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Classification of simple weight modules over the Schrödinger algebra

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Abstract

A classification of simple weight modules over the Schrödinger algebra is given. The Krull and the global dimensions are found for the centralizer $C_{\mathcal{S}}(H)$ (and some of its prime factor algebras) of the Cartan element H in the universal enveloping algebra $\mathcal{S}$ of the Schrödinger (Lie) algebra. The simple $C_{\mathcal{S}}(H)$-modules are classified. The Krull and the global dimensions are found for some (prime) factor algebras of the algebra $\mathcal{S}$ (over the centre). It is proved that some (prime) factor algebras of $\mathcal{S}$ and $C_{\mathcal{S}}(H)$ are tensor homological/Krull minimal.

Key Words: weight module, simple module, centralizer, Krull dimension, global dimension, tensor homological minimal algebra, tensor Krull minimal algebra.

Mathematics subject classification 2010: 17B10, 17B20, 17B35, 16E10, 16P90, 16P40, 16P50.

Contents

1 Introduction

In this paper, module means a left module, $\mathbb{K}$ is a field of characteristic zero, $\mathbb{K}^* = \mathbb{K}\setminus\{0\}$, $\mathbb{N} = \{0,1,2,\ldots\}$ and $\mathbb{N}_+ = \{1,2,\ldots\}$.

The *Schrödinger (Lie) algebra* $\mathfrak{s} = \mathfrak{sl}_2 \ltimes \mathcal{H}$ is a semidirect product of the Lie algebras

$$\mathfrak{sl}_2 = \mathbb{K}\langle\, E,\, F,\, H \mid [H,E]=2E,\;\; [H,F]=-2F,\;\; [E,F]=H\,\rangle,$$
$$\mathcal{H} = \mathbb{K}\langle\, X,\, Y,\, Z \mid [X,Y]=Z,\;\; [Z,X]=0,\;\; [Z,Y]=0\,\rangle,$$

where $\mathcal{H}$ is the 3-dimensional *Heisenberg (Lie) algebra*. The ad-action of the Lie algebra $\mathfrak{sl}_2$ on $\mathcal{H}$ is given by the rule:

$$\begin{array}{llll} [H,X]=X, & [E,X]=0, & [F,X]=Y, & [\mathfrak{s},Z]=0, \\ [H,Y]=-Y, & [E,Y]=X, & [F,Y]=0. & \end{array}$$

So, by definition, Z is a central element of the Lie algebra $\mathfrak{s}$. The relations above together with the defining relations of the Lie algebras $\mathfrak{sl}_2$ and $\mathcal{H}$ are defining relations of the Lie algebra $\mathfrak{s}$. Let $\mathcal{S} = U(\mathfrak{s})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{s}$.

An $\mathfrak{s}$-module M is called a *weight* $\mathfrak{s}$-module if $M = \bigoplus_{\lambda\in\mathbb{K}} M_\lambda$ where $M_\lambda := \{m\in M \,|\, Hm = \lambda m\}$ is called the *weight subspace/component of weight* μ provided $M_\mu \neq 0$. The aim of the paper is to classify simple weight $\mathfrak{s}$-modules (Proposition 3.3.(2) and Theorem 3.4). A first step was done in [20] where simple highest/lowest weight $\mathcal{S}$-modules were classified. In [18], a classification of simple weight $\mathcal{S}$-modules with finite dimensional weight spaces were classified over $\mathbb{C}$. Every weight component $M_\mu \neq 0$ of a weight $\mathcal{S}$-module M is a module over the *centralizer* $C_{\mathcal{S}}(H) := \{a\in\mathcal{S}\,|\, aH = Ha\}$ of the *Cartan* element H in $\mathcal{S}$. If, in addition, the $\mathcal{S}$-module M is simple then every weight component M_μ is a simple $C_{\mathcal{S}}(H)$-module. So, the problem of classification of simple weight $\mathcal{S}$-modules consists of three steps: Step 1: To classify *all* the simple $C_{\mathcal{S}}(H)$-modules. Step 2: How to reassemble some of the simple $C_{\mathcal{S}}(H)$-modules into a single simple $\mathcal{S}$-module. Step 3: To decide whether two simple weight $\mathcal{S}$-modules are isomorphic. What have been just said is true in a more general situation: *a Lie algebra and its abelian subalgebras* or *an algebra A and its commutative (finitely generated) subalgebra $\mathscr{H}$* where an A-module is called weight if it is a semisimple $\mathscr{H}$-module (e.g., $\mathcal{S}$ and $\mathscr{H} = \mathbb{K}[H]$). As a rule, the centralizer $C_A(\mathscr{H})$ has very rich and complex structure which is a reflection of richness of the category of weight modules. The problem of finding explicit generators and defining relations for the centralizer $C_A(\mathscr{H})$ is a challenging one. A reason for that is that generators for $C_A(\mathscr{H})$ are (linear combinations of) products of generators of the algebra A of high degree and, as a result, defining relations are also of high degree and are very complex (it is a real Noncommutative Geometry).

A problem of classification of simple weight $\mathcal{S}$-modules is essentially reduced to the one for its factor algebras $\mathcal{S}(\lambda) := \mathcal{S}/(Z-\lambda)$ where $\lambda \in \mathbb{K}$. A classification of simple weight $\mathcal{S}(0)$-modules was given in [15]. The case $\lambda \neq 0$ is considered in this paper. For the algebra $\mathcal{S}(\lambda)$ where $\lambda \in \mathbb{K}^*$, the centralizer $C_{\mathcal{S}(\lambda)}(H)$ turns out to be a generalized Weyl algebra (which is a Noetherian domain of Gelfand-Kirillov dimension 4) and the centre of $C_{\mathcal{S}(\lambda)}(H)$ is a polynomial algebra in two variables H and Δ'_λ (Proposition 2.7.(2)). So, the problem of classification of simple $C_{\mathcal{S}(\lambda)}(H)$-modules is reduced to the problem of classification of simple modules over the factor algebras $C_\lambda^{\mu,\nu} := C_{\mathcal{S}(\lambda)}(H)/(H-\mu, \Delta'_\lambda - \nu)$ where $\mu, \nu \in \mathbb{K}$. The algebras $C_\lambda^{\mu,\nu}$ are generalized Weyl algebras with coefficients from a Dedekind domain (more precisely, $\mathbb{K}[H]$). A classification of all simple modules for such generalized Weyl algebras was obtained in [3, 6]. Then the set of simple weight $\mathcal{S}$-modules are partitioned into several classes and each of them is dealt separately with different techniques, see Section 3.

In Section 2, we compute the Krull and global dimensions of the algebra $C_{\mathcal{S}(\lambda)}(H)$ (Proposition 2.7) and some of its (prime) factor algebras $C_\lambda^\nu := C_{\mathcal{S}(\lambda)}(H)/(\Delta'_\lambda - \nu)$ (Lemma 2.10) and $C_\lambda^{\mu,\nu}$ (Corollary 2.8). In more detail ($\mathcal{K}$ denotes the Krull dimension),

$$\mathcal{K}(C_{\mathcal{S}(\lambda)}(H)) = 3 \quad \text{and} \quad \operatorname{gldim} C_{\mathcal{S}(\lambda)}(H) = 4.$$

$$\mathcal{K}(C_\lambda^\nu) = 2 \qquad \text{and} \quad \operatorname{gldim} C_\lambda^\nu = \begin{cases} \infty, & \text{if } \nu = -1, \\ 3, & \text{if } \nu \in \{n(n+2) \mid n = 0,1,2,\ldots\}, \\ 2, & \text{otherwise.} \end{cases}$$

$$\mathcal{K}(C_\lambda^{\mu,\nu}) = 1 \qquad \text{and} \quad \operatorname{gldim} C_\lambda^{\mu,\nu} = \begin{cases} \infty, & \text{if } \nu \in \Lambda^\infty(\mu), \\ 2, & \text{if } \nu \in \Lambda^f(\mu) \setminus \Lambda^\infty(\mu), \\ 1, & \text{otherwise,} \end{cases}$$

where $\Lambda^f(\mu) = \{(2i+\mu-\frac{1}{2})(2i+\mu+\frac{3}{2}),\ (2i+\mu+\frac{1}{2})(2i+\mu-\frac{3}{2}),\ j^2-1 \mid i \in \mathbb{Z}\setminus\{0\},\ j = 1,2,\ldots\}$ and $\Lambda^\infty(\mu) = \{(\mu-\frac{1}{2})(\mu+\frac{3}{2}),\ (\mu+\frac{1}{2})(\mu-\frac{3}{2}),\ -1\}$. Similarly (see Proposition 2.7.(3,4) and Lemma 2.9.(3,4)),

$$\mathcal{K}(\mathcal{S}(\lambda)) = 3 \qquad \text{and} \quad \operatorname{gldim} \mathcal{S}(\lambda) = 4.$$

$$\mathcal{K}(\mathcal{S}(\lambda,\nu)) = 2 \quad \text{and} \quad \operatorname{gldim} \mathcal{S}(\lambda,\nu) = \begin{cases} \infty, & \text{if } \nu = -1, \\ 3, & \text{if } \nu \in \{n(n+2) \mid n = 0,1,2,\ldots\}, \\ 2, & \text{otherwise.} \end{cases}$$

It follows directly from the classification of simple weight $\mathcal{S}$-modules (given in this paper) that the *Finite-Infinite Dimensional Dichotomy* holds for them (Theorem 3.10): *For a simple weight $\mathcal{S}$-module all its weight spaces are either finite or infinite dimensional.* As a corollary, we obtain a short different proof of the result of Dubsky about classification of simple weight $\mathcal{S}$-modules with finite dimensional weight spaces (Theorem 3.12) over an arbitrary algebraically closed field $\mathbb{K}$ not necessarily $\mathbb{K} = \mathbb{C}$ as in [18]. Corollary 3.13.(1) gives a classification of simple weight $\mathcal{S}$-modules where all the weight components have the same finite dimensions. This result strengthens the result obtained in [29] which states: Let V be a simple $\mathcal{S}$-module but not a simple $\mathfrak{sl}_2$-module, if V is neither a highest weight nor a lowest weight module then $\mathrm{Wt}(V) = \mu + \mathbb{Z}$ for any $\mu \in \mathrm{Wt}(V)$ and all the weight spaces of V have the same dimension. Corollary 3.13.(2) gives a classification of simple weight $\mathcal{S}$-modules where all the weight components are uniformly bounded (by a constant). In [19], the category $\mathcal{O}$ of the Schrödinger algebra was studied. In [30], a classification of simple Whittaker $\mathcal{S}$-module was given.

A classification of simple weight modules over the spatial ageing algebra is given by Lü, Mazorchuk and Zhao, [22]. Classification of simple weight modules and various classes of torsion simple modules over the quantum spatial ageing algebra are given in [12] and [14], resp. Classification of prime ideals and simple weight modules over the Euclidean algebra are obtained in [13].

2 The global and Krull dimensions

The aim of this section is to study the centralizer $C_{\mathcal{S}(\lambda)}(H)$ of the Cartan element H in the algebra $\mathcal{S}(\lambda) = \mathcal{S}/(Z-\lambda)$ where $\lambda \in \mathbb{K}^*$ and the (prime) factor algebra $\mathcal{S}(\lambda,\nu)$ where $\nu \in \mathbb{K}$. The case $\lambda = 0$ was done in [15], and the cases $\lambda \neq 0$ and $\lambda = 0$ are quite different. We find the Krull and the global dimensions of the algebras $C_{\mathcal{S}(\lambda)}(H)$ (Proposition 2.7.(5,6)), C_λ^ν (Lemma 2.10.(3,4)), $C_\lambda^{\mu,\nu}$ (Lemma 2.8.(4,5)), $\mathcal{S}(\lambda)$ (Proposition 2.7.(3,4)) and $\mathcal{S}(\lambda,\nu)$ (Lemma 2.9.(3,4)). We show that the algebras $C_{\mathcal{S}(\lambda)}(H)$ (Proposition 2.7.(1)), C_λ^ν (Lemma 2.10.(1)) and $C_\lambda^{\mu,\nu}$ (Corollary 2.8.(1)) are generalized Weyl algebras and find their centres. We also show that some of these algebras are tensor homological minimal and tensor Krull minimal with respect to some classes of left Noetherian algebras.

