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Theory of channel simulation and bounds for private communication

Stefano Pirandola,[1] Samuel L. Braunstein,[1] Riccardo Laurenza,[1] Carlo Ottaviani,[1]
Thomas P. W. Cope,[1] Gaetana Spedalieri,[1,2] and Leonardo Banchi[3,4]

[1] *Computer Science and York Centre for Quantum Technologies, University of York, York YO10 5GH, UK*
[2] *Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3] *Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, UK*
[4] *QOLS, Blackett Laboratory, Imperial College London, London SW7 2AZ, UK*

We review recent results on the simulation of quantum channels, the reduction of adaptive protocols (teleportation stretching), and the derivation of converse bounds for quantum and private communication, as established in PLOB [Pirandola, Laurenza, Ottaviani, Banchi, arXiv:1510.08863]. We start by introducing a general weak converse bound for private communication based on the relative entropy of entanglement. We discuss how combining this bound with channel simulation and teleportation stretching, PLOB established the two-way quantum and private capacities of several fundamental channels, including the bosonic lossy channel. We then provide a rigorous proof of the strong converse property of these bounds by adopting a correct use of the Braunstein-Kimble teleportation protocol for the simulation of bosonic Gaussian channels. This analysis provides a full justification of claims presented in the follow-up paper WTB [Wilde, Tomamichel, Berta, arXiv:1602.08898] whose upper bounds for Gaussian channels would be otherwise infinitely large. Besides clarifying contributions in the area of channel simulation and protocol reduction, we also present some generalizations of the tools to other entanglement measures and novel results on the maximum excess noise which is tolerable in quantum key distribution.

I. INTRODUCTION

In quantum information [1–9], the area of quantum and private communications is the subject of an intense theoretical study, also driven by an increasing number of experimental implementations. This hectic field ranges from point-to-point protocols [10–41] to the development of quantum repeaters [42–59], untrusted relays [60–63] and, more generally, a quantum network or quantum Internet [64–68]. In this wide scenario, it is important to know the fundamental limits imposed by quantum mechanics, which also serve as benchmarks to test the performance of practical proposals and new technologies. However, the exploration of the ultimate limits is not easy, especially when it comes to considering quantum and private communication protocols which involve feedback strategies, where the parties may interactively update their quantum systems in a real time fashion.

The most important point-to-point quantum communication scenario involves two remote parties, Alice and Bob, which are connected by a (memoryless) quantum channel without pre-sharing any entanglement. By using this channel, the two parties may achieve various quantum tasks, including the reliable transmission of qubits, the distillation of entanglement bits (ebits) and, finally, the communication or generation of secret bits. The most general protocols are based on transmissions through the quantum channel which are interleaved by local operations (LO) assisted by unlimited and two-way classical communication (CC), briefly called adaptive LOCCs.

The first approach to simply quantum communication protocols dates back to Bennett, DiVincenzo, Smolin and Wootters (BDSW) [69]. These authors introduced the simulation of discrete-variable (DV) Pauli channels via quantum teleportation, and exploited this tool to reduce a quantum communication protocol through a Pauli channel into an entanglement distillation protocol over mixed isotropic states. This transformation was explicitly discussed for non-adaptive protocols based on one-way CCs but the extension to two-way CCs is easy. Since then we have witnessed a number of advances [70–76].

Most recently, Pirandola, Laurenza, Ottaviani and Banchi (PLOB) [77] generalized these precursory ideas in several ways. First of all, PLOB introduced the most general form of channel simulation in a communication scenario, where the quantum channel is replaced by an LOCC (not necessarily teleportation [78–80]) applied to the input and some resource state. Furthermore these elements (LOCC and resource state) may be asymptotic, i.e., defined as limit of suitable sequences. In this way, any quantum channel can be simulated at any dimension, i.e., both DV channels and continuous-variable (CV) channels. For instance, this approach allows one to deterministically simulate the amplitude damping channel, which was impossible with any of the previous approaches, including the one formulated in Ref. [70], whose limitation was due to the use of finite-dimensional and non-asymptotic LOCCs (see Eq. (11) in Ref. [70]).

The second advance brought by PLOB was teleportation stretching. This technique is based on the channel simulation and allows one to re-order an arbitrary adaptive protocol into a much simpler block version, where the output state is simply expressed in terms of tensor-product states up to a single LOCC. This technique works for any channel, at any dimension and for any type of adaptive protocol, i.e., for any task. In contrast with the BDSW reduction into entanglement distillation, teleportation stretching maintains the original task of the protocol, so that adaptive key generation is reduced into block key generation. This feature is crucial in order to

apply the technique to many different contexts, including the reduction of adaptive quantum metrology and quantum channel discrimination [81, 82], and the extension to multi-party protocols [83] and quantum networks [66].

By using teleportation stretching and extending the notion of relative entropy of entanglement (REE) [84–86] from states to channels, PLOB derived simple single-letter upper bounds for the two-way quantum and private capacities of an arbitrary quantum channel. In particular, these capacities were established for dephasing channels, erasure channels (see also Refs. [87, 88]), quantum-limited amplifiers, and bosonic lossy channels. The two-way capacity of the lossy channel, also known as PLOB bound, closes a long-standing investigation started in 2009 [89–91], and finally sets the ultimate limit for optical quantum communications in the absence of repeaters.

In this manuscript, not only we review these techniques and results, but we also show some generalizations. We study the general conditions that an entanglement measure needs to satisfy in order to be used for the derivation of single-letter upper bounds for two-way assisted capacities. We then consider a problem which is complementary to that presented in PLOB. Instead of analyzing the optimal achievable rates in quantum key distribution (QKD), we investigate the maximum excess noise which is tolerable by QKD protocols. As we will see, this characterization is and remains an open problem.

Finally, we also investigate strong converse properties. In fact, directly building on the methods described above (channel's REE and teleportation stretching), a follow-up work by Wilde, Tomamichel and Berta (WTB) [92] studied the strong converse property of the upper bounds established in PLOB. Here we re-consider this later refinement while fixing its technical mathematical issues. In fact, we show that the strong converse bounds for bosonic Gaussian channels presented in WTB technically explode to infinity, due to an imprecise use of the Braunstein-Kimble (BK) protocol for CV teleportation [79, 93].

The optimal teleportation simulation of bosonic channels is asymptotic and, for this reason, must be handled with a careful control on the simulation error (between the actual and the simulated channel). Such error needs to be rigorously propagated through the protocol and then bounded via a correct use of the BK teleportation protocol. While this technique is fully taken into account in the weak converse bounds of PLOB, it is absent in the derivations of WTB for bosonic Gaussian channels, whose strong converse bounds become therefore "unbounded".

Structure of the paper

The paper is organized as follows. In Sec. II, we define adaptive protocols and two-way capacities. In Sec. III, we provide a general weak converse upper bound for these capacities. To simplify this bound, we describe channel simulation in Sec. IV and teleportation stretching in Sec. V. We combine all these elements in Sec. VI to derive single-letter upper bounds and the formulas for the two-way capacities. In Sec. VII we specify some of the results to establish the maximum excess noise which is tolerable in QKD. In Sec. VIII, we discuss how the recipe introduced by PLOB is general and can be formulated for other entanglement measures. Sec. IX contains a detailed discussion on the main advances in the field of channel simulation and protocol reduction before the full generalization in PLOB. Then, in Sec. X, we review aspects of WTB and we provide a complete proof of its strong converse claims for bosonic Gaussian channels. Finally, Sec. XI is for conclusions.

II. ADAPTIVE QUANTUM PROTOCOLS AND TWO-WAY CAPACITIES

A. General formulation and definitions

Let us start with the description of the most general adaptive protocol for quantum or private communication over an arbitrary quantum channel $\mathcal{E}$. We adopt the notation introduced in PLOB, where Alice and Bob have local registers, $\mathbf{a}$ and $\mathbf{b}$, each composed of a (countable) number of systems. The adaptive protocol goes as follows [77] (see also Fig. 1 for a schematic).

- Alice and Bob prepare an initial state $\rho^0_{\mathbf{ab}}$ applying an adaptive LOCC Λ_0 to their registers $\mathbf{a}$ and $\mathbf{b}$.
- Alice picks a system $a_1 \in \mathbf{a}$ and sends it through the channel $\mathcal{E}$; Bob receives the output system b_1 which becomes part of his register, $b_1\mathbf{b} \to \mathbf{b}$. Another adaptive LOCC Λ_1 is then applied to the local registers, with output $\rho^1_{\mathbf{ab}}$.
- Then, there is the second transmission $\mathbf{a} \ni a_2 \to b_2$ through $\mathcal{E}$, which is followed by another adaptive LOCC Λ_2. This procedure goes so on for n uses of the channel. The sequence of adaptive LOCCs $\mathcal{P} = \{\Lambda_0, \ldots, \Lambda_n\}$ characterizes the protocol and provides the output state $\rho^n_{\mathbf{ab}}$.

The output state $\rho^n_{\mathbf{ab}}$ is epsilon-close to some target state ϕ^n with nR_n bits, where R_n is the rate. In other words, we have $||\rho^n_{\mathbf{ab}} - \phi^n|| \leq \varepsilon$ in trace norm [94]. Depending on the task of the protocol, the target bits are of different types, e.g., qubits, ebits or private bits. The *generic* two-way capacity is taking the limit of large n and optimizing over the adaptive protocols

$$\mathcal{C}(\mathcal{E}) := \sup_{\mathcal{P}} \lim_{n} R_n. \tag{1}$$

If the target ϕ^n is a maximally-entangled state, then $\mathcal{C}$ is the two-way entanglement-distribution capacity D_2. Under two-way CCs, D_2 is equal to the quantum capacity Q_2 (transmission of qubits). If the target ϕ^n is a private

state [95], then $\mathcal{C}$ is the secret key capacity K (generation of secret bits), which is equal to the two-way private capacity P_2 (private transmission of classical bits). Since a maximally-entangled state is a specific type of private state, entanglement distillation is a particular protocol of key distillation and we may write the hierarchy

$$Q_2 = D_2 \leq K = P_2. \tag{2}$$

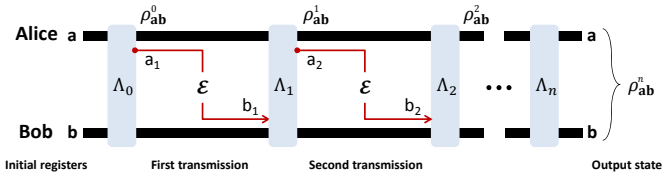

FIG. 1: Adaptive protocol through a quantum channel $\mathcal{E}$. Each transmission $a_i \to b_i$ is interleaved by adaptive LOCCs, Λ_{i-1} and Λ_i, applied to the local registers $\mathbf{a}$ and $\mathbf{b}$. After n transmissions, we have a sequence of adaptive LOCCs $\mathcal{P} = \{\Lambda_0, \ldots, \Lambda_n\}$ characterizing the protocol and a corresponding output state $\rho^n_{\mathbf{ab}}$ for Alice and Bob.

