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Ambrozio, L, Carlotto, A and Sharp, B orcid.org/0000-0002-7238-4993 (2016) Compactness of the Space of Minimal Hypersurfaces with Bounded Volume and p-th Jacobi Eigenvalue. Journal of Geometric Analysis, 26 (4). pp. 2591-2601. ISSN 1050-6926

https://doi.org/10.1007/s12220-015-9640-4

COMPACTNESS OF THE SPACE OF MINIMAL HYPERSURFACES WITH BOUNDED VOLUME AND p–TH JACOBI EIGENVALUE

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Abstract. Given a closed Riemannian manifold of dimension less than eight, we prove a compactness result for the space of closed, embedded minimal hypersurfaces satisfying a volume bound and a uniform lower bound of the first eigenvalue of the stability operator. When the latter assumption is replaced by a uniform lower bound on the p–th Jacobi eigenvalue for $p \geq 2$ one gains strong convergence to a smooth limit submanifold away from at most $p-1$ points.

1. Introduction

Let (N^{n+1}, g) be a closed Riemannian manifold and let us denote by $\mathfrak{M}^n(N)$ the class of closed, smooth and embedded minimal hypersurfaces[1] $M \subset N$. By the seminal work of Almgren-Pitts [7] (and Schoen-Simon [8]) we know that such a set $\mathfrak{M}^n(N)$ is not empty whenever $2 \leq n \leq 6$, the higher-dimensional counterpart of their method being obstructed by the occurrence of singularities of mass-minimizing currents. Over the last three decades, several existence results have been proven by means of equivariant constructions, desingularization, gluing and more recently, high-dimensional min-max techniques that ensure that in many cases of natural geometric interest the set $\mathfrak{M}^n(N)$ contains plenty of elements. Most remarkably, it was proven by Marques and Neves in [6] that when $2 \leq n \leq 6$ and the Ricci curvature of g is positive, then N contains at least countably many closed, embedded minimal hypersurfaces. Thus, one is led to investigate the *global structure* of the class $\mathfrak{M}^n(N)$ and the most basic question in this sense is perhaps that of finding geometrically natural and meaningful conditions that ensure the compactness of subsets of this space. In the three-dimensional scenario, namely for $n = 2$, and under an assumption on the positivity of the ambient Ricci curvature a prototypical statement was obtained, in 1985, by Choi and Schoen:

Theorem 1.1. *[3] Let N be a compact 3-dimensional manifold with positive Ricci curvature. Then the space of compact embedded minimal surfaces of fixed topological type in N is compact in the C^k topology for any $k \geq 2$. Furthermore, if N is real analytic, then this space is a compact finite-dimensional real analytic variety.*

Roughly speaking, the idea behind this result is that a uniform bound on the genus suffices for controlling both the area (as had already been observed in [4]) and the second fundamental form of the minimal surfaces in question, in a uniform fashion.

When $n \geq 3$ new phenomena appear and a statement of this type cannot possibly be expected. Indeed, it was shown by Hsiang [5] that in $S^{n+1}, n = 3, 4, 5$ there exists a sequence of embedded minimal hyperspheres M^n that have uniformly bounded volume and converge, in the sense of varifolds, to a singular minimal subvariety with two conical singularities located at antipodal points of the ambient manifold. Based on the seminal works of

[1]Throughout this paper, we shall always tacitly assume all hypersurfaces to be connected.

Schoen-Simon-Yau [9] and Schoen-Simon [8] concerning stable minimal hypersurfaces, one is naturally led to conjecture that some sort of control on the spectrum of the Jacobi operator (together with a volume bound) should indeed suffice to obtain compactness.

A first result of this flavour, that holds true up to (and including) ambient dimension seven, was recently proven by the third-named author.

Theorem 1.2. *[11] Let N^{n+1} be a smooth, closed Riemannian manifold with $Ric_N > 0$ and $2 \leq n \leq 6$. Then given any $0 < \Lambda < \infty$ and $I \in \mathbb{N}$ the class*

$$\mathcal{I}(\Lambda, I) := \{M \in \mathfrak{M}^n(N) : \ \mathcal{H}^n(M) \leq \Lambda, \ index(M) \leq I\}$$

is compact in the C^k topology for all $k \geq 2$.