At the beginning of this section, we collect some known results about the universal enveloping algebra $\mathcal{S} = U(\mathfrak{s})$ of the Lie algebra $\mathfrak{s}$. Let $\mathcal{S}_Z$ be the localization of the algebra $\mathcal{S}$ at the powers of the central element Z of $\mathcal{S}$. The algebra $\mathcal{S}_Z$ contains the Weyl algebra $A_1 := \mathbb{K}\langle \mathscr{X}, Y \,|\, [\mathscr{X}, Y] = 1\rangle$ where $\mathscr{X} := Z^{-1}X$.

Lemma 2.1. *([16, Lemma 2.2].)*

1. *Let $E' := E - \frac{1}{2}Z^{-1}X^2$, $F' := F + \frac{1}{2}Z^{-1}Y^2$ and $H' := H + Z^{-1}XY - \frac{1}{2}$. Then the following commutation relations hold in the algebra $\mathcal{S}_Z$:*

$$[H', E'] = 2E', \quad [H', F'] = -2F', \quad [E', F'] = H',$$

   i.e., the Lie algebra $\mathbb{K}F' \oplus \mathbb{K}H' \oplus \mathbb{K}E'$ is isomorphic to $\mathfrak{sl}_2$. Moreover, the subalgebra U' of $\mathcal{S}_Z$ generated by H', E' and F' is isomorphic to the enveloping algebra $U(\mathfrak{sl}_2)$. Furthermore, the elements E', F' and H' commute with X and Y.

2. *The localization $\mathcal{S}_Z$ of the algebra $\mathcal{S}$ at the powers of Z is $\mathcal{S}_Z = \mathbb{K}[Z^{\pm 1}] \otimes U' \otimes A_1$.*

The algebra $U' \simeq U(\mathfrak{sl}_2)$ in Lemma 2.1.(1) is called the *hidden* $U(\mathfrak{sl}_2)$. The centre $Z(U')$ of the algebra U' is a polynomial algebra $\mathbb{K}[\Delta']$ where $\Delta' := 4F'E' + H'^2 + 2H'$ is the *Casimir element*. One can check that

$$\Delta' = 4FE + H^2 + H + 2Z^{-1}(EY^2 + HXY - FX^2) - \frac{3}{4}. \tag{1}$$

Let

$$C := Z\Delta' + \frac{3}{4}Z = Z(4FE + H^2 + H) + 2(EY^2 + HXY - FX^2). \tag{2}$$

By Lemma 2.1.(2), $Z(\mathcal{S}) = \mathbb{K}[Z, C]$ is a polynomial algebra (see [16, Proposition 2.5]). This result was known before with various degrees of details (for example, the element C appeared in [24]). It seems that a complete proof was given in [19] where a different approach was taken (the proof is much more involved).

The factor algebra $\mathcal{S}/(Z)$. The 1-dimensional space $\mathbb{K}Z$ is an ideal of the Schrödinger (Lie) algebra. The Lie algebra $\mathfrak{s}/\mathbb{K}Z$ is canonically isomorphic to the semidirect product $\mathfrak{sl}_2 \ltimes V_2$ of the Lie algebra $\mathfrak{sl}_2$ with its (unique) 2-dimensional simple $\mathfrak{sl}_2$-module V_2 (treated as an abelian Lie algebra). By (2), the element $\mathsf{c} := FX^2 - HXY - EY^2$ belongs to the centre of the universal enveloping algebra $A := U(\mathfrak{sl}_2 \ltimes V_2)$ of the Lie algebra $\mathfrak{sl}_2 \ltimes V_2$. In fact, $Z(A) = \mathbb{K}[\mathsf{c}]$ (see, [15]).

Generalized Weyl algebra. *Definition*, [2, 6]. Let D be a ring, σ be an automorphism of D and a is an element of the centre of D. *The generalized Weyl algebra* $A := D(\sigma, a) := D[X, Y; \sigma, a]$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y \text{ for all } \alpha \in D, \quad YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded where $A_n = Dv_n$, $v_n = X^n$ for $n > 0$, $v_n = Y^{-n}$ for $n < 0$ and $v_0 = 1$.

Global dimension of GWAs. Let R be a commutative Noetherian ring and σ be its automorphism. An ideal $\mathfrak{p}$ of R is called a *σ-semistable ideal* if $\sigma^n(\mathfrak{p}) = \mathfrak{p}$ for some $n \geqslant 1$. If there is no such n, the ideal $\mathfrak{p}$ is called *σ-unstable*.

Theorem 2.2. *[7, Theorem 3.7] Let R be a commutative Noetherian ring of global dimension $n < \infty$, $T = R(\sigma, a)$ be a GWA and let a be a regular element of R that* $\mathrm{gldim}\,(T) < \infty$. *Then* $\mathrm{gldim}\,(T) = \sup\{\mathrm{gldim}\, T, \mathrm{ht}\,\mathfrak{p}+1, \mathrm{ht}\,\mathfrak{q}+1 \mid \mathfrak{p}$ *is a σ-unstable prime ideal of R for which there exist distinct integers i and j with $a \in \sigma^i(\mathfrak{p})$ and $a \in \sigma^j(\mathfrak{p})$; $\mathfrak{q}$ is a σ-semistable prime ideal of R*$\}$.

In this paper, the following theorem is used in many proofs about the global dimension of algebras.

Theorem 2.3. *[8, Theorem 1.6] Let $A = D(\sigma, a)$ be a GWA, D be a commutative Dedekind ring, $Da = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}$ (if $a \neq 0$) where $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are distinct maximal ideals of D. Then the global dimension of the algebra A is*

$$\mathrm{gldim}\, A = \begin{cases} \infty, & \textit{if } a = 0 \textit{ or } n_i \geqslant 2 \textit{ for some } i, \\ 2, & \textit{if } a \neq 0,\ n_1 = \cdots = n_s = 1,\ s \geqslant 1 \textit{ or } a \textit{ is a unit and } \sigma^k(\mathfrak{p}_i) = \mathfrak{p}_j \textit{ for some } k \geqslant 1 \\ & \ \textit{and } i, j \textit{ or } \sigma^k(\mathfrak{q}) = \mathfrak{q} \textit{ for some maximal ideal } \mathfrak{q} \textit{ of } D, \\ 1, & \textit{otherwise.} \end{cases}$$

Example. The Weyl algebra A_1 is a GWA $\mathbb{K}[h](\sigma, a = h)$ where $\sigma(h) = h - 1$. Hence,

$$\mathrm{gldim}\, A_1 = \begin{cases} 1, & \text{if } \mathrm{char}\mathbb{K} = 0, \\ 2, & \text{if } \mathrm{char}\mathbb{K} \neq 0. \end{cases}$$

This result is due to Reinhart [25], his proof is different from this one.

Corollary 2.4. *[2, 6, 21] Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[H](\sigma, a)$ be a GWA where $\sigma(H) = H - 1$ and $\lambda_1, \dots, \lambda_s$ be the roots of the polynomial $a \in \mathbb{K}[H]$ provided $a \notin \mathbb{K}$. Then*

$$\mathrm{gldim}\, A = \begin{cases} \infty, & \textit{if } a = 0 \textit{ or } a \textit{ has a repeated root,} \\ 2, & \textit{if the roots of } a \neq 0 \textit{ are distinct and } \lambda_i - \lambda_j \in \mathbb{Z} \textit{ for some } i \neq j, \\ 1, & \textit{otherwise.} \end{cases}$$

The algebra $U(\nu) := U(\mathfrak{sl}_2)/(\Delta - \nu) = \mathbb{K}[H](\sigma, a = \frac{1}{4}(\Delta - H(H+2)))$ (where $\sigma(H) = H - 2$) is a particular example of the GWA in Theorem 2.3. Applying Theorem 2.3 we obtain the result of Stafford [28] (his proof is different),

$$\mathrm{gldim}\, U(\nu) = \begin{cases} \infty, & \text{if } \nu = -1, \\ 2, & \text{if } \nu \in \{n(n+1) \mid n = 0, 1, 2, \dots\}, \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

Tensor homological/Krull minimal algebras. Let d be one of the following dimensions: the *weak* (homological) dimension wd, the *left homological* dimension lgd or the *left Krull dimension* $\mathcal{K}$. For $d = \mathrm{wd}, \mathcal{K}$ (resp., lgd), $d(A \otimes B) \geqslant d(A) + d(B)$ for all (resp., left Noetherian) algebras A and B, see [1, 23]. In general, a strict inequality holds: Let $Q_n = \mathbb{K}(x_1, \dots, x_n)$ be the field of rational functions and $d = \mathrm{lgd}\,, \mathcal{K}$. Then $n = d(Q_n \otimes Q_n) > d(Q_n) + d(Q_n) = 0 + 0 = 0$.

Definition, [8]. An algebra A is called a *tensor d-minimal algebra* with respect to some class of algebras Ω if

$$d(A \otimes B) = d(A) + d(B) \text{ for all } B \in \Omega.$$

For $d = \mathrm{lgd}$ (resp., $d = \mathcal{K}$), we say that the algebra A is a *tensor homological minimal*, THM (resp., a *tensor Krull minimal*, TKM).

Example. Let $\mathbb{K}$ be an algebraically closed uncountable field of characteristic zero. Then the GWA $\mathbb{K}[H](\sigma, a)$ where $\sigma(H) = H - \mu$ (where $\mu \in \mathbb{K}^*$) is a tensor homological minimal algebra with respect to the class $\mathcal{LFN}$ of left Noetherian, finitely generated algebras, [8, Corollary 1.5.(1)]. The Weyl algebra A_1 and all factor algebras $U(\nu) = U(\mathfrak{sl}_2)/(\Delta - \nu)$ (where $\nu \in \mathbb{K}$ and Δ is the Casimir element) are examples of such GWAs. In particular, they are THM with respect to the class $\mathcal{LFN}$.

Krull dimension of GWAs.

Theorem 2.5. *[9, Theorem 1.2] Let R be a commutative Noetherian ring with $\mathcal{K}(R) < \infty$ and $T = R(\sigma, a)$ be a GWA. Then $\mathcal{K}(T) = \sup\{\mathcal{K}(R), \mathrm{ht}\,\mathfrak{p} + 1, \mathrm{ht}\,\mathfrak{q} + 1 \mid \mathfrak{p}$ is a σ-unstable prime ideal of R for which there exists infinitely many $i \in \mathbb{Z}$ with $a \in \sigma^i(\mathfrak{p})$; $\mathfrak{q}$ is a σ-semistable prime ideal of $R\}$.*

Note. The ideals $\mathfrak{p}$ and $\mathfrak{q}$ in Theorem 2.5 can be assumed to be maximal of height $\mathcal{K}(R)$. The case when the ring R is not necessarily commute is considered in [10] where explicit formulae for the Krull dimension are obtained.

Example. The algebra $U(\mathfrak{sl}_2)$ is the GWA $\mathbb{K}[H, \Delta](\sigma, a = \frac{1}{4}(\Delta - H(H+2)))$. Clearly, there are no maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$ as in Theorem 2.5. Hence, $\mathcal{K}(U(\mathfrak{sl}_2)) = \mathcal{K}(\mathbb{K}[H, \Delta]) = 2$. The result is due to Smith [27], his proof is based on a different approach.

Example. The Weyl algebra $A_1 = \mathbb{K}\langle \partial, X \mid [\partial, X] = 1\rangle$ is a GWA $\mathbb{K}[h](\sigma, a = h)$. Similarly, $\mathcal{K}(A_1) = \mathcal{K}(\mathbb{K}[h]) = 1$ as there are no maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$ as in Theorem 2.5. The result is due to Rentschler and Gabriel [26], they used a different approach.

The next result shows that many GWAs are THM with respect to the class of countably generated left Noetherian algebras. This fact allows one to compute effectively their Krull dimension as well as their tensor products.

Theorem 2.6. *[11, Theorem 2.2] Let $T = \bigotimes_{i=1}^n T_i$ be a tensor product of GWAs of the form $T_i = D_i(\sigma_i, a_i)$ where each D_i is an affine commutative algebra over an algebraically closed uncountable field $\mathbb{K}$. Then T is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras, that is*

$$\mathcal{K}(T \otimes B) = \mathcal{K}(T) + \mathcal{K}(B) = \sum_{i=1}^n \mathcal{K}(T_i) + \mathcal{K}(B)$$

for any countable dimensional left Noetherian algebra B. In particular, $\mathcal{K}(\bigotimes_{i=1}^n T_i) = \sum_{i=1}^n \mathcal{K}(T_i)$.

