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Some remarks on inp-minimal and finite burden groups

Jan Dobrowolski *and John Goodrick

Abstract

We prove that any left-ordered inp-minimal group is abelian and we provide an example of a non-abelian left-ordered group of dp-rank 2. Furthermore, we establish a necessary condition for group to have finite burden involving normalizers of definable sets, reminiscent of other chain conditions for stable groups.

0 Introduction and preliminaries

One of the model-theoretic properties that gained a lot of interest recently is dp-minimality, which, on one hand, significantly strengthens NIP, and on the other hand, is satisfied by all strongly minimal theories, all (weakly) o-minimal theories, algebraically closed valued fields (more generally, by all C-minimal structures), and the valued field of p-adics. Several interesting results were obtained for dp-minimal structures in the algebraic contexts of groups and fields (sometimes with additional structure), see for example [12, 5, 4, 8].

Throughout this note, we work in the context of a complete first-order theory T , and “formula” means a first-order formula in the language of T .

We recall some key definitions, which are originally due to Shelah [11], though the precise form of the definitions which we give below seems to come from Usvyatsov [13].

Definition 0.1. 1. An inp-pattern of depth κ (in the partial type $\pi(\bar{x})$) is a sequence $\langle \varphi_i(\bar{x}; \bar{y}_i) : i < \kappa \rangle$ of formulas and an array $\{\bar{a}_{i,j} : i < \kappa, j < \omega\}$ of parameters (from some model of T) such that:

- (a) For each $i < \kappa$, there is some $k_i < \omega$ such that $\{\varphi_i(\bar{x}; \bar{a}_{i,j}) : j < \omega\}$ is k_i -inconsistent; and
- (b) For each $\eta : \kappa \rightarrow \omega$, the partial type

$$\pi(\bar{x}) \cup \{\varphi_i(\bar{x}; \bar{a}_{i,\eta(i)}) : i < \kappa\}$$

is consistent.

- 2. The inp-rank (or burden) of a partial type $\pi(\bar{x})$ is the maximal κ such that there is an inp-pattern of depth κ in $\pi(\bar{x})$, if such a maximum exists. In case there are inp-patterns of depth λ in $\pi(\bar{x})$ for every cardinal $\lambda < \kappa$ but no inp-pattern of depth κ , we say that the inp-rank of $\pi(\bar{x})$ is κ_- .
- 3. The inp-rank of T is the inp-rank of $x = x$, and T is inp-minimal if its inp-rank is 1.

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4. An ict-pattern of depth κ (in the partial type $\pi(\bar{x})$) is a sequence $\langle \varphi_i(\bar{x}; \bar{y}_i) : i < \kappa \rangle$ of formulas and an array $\{\bar{a}_{ij} : i < \kappa, j < \omega\}$ of parameters (from some model of T) such that for each $\eta : \kappa \rightarrow \omega$, the partial type

$$\pi(\bar{x}) \cup \{\varphi_i(\bar{x}; \bar{a}_{i, \eta(i)}) : i < \kappa\} \cup \{\neg \varphi_i(\bar{x}; \bar{a}_{i, j}) : i < \kappa, j \neq \eta(i)\}$$

is consistent.

5. The dp-rank of a partial type $\pi(\bar{x})$ is the maximal κ such that there is an ict-pattern of depth κ in $\pi(\bar{x})$, if such a maximum exists (if not, it is “ κ_- ” exactly as in 2 above). The dp-rank of T is the dp-rank of $x = x$, and T is dp-minimal if its dp-rank is 1.

In spite of its name, dp-rank is really more like a cardinal-valued *dimension* than an ordinal-valued rank such as $SU(p)$, and in the context of stable theories, dp-minimality is equivalent to every nonalgebraic 1-type having weight 1, as observed in [10]. It turns out that a theory is dp-minimal just in case it is both inp-minimal and NIP (see [1]).

One of the context investigated in [12] is that of (bi)-ordered groups.

Definition 0.2. A left-ordering on a group (G, \cdot) is a total ordering $<$ on G such that for any $f, g, h \in G$, whenever $g < h$, we have that $f \cdot g < f \cdot h$. A right-ordering is defined similarly, and a bi-ordering on G is an ordering which is simultaneously a left-ordering and a right-ordering.