B. Private states

Let us explain in detail the structure of a private state [95]. Decompose the local registers as $\mathbf{a} = AA'$ and $\mathbf{b} = BB'$, where A and B are the local "key systems" with individual dimension $d_{\rm K}$, while A' and B' are used to protect the key and form the so-called "shield system", whose dimension $d_{\rm S}$ is in principle arbitrary (even infinite for CV systems). The total dimension of the registers is therefore $d = d_{\rm K}^2 d_{\rm S}$. A generic private state has the form

$$\phi_{AA'BB'} = U(\Phi_{AB} \otimes \chi_{A'B'})U^\dagger, \tag{3}$$

where U is a "twisting unitary" [95], $\chi_{A'B'}$ is the state of the shield, and Φ_{AB} is the maximally entangled state

$$\Phi_{AB} = |\Phi\rangle_{AB}\langle\Phi|, \quad |\Phi\rangle_{AB} := d_{\rm K}^{-1/2} \sum_{i=0}^{d_{\rm K}-1} |i\rangle_A |i\rangle_B. \tag{4}$$

By making local measurements on AB and tracing out the shield $A'B'$, Alice and Bob share an ideal private state which is completely factorized from the eavesdropper (Eve), i.e., of the form [95]

$$\tau_{AB\mathbf{e}} = \left(d_{\rm K}^{-1} \sum_{i=0}^{d_{\rm K}-1} |i\rangle_A \langle i| \otimes |i\rangle_B \langle i| \right) \otimes \tau_{\mathbf{e}}, \tag{5}$$

with $\tau_{\mathbf{e}}$ is an arbitrary state for Eve's system $\mathbf{e}$. It is important to note that system $\mathbf{e}$ can also be embedded in an infinite-dimensional Hilbert space. In fact, even if Alice and Bob employ DV systems, Eve may resort to hybrid DV-CV interactions with a CV environment under her control. However, let us also notice that, even if Eve resorts to a CV environment, its effective Hilbert space will still be finite-dimensional, just because the minimum purification of Alice and Bob's state needs a DV system.

The shared randomness in the classical systems A and B provides $\log_2 d_{\rm K}$ secret bits. Thus, the n-use target state ϕ^n in the previous adaptive protocol is such that

$$\log_2 d_{\rm K} := nR_n. \tag{6}$$

The local dimension $d_{\rm K}$ defines the number of secret bits and is exponential in n for both DV and CV systems. On the other hand, the dimension $d_{\rm S}$ of the shield system is not specified and may be super-exponential in DVs or even infinite in CVs. For DV systems, it is well known that we may restrict the effective size of $d_{\rm S}$ to be at most exponential in n. In fact, there is the following result.

Lemma 1 (Shield [96, 97]) *The effective increase of the shield size $d_{\rm S}$ is at most exponential in the number n of copies or channel uses, i.e., $\log_2 d_{\rm S} \leq \kappa n$ for some constant κ. More precisely, for any protocol, we can design an approximate protocol with the same asymptotic rate while having an at most exponential increase of $d_{\rm S}$.*

The proof is based on the fact that, for any protocol, one can consider an approximate protocol with the same asymptotic rate but split into $m = n/n_0$ identical and independent blocks of size n_0. These blocks provide m copies which are subject to key distillation via one-way CCs. This distillation procedure has a classical communication cost (number of bits exchanged in the CCs) which is linear in the number m of copies [98]. Using arguments from Ref. [95], this implies that the shield size $d_{\rm S}$ increases at most exponentially in $m < n$. See [99] for more mathematical details.

Thus, for DV systems, the previous lemma allows us to restrict Eq. (1) to adaptive protocols $\mathcal{P}$ for which the shield size grows at most exponentially. For CV systems, this lemma can still be used after a suitable truncation of the Hilbert space, as explained in the next section.

III. GENERAL WEAK-CONVERSE BOUND

A. Relative entropy of entanglement

In order to bound the various two-way capacities in Eq. (2), one can resort to the REE. Let us recall that the REE of a quantum state ρ is defined as [84–86]

$$E_R(\rho) = \inf_{\sigma_s} S(\rho||\sigma_s), \tag{7}$$

where σ_s is a separable state and

$$S(\rho||\sigma_s) := \mathrm{Tr}\left[\rho(\log_2 \rho - \log_2 \sigma_s)\right] \tag{8}$$

is the quantum relative entropy [84]. Note that we may also consider the regularized version

$$E_R^\infty(\rho) := \lim_n \; n^{-1} E_R(\rho^{\otimes n}) \leq E_R(\rho) \; . \tag{9}$$

Consider now a DV quantum channel $\mathcal{E}$, with Choi matrix $\rho_{\mathcal{E}} := \mathcal{I} \otimes \mathcal{E}(\Phi)$, where $\mathcal{I}$ is the identity channel and Φ is a maximally-entangled state (e.g., a Bell state for qubits). Then, we may consider the entanglement flux of the channel as the REE of its Choi matrix [77]

$$\Phi(\mathcal{E}) := E_R(\rho_{\mathcal{E}}). \tag{10}$$

This is a measure of the entanglement (REE) which may be transmitted via a single use of the channel.

B. Extension to asymptotic states

Let us now extend the definition of REE to asymptotic states. This is a step which is introduced to simplify the notation in following formulas. Recall that an asymptotic state σ is the limit of a sequence of bona-fide states σ^{μ}, i.e., $\sigma := \lim_{\mu} \sigma^{\mu}$. This formulation is very natural for CV systems where the maximally-entangled state Φ is itself asymptotic. In fact, this is the ideal Einstein-Podolsky-Rosen (EPR) state which is realized as the limit of two-mode squeezed vacuum (TMSV) states Φ^{μ}, i.e., $\Phi := \lim_{\mu} \Phi^{\mu}$. Here the parameter $\mu := \bar{n} + 1/2$ quantifies both the amount of squeezing (entanglement) between the two modes and the local energy, i.e., the mean number of thermal photons in each mode. Also note that the Choi matrix of a bosonic channel $\mathcal{E}$ is an asymptotic state which is defined by the limit

$$\rho_{\mathcal{E}} := \lim_{\mu} \rho_{\mathcal{E}}^{\mu}, \quad \rho_{\mathcal{E}}^{\mu} := \mathcal{I} \otimes \mathcal{E}(\Phi^{\mu}). \tag{11}$$

Now, recall that, given two sequences of states σ_1^{μ} and σ_2^{μ}, such that $||\sigma_k^{\mu} - \sigma_k|| \overset{\mu}{\to} 0$ for $k = 0$ or 1, we may write the relative entropy between the limit states σ_1 and σ_2 as the following

$$S(\sigma_1||\sigma_2) \leq \liminf_{\mu \to \infty} S(\sigma_1^{\mu}||\sigma_2^{\mu}). \tag{12}$$

This is known as the lower semi-continuity of the relative entropy, a property which is valid at any dimension [2, Theorem 11.6]. Following this property, we extend the definition of REE to an asymptotic state $\sigma := \lim_{\mu} \sigma^{\mu}$ as follows [77]

$$E_R(\sigma) := \inf_{\sigma_s^{\mu}} \liminf_{\mu \to \infty} S(\sigma^{\mu}||\sigma_s^{\mu}), \tag{13}$$

where σ_s^{μ} is a sequence of separable states converging in trace norm, i.e., $||\sigma_s^{\mu} - \sigma_s|| \to 0$ for separable σ_s. Thanks to Eq. (13), we may extend the definition of entanglement flux in Eq. (10) to bosonic channels, so that [77]

$$\Phi(\mathcal{E}) := \inf_{\sigma_s^{\mu}} \liminf_{\mu \to \infty} S(\rho_{\mathcal{E}}^{\mu}||\sigma_s^{\mu}). \tag{14}$$

C. Weak-converse upper bound for private communication

Once we have clarified how REE is generally defined for states and asymptotic states, including Choi matrices, we may provide the following result, which bounds the two-way capacities of an arbitrary quantum channel.

Theorem 2 ([77]) *For any quantum channel $\mathcal{E}$ (at any dimension, finite or infinite), the generic two-way capacity $\mathcal{C}(\mathcal{E})$ of Eq. (1) satisfies the weak converse bound*

$$\mathcal{C}(\mathcal{E}) \leq E_R^{\bigstar}(\mathcal{E}) := \sup_{\mathcal{P}} \lim_{n} n^{-1} E_R(\rho_{\mathbf{ab}}^n) \ , \tag{15}$$

where $\rho_{\mathbf{ab}}^n$ is the output of an n-use protocol $\mathcal{P}$.

The first and complete proof of this Theorem first appeared in the second arXiv version of PLOB back in 2015 [100]. It is repeated here for the sake of completeness so to avoid misinterpretations. Let us start with the case of DV systems and then we show the extension to CV systems via truncation arguments.

Assume that the total dimension of Alice's and Bob's registers $\mathbf{a}$ and $\mathbf{b}$ is equal to d. Even though these registers may be generally composed by a countable number of quantum systems, after n uses of the channel, only a finite number of systems will effectively contribute to the generation of a secret key. In fact, we know that the target private state ϕ^n, and the effective output state $\rho_{\mathbf{ab}}^n$ (ε-close to the target), has dimension $d = d_{\mathrm{K}}^2 d_{\mathrm{S}}$ which is at most exponential in the number of uses n (see Lemma 1 on the "shield"). In other words, any protocol can be replaced with an approximate protocol with this exponential scaling. Thus, we may write

$$\log_2 d \leq \alpha n R_n, \tag{16}$$

for some constant α. See also Eq. (21) of Ref. [100].

Because $\|\rho_{\mathbf{ab}}^n - \phi^n\| \leq \varepsilon$, we may then write the Fannes-type inequality [101]

$$|E_R(\rho_{\mathbf{ab}}^n) - E_R(\phi^n)| \leq 4\varepsilon \log_2 d + 2H_2(\varepsilon), \tag{17}$$

where H_2 is the binary Shannon entropy [102]. Using Eq. (16) and $nR_n \leq E_R(\phi^n)$ [95], the previous inequality implies [100]

$$R_n \leq \frac{E_R(\rho_{\mathbf{ab}}^n) + 2H_2(\varepsilon)}{(1 - 4\varepsilon\alpha)n}. \tag{18}$$

Taking the limit for $n \to \infty$ (asymptotic rate) and $\varepsilon \to 0$ (weak converse), we derive

$$\lim_{n} R_n \leq \lim_{n} n^{-1} E_R(\rho_{\mathbf{ab}}^n) \ . \tag{19}$$

Optimizing over all protocols $\mathcal{P}$, we find Eq. (15). It is clear that, without loss of generality, the optimization in Eq. (15) can be implicitly reduced to protocols with exponential scaling of the shield system.