The Weyl algebra $A_1 = \mathbb{K}\langle \mathscr{X}, Y \mid [\mathscr{X}, Y] = 1\rangle$ is a GWA,

$$A_1 = \mathbb{K}[h][Y, \mathscr{X}; \sigma, h], \tag{4}$$

where $\sigma(h) = h - 1$. In particular, $h = \mathscr{X} Y$. The Weyl algebra $A_1 = \bigoplus_{i \in \mathbb{Z}} A_{1,i}$ is a $\mathbb{Z}$-graded algebra where $A_{1,i} = \mathbb{K}[h] v_i$ and

$$v_i = \begin{cases} Y^i, & \text{if } i \geqslant 1, \\ 1, & \text{if } i = 0, \\ \mathscr{X}^{-i}, & \text{if } i \leqslant -1. \end{cases}$$

The algebra $U = U(\mathfrak{sl}_2)$ is a GWA, $U = \mathbb{K}[H, \Delta][E, F; \sigma, a = \frac{1}{4}(\Delta - H(H+2))$, where $\mathbb{K}[H, \Delta]$ is a polynomial algebra and $\sigma(H) = H - 2$ and $\sigma(\Delta) = \Delta$. Furthermore, $\Delta = 4FE + H(H+2)$ is the Casimir element of the algebra U and the centre of U is equal to $Z(U) = \mathbb{K}[\Delta]$.

For $\lambda \in \mathbb{K}$, let $\mathcal{S}(\lambda) := \mathcal{S}/\mathcal{S}(Z-\lambda)$. Clearly, $\mathcal{S}(0) \simeq A$. If $\lambda \neq 0$ then, by Lemma 2.1.(2), the algebra

$$\mathcal{S}(\lambda) = \mathcal{S}_Z/\mathcal{S}_Z(Z-\lambda) = U'_\lambda \otimes A_1(\lambda) \tag{5}$$

is a tensor product of algebras U'_λ and $A_1(\lambda)$ which are the images of the algebra U' and A_1 in $\mathcal{S}(\lambda)$ under the epimorphism $\mathcal{S}_Z \to \mathcal{S}_Z/\mathcal{S}_Z(Z-\lambda) = \mathcal{S}(\lambda)$. The algebra U'_λ is canonically isomorphic to the algebra $U = U(\mathfrak{sl}_2)$. The elements

$$H'_\lambda := H + \lambda^{-1}XY - \frac{1}{2}, \quad E'_\lambda := E - \frac{1}{2}\lambda^{-1}X^2, \quad F'_\lambda := F + \frac{1}{2}\lambda^{-1}Y^2, \tag{6}$$

which are the images of the elements H', E' and F', respectively, are canonical generators for the algebra U'_λ. The algebra $A_1(\lambda) = \mathbb{K}\langle \mathscr{X}_\lambda, Y \,|\, [\mathscr{X}_\lambda, Y] = 1\rangle$ is isomorphic to the Weyl algebra A_1 where $\mathscr{X}_\lambda = \lambda^{-1}X$ and Y are the images of the elements $\mathscr{X} = Z^{-1}X$ and Y in $\mathcal{S}(\lambda)$. By (4), the algebra $A_1(\lambda)$ is a GWA,

$$A_1(\lambda) = \mathbb{K}[h_\lambda][Y, \mathscr{X}_\lambda; \sigma, h_\lambda], \tag{7}$$

where $\sigma(h_\lambda) = h_\lambda - 1$ and $h_\lambda = \mathscr{X}_\lambda Y = \lambda^{-1}XY$. In particular, $A_1(\lambda) = \bigoplus_{i\in\mathbb{Z}} A_1(\lambda)_i$ is a $\mathbb{Z}$-graded algebra where $A_1(\lambda)_i = \mathbb{K}[h_\lambda]v(\lambda)_i$ where

$$v(\lambda)_i = \begin{cases} Y^i, & \text{if } i \geqslant 1, \\ 1, & \text{if } i = 0, \\ \mathscr{X}_\lambda^{-i}, & \text{if } i \leqslant -1. \end{cases} \tag{8}$$

The algebra U'_λ is a GWA,

$$U'_\lambda = \mathbb{K}[H'_\lambda, \Delta'_\lambda][E'_\lambda, F'_\lambda; \sigma', a'_\lambda = \frac{1}{4}(\Delta'_\lambda - H'_\lambda(H'_\lambda + 2))], \tag{9}$$

where $\sigma'(H'_\lambda) = H'_\lambda - 2$, $\sigma(\Delta'_\lambda) = \Delta'_\lambda$ and $\Delta'_\lambda := 4F'_\lambda E'_\lambda + H'_\lambda(H'_\lambda + 2)$ is the image of the Casimir element Δ' in $\mathcal{S}(\lambda)$. The algebra $U'_\lambda = \bigoplus_{i\in\mathbb{Z}} U'_{\lambda,i}$ is a $\mathbb{Z}$-graded algebra where $U'_{\lambda,i} = \mathbb{K}[H'_\lambda, \Delta'_\lambda]v'_i$ and

$$v'_i = \begin{cases} E'_\lambda, & \text{if } i \geqslant 1, \\ 1, & \text{if } i = 0, \\ F'^{-i}_\lambda, & \text{if } i \leqslant -1. \end{cases}$$

The centralizer $C_{\mathcal{S}(\lambda)}(H)$. Recall that for an element a of an algebra A, we denote by $C_A(a) := \{b \in A \,|\, ab = ba\}$ the *centralizer* of a in A. The next proposition is about generators and defining relations of the centralizer $C_{\mathcal{S}(\lambda)}(H)$ of the element H in the algebra $\mathcal{S}(\lambda)$, the global and Krull dimensions of the algebra $\mathcal{S}(\lambda)$. If, for an algebra A, the left and right global dimension are equal, the common value is denoted by $\mathrm{gldim}\,(A)$.

Proposition 2.7. *Suppose that $\lambda \in \mathbb{K}^*$. Let $x := E'_\lambda Y^2$ and $y := F'_\lambda \mathscr{X}_\lambda^2$. Then*

1. *The algebra $C_{\mathcal{S}(\lambda)}(H)$ is a GWA,*

$$C_{\mathcal{S}(\lambda)}(H) = D_\lambda[x, y;\, \tau,\, \mathsf{a}_\lambda = a'_\lambda \cdot h_\lambda(h_\lambda + 1)],$$

where $D_\lambda = \mathbb{K}[H, \Delta'_\lambda, h_\lambda]$ is a polynomial algebra and $\tau(H) = H$, $\tau(\Delta'_\lambda) = \Delta'_\lambda$ and $\tau(h_\lambda) = h_\lambda - 2$. The algebra $C_{\mathcal{S}(\lambda)}(H)$ is a Noetherian domain of Gelfand-Kirillov dimension 4.

2. *The centre of the algebra $C_{\mathcal{S}(\lambda)}(H)$ is the polynomial algebra $\mathbb{K}[H, \Delta'_\lambda]$.*
3. *($\mathbb{K}$ is an algebraically closed uncountable field). The (left or right) global dimension of the algebra $\mathcal{S}(\lambda)$ is equal to* $\mathrm{gldim}\, S(\lambda) = 4$.
4. *($\mathbb{K}$ is an algebraically closed uncountable field). The Krull dimension of the algebra $\mathcal{S}(\lambda)$ is 3. The algebra $\mathcal{S}(\lambda)$ is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras.*
5. *($\mathbb{K}$ is an algebraically closed field)* $\mathrm{gldim}\, C_{\mathcal{S}(\lambda)}(H) = 4$.

6. ($\mathbb{K}$ *is an algebraically closed field*) $\mathcal{K}(C_{\mathcal{S}(\lambda)}(H)) = 3$.

Proof. 1. Using the fact that the algebras U'_λ and $A_1(\lambda)$ are GWA's (see (7) and (9)), we have

$$\mathcal{S}(\lambda) = U'_\lambda \otimes A_1(\lambda) = \bigoplus_{i,j\in\mathbb{Z}} D_\lambda v'_i v(\lambda)_j.$$

Using the equalities $[H, v'_i] = 2iv'_i$ and $[H, v(\lambda)_j] = -jv(\lambda)_j$ we see that

$$C_{\mathcal{S}(\lambda)}(H) = \bigoplus_{i\in\mathbb{Z}} D_\lambda v'_i v(\lambda)_{2i} = \bigoplus_{i\geqslant 1} D_\lambda y^i \oplus D_\lambda \oplus \bigoplus_{i\geqslant 1} D_\lambda x^i = D_\lambda[x, y; \tau, \mathsf{a}_\lambda].$$

2. Statement 2 follows from statement 1.

3. The Weyl algebra $A_1(\lambda)$ is a THM with respect to the class $\mathcal{LFN}$ and $U'_\lambda \in \mathcal{LFN}$. Hence, by (5), $\mathrm{gldim}\,\mathcal{S}(\lambda) = \mathrm{gldim}\,U'_\lambda \otimes A_1(\lambda) = \mathrm{gldim}\,U'_\lambda + \mathrm{gldim}\,A_1(\lambda) = 3 + 1 = 4$.

4. By Theorem 2.6, $\mathcal{K}(\mathcal{S}(\lambda)) = \mathcal{K}(U'_\lambda \otimes A_1(\lambda)) = \mathcal{K}(U'_\lambda) + \mathcal{K}(A_1(\lambda)) = 2 + 1 = 3$ and the algebra $\mathcal{S}(\lambda)$ is a TKM algebra with respect to the class of countably generated left Noetherian algebras.

5. The algebra $\mathcal{S}(\lambda) = \bigoplus_{i\in\mathbb{Z}} \mathcal{S}(\lambda)_i$ is a $\mathbb{Z}$-graded algebra where $\mathcal{S}(\lambda)_i := \{a \in \mathcal{S}(\lambda) \,|\, [H, a] = ia\}$ and $C_{\mathcal{S}(\lambda)}(H) = \mathcal{S}(\lambda)_0$. Therefore, $\mathrm{gldim}\,C_{\mathcal{S}(\lambda)}(H) \leqslant \mathrm{gldim}\,\mathcal{S}(\lambda) = 4 < \infty$, by statement 3. Notice that $\mathrm{gldim}\,D_\lambda = 3$. By Theorem 2.2, $\mathrm{gldim}\,C_{\mathcal{S}(\lambda)}(H) = 4$ as there is a maximal ideal $\mathfrak{p}$ of D_λ that satisfies the conditions of Theorem 2.2, e.g., $\mathfrak{p} = (H - \mu,\, \Delta'_\lambda - (i^2 - 1),\, h_\lambda + \mu + \frac{1}{2} + i)$ for $i \in \mathbb{Z} \setminus \{0\}$, see Case 1 of the proof of Corollary 2.8.(3).

6. By Theorem 2.5, $\mathcal{K}(C_{\mathcal{S}(\lambda)}(H)) = \mathcal{K}(D_\lambda) = 3$ as there are no maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$ that satisfy the conditions of Theorem 2.5. □

The algebras $C_\lambda^{\mu,\nu}$. By Proposition 2.7, for every pair $\mu, \nu \in \mathbb{K}$, we can consider the factor algebra

$$C_\lambda^{\mu,\nu} := C_{\mathcal{S}(\lambda)}(H)/(H - \mu, \Delta'_\lambda - \nu).$$

The algebra $C_\lambda^{\mu,\nu}$ and *all* their simple modules play an important role in a classification of the simple weight modules over the Schrödinger algebra. Roughly speaking, the problem of classification of simple weight $\mathcal{S}(\lambda)$-modules is reduced to the problem of classification of *all* simple modules for the algebras $C_\lambda^{\mu,\nu}$. In general, there is little connection between the global dimension of an algebra and its factor algebras. The next corollary is an example of this fact.

The next corollary presents a simplicity criterion for the algebra $C_\lambda^{\mu,\nu}$, it also computes values for the Krull and global dimensions of the algebra $C_\lambda^{\mu,\nu}$.