While Pierre Simon claimed that all inp-minimal “ordered groups” are abelian [12], his proof only applies to groups with a definable bi-ordering: his argument uses the fact that for any a, b in a bi-orderable group and any positive $n \in \mathbb{N}$, if $a^n = b^n$ then $a = b$. But in left-orderable groups (such as in the example of the Klein bottle group below), one may have that $a^2 = b^2$ but $a \neq b$.

The main result of Section 2 of this note is that every inp-minimal left-ordered group is abelian (Theorem 2.6), which strengthens the result mentioned above from [12]. We also show that this conclusion fails already in the dp-rank 2 case by providing a suitable example (Section 1). Finally, in Section 3 we consider necessary conditions for an arbitrary (not necessarily ordered) group to have finite burden. In the stable case, this gives a simple and apparently new condition on stable groups G of finite weight: such a group must contain finitely many definable abelian subgroups A_0, \dots, A_k such that $G/N[A_0] \dots N[A_k]$ has finite exponent (Corollary 3.5).

1 A non-abelian left-ordered group of dp-rank 2

In this section, we define the “Klein bottle group” (the fundamental group of a Klein bottle) which is presented as $G = \langle x, y : x^{-1}yx = y^{-1} \rangle$. In other words, $y^{-1}x = xy$, and routine algebraic manipulation shows:

1. Every $g \in G$ can be uniquely written as $g = x^n y^m$ for some $n, m \in \mathbb{Z}$,
2. $(x^n y^m) \cdot (x^{n'} y^{m'}) = x^{n+n'} y^{m'+(-1)^{n'} m}$, and
3. $(x^n y^m)^{-1} = x^{-n} y^{(-1)^{n+1} m}$.

We can define a left ordering \leq on G lexicographically on the exponents: $x^n y^m \leq x^{n'} y^{m'}$ iff either $n < n'$ or else $n = n'$ and $m \leq m'$. The subgroup generated by y is the minimal nontrivial convex subgroup of G , and the order type of G is $\mathbb{Z} \times \mathbb{Z}$.

We note in passing that while G is non-abelian, it is abelian-by-*finite*: a simple calculation shows that the centralizer $C(y)$ of y is $\{x^{2n} y^m : n, m \in \mathbb{Z}\}$, which is abelian, and for any $g \in G$, we have $g^2 \in C(y)$.

Also note that in $(xy)^2 = x^2$, so elements the Klein bottle group G do not always have unique “square roots,” unlike elements in a bi-orderable groups.

Proposition 1.1. *The structure (G, \cdot, \leq) is dp-rank 2.*

Proof. To show that it is NIP and dp-rank *at most* 2, it suffices (thanks to the additivity of the dp-rank proved in [6]) to note that an isomorphic copy of G is definable in the dp-minimal structure $(\mathbb{Z}, <, +)$ with $\mathbb{Z} \times \mathbb{Z}$ as the the universe of the group; the definition of the group operation depends on the parity of one of the coordinates, but of course $2\mathbb{Z}$ is a definable subgroup of \mathbb{Z} .

Now we compute the centralizer $C(x)$ of the generator x . For any $a, b \in \mathbb{Z}$, we have

$$x \cdot (x^a y^b) = x^{a+1} y^b$$

and

$$(x^a y^b) \cdot x = x^{a+1} y^{-b},$$

so we conclude that $C(x) = \langle x \rangle$.

For any $n \in \mathbb{N}$, there are pairwise disjoint intervals $\langle I_{n,k} : k < \omega \rangle$ such that for any k , $x^k \in I_{n,k}$ and $I_{n,k}$ intersects every right coset $C(x), C(x)y, \dots, C(x)y^n$. So by compactness, in an ω -saturated extension of G , we can find pairwise disjoint intervals $\langle I_k : k < \omega \rangle$ such that $x^k \in I_k$ and each I_k intersects every right coset $C(x)y^n$. Therefore the formulas expressing $z \in I_k$ and $z \in C(x)y^j$ (in the free variable z) give an inp-pattern of depth 2, so G is not inp-minimal, hence by NIP it is not dp-minimal. □

Remark 1.1. *In a previous version of this paper, we asked whether the group G above is dp-minimal in the pure language of groups. This was answered negatively by Halevi and Hasson [14].*

2 Inp-minimal left-ordered groups

In this section, we prove that every inp-minimal left-ordered group is abelian. For a left-ordered group G and a subset $A \subseteq G$, by $h(A)$ we will denote the convex hull of A in G .