Let us extend the proof to CV systems. Assume that, after the last LOCC Λ_n, Alice and Bob apply a trace-preserving LOCC $\mathbb{T}_d$ so that the protocol becomes $\mathbb{T}_d \circ \mathcal{P} = \{\Lambda_0, \Lambda_1, \cdots, \Lambda_n, \mathbb{T}_d\}$ whose truncated d-dimensional output state $\rho_{\mathbf{ab}}^{n,d} = \mathbb{T}_d(\rho_{\mathbf{ab}}^n)$ is ε-close to a DV private

state with $nR_{n,d}$ bits. We may then repeat the previous derivation for DVs, which here leads to

$$R_{n,d} \leq \frac{E_R(\rho_{\mathbf{ab}}^{n,d}) + 2H_2(\varepsilon)}{(1-4\varepsilon\alpha)n}. \tag{20}$$

It is pedantic to say that Lemma 1 still applies. In fact, the truncated protocol $\mathbb{T}_d \circ \mathcal{P}$ can be stopped after n_0 uses, and then repeated m times in an i.i.d. fashion, with $n = n_0 m$. One-way key distillation is then applied to the m DV output copies $(\rho_{\mathbf{ab}}^{n_0,d})^{\otimes m}$. This implies a number of bits of CC which is linear in m which, in turn, leads to an (at most) exponential scaling of the shield size d_S in m. In other words, we may write $\log_2 d_S(m) \leq \kappa m$ for constant κ. This automatically implies $\log_2 d_S(n) \leq \kappa_n n$ where $\liminf_n \kappa_n = \kappa$, because it is always possible to find sub-sequences $(n_0, 2n_0, 3n_0, \ldots)$ of n achieving the lower limit κ. As a result, we may always impose the condition in Eq. (16) for the total dimension d of the private state, because one can always find sub-sequences of n that make it valid as a lower limit.

Now, because $\mathbb{T}_d$ is a trace-preserving LOCC, we exploit the monotonicity of the REE

$$E_R(\rho_{\mathbf{ab}}^{n,d}) \leq E_R(\rho_{\mathbf{ab}}^{n}), \tag{21}$$

and rewrite Eq. (20) as

$$R_{n,d} \leq \frac{E_R(\rho_{\mathbf{ab}}^{n}) + 2H_2(\varepsilon)}{(1-4\alpha\varepsilon)n}\ . \tag{22}$$

For large n and small ε, this leads to

$$\lim_n R_{n,d} \leq \lim_n n^{-1} E_R(\rho_{\mathbf{ab}}^{n}). \tag{23}$$

An important observation is that the upper bound does no longer depend on d. As a consequence, in the optimization of $R_{n,d}$ over all protocols $\mathbb{T}_d \circ \mathcal{P}$ we can implicitly remove the truncation. Explicitly, we may write

$$\begin{aligned} K(\mathcal{E}) &= \sup_d \sup_{\mathbb{T}_d \circ \mathcal{P}} \lim_n R_{n,d} \\ &\leq \sup_{\mathcal{P}} \lim_n n^{-1} E_R(\rho_{\mathbf{ab}}^{n}) := E_R^{\bigstar}(\mathcal{E}). \end{aligned} \tag{24}$$

Remark 3 (Original 2015 proof) *The steps of this proof are the same as those in the original 2015 proof [100]. The truncation argument is described after Eq. (23) of Ref. [100], where we introduced a cut-off for the total Hilbert space at the output (which therefore applies to both the key and shield systems). Under this cut-off, we repeated the derivation for DV systems. Exactly as here we used the monotonicity of the REE to write an upper bound which is independent from the truncated dimension. Exploiting this independence, the cut-off was relaxed in the final expression following the same reasoning as above. Note that the published version of PLOB [77] contains other equivalent proofs which have been given for the sake of completeness. One of these proofs is completely independent from the details of the shield system.*

D. Rebuttal of some unfounded claims

Unfortunately, our truncation argument has been misunderstood. Believing that the truncation was not applied to the shield system, an author recently claimed that the shield size was unbounded in our 2015 proof for CV channels [103]. This is clearly not the case because a truncation is applied to the total output state (key plus shield system). As a result of this misunderstanding, this author started to claim "rigorous proofs" of results in PLOB (e.g., see Ref. [104]). Not only these claims are unfounded, but also in stark contradiction with other statements made by the same author [105, 106].

We also noticed that Ref. [105] would claim "full justification" of the "statements" presented in other works [67, 88, 91] which are based on the squashed entanglement. Refs. [67, 88, 91] would be "wrong" because of the potential unboundedness of the shield size in CV systems. Let us stress that these proofs are to be considered correct, because it is easily and implicitly understood that a truncation argument as the one discussed above applies and reduces the private state to an effective DV state. Such a truncation can then be released in all the final bounds derived in Refs. [67, 88, 91].

IV. SIMULATION OF QUANTUM CHANNELS

To simplify the upper bound of Eq. (15) into a single-letter quantity, PLOB [77] has devised the technique of teleportation stretching, which reduces an adaptive protocol (with any communication task) into a corresponding block protocol (with the same original task). The first ingredient in this technique is the LOCC simulation of a quantum channel, which allows one to "stretch" a channel into a quantum state. The second step is the exploitation of this simulation in the adaptive protocol, so that all channel transmissions are replaced by a tensor product of quantum states. Let us start with the review of the first step, i.e., channel simulation.

A. LOCC simulation of a quantum channel

For any quantum channel $\mathcal{E}$, we may consider an LOCC simulation. This consists of an LOCC $\mathcal{T}$ and a resource state σ such that, for any input state ρ, the output of the channel can be expressed as [77]

$$\mathcal{E}(\rho) = \mathcal{T}(\rho \otimes \sigma). \tag{25}$$

A channel $\mathcal{E}$ which is LOCC-simulable with a resource state σ as in Eq. (25) is also called "σ-stretchable" [77]. For the same channel $\mathcal{E}$ there may be different choices for $\mathcal{T}$ and σ, so that the simulation may be optimized depending on the task under study. Furthermore, the simulation can also be asymptotic. This means that we may

consider sequences of resource states σ^{μ} such that [107]

$$\mathcal{E}^{\mu}(\rho)=\mathcal{T}(\rho\otimes\sigma^{\mu}). \tag{26}$$

and define a quantum channel as a point-wise limit

$$\mathcal{E}(\rho)=\lim_{\mu}\mathcal{E}^{\mu}(\rho). \tag{27}$$

This can be expressed in terms of the Bures fidelity as

$$\lim_{\mu}F\left[\mathcal{E}^{\mu}(\rho),\mathcal{E}(\rho)\right]=1, \tag{28}$$

where $F(\rho,\rho')\coloneqq\mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}$ for states ρ and ρ'.

A simple criterion that enables us to identify a good LOCC simulation for a quantum channel is that of teleportation covariance. A quantum channel $\mathcal{E}$ is said to be teleportation covariant if, for any teleportation unitary U, i.e., Pauli operators in DVs and phase-space displacements in CVs [80], we may write the following

$$\mathcal{E}(U\rho U^{\dagger})=V\mathcal{E}(\rho)V^{\dagger}\ , \tag{29}$$

for some other unitary V [77]. Note that this is a property of many channels, including Pauli and erasure channels in DVs, and bosonic Gaussian channels in CVs. Channels with this property are "Choi-stretchable", which means that they can be simulated by using their Choi matrix as resource state. More precisely, we can state the following

Criterion 4 (Tele-covariance/Choi-stretchability) *A teleportation-covariant channel $\mathcal{E}$ is Choi-stretchable via teleportation, i.e., it can be simulated by teleporting input states ρ over its Choi matrix $\rho_{\mathcal{E}}$. For a DV channel, this means*

$$\mathcal{E}(\rho)=\mathcal{T}(\rho\otimes\rho_{\mathcal{E}}), \tag{30}$$

where $\mathcal{T}$ is teleportation. For a CV channel, this means

$$\mathcal{E}(\rho)=\lim_{\mu}\mathcal{E}^{\mu}(\rho),\quad \mathcal{E}^{\mu}(\rho)=\mathcal{T}(\rho\otimes\rho_{\mathcal{E}}^{\mu}), \tag{31}$$

where $\mathcal{T}$ is the LOCC of a (modified) BK teleportation protocol and the sequence $\rho_{\mathcal{E}}^{\mu}$ defines the asymptotic Choi matrix for large μ.

B. Error in the simulation of bosonic channels

In order to better clarify the previous criterion for CV bosonic channels, let us recall the details of the BK teleportation protocol. In the standard formulation, the protocol is implemented by using a TMSV state Φ^{μ} as resource state. This means that Alice's input state ρ_a and part a' of a shared TMSV state $\Phi^{\mu}_{a'B}$ are detected by a CV Bell detection (composed of a balanced beamsplitter whose output ports are measured by two conjugate homodyne detectors). The complex outcome α of the Bell detection is communicated to Bob, who applies the conditional phase-space displacement $D(-\alpha)$ on his mode B. In this way, Bob obtains the output state ρ_B which is a teleported version ρ_a^{μ} of the input one ρ_a.

Therefore, let us call $\mathcal{T}$ the BK teleportation LOCC, i.e., the LOs given by the Bell POVM and the conditional displacements, suitably averaged over all the Bell outcomes. The output state can be written as

$$\rho_a^{\mu}=\mathcal{T}(\rho_a\otimes\Phi^{\mu})=\mathcal{I}^{\mu}(\rho_a), \tag{32}$$

where $\mathcal{I}^{\mu}$ is the BK teleportation channel. This channel can be locally (i.e., point-wise) described by an additive-noise Gaussian channel with added noise [114, 116]

$$\xi=2\mu-\sqrt{4\mu^2-1}\ . \tag{33}$$

As a result, one has the point-wise convergence of the BK protocol [79, 93]: for any energy-bounded state (i.e., a 'point') ρ_a, we may write

$$\lim_{\mu}F(\rho_a^{\mu},\rho_a)=1. \tag{34}$$

The discussion can be automatically extended to considering ancillary systems, so that the input can be taken as a bipartite state ρ_{Aa} whose part a is teleported while part A is just subject to the identity channel $\mathcal{I}_A$. In this case, we have the output state

$$\rho_{Aa}^{\mu}=\mathcal{I}_A\otimes\mathcal{T}(\rho_{Aa}\otimes\Phi^{\mu})=\mathcal{I}_A\otimes\mathcal{I}^{\mu}(\rho_{Aa}), \tag{35}$$

and we may write the limit

$$\lim_{\mu}F(\rho_{Aa}^{\mu},\rho_{Aa})=1. \tag{36}$$

We may formulate this limit in an equivalent way. In fact, for any input state ρ_{Aa} (or 'point'), let us define the corresponding teleportation infidelity at energy μ as

$$\varepsilon_{\mathrm{BK}}(\mu,\rho_{Aa})\coloneqq 1-F(\rho_{Aa}^{\mu},\rho_{Aa}). \tag{37}$$