Corollary 2.8. *Let $\lambda \in \mathbb{K}^*$ and $\mu, \nu \in \mathbb{K}$. Then*

1. *The algebra $C_\lambda^{\mu,\nu}$ is isomorphic to the algebra $C_{\mathcal{S}(\lambda)}(H)/(H - \mu, \Delta'_\lambda - \nu)$ which is a GWA,*

$$C_\lambda^{\mu,\nu} = \mathbb{K}\big[h_\lambda\big]\big[x, y; \tau, \mathsf{a}_\lambda^{\mu,\nu} = \frac{1}{4}\Big(\nu - (h_\lambda + \mu - \frac{1}{2})(h_\lambda + \mu + \frac{3}{2})\Big)h_\lambda(h_\lambda + 1)\big],$$

where $\tau(h_\lambda) = h_\lambda - 2$ and $\mathsf{a}_\lambda^{\mu,\nu} \equiv \mathsf{a}_\lambda \mod (H - \mu, \Delta'_\lambda - \nu)$.

2. *The algebra $C_\lambda^{\mu,\nu}$ is a central Notherian domain of Gelfand-Kirillov dimension 2.*
3. *The algebra $C_\lambda^{\mu,\nu}$ is simple iff $\nu \notin \Lambda^f(\mu) := \{(2i + \mu - \frac{1}{2})(2i + \mu + \frac{3}{2}),\, (2i + \mu + \frac{1}{2})(2i + \mu - \frac{3}{2}),\, j^2 - 1 \,|\, i \in \mathbb{Z} \setminus \{0\},\, j = 1, 2, \ldots\}$.*
4. *The (left or right) global dimension of the algebra $C_\lambda^{\mu,\nu}$ is equal to*

$$\mathrm{gldim}\, C_\lambda^{\mu,\nu} = \begin{cases} \infty, & \text{if } \nu \in \Lambda^\infty(\mu), \\ 2, & \text{if } \nu \in \Lambda^f(\mu) \setminus \Lambda^\infty(\mu), \\ 1, & \text{otherwise,} \end{cases}$$

where $\Lambda^\infty(\mu) := \{(\mu - \frac{1}{2})(\mu + \frac{3}{2}),\, (\mu + \frac{1}{2})(\mu - \frac{3}{2}),\, -1\}$. The algebra $C_\lambda^{\mu,\nu}$ is a tensor homological minimal algebra with respect to the class of left Noetherian, finitely generated algebras (provided $\mathbb{K}$ is an algebraically closed and uncountable field).

5. *The (left or right) Krull dimension of the algebra $C_\lambda^{\mu,\nu}$ is 1. The algebra $C_\lambda^{\mu,\nu}$ is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras (provided the field $\mathbb{K}$ is algebraically closed and uncountable).*

Proof. 1 and 2. Statements 1 and 2 follow from Proposition 2.7.

3. By [6, 5], the GWA $C_\lambda^{\mu,\nu}$ is not simple iff there are two distinct roots of the polynomial $\mathsf{a}_\lambda^{\mu,\nu}$, say λ_1 and λ_2, such that $\lambda_2 = \lambda_1 + 2i$ for some $i \in \mathbb{Z} \setminus \{0\}$. There are three cases to consider.

Case 1: λ_1 and λ_2 are roots of the polynomial $P = (h_\lambda + \mu - \frac{1}{2})(h_\lambda + \mu + \frac{3}{2}) - \nu$, i.e., $P = (h_\lambda - \lambda_1)(h_\lambda - \lambda_1 - 2i)$. This is possible iff

$$\begin{cases} 2\lambda_1 + 2i = -2\mu - 1, \\ \nu = (\mu - \frac{1}{2})(\mu + \frac{3}{2}) - \lambda_1(\lambda_1 + 2i), \end{cases}$$

iff $\lambda_1 = -\mu - \frac{1}{2} - i$ and $\nu = i^2 - 1$.

Case 2: $\lambda_1 = 0$ and λ_2 is a root of P, i.e,

$$0 = P(\lambda_2) = P(0 + 2i) \Leftrightarrow \nu = (2i + \mu - \frac{1}{2})(2i + \mu + \frac{3}{2}).$$

Case 3: $\lambda_1 = -1$ and λ_2 is a root of P, i.e.,

$$0 = P(\lambda_2) = P(-1 + 2i) \Leftrightarrow \nu = (2i + \mu - \frac{3}{2})(2i + \mu + \frac{1}{2}).$$

4. Let $\{\lambda_i \,|\, i = 1, \ldots, s\}$ be the roots of the polynomial $\mathsf{a}_\lambda^{\mu,\nu}$. By Theorem 2.3,

$$\mathrm{gldim}\, C_\lambda^{\mu,\nu} = \begin{cases} \infty, & \text{if } \mathsf{a}_\lambda^{\mu,\nu} \text{ has a repeated root,} \\ 2, & \text{if } \mathsf{a}_\lambda^{\mu,\nu} \text{ has no repeated root, } \lambda_i - \lambda_j \in 2\mathbb{Z} \setminus \{0\} \text{ for some } i \neq j, \\ 1, & \text{otherwise.} \end{cases}$$

By [8, Corollary 1.5.(1)], the algebra $C_\lambda^{\mu,\nu}$ is a tensor homological minimal algebra with respect to the class of left Noetherian, finitely generated algebras (provided $\mathbb{K}$ is an algebraically closed and uncountable field).

(i) *The polynomial $\mathsf{a}_\lambda^{\mu,\nu}$ has a repeated root iff $\nu \in \Lambda^\infty(\mu)$*: We have to consider the cases 1–3 in the proof of statement 3 where $i = 0$, i.e., $\lambda_1 = \lambda_2$. This gives $\nu \in \Lambda^\infty(\mu)$.

(ii) By the proof of statement 2, $\mathrm{gldim} C_\lambda^{\mu,\nu} = 2$ iff $\nu \in \Lambda^f(\mu) \setminus \Lambda^\infty(\mu)$.

5. Statement 5 follows from Theorems 2.5 and 2.6. $\square$

The algebra $\mathcal{S}(\lambda, \nu)$. Let $\lambda \in \mathbb{K}^*$ and $\nu \in \mathbb{K}$. By (5), the factor algebra

$$\mathcal{S}(\lambda, \nu) := \mathcal{S}(\lambda)/(\Delta'_\lambda - \nu) \simeq \mathcal{S}/(Z - \lambda, \Delta' - \nu) \simeq U'_\lambda(\nu) \otimes A_1(\lambda) \tag{10}$$

is a tensor product of algebras where

$$U'_\lambda(\nu) := U'_\lambda/(\Delta'_\lambda - \nu) = \mathbb{K}[H'_\lambda][E'_\lambda, F'_\lambda; \sigma', a'^{\nu}_\lambda := \frac{1}{4}(\nu - H'_\lambda(H'_\lambda + 2))] \tag{11}$$

is a GWA where $\sigma'(H'_\lambda) = H'_\lambda - 2$. The algebra $\mathcal{S}(\lambda, \nu)$ is a Noetherian domain of Gelfand-Kirillow dimension 4. The algebra $\mathcal{S}(\lambda, \nu)$ is a GWA of rank 2 as it is a tensor product of two GWAs U'_λ and $A_1(\lambda)$. The problem of classification of weight $\mathcal{S}$-modules are essentially about the problem of classification of simple weight $\mathcal{S}(\lambda, \nu)$-modules. The next lemma gives a simplicity criterion for the algebra $\mathcal{S}(\lambda, \nu)$ and computes the Krull and global dimensions of the algebra $\mathcal{S}(\lambda, \nu)$.

Lemma 2.9. *Let $\lambda \in \mathbb{K}^*$ and $\nu \in \mathbb{K}$.*

1. *The algebra $\mathcal{S}(\lambda, \nu)$ is a central Noetherian domain of Gelfand-Kirillov dimension 4.*
2. *The algebra $\mathcal{S}(\lambda, \nu)$ is simple iff the algebra $U'_\lambda(\nu)$ is simple iff $\nu \notin \{n(n+2) \,|\, n = 0, 1, 2, \ldots\}$.*

3. ($\mathbb{K}$ *is an algebraically closed and uncountable field). The (left or right) global dimension of the algebra* $\mathcal{S}(\lambda,\nu)$ *is equal to*

$$\mathrm{gldim}\,\mathcal{S}(\lambda,\nu)=\mathrm{gldim}\,U'_\lambda(\nu)+\mathrm{gldim}\,A_1(\lambda)=\begin{cases}\infty, & \text{if } \nu=-1,\\ 3, & \text{if } \nu\in\{n(n+2)\,|\,n=0,1,2,\ldots\},\\ 2, & \text{if } \nu\notin\{n(n+2)\,|\,n=-1,0,1,2,\ldots\}.\end{cases}$$

The algebra $\mathcal{S}(\lambda,\nu)$ *is a tensor homological minimal algebra with respect to the class of left Noetherian, finitely generated algebras.*

4. ($\mathbb{K}$ *is an algebraically closed and uncountable field). The (left or right) Krull dimension of the algebra* $\mathcal{S}(\lambda,\nu)$ *is 2. The algebra* $\mathcal{S}(\lambda,\nu)$ *is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras.*

Proof. 1. The algebra $\mathcal{S}(\lambda,\nu)$ is a central algebra as a tensor product of central algebras, by (10).

2. Statement 2 follows from (10) and the fact that the Weyl algebra $A_1(\lambda)$ is a central simple algebra.

3. The Weyl algebra $A_1(\lambda)$ is a tensor homological minimal algebra with respect to the class $\mathcal{LFN}$ [8, Corollary 1.5.(1)], hence

$$\mathrm{gldim}\,\mathcal{S}(\lambda,\nu)=\mathrm{gldim}\,U'_\lambda(\nu)\otimes A_1(\lambda)=\mathrm{gldim}\,U'_\lambda(\nu)+\mathrm{gldim}\,A_1(\lambda)=\mathrm{gldim}\,U'_\lambda(\nu)+1.$$

Now, the result follows from (3).

4. The Weyl algebra $A_1(\lambda)$ is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras ([11, Theorem 2.2]), hence

$$\mathcal{K}(\mathcal{S}(\lambda,\nu))=\mathcal{K}(U'_\lambda(\nu)\otimes A_1(\lambda))=\mathcal{K}(U'_\lambda(\nu))+\mathcal{K}(A_1(\lambda))=1+1=2.$$

By Theorem 2.6, the algebra $\mathcal{S}(\lambda,\nu)$ is a tensor Krull minimal algebra with respect to the class of countably generated left Noetherian algebras. $\square$

The algebras C^ν_λ. Let $C^\nu_\lambda:=C_{\mathcal{S}(\lambda,\nu)}(H)$. The next lemma describes the centre of the algebra C^ν_λ and computes the Krull and global dimensions of C^ν_λ.

Lemma 2.10. *Suppose that* $\lambda\in\mathbb{K}^*$ *and* $\nu\in\mathbb{K}$.

1. *The algebra* C^ν_λ *is isomorphic to the algebra* $C_{\mathcal{S}(\lambda)}(H)/(\Delta'_\lambda-\nu)$ *which is a GWA,*

$$C^\nu_\lambda=\mathbb{K}[H,h_\lambda][x,y;\tau,a'^\nu_\lambda\cdot h_\lambda(h_\lambda+1)]$$

where $a'^\nu_\lambda:=\frac{1}{4}(\nu-H'_\lambda(H'_\lambda+2))=\frac{1}{4}(\nu-(H+h_\lambda-\frac{1}{2})(H+h_\lambda+\frac{3}{2}))$, $\tau(H)=H$ *and* $\tau(h_\lambda)=h_\lambda-2$.

2. *The centre of the algebra* C^ν_λ *is* $\mathbb{K}[H]$ *and the algebra* C^ν_λ *is a Noetherian domain of Gelfand-Kirillov dimension 3.*

3. *The (left or right) global dimension of* C^ν_λ *is equal to*

$$\mathrm{gldim}\,C^\nu_\lambda=\begin{cases}\infty, & \text{if } \nu=-1,\\ 3, & \text{if } \nu\in\{n(n+2)\,|\,n=0,1,2,\ldots\},\\ 2, & \text{otherwise.}\end{cases}$$

4. *The (left or right) Krull dimension of* C^ν_λ *is 2. The algebra* C^ν_λ *is a tensor Krull minimal algebra with respect to the class of countably generated, left Noetherian algebras (provided* $\mathbb{K}$ *is an algebraically closed uncountable field).*

Proof. 1. Statement 1 follows from Proposition 2.7.(1).

