Conway–Coxeter friezes and mutation: a survey

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Abstract In this survey article we explain the intricate links between Conway-Coxeter friezes and cluster combinatorics. More precisely, we provide a formula, relying solely on the shape of the frieze, describing how each individual entry in the frieze changes under cluster mutation. Moreover, we provide a combinatorial formula for the number of submodules of a string module, and with that a simple way to compute the frieze associated to a fixed cluster tilting object in a cluster category of Dynkin type *A* in the sense of Caldero and Chapoton.

Key words: AR-quiver, Cluster category, cluster mutation, cluster-tilted algebra, frieze pattern, Caldero-Chapoton map, string module.

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [13]. A key motivation was to provide an algebraic framework for phenomena observed in the study of dual canonical bases for quantised enveloping algebras and in total positivity for reductive groups.

Cluster categories were introduced in 2005, [4], [7] to give a categorical interpretation of cluster algebras. The following table shows the beautiful interplay and correspondences between cluster algebras and cluster categories in type *A*. Note that the correspondences between the first and second column hold more generally, not only in type *A*: Caldero and Chapoton [6] have provided a formal link between cluster categories and cluster algebras by introducing what is now most commonly known as the *Caldero Chapoton map* (short: CC-map) or *cluster character*. Fixing a cluster tilting object (which takes on the role of the initial cluster), it associates to each indecomposable in the

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cluster category a unique cluster variable in the associated cluster algebra, sending the indecomposable summands of the cluster tilting object to the initial cluster.

Cluster algebra	$\leftarrow$	Cluster category	Polygon
cluster variables	CC-map	indecomposable objects	diagonals
clusters		cluster tilting objects	triangulations
mutations		mutations	flip

In the 70s, Coxeter and Conway first studied frieze patterns of numbers ([9] and [8]). When these numbers are positive integers, they showed that the frieze patterns arise from triangulations of polygons. Thus we can extend this table by a further column:

...	Polygon	Frieze
	diagonals	integers
	triangulations	sequences of 1's
	flip	??

Here the last entry is missing: the meaning of mutation or flip on the level of frieze patterns was not known until now. The purpose of this survey article is to show how to complete the picture of cluster combinatorics in the context of friezes. It is based on the paper [2] where more background on cluster categories can be found and where the proofs are included. More precisely, we determine how mutation of a cluster affects the associated frieze, thus effectively introducing the notion of a mutation of friezes that is compatible with mutation in the associated cluster algebra. This provides a useful new tool to study cluster combinatorics of Dynkin type A.

In order to deal with the mutations for friezes we will use cluster categories and generalized cluster categories as introduced by Buan, Marsh, Reineke, Reiten and Todorov [4] for hereditary algebras and by Amiot [1] more generally. In both cases, cluster categories are triangulated categories in which the combinatorics of cluster algebras receives a categorical interpretation: cluster variables correspond to rigid indecomposable objects and clusters correspond to cluster tilting objects. One of the essential features in the definition of cluster algebras is the process of mutation, which replaces one element of the cluster by another unique element such that a new cluster is created. The corresponding categorical mutation replaces an indecomposable summand of a cluster tilting object by another unique indecomposable object using approximations in the triangulated categories; this process creates another cluster tilting object which corresponds to the mutated cluster.

We now explain the different players appearing in the table above.

1.1 Frieze patterns

The notion of friezes was introduced by Coxeter [10]; it was Gauss's *pentagramma mirificum* which was the original inspiration. We recall that a *frieze* is a grid of positive integers consisting of a finite number of infinite rows: the top and bottom rows are infinite rows of 0s and the second to top and second to bottom are infinite rows of 1s as one can see on the following diagram

$$
\begin{array}{ccccccccccc}
\ldots & & 0 & & 0 & & 0 & & 0 & & \ldots \\
& 1 & & 1 & & 1 & & 1 & & 1 & \\
\ldots & & m_{-1,-1} & & m_{00} & & m_{11} & & m_{22} & & \ldots \\
& m_{-2,-1} & & m_{-1,0} & & m_{01} & & m_{12} & & m_{23} & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
& 1 & & 1 & & 1 & & 1 & & 1 & \\
\ldots & & 0 & & 0 & & 0 & & 0 & & \ldots
\end{array}
$$

The entries of the frieze satisfy the *frieze rule*: for every set of adjacent numbers arranged in a *diamond*

$$b$$
$$a \qquad d$$
$$c$$

we have

$$ad - bc = 1.$$

The sequence of integers in the first non-trivial row, $(m_{ii})_{i \in \mathbb{Z}}$, is called a *quiddity sequence*. This sequence completely determines the frieze. Each frieze is also periodic, since it is invariant under glide reflection. The order of the frieze is defined to be the number of rows minus one. It follows that each frieze of order n is n-periodic.

Among the famous results about friezes is the bijection between the friezes of order n and triangulations of a convex n-gon, which was proved by Conway and Coxeter in [9] and [8]. This was used to set the first link with cluster combinatorics using [7] and [6] by Caldero and Chapoton. Recently, frieze patterns have been generalized in several directions and found applications in various areas of mathematics, for an overview see [17].

1.2 Cluster algebras

Fomin and Zelevinsky introduced the notion of cluster algebras in [13]. Cluster algebras are commutative algebras generated by *cluster variables*; cluster variables are obtained from an *initial cluster* (of variables) by replacing one element at a time according to a prescribed rule, where the rule is given either by a skew-symmetric (or more generally skew-symmetrizable) matrix or, equivalently by a quiver with no loops nor 2-cycles. The process of replacing one element of a cluster by another unique element in order to obtain another cluster, together with the prescribed change of the quiver, is called *mutation*. Finite sequences of iterated mutations create new clusters and new cluster variables; all cluster variables are obtained in such a way.

The process of such mutations may never stop, however if the quiver is of Dynkin type, then by a theorem of Fomin and Zelevinsky, this process stops and one obtains a finite number of cluster variables [14]. Among those cluster algebras, the best behaved and understood are the cluster algebras of type A. The clusters of the cluster algebra of type A_{n-3} are in bijection with the triangulations of a convex n-gon, for $n \geq 3$. This is exactly what is employed in this work in order to relate and use cluster categories, via triangulations of an n-gon, so that we can describe the mutations of friezes of order n. Since we will also be dealing with quivers Q' which are mutation equivalent to the quivers of type A_n and may have nontrivial potential, we need to consider generalized cluster categories $\mathscr{C}_{(Q',W)}$, which are shown to be triangle equivalent to $\mathscr{C}_Q$ [1].

1.3 Cluster categories

Let Q be an acyclic quiver with n vertices, over an algebraically closed field. We consider the category $\mathrm{mod}\, kQ$ of (finitely generated) modules over kQ, or, equivalently, the category $\mathrm{rep}\, Q$ of representations of the quiver Q. The bounded derived category $D^b(kQ)$ can be viewed as $\cup_{i \in \mathbb{Z}} \mathrm{mod}(kQ)[i]$, with connecting morphisms.

As an example, consider the quiver

$$Q: \qquad 1 \longleftarrow 2 \longleftarrow 3$$

The module category of the path algebra kQ has six indecomposable objects up to isomorphisms, with irreducible maps between them as follows:

$$P_3$$

$$P_2 \qquad I_2$$

$$P_1 \qquad S_2 \qquad I_3$$
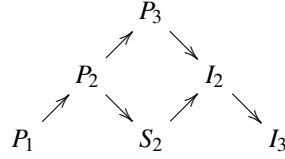
The modules P_i are indecomposable projective, the I_i are indecomposable injectives and the S_i are the simple modules, with $I_1 = P_3$, $S_1 = P_1$ and $S_3 = I_3$. The bounded derived category then looks as follows (the arrows indicate the connecting morphisms):

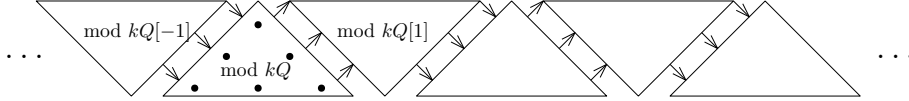Let Q be a Dynkin quiver of type A. Let $\mathscr{C}$ be the associate cluster category, which by definition is $\mathscr{C} = \mathscr{C}_Q = D^b(kQ)/\tau^{-1}[1]$ where $D^b(kQ)$ is the bounded derived category of the path algebra kQ with the suspension functor $[1]$ and the Auslander-Reiten functor τ. In this case, the specialized CC map gives a direct connection between the Auslander-Reiten quiver of the cluster category $\mathscr{C}$ with a fixed cluster tilting object T, and the associated frieze $F(T)$ in the following way: recall that each vertex of the Auslander-Reiten quiver corresponds to an isomorphism class of an indecomposable object in the cluster category. When the specialized CC map is applied to a representative of each isomorphism class and the vertex is labeled by that value, one only needs to complete those rows by the rows of 1s and 0s at the top and bottom in order to obtain a frieze, cf. [6, Proposition 5.2].