Then, we may write the point-wise limit

$$\lim_{\mu}\varepsilon_{\mathrm{BK}}(\mu,\rho_{Aa})=0. \tag{38}$$

Consider now a teleportation-covariant bosonic channel $\mathcal{E}$, i.e., satisfying [77]

$$\mathcal{E}\left[D(\alpha)\rho D(-\alpha)\right]=V_{\alpha}\mathcal{E}(\rho)V_{\alpha}^{\dagger}\ , \tag{39}$$

for a set of output unitaries V_{α}. Then consider its μ-energy simulation $\mathcal{E}^{\mu}$. This can be realized as in Eq. (31) where the resource state is the quasi-Choi matrix $\rho_{\mathcal{E}}^{\mu}\coloneqq\mathcal{I}\otimes\mathcal{E}(\Phi^{\mu})$ and $\mathcal{T}$ is the LOCC of a modified BK protocol where the output correction unitaries are given by $V_{\alpha}^{\dagger}$. For any energy-constrained input state ρ_a, the simulated output state can be written as

$$\mathcal{E}^{\mu}(\rho_a)=\mathcal{E}\circ\mathcal{I}^{\mu}(\rho_a)=\mathcal{T}(\rho_a\otimes\rho_{\mathcal{E}}^{\mu}), \tag{40}$$

as also shown in Fig. 2. Because $\mathcal{E}^{\mu}=\mathcal{E}\circ\mathcal{I}^{\mu}$ and $\mathcal{E}=\mathcal{E}\circ\mathcal{I}$, we may write

$$\lim_{\mu}F\left[\mathcal{E}^{\mu}(\rho_a),\mathcal{E}(\rho_a)\right]\geq\lim_{\mu}F\left[\mathcal{I}^{\mu}(\rho_a),\mathcal{I}(\rho_a)\right]=1, \tag{41}$$

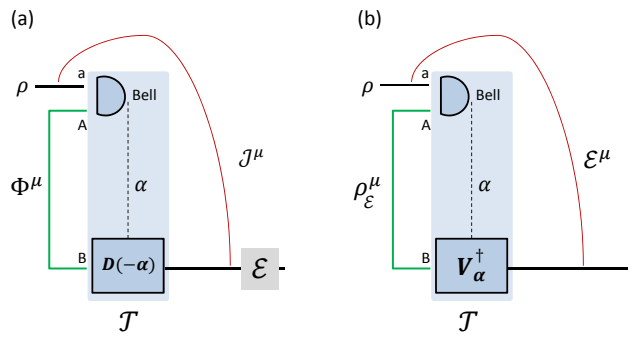

FIG. 2: BK protocol and teleportation simulation. (a) We represent the standard protocol where a TMSV Φ^μ and a teleportation LOCC $\mathcal{T}$ (Bell detection and conditional displacements) are used to teleport an input state ρ_a. The input state is equal to $\mathcal{I}^\mu(\rho_a)$ where $\mathcal{I}^\mu$ is the BK teleportation channel. In general we may consider another bosonic channel $\mathcal{E}$ at the output so that we have the composition $\mathcal{E}^\mu = \mathcal{E} \circ \mathcal{I}^\mu$, as in Eq. (40). (b) If the channel is teleportation-covariant, then we may commute it with the displacement operators $D(-\alpha)$ up to introducing the modified corrections $V_\alpha^\dagger$. As a result the resource state will become the quasi-Choi matrix $\rho_{\mathcal{E}}^\mu$ and the teleportation LOCC $\mathcal{T}$ will be re-defined over the new correction operators $V_\alpha^\dagger$. The channel $\mathcal{E}^\mu$ can be represented as $\mathcal{E}^\mu(\rho_a) = \mathcal{T}(\rho_a \otimes \rho_{\mathcal{E}}^\mu)$ as in Eq. (40).

i.e., we have the point-wise limit promised in Eq. (31).

This can be extended to the presence of ancillary systems A. In fact, given the teleportation-covariant bosonic channel $\mathcal{I}_A \otimes \mathcal{E}$, its output $\rho_{AB} := \mathcal{I}_A \otimes \mathcal{E}(\rho_{Aa})$ can be simulated by

$$\rho_{AB}^\mu := \mathcal{I}_A \otimes \mathcal{E}^\mu(\rho_{Aa}) \tag{42}$$

$$= \mathcal{I}_A \otimes \mathcal{E} \circ \mathcal{I}^\mu(\rho_{Aa}) = \mathcal{I}_A \otimes \mathcal{T}(\rho_{Aa} \otimes \rho_{\mathcal{E}}^\mu). \tag{43}$$

In other words, we may write the point-wise limit

$$\lim_\mu F(\rho_{AB}^\mu, \rho_{AB}) = 1. \tag{44}$$

Alternatively, we may write this limit in terms of the infidelity

$$\lim_\mu \varepsilon_{\rm BK}(\rho_{AB}^\mu, \rho_{AB}) = 0 \tag{45}$$

$$\varepsilon_{\rm BK}(\rho_{AB}^\mu, \rho_{AB}) := 1 - F(\rho_{AB}^\mu, \rho_{AB}). \tag{46}$$

C. Considerations for bosonic Gaussian channels

It is very important to note that the previous limit are point-wise and performed over energy-constrained input states. The situation can be completely different when, at the input, we also include asymptotic states, defined as the limit of sequences of states for increasing energy. In fact, in the presence of an unbounded input alphabet, one needs to take special care on how the limits are performed. To explore this issue, consider the case of single-mode bosonic Gaussian channels.

Consider a bosonic mode with vectorial quadrature $\hat{\mathbf{x}} = (\hat{q}, \hat{p})^T$ with $[\hat{q}, \hat{p}] = i$. Then, consider a single-mode Gaussian channel $\mathcal{E}$ acting on an input state with mean value $\bar{\mathbf{x}} = (\bar{q}, \bar{p})^T$ and covariance matrix (CM) $\mathbf{V}$. Its action is described by the transformation

$$\bar{\mathbf{x}} \to \mathbf{T}\bar{\mathbf{x}} + \mathbf{d}, \quad \mathbf{V} \to \mathbf{T}\mathbf{V}\mathbf{T}^T + \mathbf{N}, \tag{47}$$

where $\mathbf{d} \in R^2$ is a displacement, while transmission matrix $\mathbf{T}$ and the noise matrix $\mathbf{N}$ are 2×2 real matrices, with $\mathbf{N}^T = \mathbf{N} \geq 0$ and

$$\det \mathbf{N} \geq (\det \mathbf{T} - 1)^2 . \tag{48}$$

Up to input/output unitary transformations, any such channel can be reduced to a canonical form [5, 108–110] characterized by zero displacement ($\mathbf{d} = 0$) and diagonal matrices $\mathbf{T}$ and $\mathbf{N}$. Among these forms the phase-insensitive ones are the thermal-loss channel, the quantum amplifier and the additive-noise Gaussian channel. These channels can be described by

$$\bar{\mathbf{x}} \to \sqrt{\eta}\bar{\mathbf{x}}, \quad \mathbf{V} \to \eta\mathbf{V} + \nu\mathbf{I}, \tag{49}$$

where $\eta \in \mathbb{R}$, $\nu \geq 0$, and $\mathbf{I} := \mathrm{diag}(1, 1)$. In particular, we have the following specifications of Eq. (49):

- Thermal-loss channel $\mathcal{E}_{\eta,\bar{n}}$ is characterized by transmissivity $\eta \in [0, 1]$ and mean thermal number $\bar{n} \geq 0$, so that $\nu = (1 - \eta)(\bar{n} + 1/2)$. The lossy channel, or pure-loss channel, corresponds to $\bar{n} = 0$.

- Quantum amplifier $\mathcal{E}_{g,\bar{n}}$ is characterized by gain $\eta = g > 1$ and mean thermal number $\bar{n} \geq 0$, so that $\nu = (g - 1)(\bar{n} + 1/2)$. A quantum-limited amplifier corresponds to the specific case $\bar{n} = 0$.

- Additive-noise Gaussian channel $\mathcal{E}_\xi$ has transmissivity $\eta = 1$ and additive noise $\nu = \xi \geq 0$.

There are other canonical forms which are instead sensitive to phase. These are the forms in the classes A_2, B_1 and D according to the terminology introduced by Holevo [108] and summarized in Table I of Ref. [5]. For instance, a B_1 canonical form is described by

$$\bar{\mathbf{x}} \to \bar{\mathbf{x}}, \quad \mathbf{V} \to \mathbf{V} + \mathrm{diag}(0, 1), \tag{50}$$

so that a vacuum noise unit is added to the momentum quadrature only.

Given a Gaussian channel $\mathcal{E}$ and its simulation $\mathcal{E}^\mu$ based on the BK protocol, we may consider the corresponding output states $\rho_{AB} := \mathcal{I}_A \otimes \mathcal{E}(\rho_{Aa})$ and $\rho_{AB}^\mu := \mathcal{I}_A \otimes \mathcal{E}^\mu(\rho_{Aa})$ for an energy-constrained input state ρ_{Aa}. We may then write the limit in Eq. (44). However, the limit in Eq. (44) becomes ambiguous and problematic if we allow for an energy-unbounded alphabet, which means to include asymptotic states with diverging energy at the input.

For instance, consider an input sequence of TMSV states $\Phi_{Aa}^{\tilde{\mu}}$ with increasing squeezing $\tilde{\mu}$. Compute the actual output $\rho_{AB}(\tilde{\mu}) = \mathcal{I}_A \otimes \mathcal{E}(\Phi_{Aa}^{\tilde{\mu}})$ and the simulated output $\rho_{AB}^{\mu}(\tilde{\mu}) := \mathcal{I}_A \otimes \mathcal{E}^{\mu}(\Phi_{Aa}^{\tilde{\mu}})$ for some simulation energy μ. It is easy to find Gaussian channels, such as the identity channel (see Appendix A) or the B_1 canonical form (see Ref. [111]), such that the fidelity tends to zero in the limit of $\tilde{\mu} \to \infty$ for any finite μ, i.e.,

$$\lim_{\tilde{\mu}} F[\rho_{AB}^{\mu}(\tilde{\mu}), \rho_{AB}(\tilde{\mu})] = 0 \ . \tag{51}$$

By comparing Eqs. (44) and (51), we see that the joint limit in $\tilde{\mu}$ (energy of the input) and μ (energy of the simulation) is not mathematically defined. This issue can be solved in two ways:

- Specifying a precise order of the limits, i.e., first in the simulation energy μ and then in the input energy $\tilde{\mu}$ or size of the alphabet. This exploits the fact that the underlying BK teleportation protocol strongly converges to the identity channel (as we further discuss in Sec. IV D).
- Bounding the size of the input alphabet imposing an energy constraint. For this more elegant solution we need to introduce the energy-constrained diamond distance. This exploits the fact that the BK teleportation protocol converges to the identity channel according to the energy-constrained uniform topology (as we further discuss in Sec. IV E).