Here and above the word *compactness* is understood as single-sheeted graphical convergence to some limit $M \in \mathcal{I}(\Lambda, I)$. As the reader can see, in analogy with Theorem 1.1 here one does also need to assume positivity of the ambient Ricci curvature to derive a compactness theorem, for otherwise smooth, graphical convergence can only be ensured away from at most I points (cmp. Theorem 2.3 in [11]).

In this article, we derive the rather surprising conclusion that *no assumption on the ambient manifold* is needed in proving a strong convergence theorem provided an upper bound on the Morse index is replaced by a lower bound on the first eigenvalue of the Jacobi operator. To state our result, we need to recall a definition: in the setting described above, we will say that $M_k \to M$ in the sense of smooth graphs at $p \in M$ if there exists $\rho > 0, \eta > 0$ such that *in normal coordinates centered at p* the intersection of M_k with $B^n_\rho(0) \times B^1_\eta(0)$ consists, for k large enough, of the collection of the graphs of smooth defining functions $u^1_k, \ldots, u^l_k$ with $u^j_k \to 0$ in C^m for all $m \geq 2$ and $1 \leq j \leq l$. We remark that if $M_k \to M$ in the sense of smooth graphs away from a finite set $\mathcal{Y}$ and M is connected (and embedded) then the number of leaves of the convergence is constant.

Theorem 1.3. *Let $2 \leq n \leq 6$ and N^{n+1} a smooth, closed Riemannian manifold. Denote by $\mathfrak{M}^n(N)$ the class of closed, smooth and embedded minimal hypersurfaces $M \subset N$. Let $\lambda_p(M)$ denote the p-th eigenvalue of the Jacobi operator for $M \in \mathfrak{M}^n(N)$. Given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$, define the class*

$$\mathcal{M}_p(\Lambda, \mu) := \{M \in \mathfrak{M}^n(N) : \mathcal{H}^n(M) \leq \Lambda, \ \lambda_p(M) \geq -\mu\}.$$

Given a sequence $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$ there exists $M \in \mathcal{M}_p(\Lambda, \mu)$ such that $M_k \to M$ in the varifold sense and furthermore:

(1) if $p = 1$ then $M_k \to M$ locally in the sense of smooth graphs;
(2) if $p \geq 2$ then there exists a finite set $\mathcal{Y} = \{y_i\}_{i=1}^P$ with $P \leq p - 1$ such that the convergence $M_k \to M$ is smooth and graphical for all $x \in M \setminus \mathcal{Y}$; if the number of leaves of the convergence is one then $\mathcal{Y} = \emptyset$.

From such general assertion we can derive a strong compactness result under a purely topological assumption on the ambient manifold N.

Corollary 1.4. *Let $2 \leq n \leq 6$ and N^{n+1} a smooth, closed Riemannian manifold not containing any one-sided minimal hypersurface (which holds true, for instance, if N is simply connected). Then given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$ the class $\mathcal{M}_1(\Lambda, \mu)$ is compact in the C^k topology for all $k \geq 2$.*

Indeed, in such scenario it is of course the case that the limit hypersurface M (whose existence is ensured by Theorem 1.3) is itself two-sided and thus for k large enough M_k is a finite covering thereof, hence the conclusion comes via an elementary topological argument due to the connectedness assumption on M_k.

In particular, we deduce from this statement that (in presence of a volume bound) the Morse index of an element in $\mathcal{M}_1(\Lambda,\mu)$ is uniformly bounded from above, which seems a rather unexpected conclusion from a purely analytic viewpoint, as we are considering an infinite family of elliptic operators parametrized by minimal hypersurfaces in $\mathcal{M}_1(\Lambda,\mu)$. On the other hand, we know that a bound on index does not give a bound on λ_1 (since even with bounded index we could have non-smooth convergence: to see this one only needs to consider a sequence of catenoids $M_k = \frac{1}{k}M$ in $\mathbb{R}^n$, all centred at the origin, and blowing down to a double plane. One gets that eventually the catenoid has index one in the unit ball (when it is scaled down sufficiently far, and only considering compactly supported variations), moreover that $\lambda_1(M_k \cap B_1(0)) \to -\infty$.) In particular, given a sequence with only volume and index bounds, we must have that $\lambda_1 \to -\infty$ in general - moreover the gaps between the eigenvalues must also diverge (since the index is bounded).