2. Statement 2 follows from statement 1.

3. The algebra $\mathcal{S}(\lambda,\nu)=\bigoplus_{i\in\mathbb{Z}}\mathcal{S}(\lambda,\nu)_i$ is a $\mathbb{Z}$-graded algebra where $\mathcal{S}(\lambda,\nu)_i:=\{a\in\mathcal{S}(\lambda,\nu)\,|\,[H,a]=ia\}$ and $C^\nu_\lambda=\mathcal{S}(\lambda,\nu)_0$. Therefore, $\mathrm{gldim}\,C^\nu_\lambda\leqslant\mathrm{gldim}\,\mathcal{S}(\lambda,\nu)$. By Lemma 11.(3), $\mathrm{gldim}\,C^\nu_\lambda<\infty$ if $\nu\neq-1$. If $\nu\neq-1$ then, by Theorem 2.2,

$$\mathrm{gldim}\,C^\nu_\lambda=\begin{cases}3, & \text{if } \nu\in\{n(n+2)\,|\,n=0,1,2,\ldots\},\\ 2, & \text{if } \nu\notin\{n(n+2)\,|\,n=-1,0,1,2,\ldots\}.\end{cases}$$

Claim. $\operatorname{gldim} C_\lambda^{-1} = \infty$. The set $S = \mathbb{K}[h_\lambda] \setminus \{0\}$ is an Ore set of the domain C_λ^{-1} such that the localization

$$S^{-1}C_\lambda^{-1} = \mathbb{K}(h_\lambda)[H][x, y; \tau, a_\lambda'^\nu \cdot h_\lambda(h_\lambda + 1)]$$

is a GWA and the algebra $\mathbb{K}(h_\lambda)[H]$ is a Dedekind ring. For $\nu = -1$, $a_\lambda'^\nu = -\frac{1}{4}(H + h_\lambda + \frac{1}{2})^2$, hence $\operatorname{gldim} S^{-1}C_\lambda^{-1} = \infty$, by Theorem 2.3. Since $\operatorname{gldim} C_\lambda^{-1} \geqslant \operatorname{gldim} S^{-1}C_\lambda^{1}$, we must have $\operatorname{gldim} C_\lambda^{-1} = \infty$.

4. By Theorem 2.5, $\mathcal{K}(C_\lambda^\nu) = \mathcal{K}(\mathbb{K}[H, h_\lambda]) = 2$. If, in addition, the field $\mathbb{K}$ is algebraically closed and uncountable then the algebra C_λ^ν is a TKM algebra with respect to the class of countably generated, left Noetherian algebras. $\square$

3 Classification of simple weight $\mathcal{S}$-modules with nonzero central charge

In this section, $\mathbb{K}$ is an algebraically closed field of characteristic zero. In this section, a classification of simple weight $\mathcal{S}$-modules is obtained. The set $\widehat{S}\,(\text{weight})$ of isomorphism classes of simple weight $\mathcal{S}$-modules is presented as a disjoint union of subsets each of which is dealt separately.

For an algebra A, we denote by $\widehat{A}$ the set of isomorphism classes of simple A-modules and for an A-module M we denote by $[M]$ its isomorphism classes. If P is a property of simple modules which is invariant under isomorphisms of modules (e.g., 'being weight') then $\widehat{A}\,(P)$ stands for the set of all isomorphism classes of simple A-modules that satisfy P. Clearly,

$$\begin{aligned}\widehat{\mathcal{S}}\,(\text{weight}) &= \widehat{\mathcal{S}(0)}\,(\text{weight}) \;\sqcup \bigsqcup_{\lambda\in\mathbb{K}^*} \widehat{\mathcal{S}(\lambda)}\,(\text{weight}),\\ \widehat{\mathcal{S}(\lambda)}\,(\text{weight}) &= \bigsqcup_{\nu\in\mathbb{K}} \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}). \qquad (12)\end{aligned}$$

The set $\widehat{\mathcal{S}(0)}\,(\text{weight})$ was described in [15]. So, in this section we assume that $\lambda \neq 0$. In order to finish the classification of simple weight $\mathcal{S}$-modules it remains to classify simple weight $\mathcal{S}(\lambda,\nu)$-modules for all $\nu \in \mathbb{K}$.

The sets $\widehat{\mathcal{S}(\lambda,\nu)}\,(X$-torsion) and $\widehat{\mathcal{S}(\lambda,\nu)}\,(Y$-torsion). The sets $S_X := \{X^i \,|\, i \in \mathbb{N}\}$ and $S_Y := \{Y^i \,|\, i \in \mathbb{N}\}$ are Ore sets of the domain $\mathcal{S}$. Each $\mathcal{S}$-module M contains the, so-called, *X-torsion* and *Y-torsion* submodules $\operatorname{tor}_{S_X}(M) := \{m \in M \,|\, X^i m = 0 \text{ for some } i \in \mathbb{N}\}$ and $\operatorname{tor}_{S_Y}(M) := \{m \in M \,|\, Y^i m = 0 \text{ for some } i \in \mathbb{N}\}$, respectively. The module M is called *X-torsion* (resp., *Y-torsion*) if $M = \operatorname{tor}_{S_X}(M)$ (resp., $M = \operatorname{tor}_{S_Y}(M)$). First, we classify all the simple X-torsion/Y-torsion $\mathcal{S}(\lambda,\nu)$-modules (Theorem 3.1) and as a result we obtain a classification of simple weight X-torsion/Y-torsion $\mathcal{S}(\lambda,\nu)$-modules (Corollary 3.2).

For $\lambda \neq 0$, the following theorem gives classifications of simple $\mathcal{S}(\lambda,\mu)$-modules that are either X-torsion or Y-torsion.

Theorem 3.1. *Let $\lambda \in \mathbb{K}^*$ and $\nu \in \mathbb{K}$. Then*

1. *$\widehat{\mathcal{S}(\lambda,\nu)}\,(X\text{-torsion}) = \widehat{U_\lambda'(\nu)} \otimes \left[\frac{A_1(\lambda)}{A_1(\lambda)X}\right] = \left\{\left[M \otimes \frac{A_1(\lambda)}{A_1(\lambda)X}\right] \,|\, [M] \in \widehat{U_\lambda'(\nu)}\right\}$ and $\mathcal{S}(\lambda,\nu)$-modules $M \otimes A_1(\lambda)/A_1(\lambda)X$ and $M' \otimes A_1(\lambda)/A_1(\lambda)X$ are isomorphic iff $[M] = [M']$.*
2. *$\widehat{\mathcal{S}(\lambda,\nu)}\,(Y\text{-torsion}) = \widehat{U_\lambda'(\nu)} \otimes \left[\frac{A_1(\lambda)}{A_1(\lambda)Y}\right] = \left\{\left[M \otimes \frac{A_1(\lambda)}{A_1(\lambda)Y}\right] \,|\, [M] \in \widehat{U_\lambda'(\nu)}\right\}$ and $\mathcal{S}(\lambda,\nu)$-modules $M \otimes A_1(\lambda)/A_1(\lambda)Y$ and $M' \otimes A_1(\lambda)/A_1(\lambda)Y$ are isomorphic iff $[M] = [M']$.*
3. *$\widehat{\mathcal{S}(\lambda,\nu)}\,(X\text{-torsion}) \cap \widehat{\mathcal{S}(\lambda,\nu)}\,(Y\text{-torsion}) = \emptyset$.*

Proof. 1. The $A_1(\lambda)$-module $V := A_1(\lambda)/A_1(\lambda)X$ is a simple $A_1(\lambda)$-module with $\operatorname{End}_{A_1(\lambda)}(V) = \mathbb{K}$. Recall that $\mathcal{S}(\lambda,\nu) = U_\lambda' \otimes A_1(\lambda)$. Every simple X-torsion $\mathcal{S}(\lambda,\nu)$-module $\mathcal{M}$ is an epimorphic image of the $\mathcal{S}(\lambda,\nu)$-module $\mathcal{V} := \mathcal{S}(\lambda,\nu)/\mathcal{S}(\lambda,\nu)X = U_\lambda'(\nu) \otimes V$. Each $\mathcal{S}(\lambda,\nu)$-submodule of $\mathcal{V}$ is equal to $I \otimes V$ for some left ideal I of the algebra $U_\lambda'(\nu)$, and so $\mathcal{M} \simeq M \otimes V$ for some simple $U_\lambda(\nu)$-module M, and statement 1 follows.

2. Statement 2 can be proved in a similar way as statement 1 (by replacing X by Y).

3. Statement 3 follows from statements 1 and 2 since the $A_1(\lambda)$-modules $A_1(\lambda)/A_1(\lambda)X$ and $A_1(\lambda)/A_1(\lambda)Y$ are not isomorphic. $\square$

As a corollary of Theorem 3.1, we obtain classifications of simple weight $\mathcal{S}(\lambda,\nu)$-modules that are X-torsion or Y-torsion.

Corollary 3.2. *Let $\lambda \in \mathbb{K}^*$ and $\nu \in \mathbb{K}$. Then*

1.

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } X\text{-torsion}) = \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight}) \otimes \Big[\frac{A_1(\lambda)}{A_1(\lambda)X}\Big]$$
$$= \Big\{\Big[M \otimes \frac{A_1(\lambda)}{A_1(\lambda)X}\Big] \mid [M] \in \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight})\Big\}$$

and $\mathcal{S}(\lambda,\nu)$-modules $M \otimes \frac{A_1(\lambda)}{A_1(\lambda)X}$ and $M' \otimes \frac{A_1(\lambda)}{A_1(\lambda)X}$ are isomorphic iff $[M] = [M']$.

2.

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } Y\text{-torsion}) = \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight}) \otimes \Big[\frac{A_1(\lambda)}{A_1(\lambda)Y}\Big]$$
$$= \Big\{\Big[M \otimes \frac{A_1(\lambda)}{A_1(\lambda)Y}\Big] \mid [M] \in \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight})\Big\}$$

and $\mathcal{S}(\lambda,\nu)$-modules $M \otimes \frac{A_1(\lambda)}{A_1(\lambda)Y}$ and $M' \otimes \frac{A_1(\lambda)}{A_1(\lambda)Y}$ are isomorphic iff $[M] = [M']$.

3. $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } X\text{-torsion}) \cap \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } Y\text{-torsion}) = \emptyset$.

Proof. 1. Recall that $H'_\lambda = H + \lambda^{-1}X\cdot Y - \frac{1}{2} = H + Y\cdot\lambda^{-1}X + \frac{1}{2}$. Every simple, weight, X-torsion $\mathcal{S}(\lambda,\nu)$-module is an epimorphic image of the $\mathcal{S}(\lambda,\nu)$-module (for some $\mu \in \mathbb{K}$)

$$\mathcal{S}(\lambda,\nu)/\mathcal{S}(\lambda,\nu)(H-\mu, X) = \mathcal{S}(\lambda,\nu)/\mathcal{S}(\lambda,\nu)(H'_\lambda - \mu - \frac{1}{2}, X)$$
$$\simeq \frac{U'_\lambda(\nu)}{U'_\lambda(\nu)(H'_\lambda - \mu - \frac{1}{2})} \otimes \frac{A_1(\lambda)}{A_1(\lambda)X} =: \mathcal{M}(\mu)$$

which is a H'_λ-weight module. Conversely, any module from $\widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight}) \otimes \big[\frac{A_1(\lambda)}{A_1(\lambda)X}\big]$ is an epimorphic image of the weight $\mathcal{S}(\lambda,\nu)$-module $\mathcal{M}(\mu)$ for some μ. Now statement 1 follows from Theorem 3.1.(1).