D. Topologies of convergence in the BK protocol and teleportation simulation of bosonic channels

The previous considerations can be formalized in terms of different topologies of convergence associated with the BK teleportation protocol. Consider a μ-energy BK protocol, where a teleportation LOCC $\mathcal{T}$ (Bell plus conditional displacements) is performed over a TMSV state Φ^{μ} with finite energy μ. Given an input energy-constrained state ρ_{Aa}, we may write its teleported version ρ_{Aa}^{μ} as in Eq. (35). We also know [79, 93] that we may write the point-wise limit of Eq. (38) for the infidelity $\varepsilon_{\text{BK}}(\mu, \rho_{Aa}) := 1 - F(\rho_{Aa}^{\mu}, \rho_{Aa})$. By taking the supremum, it is trivial to write the strong convergence limit

$$\sup_{\rho_{Aa}} \left[\lim_{\mu} \varepsilon_{\text{BK}}(\mu, \rho_{Aa}) \right] = 0. \tag{52}$$

This is also trivially extended to teleportation simulation. In fact, for the finite-energy simulation $\mathcal{E}^{\mu}$ of a teleportation-covariant bosonic channel $\mathcal{E}$, we may write the point-wise limit of Eq. (44) that leads to

$$\sup_{\rho_{Aa}} \left[\lim_{\mu} \varepsilon_{\text{BK}}(\rho_{AB}^{\mu}, \rho_{AB}) \right] = 0. \tag{53}$$

It is clear, from the reasonings on the order of the limits in Sec. IV C, that the BK protocol does not converge *uniformly* to the identity channel. In fact, Eq. (51) written for $\mathcal{E} = \mathcal{I}$ implies that, for any finite μ, we have

$$F[\mathcal{I}_A \otimes \mathcal{I}^{\mu}(\Phi_{Aa}^{\tilde{\mu}}), \Phi_{Aa}^{\tilde{\mu}}] \overset{\tilde{\mu}}{\to} 0 \ , \tag{54}$$

so that $\varepsilon_{\text{BK}}(\mu, \Phi_{Aa}^{\tilde{\mu}}) \overset{\tilde{\mu}}{\to} 1$. Because of this limit, we have

$$\sup_{\rho_{Aa}} \varepsilon_{\text{BK}}(\mu, \rho_{Aa}) = 1, \quad \text{for any } \mu, \tag{55}$$

and, therefore,

$$\lim_{\mu} \left[\sup_{\rho_{Aa}} \varepsilon_{\text{BK}}(\mu, \rho_{Aa}) \right] = 1. \tag{56}$$

In diamond distance, this is equivalently to state that

$$\| \mathcal{I} - \mathcal{I}^{\mu} \|_{\diamond} = 2, \quad \text{for any } \mu. \tag{57}$$

In fact, recall that the diamond distance between two quantum channels $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined as

$$\| \mathcal{E}_1 - \mathcal{E}_2 \|_{\diamond} := \sup_{\rho_{Aa}} \| \mathcal{I}_A \otimes \mathcal{E}_1(\rho_{Aa}) - \mathcal{I}_A \otimes \mathcal{E}_2(\rho_{Aa}) \| \, , \tag{58}$$

where $\| \cdot \|$ is the trace norm. For any two states ρ and ρ', we may then write the Fuchs-van de Graaf inequality

$$\| \rho - \rho' \| \geq 2 \left[1 - F(\rho, \rho') \right] . \tag{59}$$

Therefore, it is easy to see that Eq. (54) implies Eq. (57), by using Eqs. (58) and (59).

This non-convergence is also true for the teleportation simulation of a generic bosonic channel. However, for most of the single-mode bosonic Gaussian channels, the teleportation simulation uniformly converges to the channels. In fact, given an arbitrary single-mode Gaussian channel $\mathcal{E}$ with teleportation simulation $\mathcal{E}^{\mu}$, we may write

$$\lim_{\mu} \| \mathcal{E} - \mathcal{E}^{\mu} \|_{\diamond} = 0, \tag{60}$$

if and only if its noise matrix $\mathbf{N}$ has full rank, i.e., $\text{rank}(\mathbf{N}) = 2$ [111]. For Gaussian channels with $\text{rank}(\mathbf{N}) < 2$ and other bosonic channels, we need to replace the uniform convergence of Eq. (60) with a notion of bounded-uniform convergence which is based on an energy-constrained version of the diamond distance.

E. Energy-constrained diamond distance

A more useful definition of diamond distance for bosonic channels involves the introduction of an energy constraint at the input [77, 112]. Following PLOB, we impose an energy constraint on the entire input space, including the ancillas. In fact, consider the following set of energy-constrained bipartite states

$$\mathcal{D}_N := \{ \rho_{Aa} \mid \text{Tr}(\hat{N} \rho_{Aa}) \leq N \}, \tag{61}$$

where $\hat{N}$ is the total number operator associated to the input a and all the ancillas A. One can check that $\mathcal{D}_N$ is a compact set [113]. Then, for two bosonic channels, $\mathcal{E}_1$ and $\mathcal{E}_2$, we may define the energy-constrained diamond distance as

$$\|\mathcal{E}_1 - \mathcal{E}_2\|_{\diamond N} := \sup_{\rho_{Aa}\in\mathcal{D}_N} \|\mathcal{I}_A \otimes \mathcal{E}_1(\rho_{Aa}) - \mathcal{I}_A \otimes \mathcal{E}_2(\rho_{Aa})\| \ . \tag{62}$$

Now, for any bounded alphabet $\mathcal{D}_N$ with energy N, consider the energy-constrained diamond distance between a (teleportation-covariant) bosonic channel $\mathcal{E}$ and its teleportation simulation $\mathcal{E}^{\mu}$. This defines the simulation error

$$\delta(\mu, N) := \|\mathcal{E} - \mathcal{E}^{\mu}\|_{\diamond N} \ . \tag{63}$$

Because $\mathcal{D}_N$ is compact, the point-wise limit in Eq. (44) implies the following uniform limit

$$\delta(\mu, N) \overset{\mu}{\to} 0 \ \text{ for any finite } N, \tag{64}$$

or, equivalently, we may write

$$\lim_{\mu} \left[\sup_{\rho_{Aa}\in\mathcal{D}_N} \varepsilon_{\text{BK}}(\rho^{\mu}_{AB}, \rho_{AB}) \right] = 0. \tag{65}$$

As a result, when we consider the asymptotic simulation in Eq. (31) for an arbitrary bosonic channel, we may consider it either as a point-wise limit or as a uniform limit while assuming an energy-constrained alphabet $\mathcal{D}_N$ at the input (as in Ref. [77]).

F. Finite-resource simulation of Gaussian channels

Very recently, a different type of simulation has been introduced for Gaussian channels [114, 115]. As shown in Ref. [116], these simulations are not optimal as the asymptotic ones, but they may still provide very good approximations of the results in PLOB. According to Ref. [114], a phase-insensitive Gaussian channel $\mathcal{E}_{\eta,\nu}$ can be simulated as follows

$$\mathcal{E}_{\eta,\nu}(\rho) = \mathcal{T}_{\eta}(\rho \otimes \sigma_{\nu}), \tag{66}$$

where $\mathcal{T}_{\eta}$ denotes the LOCC of a (modified) BK teleportation protocol with gain $\sqrt{\eta}$ [79], and σ_{ν} is a zero-mean two-mode Gaussian state with CM

$$\mathbf{V}(\sigma_{\nu}) = \frac{1}{2} \begin{pmatrix} a\mathbf{I} & c\mathbf{Z} \\ c\mathbf{Z} & b\mathbf{I} \end{pmatrix}, \tag{67}$$

where the elements in the CM are equal to [114]

$$\begin{aligned} a &= \frac{b + (\eta - 1)e^{-2r}}{\eta}, \quad c = \frac{b - e^{-2r}}{\sqrt{\eta}}, \\ b &= \frac{-|\eta - 1| + \eta e^{2r} + e^{-2r}}{-e^{2r}|\eta - 1| + \eta + 1}, \end{aligned} \tag{68}$$

and the entanglement parameter $r \geq 0$ is connected to the channel parameter via the relation

$$\nu = \frac{e^{-2r}}{2}(\eta + 1). \tag{69}$$

For the specific case of a pure-loss channel, the previous simulation cannot be used because, for this channel, one has $\nu = (1 - \eta)/2$ and, therefore, b becomes singular in Eq. (68). For the pure loss channel, a finite-resource simulation is just provided by teleporting with gain $\sqrt{\eta}$ over a Gaussian state with CM [116, 117]

$$\sigma_{\eta} = \begin{pmatrix} a\mathbf{I} & \sqrt{a^2 - 1/4}\mathbf{Z} \\ \sqrt{a^2 - 1/4}\mathbf{Z} & a\mathbf{I} \end{pmatrix}, \ a = \frac{\eta + 1}{2(1 - \eta)}. \tag{70}$$

V. TELEPORTATION STRETCHING OF ADAPTIVE PROTOCOLS

A. Stretching with non-asymptotic simulations

Thanks to the LOCC-simulation of a quantum channel, we may completely simplify the structure of an adaptive protocol for quantum/private communication. Let us start with the simple case of non-asymptotic simulations. Consider an adaptive protocol with n transmissions over a channel $\mathcal{E}$ which admits an LOCC-simulation $(\mathcal{T}, \sigma)$. Then, we can reduce the output state $\rho^n_{\mathbf{ab}}$ into a tensor-product of resource states $\sigma^{\otimes n}$ up to a trace-preserving LOCC $\bar{\Lambda}$. In other words, we may write [77]

$$\rho^n_{\mathbf{ab}} = \bar{\Lambda}\left(\sigma^{\otimes n}\right) \ . \tag{71}$$

As depicted in Fig. 3, the procedure goes as follows:

- Each transmission through $\mathcal{E}$ is replaced by its simulation $(\mathcal{T}, \sigma)$;
- The resource state σ is stretched "back in time" while $\mathcal{T}$ is included in the LOCCs;
- All the LOCCs including the register preparation are collapsed into a single LOCC $\bar{\Lambda}$, which is trace-preserving after averaging over measurements.

B. Stretching with asymptotic simulations

Consider now an adaptive protocol with n transmissions over a quantum channel $\mathcal{E}$ which can be simulated asymptotically by using an LOCC $\mathcal{T}$ and a sequence of resource states σ^{μ}, as in Eqs. (26) and (27). The procedure is more involved because we need to carefully control the propagation of the simulation error from the channel $\mathcal{E}$ to the final output state $\rho^n_{\mathbf{ab}}$.