Combining Theorem 1.3 with Remark 3.2 we deduce the following:

Corollary 1.5. *Let N^{n+1} be a smooth, closed Riemannian manifold with $Ric_N > 0$ and $2 \leq n \leq 6$. Then given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$ and $p \geq 1$ the class $\mathcal{M}_p(\Lambda,\mu)$ is compact in the C^k topology for all $k \geq 2$.*

From a different perspective, Theorem 1.3 gives us an interesting description of *what goes wrong* when absence of a smooth limit occurs for a family $\{M_k\}$ satisfying a uniform volume bound: necessarily, *every* eigenvalue of the Jacobi operator has to diverge to $-\infty$, which somehow captures the well-known picture of M_k exhibiting some neck-pinching around finitely many points.

Corollary 1.6. *Let $\{M_k\} \subset \mathfrak{M}^n(N)$ be a sequence satisfying a uniform volume bound, so that possibly by extracting a subsequence we know [10] that $M_k \to M$ for some stationary, integral varifold M in N.*

(1) If M is not smooth, then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \geq 1$.
(2) If M has multiplicity greater than one, then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \geq 1$ provided $Ric_N > 0$.
(3) If we denote by $\mathcal{Y}$ the set of points of M where the convergence $M_k \to M$ is not smooth and graphical, then $\lambda_p(M_k) \to -\infty$ for all $1 \leq p \leq |\mathcal{Y}|$. In particular, if $|\mathcal{Y}| = \infty$ then $\lambda_p(M_k) \to -\infty$ for all $p \geq 1$.

For instance: this applies to the aforementioned Hsiang minimal hyperspheres [5] (since they have a non-smooth limit) or more generally to the hyperspheres produced in [2]. We further remark that the positivity assumption on the Ricci curvature of N in item (2) above is essential because of the example described in Remark 3.3.

When $n = 2$, the scenario we obtain by combining Theorem 1.1 with Theorem 1.2 and Theorem 1.3 is rather enlightening.

Corollary 1.7. *Let $\mathfrak{C}^n \subset \mathfrak{M}^n$ be a subclass of closed minimal hypersurfaces inside some smooth closed Riemannian manifold N^{n+1} of dimension $2 \leq n \leq 6$ satisfying $Ric_N > 0$. Then a uniform bound on any one of the following quantities for every $M \in \mathfrak{C}^n$ leads to a bound on the rest of them for every $M \in \mathfrak{C}^n$:*

- *the genus of M (when $n = 2$)*
- *index*$(M) + \mathcal{H}^n(M)$
- $\lambda_p(M) + \mathcal{H}^n(M)$
- $\sup_M |A| + \mathcal{H}^n(M)$
- $\int_M |A|^n + \mathcal{H}^n(M)$.

Lastly, our main theorem extends (with minor variations in the proof) to the case when N is a complete (not necessarily compact) Riemannian manifold, provided one replaces, in the statement, the set $\mathcal{M}_p(\Lambda, \mu)$ by the set

$$\mathcal{M}_p^{\Omega}(\Lambda, \mu) := \{M \in \mathfrak{M}^n(N) : M \subset \overline{\Omega},\ \mathcal{H}^n(M) \leq \Lambda,\ \ \lambda_p(M) \geq -\mu\}$$

for some open, bounded domain Ω. In various situations of great geometric interest one can in fact drop the requirement that the minimal hypersurfaces in questions are contained in a given, bounded domain.

Remark 1.8. The conclusions of 1.3 also hold true when

(1) (N^{n+1}, g) is a compact Riemannian manifold with mean-convex boundary;
(2) (N^{n+1}, g) is a complete Riemannian manifold such that for some compact set K each component of $M \setminus K$ is foliated by closed, mean-convex leaves (in particular: asymptotically flat, asymptotically cylindrical and asymptotically hyperbolic manifolds).