Fact 2.1 ([12]). *Let G be an inp-minimal group. Then there is a definable normal abelian subgroup H of G such that G/H has finite exponent.*

Note that (in contrast to the bi-ordered groups), in a left-ordered group G , the convex hull of a subgroup H need not be a subgroup of G :

Example 2.2. *Consider the left-ordered group $G = (\text{Aut}(\mathbb{Q}, <), \prec)$, where \prec is a standard left-order on $\text{Aut}(\mathbb{Q}, <)$ coming from a well-order on \mathbb{Q} , i.e., $f \prec g$ if $f(x) < g(x)$ for x being the smallest (in the sense of the well-order) element on which f differs from g . Choose $a < b < c < d < e \in \mathbb{Q}$, where a is the first element of \mathbb{Q} in the fixed well-order, and $f, g \in G$ such that $f(a) = c$, $f(d) = d$, $g(a) = b$ and $g(b) = e$. Then, $g^2(a) = e$, and, for any $k \in \mathbb{Z}$, $f^k(a) < e$, so $f^k \prec g^2$. So we have that $g^2 \notin h(\langle f \rangle)$, but, clearly, $g \in h(\langle f \rangle)$. So we get that $h(\langle f \rangle)$ is not a subgroup of G .*

Nevertheless, in left-ordered groups, $h(H)$ is always a union of right H -cosets, and, as an analogue of Lemma 3.2 from [12], we obtain:

Fact 2.3. *Let G be an inp-minimal left-ordered group. Let H be a definable subgroup of G and let C be the convex hull of H . Then C is a union of finitely many right H -cosets.*

Proof. All we need to repeat the proof from [12] is to prove that all cosets of H contained in C are cofinal in C . So take any $c \in C$ and fix $h \in H$. Choose $h_1 \in H$ such that $h_1 < c$. Then, by the left-invariance of the order, $h = (hh_1^{-1})h_1 < hh_1^{-1}c \in Hc$, so Hc is cofinal in C . \square

We will also use a group-theoretic fact about FC-groups.

Definition 2.4. *An FC-group is a group in which the centralizer of every element has finite index.*

Note that if $[G : Z(G)]$ is finite, then G is an FC-group. The following is Theorem 6.24 from [7]:

Fact 2.5. *Every torsion-free FC-group is abelian.*

Theorem 2.6. *Every left-ordered inp-minimal group is abelian.*

Proof. By Fact 2.5, it is enough to show that $[G : Z(G)]$ is finite. Let H be given by Fact 2.1. Notice that $G = h(H)$: if $a \in G$, say $a > e$, then for l equal to the exponent of G/H , we have $e < a < a^2 < \dots < a^l \in H$, so $a \in h(H)$. Hence, by Fact 2.3, we get that $n := [G : H]$ is finite.

Claim. For any positive $x \in G$, the interval $[e, x]$ is covered by finitely many right cosets of a central subgroup of G .

Proof of Claim: We can assume that G is non-trivial (hence it is infinite so also H is non-trivial). Notice that any coset Hg in G has a positive representative (if g is negative then one can take $g^{-(l-1)}$ as such a representative). It follows (thanks to the normality of H) that for any $y, g \in G$ we can find an element $z \in G$ such that $Hg = Hz$ and $z > y$ (by choosing $z := yw$, where w is a positive element of $Hy^{-1}g$).

Now, fix any positive $x = x_0 \in G$. Using the above observation, we can choose $x_0 < x_1 < \dots < x_{n-1} \in G$ such that $G = \bigcup_{i < n} Hx_i$. By Fact 2.3, each set $h(C(x_i))$ is covered by finitely many right $C(x_i)$ -cosets; call them $C(x_i)k_{i,0}, \dots, C(x_i)k_{i,m(i)}$. Thus if

$$C := \bigcap_{i < n} h(C(x_i)),$$

then any $y \in C$ belongs to some intersection

$$S_{j,\eta} := Hx_j \cap \bigcap_{i < n} C(x_i)k_{i,\eta(i)}$$

for some $j < n$ and some $\eta : n \rightarrow \omega$. Note that for any $z \in G$, $zy \in S_{j,\eta}$ if and only if

$$z \in A := H \cap \bigcap_{i < n} C(x_i),$$

hence $S_{j,\eta} = Ay$, and therefore $C = \bigcup S_{j,\eta}$ is covered by finitely many A -cosets