2 From cluster categories to frieze patterns

Let $\mathscr{C}$ be a cluster category, let T be a cluster tilting object and let $B_T = \mathrm{End}_{\mathscr{C}}(T)$ be the endomorphism algebra, which is also called a *cluster-tilted algebra*. The module category $\mathrm{mod}(B_T)$ is shown to be equivalent to the quotient category $\mathscr{C}/\mathrm{add}(T[1])$ of the cluster category. This result by Buan, Marsh and Reiten is used in a very essential way: each indecomposable object in $\mathscr{C}$, which is not isomorphic to a summand of $T[1]$ corresponds to an indecomposable B_T-module, preserving the structure of the corresponding Auslander-Reiten quivers; at the same time the indecomposable summands of $T[1]$ correspond to the suspensions of the indecomposable projective B_T-modules in the generalized cluster category of the algebra B_T.

When $\mathscr{C}$ is the cluster category associated to a Dynkin quiver of type A, for each cluster tilting object T, the associated specialized CC-map sends each indecomposable summand of $T[1]$ to 1 and each indecomposable B_T-module M to the number of its submodules, as we explain now. In the actual Caldero-Chapoton formula for cluster variable x_M in terms of the initial cluster variables, the coefficients are given as the Euler-Poincaré characteristics of the Grassmannians of submodules of the module M. In this expression the sum is being taken over the dimension vectors of the submodules of M. However in this set-up, since all indecomposable B_T-modules are string modules that have dimension at most one at every vertex, all the Grassmannians are just points. The *specialized Caldero-Chapoton map* is the map we get from postcomposing the CC-map associated to T with the specialization of the initial cluster variables to one. It will be denoted by ρ_T throughout the paper. It is given by the following formula:

$$\rho_T(M) = \begin{cases} 1 & \text{if } M = T_i[1] \ , \\ \sum_{\underline{e}} \chi(\mathrm{Gr}_{\underline{e}}(M)) & \text{if } M \text{ is a } B_T\text{-module.} \end{cases}$$

Here $\mathrm{Gr}_{\underline{e}}(M))$ is the Grassmannian of submodules of M with dimension vector $\underline{e}$. Hence the sum is equal to the number of submodules and the values of the specialized CC-map are positive integers. The values of the specialized CC-map are now entered in the AR-quiver of the cluster category $\mathscr{C}$ at the places of the corresponding indecomposable objects. The image of this generalized CC-map only needs to be completed with the rows of 1s and 0s above and below in order to obtain the *frieze associated to the cluster tilting object T*, denoted by $F(T)$.

Since the generalized CC-map for cluster categories of Dynkin type A is given in terms of the number of submodules of B_T-modules, the first goal of the paper is to give a formula for the number of submodules. This is determined by the following result, hence providing a combinatorial formula for the number of submodules of any given indecomposable B_T-module. Its proof can be found in [2, Section 4]. We recall that each B_T-module is a string module and hence has a description in terms of the lengths of the individual legs. Let $(k_1, \ldots, k_m)$ denote these lengths, cf. Figure 1. We further denote by $s(M)$ the number of submodules of a B_T-module M. In order to state the formula for the number of submodules, we say that a subset I of $\{1, \ldots, m\}$ is *admissible* if, assuming that $I = \{i_1, \ldots, i_l\}$ is ordered by $i_1 < \ldots < i_l$, any two consecutive numbers i_j, i_{j+1} are either of parity even-odd or odd-even.
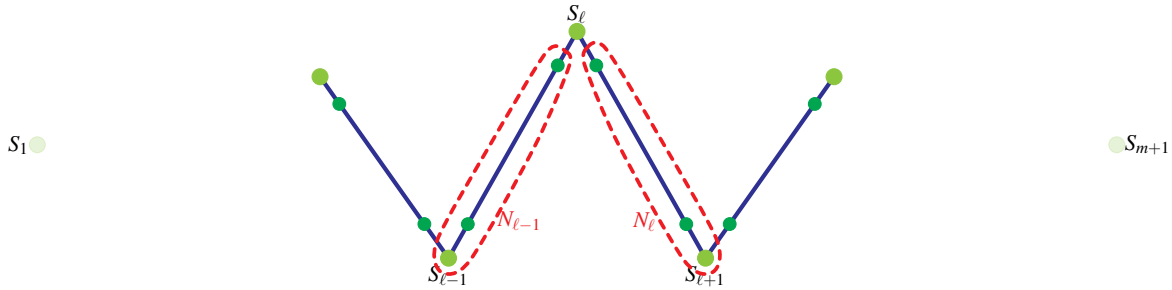

Fig. 1 A string module $M = (k_1, \ldots, k_m)$ with legs N_l.

Theorem 2.1. *Let M be an indecomposable B_T-module, of shape $(k_1, \ldots, k_m)$. Then the number of submodules of M is given as:*

$$s(M) = 1 + \sum_{j=0}^{m} \sum_{|I|=m-j} \prod_{i \in I} k_i ,$$

where the second sum runs over all admissible subsets I of $\{1, \ldots, m\}$.

Using the position of the module in the AR-quiver and the information about the positions of the indecomposable projective B_T-modules, the procedure for finding the numerical invariants $(k_1, \ldots, k_m)$ of the module is given in [2, Section 2]. This purely combinatorial way of computing the numbers of submodules is the basis for computing the associated friezes, and eventually, mutations of friezes.

Remark 2.2. Let $\mathscr{C}$ is the cluster category associated to a Dynkin quiver of type A and let T be a cluster tilting object in $\mathscr{C}$. Then for each indecomposable B_T-module M we have $\rho_T(M) = s(M)$. Theorem 2.1 thus gives us a combinatorial way to compute the specialized Caldero-Chapoton map.

We end this section by giving an example illustrating the frieze pattern obtained through the specialized CC-map.

Example 2.3. We now illustrate several notions on the example of the cluster category $\mathscr{C}_{A_{11}}$: the Auslander-Reiten quiver of $\mathscr{C}_{A_{11}}$, a cluster tilting object T, the cluster-tilted algebra B_T and the Auslander-Reiten quiver of the generalized cluster category of B_T where the modules are given by their composition factors. The Auslander-Reiten quiver of $\mathscr{C}_{A_{11}}$ is the quotient of the Auslander-Reiten quiver of $D^b(kA_{11})$ by the action of $\tau^{-1}[1]$, a fundamental domain which is depicted in black below. We pick the cluster tilting object $T = \bigoplus_{i=1}^{11} T_i$ whose indecomposable summands are marked with circles:
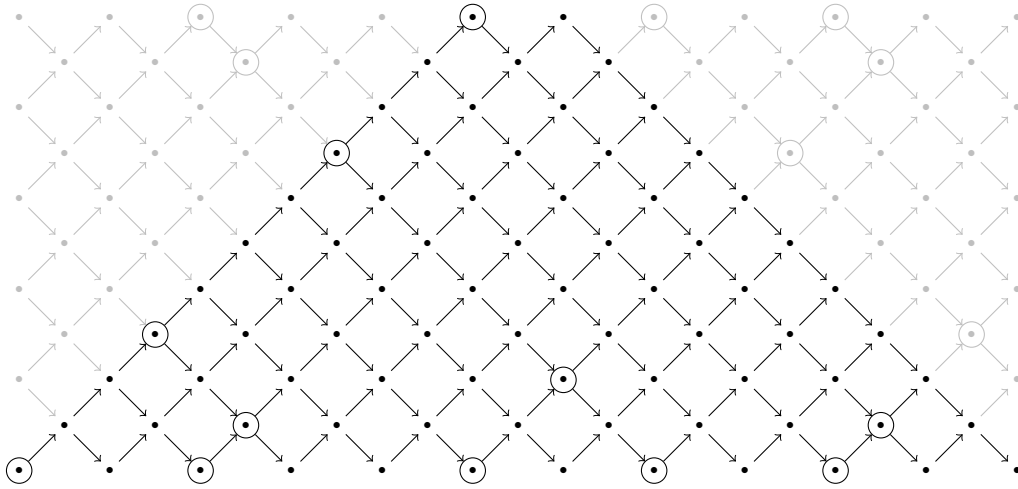
Consider the cluster-tilted algebra $B_T = \text{End}_{\mathscr{C}_{A_{11}}}(T)$. Then $B_T = kQ/I$, where Q is the quiver
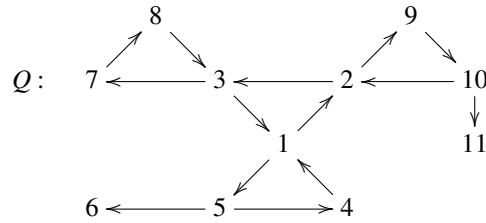
and I is the ideal generated by the directed paths of length 2 which are part of the same 3-cycle. We refer the reader to [5] for a detailed description of cluster-tilted algebras of Dynkin type A.