The proof follows along the same lines of Theorem 1.3, modulo exploiting a geometric maximum principle (see, for instance, [13]) in order to reduce our compactness analysis to a bounded domain of the manifold N.

2. Preliminaries

We shall recall here the definition of the Morse index and the Jacobi eigenvalues λ_p for general smooth minimal hypersurfaces $M \hookrightarrow N$. First of all, if M is orientable then the second variation of the area functional can be written down purely in terms of section of the normal bundle $v \in \Gamma(NM)$ by

$$Q(v, v) := \int_M |\nabla^{\perp} v|^2 - |A|^2 |v|^2 - Ric_N(v, v).$$

Standard results on the spectra of compact self-adjoint operators on separable Hilbert spaces tell us that there is an orthonormal basis $\{v_i\}_{i=1}^{\infty}$ of $L^2(\Gamma(TN))$ consisting of eigenfunctions for the operator

$$L^{\perp} v := \Delta^{\perp} v + |A|^2 v + Ric_N^{\perp}(v)$$

with associated eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of Q. Moreover, we have the following *Rayleigh characterization of the eigenvalues* due to R. Courant:

$$\lambda_k := \inf_{\dim(V)=k} \max_{v \in V} \frac{Q(v, v)}{\int |v|^2}$$

where of course V is a linear subspace of $\Gamma(NM)$.

Now, if M is non-orientable then we simply lift the problem to its orientable double cover $\tilde{M}$ via $\pi : \tilde{M} \to M$. Consider the linear subspace of smooth sections $v \in \Gamma(\pi^* NM)$ such that $v \circ \tau = v$ where $\tau : \tilde{M} \to \tilde{M}$ is the unique deck transformation of π which reverses orientation. Denote this subspace by $\tilde{\Gamma}(\pi^* NM)$. We can also pull back the quantities

$|A(x)|^2 := |A(\pi(x))|^2$ and $Ric_N(a,b) := Ric_N(\pi_* a, \pi_* b)$. Thus consider the quadratic form

$$\tilde{Q}(v,v) := \int_{\tilde{M}} |\nabla^\perp v|^2 - |A|^2|v|^2 - Ric_N(v,v)$$

over $\tilde{\Gamma}(\pi^* NM)$. As before we can define the spectrum, and therefore index of M to be that of $\tilde{M}$ with respect to $\tilde{Q}$ and $\tilde{\Gamma}(\pi^* NM)$.

If the ambient manifold N is orientable, a closed hypersurface is orientable if and only if it is two-sided, namely if there exists a global section $\nu = \nu_M$ of its normal bundle inside TN. If this is the case (namely if $M \hookrightarrow N$ is minimal and two-sided) the spectrum defined above patently coincides with the spectrum of the *scalar* Jacobi or stability operator of M, namely

$$Lu := \Delta_M u + (Ric_N(\nu,\nu) + |A|^2)u$$

when we regard $L : W^{1,2}(M) \to W^{-1,2}(M)$.

Furthermore, we shall introduce the following notation: given a minimal hypersurface M in the Riemannian manifold (N,g) and a bounded open domain $\Omega \subset M$ we shall set

$$\lambda_1^M(\Omega) = \inf\left\{ -\int_M vL^\perp v \; | v \in C_0^\infty(\Omega; NM) \text{ and } \int_M |v|^2 = 1 \right\}$$

where $C_0^\infty(\Omega; NM)$ denotes the (smooth) sections of the normal bundle whose support, projected on the base M is relatively compact in Ω.

3. Proofs

For the sake of conceptual clarity, we will separate the proof of Theorem 1.3 in the case $p = 1$ and $p \geq 2$, the former being a building block for the latter.