Let us replace each transmission through $\mathcal{E}$ with an imperfect channel $\mathcal{E}^{\mu}(\rho) := \mathcal{T}(\rho \otimes \sigma^{\mu})$ based on a finite-energy resource state σ^{μ}. Assuming that, in each ith

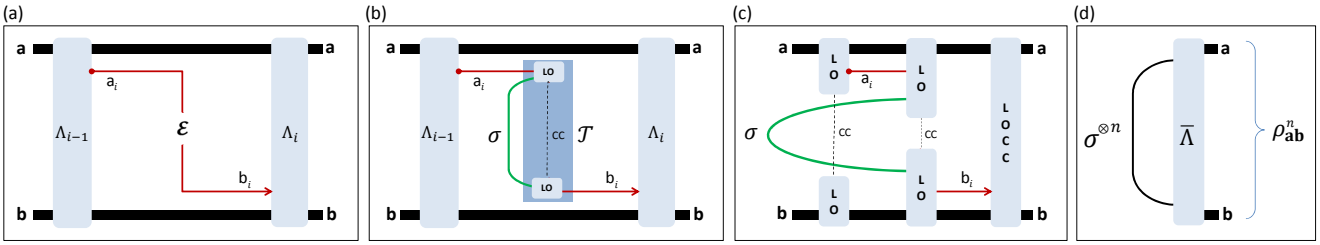

FIG. 3: Teleportation stretching of an adaptive point-to-point protocol [77]. (a) Consider the generic ith transmission through channel $\mathcal{E}$ between two adaptive LOCCs Λ_{i-1} and Λ_i. (b) The channel can be simulated by an LOCC $\mathcal{T}$ and a resource state σ. (c) The resource state σ is stretched back in time out of the adaptive LOCCs while $\mathcal{T}$ becomes part of the LOCCs of the simulated protocol. (d) By repeating the operation at point (c) for all the n transmissions, we accumulate the tensor-product state $\sigma^{\otimes n}$. All the LOCCs (and also the initial state of the registers) are collapsed into a single LOCC $\bar{\Lambda}$, which is trace-preserving after averaging over all measurements. The final result is a block protocol where the output state $\rho^n_{\mathbf{ab}}$ is obtained by applying $\bar{\Lambda}$ to the resource states $\sigma^{\otimes n}$. This is the decomposition in Eq. (71).

transmission, the local registers are bounded in energy so that the total input state $\rho_{\mathbf{a}a_i\mathbf{b}}$ belongs to a bounded alphabet $\mathcal{D}_N$, we may write the imperfect simulation with error $\delta(\mu,N) := \|\mathcal{E}-\mathcal{E}^\mu\|_{\diamond N}$ as in Eq. (63). We then need to propagate $\delta(\mu,N)$ throughout the protocol and quantify the trace distance between the actual output $\rho^n_{\mathbf{ab}} := \rho_{\mathbf{ab}}(\mathcal{E}^{\otimes n})$ and the simulated output $\rho^{n,\mu}_{\mathbf{ab}} := \rho_{\mathbf{ab}}(\mathcal{E}^{\mu\otimes n})$. For any finite N, we find [77]

$$\|\rho^n_{\mathbf{ab}}-\rho^{n,\mu}_{\mathbf{ab}}\| \leq n\delta(\mu,N)\ . \tag{72}$$

The proof exploits basic properties of the trace distance. Starting from the register state $\rho^0_{\mathbf{ab}}$, we write

$$\rho^n_{\mathbf{ab}} = \Lambda_n\circ\mathcal{E}\circ\Lambda_{n-1}\cdots\circ\Lambda_1\circ\mathcal{E}(\rho^0_{\mathbf{ab}}), \tag{73}$$
$$\rho^{n,\mu}_{\mathbf{ab}} = \Lambda_n\circ\mathcal{E}^\mu\circ\Lambda_{n-1}\cdots\circ\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}}), \tag{74}$$

where we implicitly assume that channels $\mathcal{E}$ and $\mathcal{E}^\mu$ are applied to the input system a_i in the i-th transmission, so that $\mathcal{E}^{(\mu)} = \mathcal{I}_{\mathbf{a}}\otimes\mathcal{E}^{(\mu)}_{a_i}\otimes\mathcal{I}_{\mathbf{b}}$. For simplicity, assume $n=2$. We may apply the "peeling" argument [77]

$$\begin{aligned}
&\|\rho^2_{\mathbf{ab}}-\rho^{2,\mu}_{\mathbf{ab}}\|\\
&\overset{(1)}{\leq} \|\mathcal{E}\circ\Lambda_1\circ\mathcal{E}(\rho^0_{\mathbf{ab}})-\mathcal{E}^\mu\circ\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})\|\\
&\overset{(2)}{\leq} \|\mathcal{E}\circ\Lambda_1\circ\mathcal{E}(\rho^0_{\mathbf{ab}})-\mathcal{E}\circ\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})\|\\
&+\|\mathcal{E}\circ\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})-\mathcal{E}^\mu\circ\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})\|\\
&\overset{(1)}{\leq} \|\mathcal{E}(\rho^0_{\mathbf{ab}})-\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})\|\\
&+\|\mathcal{E}[\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})]-\mathcal{E}^\mu[\Lambda_1\circ\mathcal{E}^\mu(\rho^0_{\mathbf{ab}})]\|\\
&\overset{(3)}{\leq} 2\|\mathcal{E}-\mathcal{E}^\mu\|_{\diamond N}\ ,
\end{aligned} \tag{75}$$

where we use the monotonicity of the relative entropy under maps (1), the triangle inequality (2) and the definition of energy-constrained diamond distance (3). Generalization to $n\geq 2$ provides the result in Eq. (72).

The next step is the stretching of the simulated protocol, i.e., the decomposition of the state $\rho^{\mu,n}_{\mathbf{ab}}$. By repeating the steps in Fig. 3 with $\mathcal{E}^\mu$ in the place of the original channel $\mathcal{E}$, we derive the decomposition $\rho^{\mu,n}_{\mathbf{ab}} = \bar{\Lambda}\left(\sigma^{\mu\otimes n}\right)$ for a trace-preserving LOCC $\bar{\Lambda}$. For any energy constraint N, we may therefore write

$$\left\|\rho^n_{\mathbf{ab}}-\bar{\Lambda}\left(\sigma^{\mu\otimes n}\right)\right\| \leq n\delta(\mu,N)\ . \tag{76}$$

For finite energy N and number of uses n, we may take the limit of $\mu\to\infty$ and get the asymptotic stretching

$$\left\|\rho^n_{\mathbf{ab}}-\bar{\Lambda}(\sigma^{\mu\otimes n})\right\| \overset{\mu}{\to} 0. \tag{77}$$

VI. SINGLE-LETTER UPPER BOUNDS

The most crucial insight of PLOB [77] has been the combination of the channel's REE, as expressed by the general weak converse bound in Eq. (15), with the adaptive-to-block reduction realized by teleportation stretching, as expressed by Eqs. (71) and (77). This has been the novel recipe that led PLOB to the computation of extremely simple single-letter upper bounds for all the two-way capacities of a quantum channel. This entire technique of "channel's REE and teleportation stretching" has been later used as a tool in a number of other works [66, 83, 116, 118] and is at the core of WTB [92] and other follow-up papers.

A. Bounds for channels with standard simulations

Let us start with a quantum channel having a standard (non-asymptotic) simulation with resource state σ. Let us compute the REE of the output state $\rho^n_{\mathbf{ab}}$ of an adaptive protocol over this channel. By using the decomposition in Eq. (71), we derive

$$E_R(\rho^n_{\mathbf{ab}}) \overset{(1)}{\leq} E_R(\sigma^{\otimes n}) \overset{(2)}{\leq} nE_R(\sigma)\ , \tag{78}$$

where we use the monotonicity of the REE under trace-preserving LOCCs in (1), and its subadditivity over tensor products in (2). By replacing Eq. (78) in Eq. (15), we

then find the single-letter upper bound [77, Theorem 5]

$$\mathcal{C}(\mathcal{E}) \leq E_R^{\infty}(\sigma) \leq E_R(\sigma) \ . \tag{79}$$

In particular, if the channel $\mathcal{E}$ is teleportation-covariant, it is Choi-stretchable, and we may write [77, Theorem 5]

$$\mathcal{C}(\mathcal{E}) \leq E_R(\rho_{\mathcal{E}}) = \Phi(\mathcal{E}), \tag{80}$$

so that the entanglement flux of the channel $\Phi(\mathcal{E})$ bounds all its two-way assisted capacities. The computation of the single-letter quantity $\Phi(\mathcal{E})$ is very simple.

B. Formulas for Pauli and erasure channels

Consider a qubit Pauli channel $\mathcal{P}$ whose action on an input state ρ is given by

$$\mathcal{P}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z, \tag{81}$$

where X, Y, and Z are Pauli operators [1] and $\{p_k\}$ is a probability distribution. It is easy to check that $\mathcal{P}$ is teleportation covariant and, therefore, Choi-stretchable. Computing its entanglement flux $\Phi(\mathcal{P})$, one finds that [77]

$$\mathcal{C}(\mathcal{P}) \leq \Phi(\mathcal{P}) = 1 - H_2(p_{\max}), \tag{82}$$

if $p_{\max} := \max\{p_k\} \geq 1/2$, while $\Phi = 0$ otherwise. The result can be generalized to arbitrary finite dimension [77].

A particular type of Pauli channel is the depolarizing channel $\mathcal{P}_{\text{depol}}$, which is defined by

$$\mathcal{P}_{\text{depol}}(\rho) = (1-p)\rho + pI/2, \tag{83}$$

for some probability p. Specifying Eq. (82), we find [77]

$$\mathcal{C}(\mathcal{P}_{\text{depol}}) \leq \Phi(\mathcal{P}_{\text{depol}}) = 1 - H_2\left(3p/4\right), \tag{84}$$

for $p \leq 2/3$, while $\Phi = 0$ otherwise. Another type of Pauli channel is the dephasing channel $\mathcal{P}_{\text{deph}}$, defined by

$$\mathcal{P}_{\text{deph}}(\rho) = (1-p)\rho + pZ\rho Z, \tag{85}$$

where p is the probability of a phase flip. For this channel, we compute the entanglement flux [77]

$$\Phi(\mathcal{P}_{\text{deph}}) = 1 - H_2(p) \ . \tag{86}$$

This upper bound coincides with a lower bound to the capacity which is given by the one-way distillability of the Choi matrix, i.e., $D_1(\rho_{\mathcal{P}_{\text{deph}}})$. The latter is lower bounded by the maximum between the coherent [119, 120] and reverse coherent [89, 90] information. Because $\Phi(\mathcal{P}_{\text{deph}}) = D_1(\rho_{\mathcal{P}_{\text{deph}}})$, the dephasing channel is also called "distillable" and its two-way capacity is completely determined. We have [77]

$$\mathcal{C}(\mathcal{P}_{\text{deph}}) = 1 - H_2(p) \ . \tag{87}$$

Note that this also proves $Q_2(\mathcal{P}_{\text{deph}}) = Q(\mathcal{P}_{\text{deph}})$, where the latter was derived in Ref. [121].