We can view $\text{mod}(B_T)$ as a subcategory of $\mathscr{C}_{A_{11}}$ and label the indecomposable objects in $\mathscr{C}_{A_{11}}$ by modules and shifts of projective modules respectively:
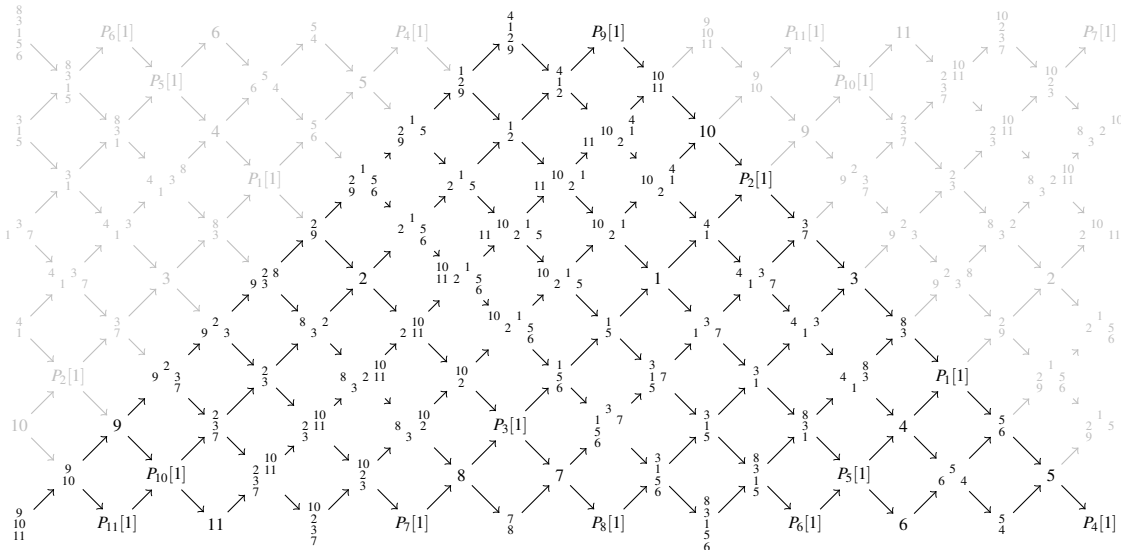
In this picture the direct summands of T correspond to the indecomposable projective modules P_i of B_T (which lie on the right of the $P_i[1]$ in the picture). The specialized CC-map replaces each vertex labelled by a module, by the number of its submodules and the shifts of projectives by 1s. Adding in the first two and last two rows of 0s and 1s gives rise to the associated frieze $F(T)$:

```
0       0       0       0       0       0       0       0       0       0       0       0
    1       1       1       1       1       1       1       1       1       1       1       1
6       1       2       3       1       5       1       4       1       2       5       1
    5       1       5       2       4       4       3       3       1       9       4       2
4       4       2       3       7       3      11       2       2       4       7       7
    3       7       1      10       5       8       7       1       7       3      12       3
5       5       3       3       7      13       5       3       3       5       5       5
    8       2       8       2      18       8       2       8       2       8       2      18
3       3       5       5       5      11       3       5       5       3       3       7
    1       7       3      12       3       4       7       3       7       1      10       5
2       2       4       7       7       1       9       4       4       2       3       7
    3       1       9       4       2       2       5       5       1       5       2       4
4       1       2       5       1       3       1       6       1       2       3       1
    1       1       1       1       1       1       1       1       1       1       1       1
0       0       0       0       0       0       0       0       0       0       0       0
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3 Description of the regions in the frieze

The quiver of a triangulation.

Let $\mathscr{T}$ be a triangulation of an $(n+3)$-gon, and let the diagonals be labeled by $1, 2, \ldots, n$. We recall that the quiver $Q_{\mathscr{T}}$ of the triangulation $\mathscr{T}$ is defined as follows: the vertices of $Q_{\mathscr{T}}$ are the labels $\{1, 2, \ldots, n\}$. There is an arrow $i \to j$ in case the diagonals share an endpoint and the diagonal i can be rotated clockwise to diagonal j (without passing through another diagonal incident with the common vertex). This is illustrated in Example 3.2 and Figure 5 below.

Let $B = B_{\mathscr{T}}$ be the path algebra of $Q_{\mathscr{T}}$ modulo the relations arising from triangles in $Q_{\mathscr{T}}$: whenever α, β are two successive arrows in an oriented triangle in $Q_{\mathscr{T}}$, their composition is 0. Let P_x be the indecomposable projective B-module associated to the vertex x and S_x its simple top. Let

$$T = \oplus_{x \in \mathscr{T}} P_x.$$

We considered T as an object of the generalized cluster category $\mathscr{C} = \mathscr{C}_B$. Then T is a cluster tilting object in $\mathscr{C}$ and $B \cong \mathrm{End}_{\mathscr{C}}(T)$. Hence B is a cluster-tilted algebra, called the cluster-tilted algebra associated to the triangulation $\mathscr{T}$. We can extend this to an object in the Frobenius category $\mathscr{C}_f$ by adding the $n+3$ projective-injective summands associated to the boundary segments $[12], [23], \ldots, [n+3, 1]$ of the polygon, with irreducible maps between the objects corresponding to diagonals/edges as follows: $[i-1, i+1] \to [i, i+1]$, $[i, i+1] \to [i, i+2]$ ([16, 3, 11]). We denote the projective-injective associated to $[i, i+1]$ by Q_{x_i}. Let

$$T_f = \left(\oplus_{x \in \mathscr{T}} P_x \right) \oplus \left(Q_{x_1} \oplus \cdots \oplus Q_{x_{n+3}} \right)$$

This is a cluster tilting object of $\mathscr{C}_f$ in the sense of [15, Section 3]. Given a B-module M, by abuse of notation, we denote the corresponding objects in $\mathscr{C}$ and $\mathscr{C}_f$ by M, that is $\mathrm{Hom}_{\mathscr{C}}(T, M) = M$. In other words, an indecomposable object of $\mathscr{C}_f$ is either an indecomposable B-module or Q_{x_i} for some $i \in \{1, \ldots, n+3\}$ or of the form $P_x[1]$ for some $x \in \mathscr{T}$.

The *frieze $F(\mathscr{T})$ of the triangulation* $\mathscr{T}$ is the frieze pattern $F(T)$ for T the cluster tilting object associated to $\mathscr{T}$.

3.1 Diagonal defines a quadrilateral