But, since H is abelian, and G is generated by H, x_0, \dots, x_{n-1} , we have that $A \subseteq Z(G)$. Also, since $x_0 < x_1 < \dots < x_{n-1}$ and $\forall_{i < n} x_i \in C(x_i)$, we get that $x = x_0 \in C$, so, by convexity of C , $[e, x_0] \subseteq C$, which proves the claim. \square

Now, suppose for a contradiction that $[G : Z(G)]$ is infinite. Note that if some coset $Z(G)g$ contains only negative elements, then the coset $Z(G)g^{-1}$ contains only positive elements, so in any case we may choose infinitely many positive representatives y_0, y_1, y_2, \dots of pairwise distinct right

cosets of $Z(G)$ in G . Without loss of generality, G is ω -saturated, and there is an element $x \in G$ greater than all the y_i 's. Then $[e, x]$ cannot be covered by finitely many right cosets of $Z(G)$, so it cannot be covered by finitely many right cosets of any central subgroup of G , contradicting the Claim. \square

Corollary 2.7. *If $(G, \cdot, <)$ is a left-ordered group which is inp-minimal (in the pure language of ordered groups), then it is dp-minimal.*

Proof. By Theorem 2.6, G is abelian, and any ordered abelian group is NIP, as shown in [3]; since NIP and inp-minimality imply dp-minimality, we are done. \square

3 Some observations on groups of finite inp-rank

The example from Section 1, as every group definable in the Presburger arithmetic, is abelian-by-finite (see [9]). It seems natural to ask the following general question:

Problem 3.1. *What can be said about ordered groups of finite inp-rank (possibly under some additional model-theoretic assumptions)?*

To apply some ideas from the proof of Theorem 2.6, it seems necessary to prove some variant of the following property, which was essentially observed in the proof of Proposition 3.1 from [12]:

Fact 3.2. *If G is an inp-minimal group and $H, K < G$ are definable, then either $[H : H \cap K]$ or $[K : H \cap K]$ is finite.*

Below, we make an observation of this kind in the context of finite inp-rank, but we need to work with normal subgroups.

If G is a group and $A \subseteq G$, then by $N[A]$ we shall denote the normal subgroup of G generated by A . If H is a subgroup of G , then we put $A/H := \{aH : a \in A\}$. For any elements $g, h \in G$, by g^h we mean the conjugate $h^{-1}gh$ of g by h .

Proposition 3.3. *If G is a group of burden $n \in \omega$, then there do not exist definable symmetric sets D_0, D_1, \dots, D_n such that, if we put $N_i = N[D_i]$, then*

$$(\forall i \leq n)(|D_i/N_0N_1 \dots N_{i-1}N_{i+1}N_{i+2} \dots N_n| \geq \omega).$$

Moreover, we can replace the above condition by: for each $i \leq n$, there is an infinite subset E_i of D_i such that

$$(\forall e_0, e_1 \in E_i)(e_0e_1^{-1} \in ((D_0D_1 \dots D_{i-1}D_{i+1}D_{i+2} \dots D_n)^G)^{2n} \implies e_0 = e_1).$$

Proof. Clearly, it is enough to prove the ‘‘moreover’’ part. Suppose for a contradiction that there exist sets $(D_i)_{i \leq n}$ and $(E_i)_{i \leq n}$ as above. For each $i \leq n$, let $(e_{i,j})_{j < \omega}$ be a sequence of pairwise distinct elements of E_i . We claim that the formulas

$$(\phi_i(x, e_{i,j}) := x \in D_0D_1 \dots D_{i-1}e_{i,j}D_{i+1}D_{i+2} \dots D_n)_{i \leq n, j < \omega}$$

form an inp-pattern of depth $n + 1$, which will contradict the assumption. Obviously, for any $\eta \in \omega^{n+1}$, the element $\prod_{i \leq n} e_{i, \eta(i)}$ satisfies $\bigwedge_{i \leq n} \phi_i(x, e_{i, \eta(i)})$. On the other hand, if there is some g satisfying both $\phi_i(x, e_{i, j_0})$ and $\phi_i(x, e_{i, j_1})$, then for some $(d_k, d'_k)_{k \in \{0, 1, \dots, n\} \setminus \{i\}}$ with $d_k, d'_k \in D_k$, we have