Proof of Theorem 1.3, case $p = 1$. Let $\{M_k\} \subset \mathcal{M}_1(\Lambda, \mu)$ be a sequence of closed, embedded minimal hypersurfaces satisfying our bounds on the volume and first Jacobi eigenvalue: we claim the existence of a constant $C = C(N, \Lambda, \mu) > 0$ such that

$$\sup_{k \geq 1} \sup_{z \in M_k} |A_k(z)| \leq C$$

where A_k denotes the second fundamental form of M_k in (N,g). For the sake of a contradiction, let us assume instead that such a uniform curvature bound does not hold. Then, we could find (for every $k \geq 1$) a sequence of points $\{z_k\} \subset N$ such that A_k attains its maximum value at $z_k \in M_k$ and, furthermore, $\lim_{k\to\infty} |A_k(z_k)| = +\infty$. Thanks to the compactness of the ambient manifold N, possibly by extracting a subsequence (which we shall not rename) we can assume that $z_k \to y$ for some point $y \in N$. Let us pick, once and for all, a small radius $r_0 > 0$ (less than the injectivity radius of (N,g) at y) and let us denote by $\{x\}$ a system of geodesic normal coordinates centered at y and by $g_{ij}(x)$ the corresponding components of the Riemannian metric g. For k large enough we know that $M_k \cap B_{r_0/2}(z_k) \subset B_{r_0}(y)$ and we can assume, without loss of generality, that $M_k \cap B_{r_0}(y)$ is two-sided. We can then consider the blown-up hypersurfaces defined by

$$\hat{M}_k := |A_k(z_k)|(M_k - z_k)$$

and the appropriately rescaled Riemannian metrics on $\hat{B}_k := B_{r_0|A_k(z_k)|/2}(0) \subset \mathbb{R}^{n+1}$

$$\hat{g}_k(x) := g\left(z_k + \frac{x}{|A_k(z_k)|}\right).$$

(For the sake of clarity we have identified, in the equation above, the hypersurface M_k with its portion in $B_{r_0/2}(z_k)$). Now, the hypersurface $\hat{M}_k$ is minimal in metric $\hat{g}_k$ and patently satisfies volume and curvature bounds, for $M_k \in \mathcal{M}_1(\Lambda,\mu)$ implies

$$\frac{\mathcal{H}^n(\hat{M}_k \cap B_r(0))}{r^n} \leq \Lambda, \text{ for any } r < \frac{r_0|A_k(z_k)|}{2}$$

and by scaling

$$\sup_{x\in\hat{B}_k} |\hat{A}_k(x)| \leq 1, \quad \hat{A}_k(0) = 1$$

where $\hat{A}_k$ denotes the second fundamental form of $\hat{M}_k$ in metric $\hat{g}_k$. It follows that the sequence $\left\{\hat{M}_k\right\}$ converges (for any m in C^m_{loc}, in the sense of smooth graphs) to a complete, embedded minimal hypersurface $\hat{M}_\infty \subset \mathbb{R}^{n+1}$ (in flat Euclidean metric). We further claim that $\hat{M}_\infty$ has to be stable. If not, we could find a smooth, compactly supported vector field u such that the second variation

$$\left[\frac{d^2}{dt^2}\right]_{t=0} \mathcal{H}^n\left((\varphi_t)_\#\hat{M}_\infty\right) < 0$$

where $\{\varphi_t\}$ is the flow of diffeomorphisms generated by u (which coincides with the identity outside of a compact set). As a result, thanks to the locally strong convergence $\hat{M}_k \to \hat{M}_\infty$ we would have

$$\left[\frac{d^2}{dt^2}\right]_{t=0} \mathcal{H}^n\left((\varphi_t)_\#\hat{M}_k\right) < 0$$

for all indices k that are large enough and, more specifically, for a fixed open, bounded set $\Omega \subset \mathbb{R}^{n+1}$ we would have $\lambda_1^{\hat{M}_k}(\Omega) \leq -\varepsilon$ for some $\varepsilon > 0$. Therefore, scaling back and keeping in mind the Rayleigh characterization of the eigenvalues of an elliptic operator we must conclude that

$$-\mu \leq \lambda_1^{M_k}(\Omega) \leq -\varepsilon|A_k(z_k)|,$$

which is impossible when k attains sufficiently large values. It follows that $\hat{M}_\infty$ is a *stable* minimal hypersurface in $\mathbb{R}^{n+1}$ (with polynomial volume growth), hence an affine hyperplane by the work of Schoen-Simon [8] and thus on the one hand $\hat{A}_\infty$ vanishes identically, while on the other $\hat{A}_\infty(0) = 1$ and this contradiction completes the proof of our initial claim. Once those uniform curvature estimates are gained, the strong convergence of $M_k \to M$ follows from a geometric counterpart of the Arzelá-Ascoli compactness theorem, and the fact that in this case the volume and first Jacobi eigenvalue of M are also controlled, namely $M \in \mathcal{M}_1(\Lambda,\mu)$ is also clear. □