Let a be a diagonal in the triangulation, $a \in \{1, 2, \ldots, n\}$. This diagonal uniquely defines a quadrilateral formed by diagonals or boundary segments. Label them b, c, d, e as in Figure 3.
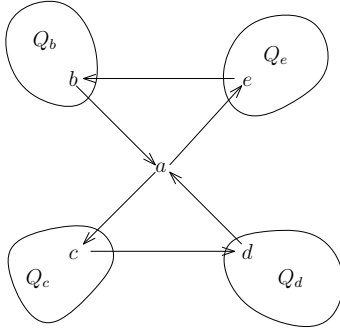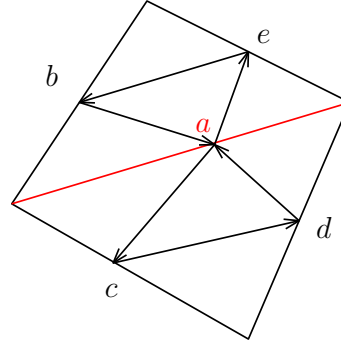

Fig. 2 Regions in quiver.

Fig. 3 Triangulation around a.

3.2 Diagonal defines two rays

Consider the entry 1 of the frieze corresponding to a. There are two rays passing through it. We go along these rays forwards and backwards until we reach the first entry 1. As the frieze has two rows of ones bounding it, we will always reach an entry 1 in each of these four directions. Going forwards and upwards: the first occurrence of 1 corresponds to the diagonal b. Down and forwards: diagonal d. Backwards down from the entry corresponding to 1: diagonal c and backwards up: diagonal e. If we compare with the coordinate system for friezes of Section 1.1, the two rays through the object corresponding to the diagonal $a = [kl]$ are the entries $m_{i,l}$ (with i varying) and $m_{k,j}$ (with j varying).

In the frieze or in the AR quiver, we give the four segments between the entry 1 corresponding to a and the entries corresponding to b, c, d and e names (see Figure 4 for a larger example containing these paths). Whereas a is always a diagonal, b, c, d, e may be boundary segments. If b is a diagonal, the ray through $P_a[1]$ goes through $P_b[1]$, and if b is a boundary segment, say $b = [i, i+1]$ (with $a = [ij]$) this ray goes through Q_{x_i}. By abuse of notation, it will be more convenient to write this projective-injective as $P_b[1]$ or as $P_{x_i}[1]$ (if we want to emphasize that it is an object of the Frobenius category $\mathscr{C}_f$ that does not live in $\mathscr{C}$).

Let $\mathfrak{e}$ and $\mathfrak{c}$ denote the unique sectional paths in $\mathscr{C}_f$ starting at $P_a[1]$ and ending at $P_b[1]$ and $P_d[1]$ respectively, but not containing $P_b[1]$ or $P_d[1]$. Similarly, let $\mathfrak{b}$ and $\mathfrak{d}$ denote the sectional paths in $\mathscr{C}_f$ starting at $P_e[1]$ and $P_c[1]$ respectively and ending at $P_a[1]$, not containing $P_e[1]$, $P_c[1]$, see Figure 4.

Note that b and d are opposite sides of the quadrilateral determined by a. In particular, the corresponding diagonals do not share endpoints. In other words, $P_b[1]$ and $P_d[1]$ do not lie on a common ray in the AR quiver. So by the combinatorics of $\mathscr{C}_f$ there exist two distinct sectional paths starting at $P_b[1], P_d[1]$. These sectional paths both go through S_a. Let $\mathfrak{c}^a, \mathfrak{e}^a$ denote these paths starting at $P_b[1]$ and at $P_d[1]$, up to S_a, but not including $P_b[1], P_d[1]$

respectively. Observe that the composition of $\mathfrak{e}$ with $\mathfrak{c}^a$ and the composition of $\mathfrak{c}$ with $\mathfrak{e}^a$ are not sectional, see Figure 4. Similarly, let $\mathfrak{d}_a, \mathfrak{b}_a$ denote the two distinct sectional paths starting at S_a and ending at $P_e[1], P_c[1]$ respectively but not including $P_e[1], P_c[1]$. Note that the composition of $\mathfrak{c}^a$ with $\mathfrak{b}_a$ and the composition of $\mathfrak{e}^a$ with $\mathfrak{d}_a$ are not sectional.

3.3 Diagonal defines subsets of indecomposables

For x a diagonal in the triangulation $\mathscr{T}$ and P_x the corresponding projective indecomposable, we write $\mathscr{X}$ for the set of indecomposable B-modules having a non-zero homomorphism from P_x into them, $\mathscr{X} = \{M \in \text{ind}\,B \mid \text{Hom}(P_x, M) \neq 0\}$. Given a B-module M, its *support* is the full subquiver supp(M) of $Q_{\mathscr{T}}$ generated by all vertices x of $Q_{\mathscr{T}}$ such that $M \in \mathscr{X}$. It is well known that the support of an indecomposable module is connected.

If x is a boundary segment, we set $\mathscr{X}$ to be the empty set (there is no projective indecomposable associated to x, so there are no indecomposables reached).

We use the notation above to describe the regions in the frieze. Thus, if x, y are diagonals or boundary segments, we write $\mathscr{X} \cap \mathscr{Y}$ for the indecomposable objects in $\mathscr{C}$ that have x and y in their support.
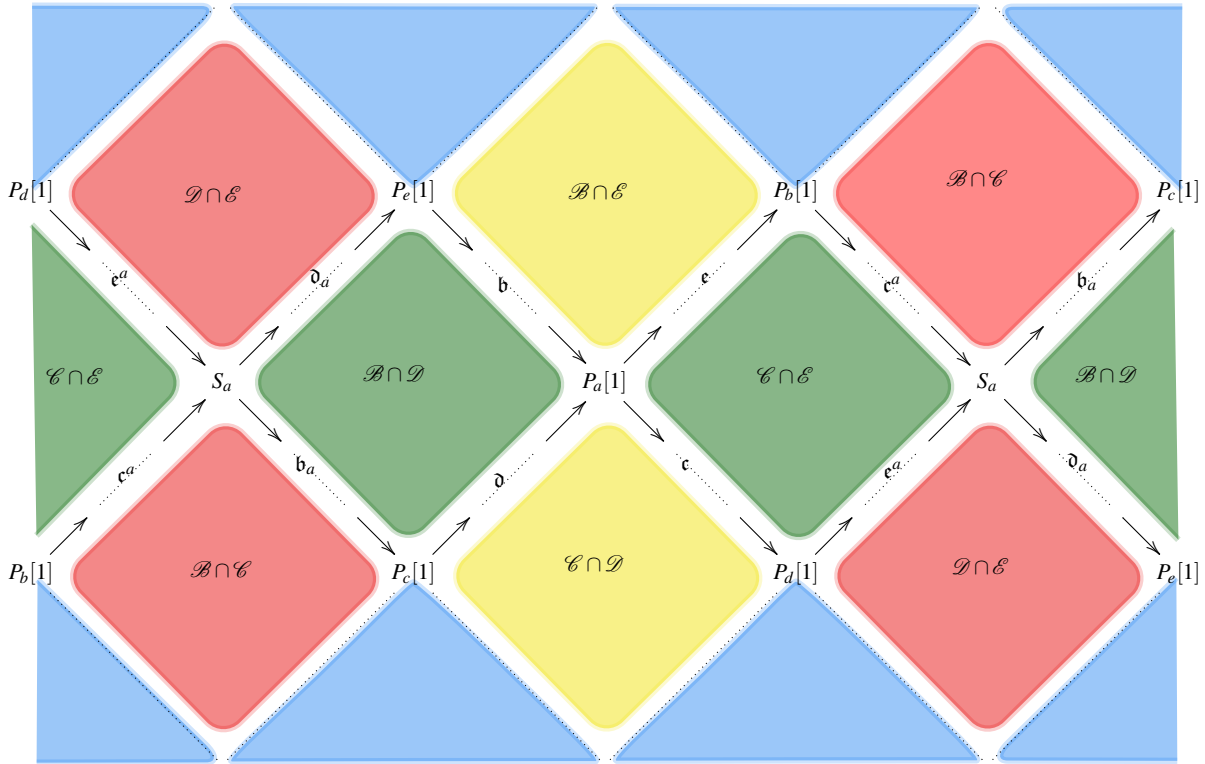

Fig. 4 Regions in the AR quiver determined by $P_a[1]$.

Remark 3.1. Let M be an indecomposable B-module in $\mathscr{X} \cap \mathscr{Y}$ such that there exists a (unique) arrow $\alpha : x \to y$ in the quiver. It follows that the right action of the element $\alpha \in B$ on M is nonzero, that is $M\alpha \neq 0$.