Consider now an erasure channel which is a non-Pauli channel described by

$$\mathcal{E}_{\text{erase}}(\rho) = (1-p)\rho + p\left|e\right\rangle\left\langle e\right|, \tag{88}$$

where p is the probability of getting an orthogonal erasure state $|e\rangle$. This channel is teleportation covariant and also distillable, therefore we may compute [77]

$$\mathcal{C}(\mathcal{E}_{\text{erase}}) = \Phi(\mathcal{E}_{\text{erase}}) = 1 - p \ . \tag{89}$$

Remark 5 *Note that only the Q_2 of the erasure channel was previously known [87], so that the novel result here is about the secret key capacity, i.e., $K(\mathcal{E}_{erase}) = P_2(\mathcal{E}_{erase}) = 1 - p$. Simultaneously with Ref. [77], an independent study of the erasure channel has been provided in Ref. [88] which computed the secret key capacity K from the squashed entanglement of this channel.*

C. Bounds for channels with asymptotic simulations

Consider now a quantum channel $\mathcal{E}$ which is described by an asymptotic simulation, with an associated sequence of resource states σ^{μ}. For any input alphabet of finite energy N and for any finite number of channel uses n, we may write the output of the adaptive protocol as $\rho_{\mathbf{ab}}^n = \lim_{\mu} \bar{\Lambda}(\sigma^{\mu\otimes n})$ according to the trace norm limit in Eq. (77). Computing the REE on the (finite-energy) output state $\rho_{\mathbf{ab}}^n$ we find [77]

$$\begin{aligned}
E_R(\rho_{\mathbf{ab}}^n) &= \inf_{\sigma_s} S(\rho_{\mathbf{ab}}^n||\sigma_s) \\
&\overset{(1)}{\leq} \inf_{\sigma_s^{\mu}} S\left[\lim_{\mu} \bar{\Lambda}(\sigma^{\mu\otimes n}) \ || \ \lim_{\mu} \sigma_s^{\mu}\right] \\
&\overset{(2)}{\leq} \inf_{\sigma_s^{\mu}} \liminf_{\mu} \ S\left[\bar{\Lambda}(\sigma^{\mu\otimes n}) \ || \ \sigma_s^{\mu}\right] \\
&\overset{(3)}{\leq} \inf_{\sigma_s^{\mu}} \liminf_{\mu} \ S\left[\bar{\Lambda}(\sigma^{\mu\otimes n}) \ || \ \bar{\Lambda}(\sigma_s^{\mu})\right] \\
&\overset{(4)}{\leq} \inf_{\sigma_s^{\mu}} \liminf_{\mu} \ S\left(\sigma^{\mu\otimes n} \ || \ \sigma_s^{\mu}\right) \\
&\overset{(5)}{=} E_R(\sigma^{\otimes n}),
\end{aligned} \tag{90}$$

where: (1) σ_s^{μ} is a sequence of separable states such that $\|\sigma_s - \sigma_s^{\mu}\| \overset{\mu}{\to} 0$ for separable σ_s; (2) we use the lower semi-continuity of the relative entropy [2]; (3) we use that $\bar{\Lambda}(\sigma_s^{\mu})$ are specific types of sequences; (4) we use the monotonicity of the relative entropy under $\bar{\Lambda}$; and (5) we use the definition of REE for asymptotic states given in Eq. (13).

By replacing in Eq. (15), we derive

$$\mathcal{C}(\mathcal{E}|N) \leq \lim_{n} n^{-1} E_R(\sigma^{\otimes n}) = E_R^{\infty}(\sigma) \leq E_R(\sigma) \ , \tag{91}$$

where we also consider the fact that the capacity is computed assuming an input alphabet with bounded energy N. Because the upper bound does no depend on N, we may extend the result to the supremum and write the final result [77, Theorem 5]

$$\mathcal{C}(\mathcal{E}) = \sup_N \mathcal{C}(\mathcal{E}|N) \leq E_R^{\infty}(\sigma) \leq E_R(\sigma) \ . \tag{92}$$

Exactly as in PLOB, the energy constraint is released at the very end of the calculations.

In particular, for a quantum channel which is teleportation covariant, we may write the simulation with $\sigma^{\mu} = \rho_{\mathcal{E}}^{\mu}$, i.e., considering a Choi sequence. Then, Eq. (92) becomes again [77, Theorem 5]

$$\mathcal{C}(\mathcal{E}) \leq E_R(\rho_{\mathcal{E}}) = \Phi(\mathcal{E}), \tag{93}$$

where the entanglement flux is defined as in Eq. (14). Note that we may simplify the upper bound by making a specific choice $\tilde{\sigma}_s^{\mu}$ for the separable sequence σ_s^{μ} in Eq. (14), so that

$$\Phi(\mathcal{E}) \leq \liminf_{\mu} \ S\left(\rho_{\mathcal{E}}^{\mu} \ || \ \tilde{\sigma}_s^{\mu}\right). \tag{94}$$

D. Formulas for Gaussian channels

A single-mode Gaussian channel is teleportation covariant and therefore admits a teleportation simulation in terms of a Choi sequence $\rho_{\mathcal{E}}^{\mu}$. We may bound the generic two-way capacity by using Eq. (94) with a suitable separable sequence $\tilde{\sigma}_s^{\mu}$. Since $\rho_{\mathcal{E}}^{\mu}$ is Gaussian, we may also choose $\tilde{\sigma}_s^{\mu}$ to be Gaussian (see PLOB on how to build this separable state following ideas in Refs. [122–124]).

The next step is to develop a formula for computing the relative entropy between two arbitrary Gaussian states. Given n modes with quadratures $\hat{\mathbf{x}} = (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^T$, consider the symplectic form

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_n, \tag{95}$$

where I_n is the $n \times n$ identity matrix. Using the Gibbs representation for Gaussian states [125], one can prove the following [77, Theorem 7]

Theorem 6 (Relative entropy for Gaussian states) *Given two arbitrary n-mode Gaussian states $\rho_1(\mathbf{x}_1, \mathbf{V}_1)$ and $\rho_2(\mathbf{x}_2, \mathbf{V}_2)$, with mean values $\mathbf{x}_i$ and CMs $\mathbf{V}_i$, their relative entropy is given by*

$$S(\rho_1||\rho_2) = -\Sigma(\mathbf{V}_1, \mathbf{V}_1) + \Sigma(\mathbf{V}_1, \mathbf{V}_2) \ , \tag{96}$$

where we have defined

$$\Sigma(\mathbf{V}_1, \mathbf{V}_2) = \frac{\ln\det\left(\mathbf{V}_2 + \frac{i\mathbf{\Omega}}{2}\right) + \mathrm{Tr}(\mathbf{V}_1\mathbf{G}_2) + \delta^T\mathbf{G}_2\delta}{2\ln 2}, \tag{97}$$

with $\delta = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{G}_2 = 2i\mathbf{\Omega}\coth^{-1}(2i\mathbf{V}_2\mathbf{\Omega})$.

Remark 7 *Note that this formula expresses the relative entropy directly in terms of the statistical moments of the Gaussian states, without the need of performing the symplectic diagonalization of the CMs. In fact, Eq. (97) enables the use of matrix functions, which are implemented in most numerical and symbolic software packages. By contrast, a full symplectic diagonalization needs to be carried out in previous formulations [126, 127]. In Refs. [126, 127], the practical problem is not the computation of the symplectic spectrum of a CM $\mathbf{V}$ (which is relatively easy) but the derivation of the symplectic matrix $\mathbf{S}$ performing the diagonalization $\mathbf{SVS}^T = \mathbf{W}$ into the diagonal Williamson form $\mathbf{W}$ [5]. For this matrix $\mathbf{S}$, we know closed formulas only in very particular cases, e.g., for specific types of two-mode Gaussian states [128] as those appearing in problems of quantum illumination [129–132] and quantum reading [133, 134].*

Using Theorem 6 for the computation of the relative entropy in Eq. (94), PLOB established the tightest known upper bounds for the two-way quantum and private capacities of all single-mode phase insensitive Gaussian channels. In fact, let us introduce the entropic function

$$h(x) := (x+1)\log_2(x+1) - x\log_2 x. \tag{98}$$

Then, we may write the following results [77].

- For a thermal-loss channel $\mathcal{E}_{\eta,\bar{n}}$ with transmissivity $\eta \in [0,1]$ and mean thermal number $\bar{n} \geq 0$, one finds

$$\mathcal{C}(\mathcal{E}_{\eta,\bar{n}}) \leq -\log_2\left[(1-\eta)\eta^{\bar{n}}\right] - h(\bar{n}), \tag{99}$$

  for $\bar{n} < \eta/(1-\eta)$, while $\mathcal{C} = 0$ otherwise.

- For a quantum amplifier $\mathcal{E}_{g,\bar{n}}$ with gain $g > 1$ and mean thermal number $\bar{n} \geq 0$, one has

$$\mathcal{C}(\mathcal{E}_{g,\bar{n}}) \leq \log_2\left(\frac{g^{\bar{n}+1}}{g-1}\right) - h(\bar{n}), \tag{100}$$

  for $\bar{n} < (g-1)^{-1}$, while $\mathcal{C} = 0$ otherwise

- For an additive-noise Gaussian channel $\mathcal{E}_{\xi}$ with additive noise $\xi \geq 0$, one writes

$$\mathcal{C}(\mathcal{E}_{\xi}) \leq \frac{\xi - 1}{\ln 2} - \log_2 \xi, \tag{101}$$

  for $\xi < 1$, while $\mathcal{C} = 0$ otherwise.

More strongly, for a bosonic lossy channel $\mathcal{E}_{\eta}$ with transmissivity η, PLOB showed that the upper bound $\Phi(\mathcal{E}_{\eta})$ coincides with the lower bound $D_1(\rho_{\mathcal{E}_{\eta}})$, the latter being already known from past computations using the reverse coherent information [90]. Therefore, a lossy channel is distillable and the two-way capacities are all equal ($D_2 = Q_2 = K = P_2$) and given by [77]

$$\mathcal{C}(\mathcal{E}_{\eta}) = -\log_2(1-\eta) \ . \tag{102}$$

In particular, the secret-key capacity K of the lossy channel gives the maximum rate achievable by point-to-point QKD protocols. At high loss $\eta \simeq 0$, one finds the optimal rate-loss scaling $K \simeq 1.44\eta$ secret bits per channel use. This result is known as repeaterless or PLOB bound, and establishes the exact benchmark that a quantum repeater must surpass in order to be effective.

This result also proves the strict separation $Q_2(\mathcal{E}_\eta) > Q(\mathcal{E}_\eta)$, where Q is the unassisted quantum capacity [119, 120]. It is then interesting to note that the capacity in Eq. (102) coincides with the maximum discord [135] that can be distributed through the lossy channel, supporting the operational interpretation of discord as a resource for quantum cryptography [136]. One can also check, using the tools in Ref. [137], that this discord corresponds to Gaussian discord [138, 139].

In conclusion, a quantum-limited amplifier $\mathcal{E}_g$ with gain $g > 1$ is also distillable, i.e., $\Phi(\mathcal{E}_g) = D_1(\rho_{\mathcal{E}_g})$. As a result, all the two-way capacities are equal and given by [77]

$$C(\mathcal{E}_g) = -\log_2\left(1 - g^{-1}\right) \ . \tag{103}$$

In particular, this also proves that $Q_2(\mathcal{E}_g)$ coincides with the unassisted quantum capacity $Q(\mathcal{E}_g)$ [140, 141].

E. Amplitude damping channel

The amplitude damping channel is a very important model of decoherence in spin chains and networks [142, 143], especially when we consider the transfer of quantum information, e.g., in a quantum chip architecture. Despite this, the inherent asymmetry of this channel makes it the hardest to study. In the qubit computational basis $\{|0\rangle, |1\rangle\}$, the action of this channel is expressed by

$$\mathcal{E}_{\rm damp}(\rho) = \textstyle\sum_{i=0,1} A_i \rho A_i^\dagger, \tag{104}$$

where p is the damping probability and

$$A_0 := |0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|\,, \ A_1 := \sqrt{p}\,|0\rangle\langle 1|\,. \tag{105}$$

One can check that $\mathcal{E}_{\rm damp}$ is not teleportation-covariant. However, it is still LOCC simulable thanks to the decomposition

$$\mathcal{E}_{\rm damp} = \mathcal{E}_{\rm CV\to DV} \circ \mathcal{E}_{\eta(p)} \circ \mathcal{E}_{\rm DV\to CV}, \tag{106}$$

where:

- $\mathcal{E}_{\rm DV\to CV}$ teleports the spin qubit into a single-rail bosonic qubit [80];
- $\mathcal{E}_{\eta(p)}$ is a lossy channel with transmissivity $\eta(p) := 1-p$;
- $\mathcal{E}_{\rm CV\to DV}$ teleports the single-rail qubit back to the original qubit.