Proof of Theorem 1.3, case $p \geq 2$. We shall start by stating the following important:

Lemma 3.1. *Let $\Omega_1, \Omega_2, \ldots, \Omega_p \subset N$ be p pairwise disjoint, bounded open sets. If we assume $M \in \mathcal{M}_p(\Lambda,\mu)$ and $M \cap \Omega_i \neq \emptyset$ for $i = 1,\ldots,p$ then there exists an index i_0 such that $\lambda_1^M(\Omega_{i_0}) \geq -\mu$.*

Indeed, suppose that were not the case: then we could find, for each index $i = 1,\ldots,p$ a section $\phi_i \in C_0^\infty(\Omega_i; NM)$ that ensures $\lambda_1^M(\Omega_i) < -\mu$ (that is to say $\int_M |\phi_i|^2 = 1$ and $Q(\phi_i,\phi_i) < -\mu$) and thus

$$Q(\phi,\phi) < -\mu \text{ for all } \phi \in W = \langle\phi_1,\ldots,\phi_p\rangle_{\mathbb{R}} \text{ such that } \int_M |\phi|^2 = 1$$

which contradicts the assumption that $\lambda_p(M) \geq -\mu$ due to the well-known min-max characterization of the eigenvalues of an elliptic operator. (In case M is not orientable, one needs to consider the space $\tilde{W} = \left\langle \tilde{\phi}_1, \ldots, \tilde{\phi}_p \right\rangle_{\mathbb{R}}$ where each $\tilde{\phi}_i$ is the lift of ϕ_i to $\tilde{M}$ and the quadratic form Q is evaluated on $\tilde{M}$ as explained in Section 2).

Now, let a sequence $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$ be given: thanks to the volume bound, we know that possibly by taking a subsequence (which we shall not rename) $M_k \to \mathbf{V}$ in the sense of varifolds, for some integral varifold $\mathbf{V}$. Furthermore, given $\varepsilon > 0$ the Lemma 3.1 we have just stated and a standard covering argument ensure that there exists a set $\mathcal{Y} = \{y_i\}_{i=1}^{P} \subset \mathrm{spt}(\mathbf{V})$ consisting of at most $p-1$ points and a subsequence $\left\{M_{l(k)}\right\}$ such that in $M \setminus \mathcal{Y}$ the hypersurfaces $M_{l(k)}$ locally converge to $\mathbf{V}$ strongly in the sense of smooth graphs. This descends from the fact that for any $z \in M \setminus \mathcal{Y}$ and $\varepsilon > 0$ the sequence $M_{l(k)}$ satisfies a uniform bound on the *first* Jacobi eigenvalue, which in turn implies

$$\sup_{k \geq 1} \sup_{x \in B_\varepsilon(z)} |A_{l(k)}(x)| \leq C = C(N, \Lambda, \mu)$$

by following the argument that has been used to prove Theorem 1.3 in the case $p = 1$. Here $A_{l(k)}$ stands for the second fundamental form of $M_{l(k)}$ in the ambient manifold (N, g). In particular, this implies that the varifold $\mathbf{V}$ is supported on a smooth submanifold M away from finitely many points, namely those points belonging to the set $\mathcal{Y}$.