By the remark above we have the following equalities. Note that none of the modules below are supported at a, because the same remark would imply that such modules are supported on the entire 3-cycle in $Q_{\mathscr{T}}$ containing a. However, this is impossible as the composition of any two arrows in a 3-cycles is zero in B. We have

$$\mathscr{B} \cap \mathscr{E} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } e \to b\}$$

$$\mathscr{C} \cap \mathscr{D} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } c \to d\}$$

Moreover, since the support of an indecomposable B-module forms a connected subquiver of Q, we also have the following equalities.

$$\mathscr{B} \cap \mathscr{C} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } b \to a \to c\}$$

$$\mathscr{D} \cap \mathscr{E} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } d \to a \to e\}$$

$$\mathscr{B} \cap \mathscr{D} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } b \to a \leftarrow d\}$$

$$\mathscr{C} \cap \mathscr{E} = \{M \in \operatorname{ind} B \mid M \text{ is supported on } c \leftarrow a \to e\}$$

Finally, using similar reasoning it is easy to see that the sets described above are disjoint. Next we describe modules lying on sectional paths defined in section 3.2. First, consider sectional paths starting or ending in $P_a[1]$, then we claim that

$$\mathfrak{i} = \{M \in \operatorname{ind} B \mid i \in \operatorname{supp}(M) \subset Q_i\} \cup \{P_a[1]\}$$

for all $i \in \{b,c,d,e\}$, for Q_i the subquiver of Q containing i, as in Figure 2. We show that the claim holds for $i = b$, but similar arguments can be used to justify the remaining cases. Note, that it suffices to show that a module $M \in \mathfrak{b}$ is supported on b but it is not supported on e or a. By construction the sectional path $\mathfrak{b}$ starts at $P_e[1]$, so $0 = \operatorname{Hom}(\tau^{-1}P_e[1], M) = \operatorname{Hom}(P_e, M)$. On the other hand, $\mathfrak{b}$ ends at $P_a[1]$, so $0 = \operatorname{Hom}(M, \tau P_a[1]) = \operatorname{Hom}(M, I_a)$, where I_a is the injective B-module at a. This shows that M is not supported at e or a. Finally, we can see from Figure 4 that M has a nonzero morphism into $\tau P_b[1] = I_b$, provided that b is not a boundary segment. However, if b is a boundary segment, then $\mathfrak{b} \cap \operatorname{Ob}(\operatorname{mod} B) = \emptyset$ and we have $\mathfrak{b} = \{P_a[1]\}$. Conversely, it also follows from Figure 4 that every module M supported on b and some other vertices of Q_b lies on $\mathfrak{b}$. This shows the claim.

Now consider sectional paths starting or ending in S_a. Using similar arguments as above we see that

$$\mathfrak{i}^a = \{M \in \operatorname{ind} B \mid a \in \operatorname{supp}(M) \subset Q_i^a\}$$

for $i \in \{c,e\}$ and

$$\mathfrak{i}_a = \{M \in \operatorname{ind} B \mid a \in \operatorname{supp}(M) \subset Q_i^a\}$$

for $i \in \{b,d\}$, where Q_i^a is the full subquiver of Q on vertices of Q_i and the vertex a.

Finally, we define $\mathscr{F}$ to be the set of indecomposable objects of $\mathscr{C}_f$ that do not belong to

$$\mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \cup \{P_a[1]\}.$$

The region $\mathscr{F}$ is a succession of wings in the AR quiver of $\mathscr{C}_f$, with peaks at the $P_x[1]$ for $x \in \{b,c,d,e\}$. That is, in the AR quiver of $\mathscr{C}_f$ consider two neighboured copies of $P_a[1]$ with the four vertices $P_b[1]$, $P_c[1]$, $P_d[1]$, $P_e[1]$. Then the indecomposables of $\mathscr{F}$ are the vertices in the triangular regions below these four vertices, including them (as their peaks). By the glide symmetry, we also have these regions at the top of the frieze. In Figure 4, the wings are the shaded unlabelled regions at the boundary. It corresponds to the diagonals inside and bounding the shaded regions in Figure 5. We will see in the next section that objects in $\mathscr{F}$ do not change under mutation of T_f at $P_a[1]$.

Example 3.2. We consider the triangulation $\mathscr{T}$ of a 14-gon, see the left hand of Figure 5 and the triangulation $\mathscr{T}' = \mu_1(\mathscr{T})$ obtained by flipping diagonal 1.

The quivers of $\mathscr{T}$ and of $\mathscr{T}'$ are given below. Note that the quiver Q is the same as in Example 2.3.
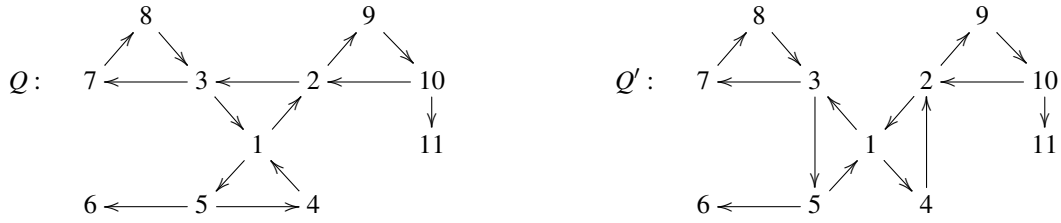
Figure 6 shows the Auslander-Reiten quiver of the cluster category $\mathscr{C}_f$ for Q.
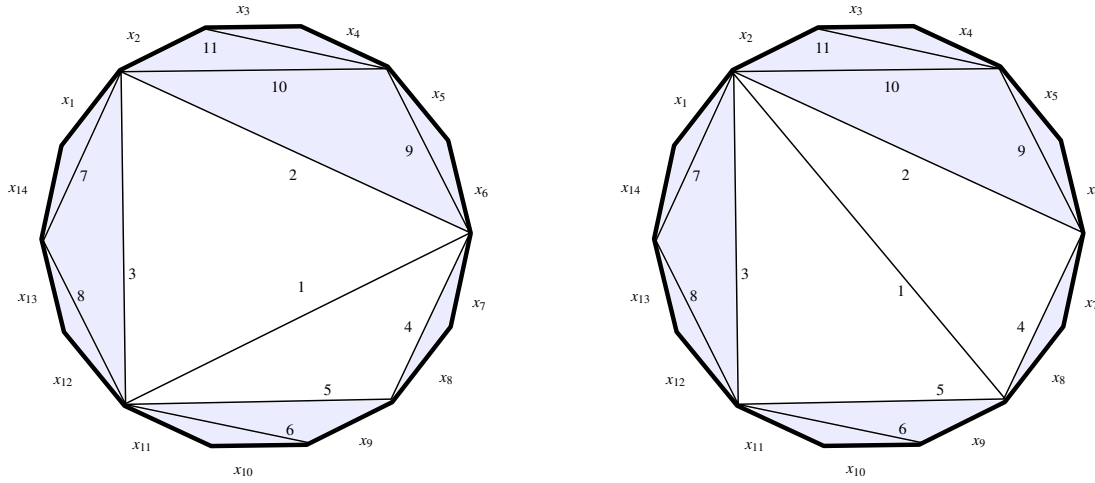In Figure 7 (Section 4.2), the frieze patterns of T and of T' are given.

Fig. 5 Triangulations $\mathscr{T}$ and $\mathscr{T}' = \mu_1(\mathscr{T})$

4 Mutating friezes