2. Statement 2 can be proved in a similar obvious way.

3. Statement 3 follows from Theorem 3.1.(3). $\square$

The set $\widehat{\mathcal{S}(\lambda,\nu)}$ (weight, T-torsion). Let $\lambda \in \mathbb{K}^*$. The set $T = \{h_\lambda - i \,|\, i \in \mathbb{Z}\}$ is an Ore set in $C_\lambda^{\mu,\nu}$ and the algebra

$$T^{-1}C_\lambda^{\mu,\nu} = \big(T^{-1}\mathbb{K}[h_\lambda]\big)\big[x, y; \sigma, \mathsf{a}_\lambda^{\mu,\nu}\big]$$

is a GWA where $\sigma(h_\lambda) = h_\lambda - 2$. By Corollary 2.8.(2), the algebra $T^{-1}C_\lambda^{\mu,\nu}$ is central with Gelfand-Kirillov dimension 2. The set $T = \{h_\lambda - i \,|\, i \in \mathbb{Z}\}$ is an Ore set in the Weyl algebra $A_1(\lambda)$ such that the localization

$$T^{-1}A_1(\lambda) = \big(T^{-1}\mathbb{K}[h_\lambda]\big)\big[Y, Y^{-1}; \sigma\big]$$

is a skew Laurent polynomial algebra where $\sigma(h_\lambda) = h_\lambda - 1$. By (5), the set T is also an Ore set of the algebras $\mathcal{S}(\lambda)$ and $\mathcal{S}(\lambda,\nu)$,

$$T^{-1}\mathcal{S}(\lambda) \simeq U'_\lambda \otimes T^{-1}A_1(\lambda) \quad \text{and} \quad T^{-1}\mathcal{S}(\lambda,\nu) \simeq U'_\lambda/(\Delta'_\lambda - \nu) \otimes T^{-1}A_1(\lambda).$$

The next proposition together with Corollary 3.2 classifies the simple (weight) T-torsion $\mathcal{S}(\lambda,\nu)$-modules.

Proposition 3.3. *1.* $\widehat{\mathcal{S}(\lambda,\nu)}\,(T\text{-torsion}) = \widehat{\mathcal{S}(\lambda,\nu)}\,(X\text{-torsion}) \sqcup \widehat{\mathcal{S}(\lambda,\nu)}\,(Y\text{-torsion})$.

2. $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } T\text{-torsion}) = \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } X\text{-torsion}) \sqcup \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight, } Y\text{-torsion})$.

Proof. 1. By Theorem 3.1.(3), the union in statement 1 is a disjoint union. The equality in statement 1 follows from (10) and the equalities $Y^iX^i = \lambda^i(h_\lambda - 1)(h_\lambda - 2)\cdots(h_\lambda - i)$ and $X^iY^i = \lambda^i h_\lambda(h_\lambda + 1)\cdots(h_\lambda + i - 1)$ for all $i \geqslant 1$.

2. Statement 2 follows from statement 1. □

We have that

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}) = \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsion}) \sqcup \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}). \tag{13}$$

Corollary 3.2.(1,2) and Proposition 3.3.(2) classify the set of simple, weight, T-torsion $\mathcal{S}(\lambda,\nu)$-modules. In order to finish a classification of simple weight $\mathcal{S}$-modules, it remains to classify elements of the set $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree})$.

The set $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree})$. Notice that $T \subseteq C_\lambda^\nu$, and so

$$T^{-1}\mathcal{S}(\lambda,\nu) = C_{T^{-1}\mathcal{S}(\lambda,\nu)}(H)[Y, Y^{-1}; \omega_Y] = \big(T^{-1}C_{\mathcal{S}(\lambda,\nu)}(H)\big)[Y, Y^{-1}; \omega_Y] = \big(T^{-1}C_\lambda^\nu\big)[Y, Y^{-1}; \omega_Y] \tag{14}$$

where $\omega_Y(c) = YcY^{-1}$. In particular, $\omega_Y(H) = H + 1$, $\omega_Y(h_\lambda) = h_\lambda - 1$, $\omega_Y(x) = x$ and $\omega_Y(y) = y(1 - 2(h_\lambda - 1)^{-1})$. In more detail,

$$\begin{aligned} \omega_Y(x) &= YxY^{-1} = YE'_\lambda Y^2Y^{-1} = E'_\lambda Y^2 = x, \\ \omega_Y(y) &= YyY^{-1} = YF'_\lambda \mathscr{X}_\lambda^2 Y^{-1} = F'_\lambda Y\mathscr{X}_\lambda^2 Y^{-1} = F'_\lambda(\mathscr{X}_\lambda^2 Y - 2\mathscr{X}_\lambda)Y^{-1} \\ &= y - 2F'_\lambda\mathscr{X}_\lambda\mathscr{X}_\lambda(Y\mathscr{X}_\lambda)^{-1} = y(1 - 2(h_\lambda - 1)^{-1}). \end{aligned}$$

The group $\mathbb{Z}$ acts in the obvious way on $\mathbb{K}$ (by addition). For each $\mu \in \mathbb{K}$, $\mathcal{O}(\mu) := \mu + \mathbb{Z}$ is the orbit of μ. Let $\mathbb{K}/\mathbb{Z}$ be the set of all $\mathbb{Z}$-orbits. For each orbit $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$, we fix a representative $\mu_\mathcal{O}$, i.e., $\mathcal{O} = \mu_\mathcal{O} + \mathbb{Z}$.

Let $[M] \in \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree})$. Then $\mathrm{Wt}\,(M) \subseteq \mathcal{O}$ for some orbit $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$. Since, in the algebra $\mathcal{S}(\lambda,\nu)$,

$$Y^iX^i = \lambda^i(h_\lambda - 1)(h_\lambda - 2)\cdots(h_\lambda - i) \text{ and } X^iY^i = \lambda^i h_\lambda(h_\lambda + 1)\cdots(h_\lambda + i - 1) \text{ for } i \geqslant 1, \tag{15}$$

the maps $X_M : M \to M$, $m \mapsto Xm$, and $Y_M : M \to M$, $m \mapsto Ym$, are injections. Therefore, $\mathrm{Wt}\,(M) = \mathcal{O}$. Hence,

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}) = \bigsqcup_{\mathcal{O}\in\mathbb{K}/\mathbb{Z}} \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}) \tag{16}$$

where the set $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O})$ contains all the isomorphism classes of simple, weight, T-torsionfree $\mathcal{S}(\lambda,\nu)$-modules M such that $\mathrm{Wt}\,(M) = \mathcal{O}$.

The next theorem (together with Theorems 3.5 and 3.6) classifies the elements of the set $\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree})$.

Theorem 3.4. *Let $\lambda \in \mathbb{K}^*$, $\nu \in \mathbb{K}$ and $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$. We fix an element $\mu_\mathcal{O} \in \mathcal{O}$, i.e., $\mathcal{O} = \mu_\mathcal{O} + \mathbb{Z}$. Then the map*

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}) \to \widehat{C_\lambda^{\mu_\mathcal{O},\nu}}\,(T\text{-torsionfree}), \quad [M] \mapsto [M_{\mu_\mathcal{O}}],$$

is a bijection with the inverse

$$[N] \mapsto \mathrm{soc}_{\mathcal{S}(\lambda,\nu)}(T^{-1}\mathcal{S}(\lambda,\nu) \otimes_{T^{-1}C_\lambda^\nu} T^{-1}N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_\lambda^\nu}(Y^iT^{-1}N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_\lambda^\nu}(X^iT^{-1}N).$$

Proof. (i) *The map* $[M] \mapsto [M_{\mu_\mathcal{O}}]$ *is well-defined*: It is obvious.

(ii) *The map* $[N] \mapsto [\mathrm{soc}_{\mathcal{S}(\lambda,\nu)}(T^{-1}\mathcal{S}(\lambda,\nu)\otimes_{T^{-1}C_\lambda^\nu} T^{-1}N)] = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_\lambda^\nu}(Y^iT^{-1}N)$ *is well defined*: By (14), the $T^{-1}\mathcal{S}(\lambda,\nu)$-module $\widetilde{N} := T^{-1}\mathcal{S}(\lambda,\nu)\otimes_{T^{-1}C_\lambda^\nu} T^{-1}N$ is a direct sum $\bigoplus_{i\in\mathbb{Z}} Y^iT^{-1}N$. Clearly, the $T^{-1}C_\lambda^{\mu_\mathcal{O},\nu}$-module $T^{-1}N$ is simple. Moreover, for each $i\in\mathbb{Z}$, the $T^{-1}C_\lambda^\nu$-module $Y^iT^{-1}N$ is isomorphic to the twisted $T^{-1}C_\lambda^\nu$-module ${}^{\omega_{Y^i}^{-1}}T^{-1}N$ and, hence, is simple. So, ${}^{\omega_{Y^i}^{-1}}T^{-1}N$ is a simple $T^{-1}C_\lambda^{\mu_\mathcal{O}+i,\nu}$-module. By [6], $\mathrm{soc}_{C_\lambda^{\mu_\mathcal{O}+i,\nu}}(Y^iT^{-1}N)$ is a simple, T-torsionfree $C_\lambda^{\mu_\mathcal{O}+i,\nu}$-module/$C_\lambda^\nu$-module, it is an essential C_λ^ν-submodule of $Y^iT^{-1}N$, and so it is contained in every nonzero C_λ^ν-submodule of $Y^iT^{-1}N$.

Claim. $\mathrm{soc}_{\mathcal{S}(\lambda,\nu)}(\widetilde{N}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_\lambda^\nu}(Y^iT^{-1}N)$: Let S be the direct sum of socles. Then it is contained in each $\mathcal{S}(\lambda,\nu)$-submodule of $\widetilde{N}$. The equality follows from the fact that S is *an* $\mathcal{S}(\lambda,\nu)$*-module*. Since S is the intersection of all nonzero $\mathcal{S}(\lambda,\nu)$-submodules of $\widetilde{N}$. Let us give more details. Clearly, S is contained in the intersection, say S'. In fact, $S=S'$ since for each $i\in\mathbb{Z}$, the $(\mu_\mathcal{O}+i)$'th weight component of the $S(\lambda,\nu)$-submodule $\mathcal{S}(\lambda,\nu)\mathrm{soc}_{C_\lambda^\nu}(Y^iT^{-1}N)$ is precisely $\mathrm{soc}_{C_\lambda^\nu}(Y^iT^{-1}N)$. The proof of the statement (ii) is complete. Clearly, the maps in the statements (i) and (ii) are mutually inverse. Notice that, for all $i\in\mathbb{Z}$, $Y^iT^{-1}N = X^{-i}T^{-1}N$ since for all $j\geqslant 1$, X^jY^j, $Y^jX^j\in T$. So, the last equality of the theorem is obvious. $\square$

Below, we give a classification of simple $C_\lambda^{\mu_\mathcal{O},\nu}$-modules (Theorems 3.5 and 3.6) and also give an explicit construction of the direct sum of socles in Theorem 3.4 (Theorem 3.8 and 3.9).

Classification of simple A-modules where $A = D(\sigma,a)$ and D is a Dedekind ring. Let $A = D(\sigma,a) = D[x,y;\sigma,a]$ be a GWA such that D is a Dedekind ring, $a\neq 0$, and the automorphism σ of D satisfies the condition that $\sigma^i(\mathfrak{p})\neq\mathfrak{p}$ for all $i\in\mathbb{Z}\setminus\{0\}$ and all maximal ideals $\mathfrak{p}$ of D.

Example. $A = \mathbb{K}[H](\sigma,a)$ where $\sigma(H) = H-\gamma$, $\gamma\in\mathbb{K}^*$ and $a\neq 0$. In particular, the algebras $C_\lambda^{\mu,\nu}$ are of this type. A classification of simple $\mathbb{K}[H](\sigma,a)$-modules is given in [3, 6].