We further claim that for any $y \in \mathcal{Y}$ there exists $\varepsilon_0 > 0$ such that

$$\lambda_1^M(B_{\varepsilon_0}(y) \setminus \{y\}) \geq -\mu \quad (*)$$

If this claim were false, then we could find a smooth, normal vector field u_0, compactly supported in $B_{\varepsilon_0}(y) \setminus \{y\}$ and hence (say) supported in $B_{\varepsilon_0}(y) \setminus B_{\varepsilon_1}(y)$ for some $0 < \varepsilon_1 < \varepsilon_0$ such that

$$\frac{Q(u_0, u_0)}{\int_M |u_0|^2} < -\mu.$$

At that stage, we shall observe that it cannot be $\lambda_1^M(B_{\varepsilon_1}(y) \setminus \{y\}) \geq -\mu$ either (for otherwise we would have gained property $(*)$ with ε_1 in lieu of ε_0) and hence, again, there is a smooth vector field u_1 that is compactly supported in $B_{\varepsilon_1}(y) \setminus \{y\}$ and hence (say) supported in $B_{\varepsilon_1}(y) \setminus B_{\varepsilon_2}(y)$ for some $0 < \varepsilon_2 < \varepsilon_1$ such that

$$\frac{Q(u_1, u_1)}{\int_M |u_1|^2} < -\mu$$

with the same notation as above. Of course, we can repeat this argument p times, hereby getting sections $u_0, \ldots, u_{p-1}$ supported on smaller and smaller annuli, specifically u_j shall be supported on $B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y)$ for $0 < \varepsilon_p < \ldots < \varepsilon_0$. But we already know that $M_{l(k)} \to M$ on (the closure of) $B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y)$ for each $j \leq p-1$ and thus we derive (for k large enough) the conclusion

$$\lambda_1^{M_{l(k)}}(B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y)) < -\mu, \text{for} \quad j = 0, 1, \ldots, p-1$$

which contradicts our preliminary Lemma 3.1. This ensures the validity of $(*)$ for some suitable choice of $\varepsilon_0 > 0$. At that stage, let $\mathbf{V}^{(i)}$ for each fixed $y_i \in \mathcal{Y}$ be a tangent cone to the varifold $\mathbf{V}$ at y_i: necessarily $\mathbf{V}^{(i)}$ has to be stable, for otherwise we could scale back and argue as in the proof of Theorem 1.3 to show that the bound $(*)$ cannot possibly hold. As a result, $\mathbf{V}^{(i)}$ is a stable minimal hypercone in $\mathbb{R}^{n+1}$ and hence an hyperplane for $2 \leq n \leq 6$ due to the classic work of J. Simons [12]. Therefore $\mathbf{V}$ is regular (in fact, smooth) in a neighborhood of each point y_i thanks to Allard's regularity theorem [1], and we can conclude that its support M is a smooth, minimal hypersurface in (N,g), as we had to prove.

Varifold convergence directly implies that $\mathcal{H}^n(M)\leq\Lambda$, while the fact that $\lambda_p(M)\geq -\mu$ is more delicate as the set $\mathcal{Y}$ may not be empty. To that aim, we argue as follows. For small $r>0$ let $\eta_r^{(j)}$ be a smooth non-negative function on M that vanishes on the geodesic ball of radius r centered at y_j and equals one outside of the ball of radius $2r$. For the sake of contradiction, suppose there exists p linearly independent (and orthonormal) sections in $W^{1,2}(M;NM)$ (in fact $\tilde{W}^{1,2}(\tilde{M};\pi^*NM)$ when M is not orientable), say $\phi_1,\ldots,\phi_p$ such that

$$\frac{Q(\phi,\phi)}{\int_M|\phi|^2}<-\mu \text{ on } V=\langle\phi_1,\ldots,\phi_p\rangle_{\mathbb{R}}$$

and set $\phi_i^r=\phi_i\prod_{j=1}^{P}\eta_r^{(j)}$. The fact that points have zero capacity in $\mathbb{R}^n$ for any $n\geq 2$ implies that on the subspace $V_r=\left\langle\phi_1^r,\ldots,\phi_p^r\right\rangle_{\mathbb{R}}$ the inequality above must also hold for r small enough, and hence (replacing each ϕ_i by ϕ_i^r and then applying the Gram-Schmidt process to the latter family, without further renaming) we can assume that the sections in question vanish on small geodesic neighborhoods of the points in the set $\mathcal{Y}$. Now, such vector fields can be extended to a tubular neighborhood of $M\hookrightarrow N$ (without renaming) and since each M_k has to be contained in that neighborhood for k large enough we can define sections $\phi_i^{r,k}\in\Gamma(M_k;NM_k)$ by projecting those extended vector fields onto the normal bundle of $M_k\hookrightarrow N$. The strong convergence of M_k to M away from the points in $\mathcal{Y}$ together with the assumption $M_k\in\mathcal{M}_p(\Lambda,\mu)$ implies that we can find real coefficients $\alpha_1^k,\ldots,\alpha_p^k$ such that