Assume now that our cluster tilting object T in $\mathscr{C}$ is of the form $T = \bigoplus_{i=1}^n T_i$, where the T_i are mutually non-isomorphic indecomposable objects. Mutating T at T_i for some $1 \leq i \leq n$ yields a new cluster tilting object $T' = T/T_i \oplus T_i'$, to which we can associate a new frieze $F(T')$. In terms of the frieze, we can think of this mutation as a mutation at an entry of value 1, namely the one sitting in the position of the indecomposable object $T_i[1]$.

We describe how, using graphic calculus, we can obtain each entry of the frieze $F(T')$ independently and directly from the frieze $F(T)$, thus effectively introducing the concept of mutations of friezes at entries of value 1 that do not lie in one of the two constant rows of 1s bounding the frieze pattern.

We are able to give an explicit formula of how each entry in the frieze $F(T)$ changes under mutation at the entry corresponding to T_i, see Theorem 4.7 below. We observe that each frieze can be divided into four separate regions, relative to the entry of value 1 at which we want to mutate. Each of these regions gets affected differently by mutation. The formula of the theorem relies solely on the shape of the frieze and the entry at which we mutate. It determines how each entry of the frieze individually changes under mutation.

In Section 4.2 we will describe the four separate regions in and introduce the necessary notation before stating the theorem.
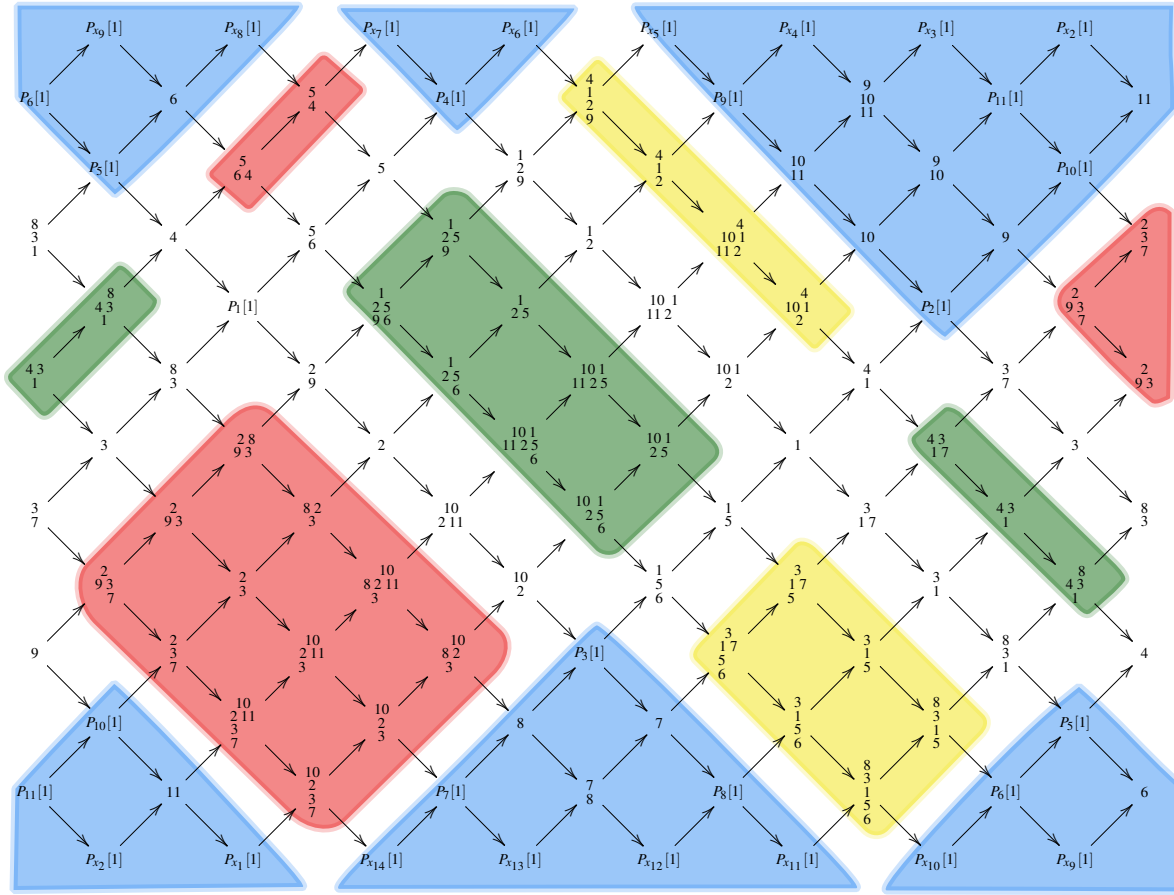
Fig. 6 AR quiver of the category $\mathscr{C}_f$ arising from Q

4.1 Frieze category

We extend ind$\mathscr{C}$ by adding an indecomposable for each boundary segment of the polygon and denote the resulting category by $\mathscr{C}_f$. Then $\mathscr{C}_f$ is the Frobenius category of maximal CM-modules categorifying the cluster algebra structure of the coordinate ring of the (affine cone of the) Grassmannian Gr(2,n) as studied in [11] and for general Grassmannians in [16, 3]. The stable category of $\mathscr{C}_f$ is equivalent to $\mathscr{C}$. We then extend the definition of ρ_T to $\mathscr{C}_f$ by setting

$$\rho_T(M) = 1 \quad \text{if } M \text{ corresponds to a boundary segment.}$$

This agrees with the extension of the cluster character to Frobenius category given by Fu and Keller, cf. Theorem [15, Theorem 3.3].

4.2 The effect of flips on friezes

The goal of this section is to describe the effect of the flip of a diagonal or equivalently the mutation at an indecomposable projective on the associated frieze. We give a formula for computing the effect of the mutation using the specialised Caldero Chapoton map. Let $\mathscr{T}$ be a triangulation of a polygon with associated quiver Q (see Sec-
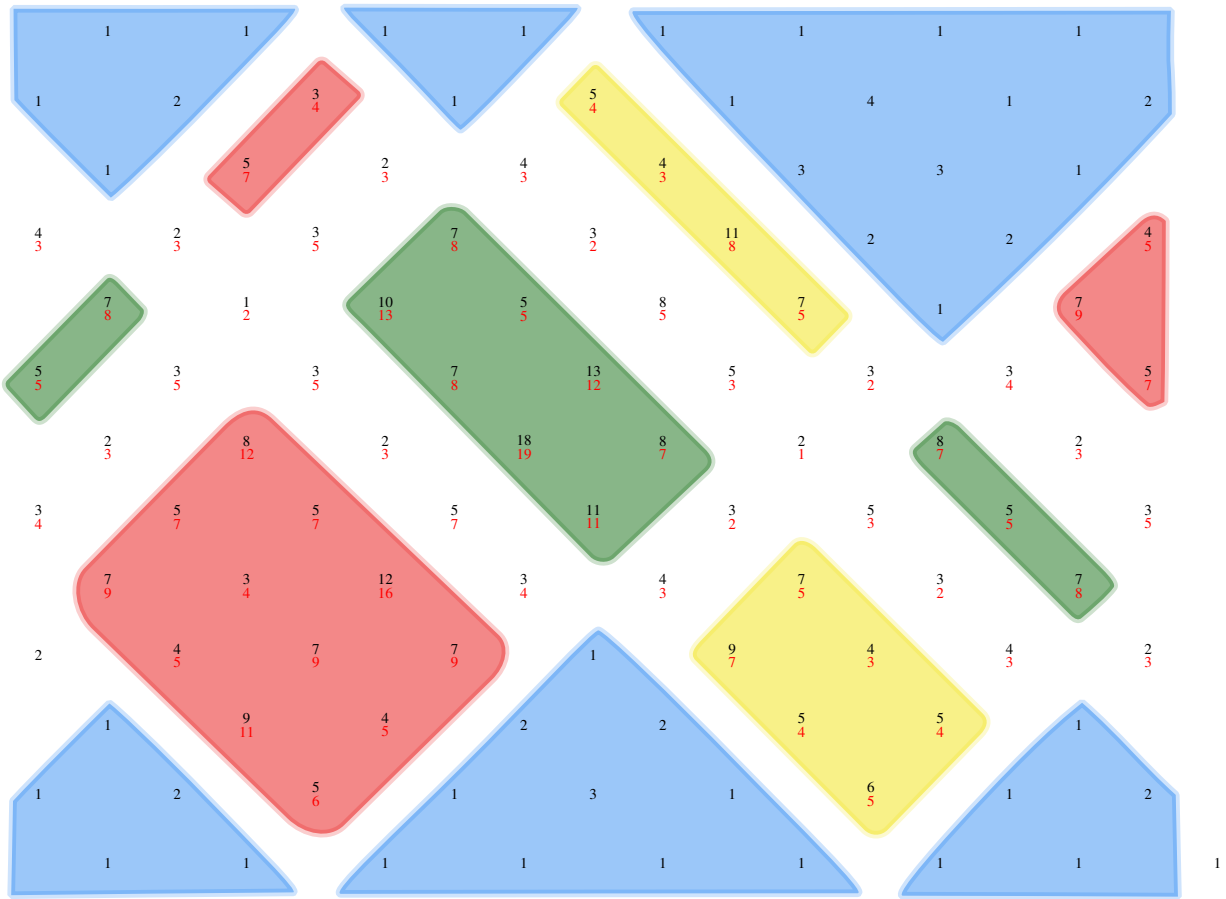
Fig. 7 Frieze pattern of Example 3.2. Red entries: after flip of diagonal 1

tion 3). The quiver Q looks as in Figure 2, where the subquivers Q_b, Q_c, Q_d, Q_e may be empty. Let $T = \bigoplus_{x \in T} P_x$ and $B = \text{End}_{\mathscr{C}} T$ be the associated cluster-tilted algebra.

Take $a \in \mathscr{T}$ and let $\mathscr{T}' = \mu_a(\mathscr{T})$ be the triangulation obtained from flipping a, with quiver $Q' = \mu_a(Q)$ (Figure 8).
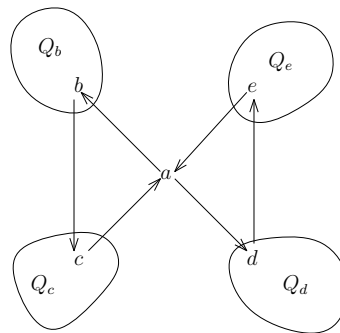

Fig. 8 Quiver after flipping diagonal a

Let B' be the algebra obtained through this, it is the cluster-tilted algebra for $T' = \oplus_{x \in \mathscr{T}'} P_x$. If M is an indecomposable B-module, we write M' for $\mu_a(M)$ in the sense of [12]. If M is an indecomposable B-module, the *entry of M* in the frieze $F(T)$ is the entry at the position of M in the frieze.

Definition 4.1. Let $\mathscr{T}$ be a triangulation of a polygon, $a \in \mathscr{T}$ and M an indecomposable object of $\mathscr{C}_f$. Then we define the *frieze difference (w.r.t. mutation at a)* $\delta_a : \operatorname{ind} \mathscr{C}_f \to \mathbb{Z}$ by

$$\delta_a(M) = \rho_{\mathscr{T}}(M) - \rho_{\mathscr{T}'}(M') \in \mathbb{Z}$$

In Section 4.3 we first describe the effect mutation has on the regions in the frieze. This gives us all the necessary tools to compute the frieze difference δ_a (Section 4.4).

4.3 Mutation of regions

Here we describe how mutation affects the regions (Section 3.3) of the frieze $F(T)$. Let $\mathscr{T}, a, B$ and $\mathscr{T}', B'$ be as above. When mutating at a, the change in support of the indecomposable modules can be described explicitly in terms of the local quiver around a. This is what we will do here. We first describe the regions in the AR quiver of $\mathscr{C}_f$ for B'.

If x is a diagonal or a boundary segment, we write

$$\mathscr{X}' = \{M \in \operatorname{ind} B' \mid \operatorname{Hom}(P_x, M) \neq 0\}$$

for the indecomposable modules supported on x.