$$d_0d_1 \dots d_{i-1}e_{i, j_0}d_{i+1} \dots d_n = g = d'_0d'_1 \dots d'_{i-1}e_{i, j_1}d'_{i+1} \dots d'_n,$$

so

$$e_{i,j_0} e_{i,j_1}^{-1} = d_{i-1}^{-1} d_{i-2}^{-1} \dots d_0^{-1} d'_0 d'_1 \dots d'_{i-1} (d'_{i+1} d'_{i+2} \dots d'_n d_n^{-1} d_{n-1}^{-1} \dots d_{i+1}^{-1}) e_{i,j_1}^{-1}$$

is an element of $((D_0 D_1 \dots D_{i-1} D_{i+1} D_{i+2} \dots D_n)^G)^{2n}$, hence, by the assumption on E_i , we get that $e_{i,j_0} = e_{i,j_1}$. This completes the proof. \square

Example 3.4. *Note that the above proposition does not follow from the (somewhat similarly looking) chain condition [2, Proposition 4.5 (2)], as the latter is satisfied in a non-abelian free group F : the only non-trivial definable proper subgroups are the cyclic groups which are not normal, and the same is true about type-definable subgroups, as in every stable group a type-definable subgroup is an intersection of definable subgroups. But the conclusion of Proposition 3.3 is not satisfied in F :*

To see this, we may assume (as the failure of the conclusion of Proposition is \wedge -expressible) that F is the free group on generators x_0, x_1, x_2, \dots . Put $D_i = \{x_i^m : m < \omega\}$. Then, clearly,

$$(\forall i \leq n)(|D_i/N[D_0]N[D_1] \dots N[D_{i-1}]N[D_{i+1}]N[D_{i+2}] \dots N[D_n]| \geq \omega).$$

Using the above chain condition we get in the stable context:

Corollary 3.5. *If G is a stable group of finite weight, then there are finitely many definable abelian subgroups A_0, \dots, A_k of G such that the quotient $G/N[A_0]N[A_1] \dots N[A_k]$ has finite exponent.*

Proof. It follows from the assumptions that G has a finite burden, say n . For any $g \in G$ put $A_g = C(C(g))$. Then A_g is a definable group containing the group generated by g , and a standard argument shows that A_g is abelian.¹ Suppose for a contradiction that the conclusion does not hold. Then, using compactness, we can find inductively a sequence $(g_i)_{i < \omega}$ of elements of G such that for each $i, m < \omega$, $g_i^m \notin N[A_{g_0}]N[A_{g_1}] \dots N[A_{g_{i-1}}]$. Since the latter is a type-definable condition on the sequence $(g_i)_{i < \omega}$ (as $N[A_{g_0}]N[A_{g_1}] \dots N[A_{g_{i-1}}]$ is \vee -definable over g_0, g_1, \dots, g_{i-1}), we can additionally assume that $(g_i)_{i < \omega}$ is an indiscernible sequence. Now, by Proposition 3.3, there is some $i \leq n$ such that for some $m < \omega$, $g_i^m \in ((A_{g_0} A_{g_1} \dots A_{g_{i-1}} A_{g_{i+1}} A_{g_{i+2}} \dots A_{g_n})^G)^{2n}$ (otherwise, putting $D_i = A_{g_i}$ and $E_i = \{g_i^m : m < \omega\}$, we contradict the conclusion of the Proposition). But this is expressible by a sentence $\phi(g_i; g_0, g_1, \dots, g_{i-1}, g_{i+1}, g_{i+2}, \dots, g_n)$, and by the choice of the g_i 's, the sentence

$\phi(g_n; g_0, g_1, \dots, g_{n-1})$ is not true in G , so the sequence $(g_i)_{i < \omega}$ is not totally indiscernible. This contradicts stability. \square

We end by stating a question about relaxing the assumption of stability in the last corollary to some settings which allow the existence of a definable order:

Question 3.6. *Is the conclusion of Corollary 3.5 true for:*

- 1) *rosy groups of finite burden? (in particular, for simple groups of finite weight and groups definable in o-minimal structures?)*
- 2) *distal groups of finite burden?*

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¹Proof: if $h, k \in A_g$, then since $g \in C(g)$ we have $h \in C(g)$, so k commutes with h by definition of $C(C(g))$.

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