sum_{n \geq 0} [\pi_n^{-1}(n[0])] t^n \right) \left(\sum_{n \geq 0} [(\pi_n^{-1}(S^n(\mathbb{C}_x \setminus \{0\})))^{\Gamma_m}] t^n \right) = \\ &= \left(\prod_{i \geq 1} \frac{1}{1-t^i} \right) \left(\prod_{i \geq 1} \frac{1}{1-t^{mi}} \right)^{\mathbb{L}-1} = \prod_{i \geq 1} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{mi}}. \end{aligned}$$

The lemma is proved. \square

The theorem is proved.

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