After mutating a, the regions in the AR quiver are still determined by the projective indecomposables corresponding to the framing diagonals (or edges) b, c, d, e. The relative positions of a, b, c, d and e have changed, however it follows from [12] that except for vertex a the support of an indecomposable module at all other vertices remains the same. Therefore, the regions are now described as follows:

$$\mathscr{B}' \cap \mathscr{E}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } e \to a \to b\}$$

$$\mathscr{C}' \cap \mathscr{D}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } c \to a \to d\}$$

$$\mathscr{B}' \cap \mathscr{C}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } b \to c\}$$

$$\mathscr{D}' \cap \mathscr{E}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } d \to e\}$$

$$\mathscr{B}' \cap \mathscr{D}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } b \leftarrow a \to d\}$$

$$\mathscr{C}' \cap \mathscr{E}' = \{M \in \operatorname{ind} B' \mid M \text{ is supported on } c \to a \leftarrow e\}$$

Under the mutation at a, if a module M lies on one of the rays $\mathfrak{b}_a$, $\mathfrak{d}_a$ $\mathfrak{c}^a$ and $\mathfrak{e}^a$ then M' is obtained from M by removing support at vertex a. The modules lying on the remaining four rays gain support at vertex a after the mutation.

4.4 Mutation of frieze

We next present the main result of this section, the effect of flip on the generalized Caldero Chapoton map, i.e. the description of the frieze difference δ_a. We begin by introducing the necessary notation.

Depending on the position of an indecomposable object M we define several projection maps sending M to objects on the eight rays from Section 3.2.

Let $M \in \mathrm{ind}\,B$, and let $\mathfrak{i}$ be one of the sectional paths defined in section 3.2. Suppose $M \notin \mathfrak{i}$, then we denote by $M_{\mathfrak{i}}$ a module on $\mathfrak{i}$ if there exists a sectional path $M_{\mathfrak{i}} \to \cdots \to M$ or $M \to \cdots \to M_{\mathfrak{i}}$ in $\mathscr{C}_f$, otherwise we let $M_{\mathfrak{i}} = 0$. If $M \in \mathfrak{i}$ then we let $M_{\mathfrak{i}} = M$. In the case when it is well-defined, we call $M_{\mathfrak{i}}$ the *projection* of M onto the path $\mathfrak{i}$.

It will be convenient to write these projections in a uniform way.

Definition 4.2 (Projections). If (x,y) is one of the pairs $\{(b,c),(d,e),(b,e),(c,d)\}$, the region $\mathscr{X} \cap \mathscr{Y}$ has two paths along its boundary and two paths further backwards or forwards met along the two sectional paths through any vertex M of $\mathscr{X} \cap \mathscr{Y}$. We call the backwards projection onto the first path $\pi_1^-(M)$ and the projection onto the second path $\pi_2^-(M)$. The forwards projection onto the first path is denoted by $\pi_1^+(M)$ and the one onto the second path $\pi_2^+(M)$.

Figure 9 illustrates these projections in the case $(x,y) \in \{(b,c),(d,e)\}$.

The remaining two regions will be treated together with the surrounding paths.

Definition 4.3. The *closure of* $\mathscr{C} \cap \mathscr{E}$ is the Hom-hammock

$$\overline{\mathscr{C} \cap \mathscr{E}} = \mathrm{ind}(\mathrm{Hom}_{\mathscr{C}_f}(P_a[1],-) \cap \mathrm{Hom}_{\mathscr{C}_f}(-,S_a))$$

in $\mathscr{C}_f$ starting at $P_a[1]$ and ending at S_a. Similarly, the *closure of* $\mathscr{B} \cap \mathscr{D}$ is the Hom-hammock

$$\overline{\mathscr{B} \cap \mathscr{D}} = \mathrm{ind}(\mathrm{Hom}_{\mathscr{C}_f}(S_a,-) \cap \mathrm{Hom}_{\mathscr{C}_f}(-,P_a[1]))$$

in $\mathscr{C}_f$ starting at S_a and ending at $P_a[1]$. For $(x,y) \in \{(c,e),(b,d)\}$, the *boundary of* $\overline{\mathscr{X} \cap \mathscr{Y}}$ (or of $\mathscr{X} \cap \mathscr{Y}$) is $\overline{\mathscr{X} \cap \mathscr{Y}} \setminus (\mathscr{X} \cap \mathscr{Y})$.

Note that $\overline{\mathscr{C} \cap \mathscr{E}}$ is the union of $\mathscr{C} \cap \mathscr{E}$ with the surrounding rays and the shifted projectives $\{P_b[1],P_d[1]\}$. Analogously, $\overline{\mathscr{B} \cap \mathscr{D}}$ contains $\{P_c[1],P_e[1]\}$.
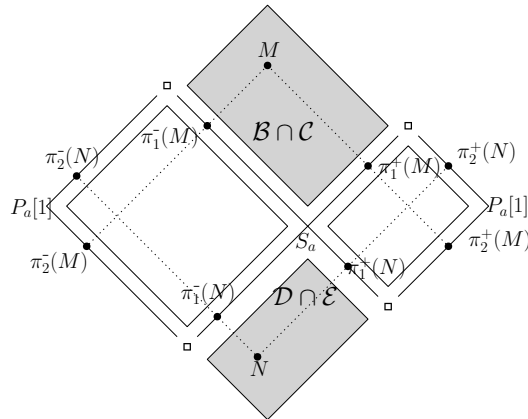

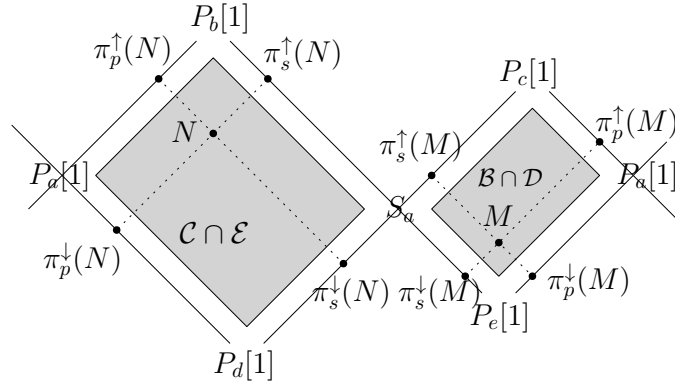**Fig. 9** Projections for $\mathscr{B} \cap \mathscr{C}$, $\mathscr{D} \cap \mathscr{E}$

Fig. 10 Projections for $\mathscr{B} \cap \mathscr{D}, \mathscr{C} \cap \mathscr{E}$

Definition 4.4 (Projections, continued). If M is a vertex of one of the two closures, we define four projections for M onto the four different "edges" of the boundary of its region: We denote the projections onto the paths starting or ending next to $P_a[1]$ by $\pi_p^\uparrow$, $\pi_p^\downarrow$ and the projections onto the paths starting or ending next to S_a by $\pi_s^\uparrow$ and $\pi_s^\downarrow$ respectively. We choose the upwards arrow to refer to the paths ending/starting near $P_b[1]$ or $P_c[1]$ and the downwards arrow to refer to paths ending/starting near $P_d[1]$ or $P_e[1]$. See Figure 10.