$$\sum_{i=1}^{p}\alpha_i^k\phi_i^{r,k}=0.$$

Possibly by dividing the coefficients by $\max_i|\alpha_i^k|$ and renaming we can assume that $|\alpha_i^k|\leq 1$ for each index i and $|\alpha_{i_0}^k|=1$ for some index i_0. Hence, squaring the previous equation and integrating over M_k we get

$$0=\int_{M_k}\left|\sum_{i=1}^{p}\alpha_i^k\phi_i^{r,k}\right|^2=\sum_{i,j}\alpha_i^k\alpha_j^k\int_{M_k}g\left(\phi_i^{r,k},\phi_j^{r,k}\right)$$

and by letting $k\to\infty$ the orhogonality of the family $\left\{\phi_1^r,\ldots,\phi_p^r\right\}$ implies that

$$\sum_{i=1}^{p}(\alpha_i)^2=0$$

where (possibly by extracting a subsequence, which we shall not rename) $\alpha_i^k\to\alpha_i$ as $k\to\infty$ for each $i=1,\ldots,p$. Thus $\alpha_i=0$ for each i but on the other hand (by construction) $\sum_{i=1}^{p}(\alpha_i)^2\geq 1$, a contradiction. This proves that $M\in\mathcal{M}_p(\Lambda,\mu)$. Lastly, the fact that single-sheeted convergence implies $\mathcal{Y}=\emptyset$ is a direct consequence of Allard's interior regularity theorem, [1]. Thereby, the proof is complete. □

Remark 3.2. When the number of leaves in the convergence of M_k to M is known, then one can deduce further information about the limit hypersurface M. Specifically:

- if the number of sheets in the convergence is one
  - if M is two-sided and $M_k\cap M=\emptyset$ eventually then M is stable
  - if M is two-sided and $M_k\cap M\neq\emptyset$ eventually then $index(M)\geq 1$
- if the number of sheets in the convergence is at least two
  - if N has $Ric_N>0$ then M cannot be one-sided

– if M is two-sided then M is stable.

All of these statements follow from variations on the same argument, which consists of conctructing a global section in the kernel of the Jacobi operator L of M (or a suitable lift, in the case of the third assertion) by appropriately renormalizing the distance function between M and M_k (first and second assertion) and two adjacent leaves of M_k (third and fourth assertion). The reader can consult pages 10-13 of [11] for detailed arguments.

Remark 3.3. The assertion given in part (1) of Theorem 1.3 is *sharp* at that level of generality in the sense that one can provide explicit examples of Riemannian manifolds (N, g) and sequences $\{M_k\} \subset \mathcal{M}_1(\Lambda, \mu)$ such that $M_k \to M$ in the sense of smooth graphs, but with multiple leaves. For instance: let (N^3, g) be gotten by taking the quotient of the product manifold $(S^2 \times \mathbb{R}, g_{round} \times dt^2)$ modulo the equivalence relation $(x, t) \sim (-x, -t)$. If we consider π the associated Riemannian projection, then $\pi(S^2 \times \{t\})$ is a totally geodesic, stable minimal sphere for any $t \neq 0$ while $\pi(S^2 \times \{0\})$ is a stable minimal $\mathbb{RP}^2$ and for any sequence $t_k \downarrow 0$ we have that $\pi(S^2 \times \{t_k\}) \to \pi(S^2 \times \{0\})$ strongly in the sense of graphical, two-sheeted convergence.

Acknowledgments. The authors wish to express their gratitude to Prof. André Neves for a number of enlightening conversations. During the preparation of this article, L. A. was supported by CNPq-Brazil, while A.C. and B. S. were supported by Prof. André Neves through his Leverhulme and European Research Council Start Grant.

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