Let us recall a classification of simple A-modules for the algebra $A = D(\sigma,a)$, see [6, 3, 4] for details. Clearly

$$\widehat{A} = \widehat{A}\,(D\text{-torsion}) \ \sqcup\ \widehat{A}\,(D\text{-torsionfree}). \tag{17}$$

The set $\widehat{A}\,(D$**-**torsion$) = \widehat{A}\,(\text{weight})$. The group $\langle\sigma\rangle\simeq\mathbb{Z}$ acts freely on the set $\mathrm{Max}\,(D)$ of maximal ideals of the Dedekind ring D. For each maximal ideal $\mathfrak{p}$ of D, $\mathcal{O}(\mathfrak{p}) = \{\sigma^i(\mathfrak{p})\,|\, i\in\mathbb{Z}\}$ is its orbit. We use the bijection $\mathbb{Z}\to\mathcal{O}(\mathfrak{p})$, $i\mapsto\sigma^i(\mathfrak{p})$, to define the order $\leqslant$ on each orbit $\mathcal{O}(\mathfrak{p})$: $\sigma^i(\mathfrak{p})\leqslant\sigma^j(\mathfrak{p})$ iff $i\leqslant j$. A maximal ideal of D is called *marked* if it contains the element a. There are only finitely many marked ideals. An orbit $\mathcal{O}$ is called *degenerated* if it contains a marked ideal. Marked ideals, say $\mathfrak{p}_1<\cdots<\mathfrak{p}_s$, of a degenerated orbit $\mathcal{O}$ partition it into $s+1$ parts,

$$\Gamma_1 = (-\infty,\mathfrak{p}_1],\quad \Gamma_2 = (\mathfrak{p}_1,\mathfrak{p}_2],\ldots,\Gamma_s = (\mathfrak{p}_{s-1},\mathfrak{p}_s],\quad \Gamma_{s+1} = (\mathfrak{p}_s,\infty).$$

Two ideals $\mathfrak{p},\mathfrak{q}\in\mathrm{Max}\,(D)$ are called *equivalent* $\mathfrak{p}\sim\mathfrak{q}$ if they belong either to a non-degenerated orbit or to some Γ_i. We denote by $\mathrm{Max}\,(D)/\!\sim$ the set of equivalence classes in $\mathrm{Max}\,(D)$.

An A-module V is called *weight* if $V = \bigoplus_{\mathfrak{p}\in\mathrm{Max}\,(D)} V_\mathfrak{p}$ where $V_\mathfrak{p} = \{v\in V\,|\,\mathfrak{p}v = 0\}$ = the sum of all simple D-submodules of V which are isomorphic to $D/\mathfrak{p}$. The set $\mathrm{Supp}\,(V) = \{\mathfrak{p}\in\mathrm{Max}\,(D)\,|\,V_\mathfrak{p}\neq 0\}$ is called the *support* of V, elements of $\mathrm{Supp}\,(V)$ are called *weights* and $V_\mathfrak{p}$ is called the *component* of V of weight $\mathfrak{p}$. Clearly, an A-module is weight iff it is a semisimple D-module. Clearly,

$$\widehat{A}\,(D\text{-torsion}) = \widehat{A}\,(\text{weight}), \tag{18}$$

i.e., a simple A-module is D-torsion iff it is weight.

Theorem 3.5. *[6, 3, 4] (Classification of simple D-torsion/weight A-modules). The map* $\mathrm{Max}\,(D)/\!\sim\ \to\widehat{A}\,(D\text{-torsion})$, $\Gamma\mapsto[L(\Gamma)]$, *is a bijection with the inverse* $[M]\mapsto\mathrm{Supp}\,(M)$ *where*

1. *If Γ is a non-degenerated orbit then $L(\Gamma) = A/A\mathfrak{p}$ where $\mathfrak{p} \in \Gamma$.*
2. *If $\Gamma = (-\infty, \mathfrak{p}]$ then $L(\Gamma) = A/A(\mathfrak{p}, x)$.*
3. *If $\Gamma = (\sigma^{-n}(\mathfrak{p}), \mathfrak{p}]$ for some $n \geqslant 1$ then $L(\Gamma) = A/A(y^n, \mathfrak{p}, x)$. The D-length of $L(\Gamma)$ is n.*
4. *If $\Gamma = (\mathfrak{p}, \infty)$ then $L(\Gamma) = A/A(\sigma(\mathfrak{p}), y)$.*

The set $\widehat{A}\,(D$-torsionfree). For elements $\alpha, \beta \in D$, we write $\alpha < \beta$ if $\mathfrak{p} < \mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathrm{Max}\,(D)$ such that $\mathcal{O}(\mathfrak{p}) = \mathcal{O}(\mathfrak{q})$, $\alpha \in \mathfrak{p}$ and $\beta \in \mathfrak{q}$. (We write $\alpha < \beta$ if there are no such ideals $\mathfrak{p}$ and $\mathfrak{q}$). Recall that the GWA $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded algebra where $A_i = Dv_i = v_iD$.

Definition, [6, 3, 4]. An element $b = v_{-m}\beta_{-m} + v_{-m+1}\beta_{-m+1} + \cdots + \beta_0 \in A$ (where $m \geqslant 1$, all $\beta_i \in D$ and $\beta_{-m}, \beta_0 \neq 0$) is called a *normal* element if $\beta_0 < \beta_{-m}$ and $\beta_0 < a$.

The set $S := D \setminus \{0\}$ is an Ore set of the domain A. Let $k := S^{-1}D$ be the field of fractions of D. The algebra $B := S^{-1}A = k[x, x^{-1}; \sigma]$ is a skew Laurent polynomial ring which is a (left and right) principle ideal domain. So, any simple B-module is of type B/Bb for some irreducible element b of B. Two simple B-modules are isomorphic, $B/Bb \simeq B/Bc$, iff the elements b and c are *similar* (i.e., there exists an element $d \in B$ such that 1 is the greatest common right divisor of c and d, and bd is a least common left multiple of c and d).

Theorem 3.6. *[6, 3, 4] (Classification of simple D-torsionfree A-modules).*
$\widehat{A}\,(D\text{-torsionfree}) = \{[M_b := A/A \cap Bb] \,|\, b$ *is a normal irreducible element of B}. The A-modules M_b and $M_{b'}$ are isomorphic iff the elements b and b' are similar.*

For all nonzero elements $\alpha, \beta \in D$, the B-modules $S^{-1}M_b$ and $S^{-1}M_{\beta b\alpha^{-1}}$ are isomorphic. If an element $b = v_{-m}\beta_{-m} + \cdots + \beta_0$ is irreducible in B but not necessarily normal the next lemma shows that there are explicit elements α and β such that the element $\beta b\alpha^{-1}$ is normal and irreducible in B.

Lemma 3.7. *[3, Lemma 13] (Normalization procedure).*
Given an element $b = v_{-m}\beta_{-m} + \cdots + \beta_0 \in A$ where $m \geqslant 1$, all $\beta_i \in D$ and $\beta_{-m}, \beta_0 \neq 0$. Fix a natural number $n \in \mathbb{N}$ such that $\sigma^{-s}(\beta_0) < \beta_{-m}$, $\sigma^{-s}(\beta_0) < \beta_0$ and $\sigma^{-s}(\beta_0) < a$. Let $\alpha = \prod_{i=0}^{s} \sigma^{-i}(\beta_0)$ and $\beta = \prod_{i=1}^{s+m} \sigma^{-i}(\beta_0)$. Then the element $\beta b\alpha^{-1}$ is a normal element which is called a normalization of b and denoted b^{norm} (we can always assume that s is the least possible).

Explicit construction of the socle in Theorem 3.4. Every $\mathcal{S}(\lambda, \nu)$-module is also a $\mathbb{K}[h_\lambda]$-module (since $\mathbb{K}[h_\lambda] \subseteq \mathcal{S}(\lambda, \nu)$). Therefore, for each orbit $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$,

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}) = \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}, \mathbb{K}[h_\lambda]\text{-torsion}) \sqcup \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, \mathcal{O}, \mathbb{K}[h_\lambda]\text{-torsionfree}), \quad (19)$$

$$\widehat{C_\lambda^{\mu_\mathcal{O},\nu}}\,(T\text{-torsionfree}) = \widehat{C_\lambda^{\mu_\mathcal{O},\nu}}\,(T\text{-torsionfree}, \mathbb{K}[h_\lambda]\text{-torsion}) \,\sqcup\, \widehat{C_\lambda^{\mu_\mathcal{O},\nu}}\,(\mathbb{K}[h_\lambda]\text{-torsionfree}). \quad (20)$$

The map $[M] \mapsto [M_{\mu_\mathcal{O}}]$ in Theorem 3.4 respects the disjoint unions (19) and (20). Recall that for each orbit $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$ we fixed its representative $\mu_\mathcal{O}$.

Theorem 3.8. *Let $\lambda \in \mathbb{K}^*$, $\nu \in \mathbb{K}$ and $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$. Then*

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}, \mathbb{K}[h_\lambda]\text{-torsion}) = \bigsqcup_{\mathcal{O}'\in\mathbb{K}/\mathbb{Z},\ \mathcal{O}'\neq\mathbb{Z}} \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight}) \otimes [W(\lambda, \mathcal{O}')],$$

where $W(\lambda, \mathcal{O}') := A_1(\lambda)/A_1(\lambda)(h_\lambda - \mu_{\mathcal{O}'})$ and $\widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight}) \otimes [W(\lambda, \mathcal{O}')] := \{[M \otimes W(\lambda, \mathcal{O}')] \,|\, M \in \widehat{U'_\lambda(\nu)}\,(H'_\lambda\text{-weight})\}$ and $S(\lambda, \nu)$-modules $M \otimes W(\lambda, \mathcal{O}')$ and $M' \otimes W(\lambda, \mathcal{O}')$ are isomorphic iff the $U'_\lambda(\nu)$-modules M and M' are isomorphic.

Proof. Recall that $H'_\lambda = H + \lambda^{-1}XY - \frac{1}{2} = H + h_\lambda - \frac{1}{2}$. Let $[\mathcal{M}] \in \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{weight}, T\text{-torsionfree}, \mathcal{O}, \mathbb{K}[h_\lambda]\text{-torsion})$. Then $\mathcal{M}$ is an epimorphic image of the $\mathcal{S}(\lambda, \nu)$-module

$$\mathcal{S}(\lambda,\nu)/\mathcal{S}(\lambda,\nu)(H - \mu, h_\lambda - \mu') \simeq \mathcal{S}(\lambda,\nu)/\mathcal{S}(\lambda,\nu)(H'_\lambda - \mu - \mu' + \frac{1}{2}, h_\lambda - \mu')$$
$$\simeq \frac{U'_\lambda(\nu)}{U'_\lambda(\nu)(H'_\lambda - \mu - \mu' + \frac{1}{2})} \otimes \frac{A_1(\lambda)}{A_1(\lambda)(h_\lambda - \mu')} \simeq \frac{U'_\lambda(\nu)}{U'_\lambda(\nu)(H'_\lambda - \mu - \mu' + \frac{1}{2})} \otimes W(\lambda, \mu_{\mathcal{O}'})$$

where $\mathcal{O}' := \mathcal{O}(\mu') \neq \mathbb{Z}$. Since $\mathrm{End}_{A_1(\lambda)} W(\lambda, \mu_{\mathcal{O}'}) = \mathbb{K}$, the result follows. $\square$

The elements X and Y are units in the algebra $T^{-1}C_\lambda^\nu$. For each $i \in \mathbb{Z}$, the inner automorphism $\omega_{X^{-i}} = \omega_{X^i}^{-1} : T^{-1}C_\lambda^\nu \to T^{-1}C_\lambda^\nu$ induces the algebra isomorphism

$$\omega_{X^i}^{-1} : C_\lambda^{\mu_\mathcal{O},\nu} \to C_\lambda^{\mu_\mathcal{O}+i,\nu}, \ \ u \mapsto X^{-i}uX^i,$$

(since $X^{-i}(h_\lambda - \mu_\mathcal{O})X^i = h_\lambda - i - \mu_\mathcal{O}$). The localization of the GWA $C_\lambda^{\mu_\mathcal{O},\nu}$ at the Ore set $S := \mathbb{K}[h_\lambda] \setminus \{0\}$ is a skew polynomial algebra $B = \mathbb{K}(h_\lambda)[Y, Y^{-1}, \sigma]$ where $\sigma(h_\lambda) = h_\lambda - 1$. Notice that $B = \mathbb{K}(h_\lambda)[X, X^{-1}; \sigma^{-1} = \omega_X]$.