Remark 4.5. The statement of Theorem 4.7 is independent of the choice of $\uparrow$ (paths near $P_b[1]$ or $P_c[1]$) and $\downarrow$ in Definition 4.4 as the formula is symmetric in these expressions.

Example 4.6. If $M \in \mathfrak{e}$, we have $\pi_p^\uparrow(M) = M$, $\pi_s^\uparrow(M) = P_b[1]$, $\pi_p^\downarrow(M) = P_a[1]$ and $\pi_s^\downarrow(M) = M_{\mathfrak{e}^a}$.

For S_a we have $\pi_s^\uparrow(S_a) = \pi_s^\downarrow(S_a) = S_a$ whereas the two modules $\pi_p^\uparrow(S_a)$ and $\pi_p^\downarrow(S_a)$ are $\{P_b[1], P_d[1]\}$ or $\{P_c[1], P_e[1]\}$ depending on whether S_a is viewed as an element of $\overline{\mathscr{C} \cap \mathscr{E}}$ or of $\overline{\mathscr{B} \cap \mathscr{D}}$.

For $P_a[1]$, we have $\pi_p^\uparrow(P_a[1]) = \pi_p^\downarrow(P_a[1]) = P_a[1]$ whereas the two modules $\pi_s^\uparrow(P_a[1])$ and $\pi_s^\downarrow(P_a[1])$ are $\{P_b[1], P_d[1]\}$ or $\{P_c[1], P_e[1]\}$ These four shifted projectives evaluate to 1 under s, and so in Theorem 4.7, this ambiguity does not play a role.

With this notation we are ready to state the theorem, proved in [2, Section 6]. Recall that $s(M)$ denotes the number of submodules of a module M, cf. also Theorem 2.1.

Theorem 4.7. *Consider a frieze associated to a triangulation of a polygon. Let a be a diagonal in the triangulation. Consider the mutation of the frieze at a. Then the frieze difference $\delta_a(M)$ at the point corresponding to the indecomposable object M in the associated Frobenius category $\mathscr{C}_f$ is given by:*

If $M \in (\mathscr{B} \cap \mathscr{C}) \cup (\mathscr{D} \cap \mathscr{E})$ then

$$\delta_a(M) = (s(\pi_1^+(M)) - s(\pi_2^+(M)))(s(\pi_1^-(M)) - s(\pi_2^-(M)));$$

If $M \in (\mathscr{B} \cap \mathscr{E}) \cup (\mathscr{C} \cap \mathscr{D})$ then

$$\delta_a(M) = -(s(\pi_2^+(M)) - 2s(\pi_1^+(M)))(s(\pi_2^-(M)) - 2s(\pi_1^-(M));$$

If $M \in \overline{\mathscr{C} \cap \mathscr{E}} \cup \overline{\mathscr{B} \cap \mathscr{D}}$ then

$$\delta_a(M) = s(\pi_s^\downarrow(M))s(\pi_p^\downarrow(M)) + s(\pi_s^\uparrow(M))s(\pi_p^\uparrow(M)) - 3\,s(\pi_p^\downarrow(M))s(\pi_p^\uparrow(M));$$

If $M \in \mathscr{F}$ then

$$\delta_a(M) = 0.$$

Note, that given a frieze and an indecomposable M in one of the six regions $\mathscr{X} \cap \mathscr{Y}$, it is easy to locate the entries required to compute the frieze difference $\delta_a(M)$. We simply need to find projections onto the appropriate rays in the frieze. In this way, we do not need to know the precise shape of the modules appearing in the formulas of Theorem 4.7.

Example 4.8. Let $\mathscr{C}_f$ be the category given in Example 3.2. We consider three possibilities for M below.

If $M = \begin{smallmatrix} & 4 & \\ 10 & & 1 \\ & 11 & 2 \end{smallmatrix}$ then we know by Figure 7 that $s(M) = 11$ and $s(M') = 8$. On the other hand, we see from Figure 6 that $M \in \mathscr{B} \cap \mathscr{C}$. Theorem 4.7 implies that

$$
\begin{aligned}
\delta_a(M) = s(M) - s(M') &= (s(M_{b_a}) - s(M_b))(s(M_{c^a}) - s(M_c)) \\
&= (s(\begin{smallmatrix}4\\1\end{smallmatrix}) - s(4))(s(\begin{smallmatrix}10\,1\\11\,2\end{smallmatrix}) - s(\begin{smallmatrix}10\\11\,2\end{smallmatrix})) \\
&= (3-2)(8-5) = 3.
\end{aligned}
$$

Similarly, if $M = \begin{smallmatrix}8\,2\\3\end{smallmatrix}$, then $M \in \mathscr{C} \cap \mathscr{D}$ with $s(M) = 5$ and $s(M') = 7$. The same theorem implies that

$$
\begin{aligned}
\delta_a(M) = s(M) - s(M') &= -(s(M_{c^a}) - 2s(M_c))(s(M_{d^a}) - 2s(M_d)) \\
&= -(s(\begin{smallmatrix}1\\2\end{smallmatrix}) - 2s(2))(s(\begin{smallmatrix}8\\3\\1\end{smallmatrix}) - 2s(\begin{smallmatrix}8\\3\end{smallmatrix})) \\
&= -(3-4)(4-6) = -2.
\end{aligned}
$$

Finally, if $M = \begin{smallmatrix}10\,1\\2\,5\\6\end{smallmatrix}$, then $M \in \mathscr{C} \cap \mathscr{E}$. We also know that $s(M) = s(M') = 11$. By the third formula in Theorem 4.7, we have

$$
\begin{aligned}
\delta_a(M) = s(M) - s(M') &= s(M_{e^a})s(M_c) + s(M_{c^a})s(M_e) - 3s(M_e)s(M_c) \\
&= s(\begin{smallmatrix}1\\5\\6\end{smallmatrix})s(\begin{smallmatrix}10\\2\end{smallmatrix}) + s(\begin{smallmatrix}10\,1\\2\end{smallmatrix})s(\begin{smallmatrix}5\\6\end{smallmatrix}) - 3s(\begin{smallmatrix}5\\6\end{smallmatrix})s(\begin{smallmatrix}10\\2\end{smallmatrix}) \\
&= 4 \cdot 3 + 5 \cdot 3 - 3 \cdot 3 \cdot 3 = 0.
\end{aligned}
$$

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