Let $M_b := C_\lambda^{\mu_\mathcal{O},\nu}/C_\lambda^{\mu_\mathcal{O},\nu} \cap Bb$ be a $\mathbb{K}[h_\lambda]$-torsionfree simple $C_\lambda^{\mu_\mathcal{O},\nu}$-module where $b = X^m\beta_{-m} + X^{m-1}\beta_{-m+1} + \cdots + \beta_0 \in C_\lambda^{\mu_\mathcal{O},\nu}$ is a normal and irreducible element in B ($m \geqslant 1$, all $\beta_i \in \mathbb{K}[h_\lambda]$, and $\beta_{-m}, \beta_0 \neq 0$). For each $i \in \mathbb{Z}$, the C_λ^ν-socle of the $T^{-1}C_\lambda^\nu$-module/C_λ^ν-module/$C_\lambda^{\mu_\mathcal{O}-i,\nu}$-module

$$X^iT^{-1}M_b = X^i\frac{T^{-1}C_\lambda^{\mu_\mathcal{O},\nu}}{T^{-1}C_\lambda^{\mu_\mathcal{O},\nu} \cap Bb} = {}^{\omega_{X^i}^{-1}}\left(\frac{T^{-1}C_\lambda^{\mu_\mathcal{O},\nu}}{T^{-1}C_\lambda^{\mu_\mathcal{O},\nu} \cap Bb}\right) \simeq \frac{T^{-1}C_\lambda^{\mu_\mathcal{O}-i,\nu}}{T^{-1}C_\lambda^{\mu_\mathcal{O}-i,\nu} \cap B(X^ibX^{-i})}$$

is equal to $C_\lambda^{\mu_\mathcal{O}-i,\nu}/C_\lambda^{\mu_\mathcal{O}-i,\nu} \cap B(X^ibX^{-i})^{\mathrm{norm}}$. Now, the next theorem follows from Theorem 3.4.

Theorem 3.9. *Let $\lambda \in \mathbb{K}^*, \nu \in \mathbb{K}$ and $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$. The map*

$$\widehat{\mathcal{S}(\lambda,\nu)}\,(\mathrm{weight}, \mathcal{O}, \mathbb{K}[h_\lambda]\text{-torsionfree}) \to \widehat{C_\lambda^{\mu_\mathcal{O},\nu}}\,(\mathbb{K}[h_\lambda]\text{-torsionfree}), \ \ [M] \mapsto [M_{\mu_\mathcal{O}}],$$

is a bijection with the inverse

$$\Big[M_b := \frac{C_\lambda^{\mu_\mathcal{O},\nu}}{C_\lambda^{\mu_\mathcal{O},\nu} \cap Bb}\Big] \to \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_\lambda^\nu}(X^iT^{-1}M_b) = \bigoplus_{i\in\mathbb{Z}} \frac{C_\lambda^{\mu_\mathcal{O}-i,\nu}}{C_\lambda^{\mu_\mathcal{O}-i,\nu} \cap B(X^ibX^{-i})^{\mathrm{norm}}}$$

where $b \in C_\lambda^{\mu_\mathcal{O},\nu}$ is a normal and irreducible element in B and $(X^ibX^{-i})^{\mathrm{norm}} \in C_\lambda^{\mu_\mathcal{O}-i,\nu}$ is the normalization in $C_\lambda^{\mu_\mathcal{O}-i,\nu}$ of the irreducible element X^ibX^{-i} in B.

The following result was proved in [29] for $\mathbb{K} = \mathbb{C}$.

Theorem 3.10. *For a simple weight $\mathcal{S}$-module all weight spaces are either finite or infinite dimensional.*

Proof. This follows directly from a classification of all simple weight $\mathcal{S}$-modules where $\lambda \neq 0$. For $\lambda = 0$, this follows from the classification of simple modules obtained in [15]. $\square$

The set of simple $\mathcal{S}(\lambda)$-modules $\widehat{U_\lambda'} \otimes \widehat{A_1(\lambda)}$. By (5), $\mathcal{S}(\lambda) = U_\lambda' \otimes A_1(\lambda)$. Given a U_λ'-module $[M]$ and an $A_1(\lambda)$-module N. Their tensor product over $\mathbb{K}$, $M \otimes N$, is an $\mathcal{S}(\lambda)$-module. If, in addition, the modules M and N are simple then $\mathrm{End}_{U_\lambda'}(M) = \mathbb{K}$ and $\mathrm{End}_{A_1(\lambda)}(N) = \mathbb{K}$, and so $M \otimes N$ is a simple $\mathcal{S}(\lambda)$-module.

Proposition 3.11. *$\widehat{U_\lambda'} \otimes \widehat{A_1(\lambda)} := \{[M \otimes N] \mid M \in \widehat{U_\lambda'}, \ N \in \widehat{A_1(\lambda)}\} \subseteq \widehat{\mathcal{S}(\lambda)}$, and $[M \otimes N] = [M' \otimes N']$ iff $[M] = [M']$ and $[N] = [N']$.*

Proposition 3.11 gives plenty of simple $\mathcal{S}(\lambda)$-modules as the simple modules over the algebras U_λ' and $A_1(\lambda)$ are classified, see [17] or Theorems 3.5 and 3.6.

Classification of simple weight $\mathcal{S}$-modules with finite dimensional weight spaces. Using the classification of simple $\mathcal{S}$-modules we can easily describe the set of isomorphism classes of simple weight $\mathcal{S}$-modules with finite dimensional weight spaces $\widehat{\mathcal{S}}\,(\mathrm{f.\,d.\,weight\,spaces})$. This was done by Dubsky for $\mathbb{K} = \mathbb{C}$ using a different approach [18]. The simple lowest weight $\mathcal{S}$-modules were classified earlier in [20]. Clearly,

$$\widehat{\mathcal{S}}\,(\mathrm{f.\,d.\,weight\,spaces}) = \widehat{\mathcal{S}(0)}\,(\mathrm{f.\,d.\,weight\,spaces}) \ \sqcup \ \bigsqcup_{\lambda\in\mathbb{K}^*} \widehat{\mathcal{S}(\lambda)}\,(\mathrm{f.\,d.\,weight\,spaces}).$$

It was shown that $\widehat{\mathcal{S}(0)}$ (f. d. weight spaces) $= \widehat{U(\mathfrak{sl}_2)}$ (weight), [18] and [15]. Clearly, for $\lambda \neq 0$,

$$\widehat{\mathcal{S}(\lambda)}\,(\text{f. d. weight spaces}) = \bigsqcup_{\nu\in\mathbb{K}} \widehat{\mathcal{S}(\lambda,\nu)}\,(\text{f. d. weight spaces}).$$

For $\lambda \in \mathbb{K}^*$, we denote by $\widehat{U'_\lambda}$ (fin. dim.) (resp., $\widehat{U'_\lambda}$ (h. w., dim $=\infty$); $\widehat{U'_\lambda}$ (l. w., dim $=\infty$)) the set of isomorphism classes of simple finite dimensional (resp., highest weight infinite dimensional; lowest weight infinite dimensional) U'_λ-modules. Let $V^+(\lambda) := A_1(\lambda)/A_1(\lambda)X$ and $V^-(\lambda) := A_1(\lambda)/A_1(\lambda)Y$. The next theorem classifies all the simple weight $\mathcal{S}$-modules with finite dimensional weight spaces.

Theorem 3.12.

$$\begin{aligned}\widehat{\mathcal{S}}\,(\text{f. d. weight spaces}) = \widehat{U(\mathfrak{sl}_2)}\,(\text{weight}) \ \sqcup \bigsqcup_{\lambda\in\mathbb{K}^*}\Big\{&\widehat{U'_\lambda}\,(\text{fin. dim.})\otimes\widehat{A_1(\lambda)}\,(\mathbb{K}[h_\lambda]\text{-torsion})\\ &\sqcup\ \widehat{U'_\lambda}\,(\text{h. w., dim}=\infty)\otimes V^+(\lambda)\\ &\sqcup\ \widehat{U'_\lambda}\,(\text{l. w., dim}=\infty)\otimes V^-(\lambda)\Big\}.\end{aligned}$$

Proof. Let $[\mathcal{M}] \in \widehat{\mathcal{S}}$ (f. d. weight spaces). If $Z\mathcal{M} = 0$ then $[\mathcal{M}] \in \widehat{U(\mathfrak{sl}_2)}$ (weight). Without loss of generality we may assume that $[\mathcal{M}] \in \widehat{\mathcal{S}(\lambda,\nu)}$ (f. d. weight spaces) for some $\lambda \in \mathbb{K}^*$ and $\nu \in \mathbb{K}$. Every simple weight $\mathcal{S}(\lambda,\nu)$-module in Theorem 3.9 has infinite dimensional weight spaces. Therefore, the module $\mathcal{M}$ must be $\mathbb{K}[h_\lambda]$-torsion, i.e., h_λ-weight since the filed $\mathbb{K}$ is algebraically closed. Then $\mathcal{M} = M \otimes N$ for some $[M] \in \widehat{U'_\lambda(\nu)}$ (H'_λ-weight) and $[N] \in \widehat{A_1(\lambda)}$ ($\mathbb{K}[h_\lambda]$-torsion) by Proposition 3.3.(2), Corollary 3.2.(2) and Theorem 3.8. By Theorem 3.5,

$$\widehat{A_1(\lambda)}\,(\mathbb{K}[h_\lambda]\text{-torsion}) = \Big\{[V^\pm(\lambda)],\ [W(\lambda,\mathcal{O}')] \mid \mathcal{O}' \in \mathbb{K}/\mathbb{Z},\ \mathcal{O}' \neq \mathbb{Z}\Big\}.$$

Notice that $V^+(\lambda) = \bigoplus_{i\geqslant 0}\mathbb{K}Y^i\bar{1}$ where $\bar{1} = 1 + A_1(\lambda)X$ and $h_\lambda Y^i\bar{1} = Y^i(h_\lambda + i)\bar{1} = (i+1)Y^i\bar{1}$. Similarly, $V^-(\lambda) = \bigoplus_{i\geqslant 0}\mathbb{K}X^i\bar{1}$ where $\bar{1} = 1 + A_1(\lambda)Y$ and $h_\lambda X^i\bar{1} = X^i(h_\lambda - i)\bar{1} = -iX^i\bar{1}$. For $\mathcal{O}' \in \mathbb{K}/\mathbb{Z}$ such that $\mathcal{O}' \neq \mathbb{Z}$, $W(\lambda,\mathcal{O}') = \bigoplus_{i\in\mathbb{Z}}\mathbb{K}v(\lambda)_i\bar{1}$ where $\bar{1} = 1 + A_1(\lambda)(h_\lambda - \mu_{\mathcal{O}'})$ and $h_\lambda v(\lambda)_i\bar{1} = (i+\mu_{\mathcal{O}'})v(\lambda)_i\bar{1}$. (The elements $v(\lambda)_i$ are defined in (8)). Recall that $H = H'_\lambda - h_\lambda + \frac{1}{2}$. Given elements $m_i \in M$ and $n_j \in N$ such that $H'_\lambda m_i = im_i$ and $h_\lambda n_j = jn_j$. Then $Hm_i \otimes n_j = (i - j + \frac{1}{2})m_i \otimes n_j$. Now the result follows easily from Theorem 3.5. $\square$

We say that a weight module has *uniformly bounded weight spaces* if their dimensions do not exceed a fixed natural number.

Corollary 3.13.
1. *The set* $\widehat{U(\mathfrak{sl}_2)}$ (weight) $\sqcup \bigsqcup_{\lambda\in\mathbb{K}^*}\widehat{U'_\lambda}$ (fin. dim.) $\otimes \big\{\widehat{A_1(\lambda)}\,(\mathbb{K}[h_\lambda]\text{-torsion}) \setminus \{V^\pm(\lambda)\}\big\}$ *contains precisely the isomorphism classes of simple weight* $\mathcal{S}$*-modules where all the weight components are finite dimensional vector spaces of the same dimension.*
2. *The set* $\widehat{U(\mathfrak{sl}_2)}$ (weight) $\sqcup \bigsqcup_{\lambda\in\mathbb{K}^*}\widehat{U'_\lambda}$ (fin. dim.) $\otimes \big\{V^+(\lambda), V^-(\lambda)\big\}$ *contains precisely the isomorphism classes of simple weight* $\mathcal{S}$*-modules with uniformly bounded finite dimensional weight spaces.*

The first statement of the corollary above is strengthening of the result due to Wu and Zhu [29] that states that if V is a simple weight $\mathcal{S}$-module which is neither a highest weight nor a lowest weight module then all its weight spaces have the same dimension.

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