For this reason, $\mathcal{E}_{\rm damp}$ is stretchable into the asymptotic Choi matrix of the lossy channel $\mathcal{E}_{\eta(p)}$ by means of a simulating LOCC which combines the local maps $\mathcal{E}_{\rm CV\to DV}$ and $\mathcal{E}_{\rm DV\to CV}$ with the BK protocol. In this way, PLOB showed that

$$\mathcal{C}(\mathcal{E}_{\rm damp}) \leq \min\{1, -\log_2 p\}. \tag{107}$$

Let us notice that squashed entanglement can beat this upper bound as shown in PLOB and Ref. [88]. The REE bound in Eq. (107) is very simple but performs well only in the regime of high damping ($p > 0.9$). Finally, notice that the amplitude damping proves the need of a dimension-*independent* theory for channel simulation even if we restrict ourself to DV channels.

VII. MAXIMUM TOLERABLE NOISE IN QUANTUM KEY DISTRIBUTION

In this section we provide a study which complements the one in Ref. [77, Figure 6], where we plotted the optimal key rate versus distance of several QKD protocols, in comparison with the PLOB bound. Here we study the optimal security thresholds which are achieved by setting the key rates equal to zero.

Consider a thermal-loss channel $\mathcal{E}_{\eta,\bar{n}}$ with transmissivity η and mean thermal number $\bar{n}$. From the variance parameter $\omega = \bar{n} + 1/2$, we define the so-called "excess noise" ε of the channel by setting

$$\omega = \frac{1}{2} + \frac{\eta\varepsilon}{1-\eta}, \tag{108}$$

which leads to

$$\varepsilon = \eta^{-1}(1-\eta)\bar{n}. \tag{109}$$

For any protocol, we may write an optimal rate in terms of these channel parameters, i.e., $R = R(\eta, \varepsilon)$. The security threshold is then achieved by setting $R = 0$ which provides the maximum tolerable excess noise as a function of the transmissivity, i.e., $\varepsilon = \varepsilon(\eta)$. Now the crucial question is the following: *What is the maximum excess noise that is tolerable in QKD? I.e., optimizing over all QKD protocols?*

It is easy to write an upper bound to the security threshold associated with the secret key capacity of the thermal-loss channel. In fact, from Eq. (99), we see that $K(\mathcal{E}_{\eta,\bar{n}}) = 0$, corresponds to the entanglement-breaking value $\bar{n} = \eta/(1-\eta)$. By replacing in Eq. (109), we find that the maximum tolerable excess noise is upper-bounded by $\varepsilon_{\rm UB} = 1$ for any value of the transmissivity η. For the lower bound, we may consider the maximum key rate achievable by using the reverse coherent information [89], which is equal to [90]

$$R_{\rm LB} = -\log_2(1-\eta) - s\left(\omega\right) \ , \tag{110}$$

where

$$s(x) := \left(x+\frac{1}{2}\right)\log_2\left(x+\frac{1}{2}\right)-\left(x-\frac{1}{2}\right)\log_2\left(x-\frac{1}{2}\right). \tag{111}$$

Using Eq. (108) in Eq. (110), we may numerically compute $R_{\rm LB}(\eta,\varepsilon)=0$ and find the lower bound $\varepsilon_{\rm LB}$. As we can see from Fig. 4, there is a huge gap between $\varepsilon_{\rm LB}$ and $\varepsilon_{\rm UB}$.

Can we reduce this gap? From Refs. [90, 144], we know that the use of trusted noise at the receiver station may improve the performance of a one-way CV-QKD protocol performed in reverse reconciliation. Such an idea has been also explored in a recent work [145]. Both Refs. [90, 145] show that rate associated with the reverse coherent information can be beaten by a CV-QKD protocol based on trusted noise when non-zero excess noise is present in the channel. Here we show an equivalent CV-QKD protocol which outperforms the security threshold $\varepsilon_{\rm LB}$ associated with the reverse coherent information.

The protocol consists of Alice preparing Gaussian-modulated squeezed states, e.g., by homodyning one part of TMSV states in her hands. It is easy to see that, for a TMSV state with parameter μ, the local homodyne in $\hat{q}$ on one mode projects the other mode into a displaced q-squeezed state with variance μ^{-1}. Alice randomly switches between q- and p-squeezed states following Ref. [146]. The squeezed states are sent through the thermal-loss channel whose output is measured by Bob. Before detection Bob applies an additive noise Gaussian channel $\mathcal{E}_\xi$, so that the output quadratures are transformed according to $\hat{\mathbf{x}}\to\hat{\mathbf{x}}+\zeta$, where ζ is a classical Gaussian variable with variance $\xi\geq 0$. Then, he performs homodyne detection, switching between the measurement of the $\hat{q}$ and $\hat{p}$ quadrature.

After the parties reconcile their bases, perform error correction and privacy amplification, they will share an asymptotic key rate $R=(I_{\rm AB}-\chi_{\rm BE})/2$, where $I_{\rm AB}$ is Alice and Bob's mutual information (ignoring the basis reconciliation), and $\chi_{\rm BE}$ is the corresponding Eve's Holevo information on Bob's outcomes. The factor $1/2$ accounts for the basis reconciliation. After some algebra, we compute

$$I_{\rm AB}=\frac{1}{2}\log_2\frac{\eta\mu+(1-\eta)\omega+\xi}{\eta\mu^{-1}+(1-\eta)\omega+\xi} \tag{112}$$

$$\overset{\mu}{\to}\frac{1}{2}\log_2\frac{\eta\mu}{(1-\eta)\omega+\xi}\,, \tag{113}$$

and, for large Gaussian modulation ($\mu\gg 1/2$), we get

$$\chi_{\rm BE}=\frac{1}{2}\log_2\frac{(1-\eta)\eta\mu}{\omega+\xi(1-\eta)}+s(\omega)-s\left(\bar{\nu}\right), \tag{114}$$

where the symplectic eigenvalue $\bar{\nu}$ is given by

$$\bar{\nu}=\sqrt{\frac{\omega\left[1+4\omega\xi(1-\eta)\right]}{4[\omega+\xi(1-\eta)]}}. \tag{115}$$

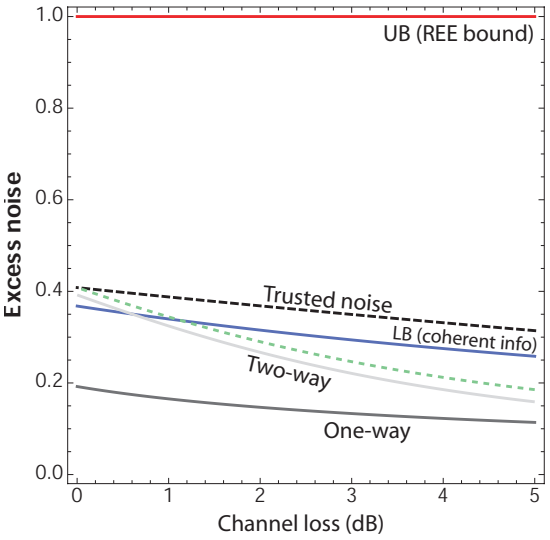

FIG. 4: Security thresholds in terms of maximum tolerable excess noise ε as a function of the loss in the channel (dB). Protocols are secure below their corresponding thresholds. The red line is the upper bound $\varepsilon_{\rm UB}=1$ coming from the entanglement flux (REE bound) of the thermal-loss channel [77]. The blue line is the lower bound $\varepsilon_{\rm LB}$ computed from the reverse coherent information [89, 90]. The black dashed line is the security threshold which is obtained from the key-rate of Eq. (116), for the one-way trusted noise protocol described in the text. This is an improved lower bound, but still far to close the gap with the upper bound. Finally, we also show the security thresholds corresponding to the one-way no-switching protocol [148] and the two-way protocols [17] with coherent states (solid line) and largely-thermal states [149] (green dashed line).

Therefore, from Eqs. (112) and (114), we compute the asymptotic and high-modulation rate

$$R=\frac{1}{4}\log_2\frac{\omega+\xi(1-\eta)}{(1-\eta)\left[(1-\eta)\omega+\xi\right]}+\frac{s\left(\bar{\nu}\right)-s(\omega)}{2}. \tag{116}$$

The previous rate is a function of the main parameters, i.e., $R=R(\eta,\varepsilon,\xi)$. Setting $R=0$, we derive the threshold $\varepsilon=\varepsilon(\eta,\xi)$ which is maximal in the limit of large trusted noise $\xi\gg 0$. The limit $\varepsilon_\infty:=\lim_\xi\varepsilon(\eta,\xi)$ beats $\varepsilon_{\rm LB}$ as shown in Fig. 4, therefore establishing the best-known lower bound. Unfortunately, this is still far from $\varepsilon_{\rm UB}$ so that it remains an open problem to establish the maximum value of tolerable excess noise in QKD.

Remark 8 *Note that the previous protocol can be implemented in a coherent fashion, where Alice distributes TMSV states whose her kept modes and Bob's output modes (from the channel) are stored in quantum memories. The parties may then agree to perform a joint random sequence of q- and p- homodyne detections, so that no basis reconciliation is needed. In this way they can reach a key rate which is the double of the one in Eq. (116). This coherent protocol is equivalent to the*

one described in Ref. [145], where the parties use quantum memories and the trusted noise is created by a beam splitter of transmissivity η_d mixing the output with a thermal mode with variance γ. One can check that the rates are equal by setting

$$\xi = \frac{1-\eta_d}{\eta_d}\gamma. \tag{117}$$

Remark 9 *As one can check, if we set Bob's trusted noise to zero ($\xi = 0$), then the rate in Eq. (116) becomes equal to half the rate $R_{\rm LB}$ in Eq. (110). Assuming a coherent implementation as discussed in the previous remark, then the rate becomes equal to the rate $R_{\rm LB}$ in Eq. (110). Taking the limit for low loss $\eta \simeq 0$ and low noise $\omega \simeq 1$, i.e., low thermal photon number $\bar{n} \simeq 0$, one may derive the expansion*

$$R_{\rm LB} \simeq (\eta - \bar{n}) \log_2 e + \bar{n} \log_2 \bar{n}, \tag{118}$$

which is the rate studied in Ref. [147]. Note that Ref. [147] also investigated the use of trusted noise at Bob's side in order to increase the security threshold of the basic squeezed-state protocol [146].

For comparison, in Fig. 4, we also show the optimal threshold of the one-way protocol based on Gaussian-modulated coherent states and heterodyne detection [148], whose ideal reverse reconciliation rate is given by the formula

$$R = \log_2 \frac{2}{e} \frac{\eta}{(1-\eta)\left[\eta + 2\omega(1-\eta)+1\right]} + s\left[\frac{1+2\omega(1-\eta)}{2\eta}\right] - s(\omega). \tag{119}$$

Then, in Fig. 4, we also show the optimal threshold of the two-way protocol [17] which is based on the Gaussian modulation of thermal states (with variance V_0) and homodyne detection at the output. Its ideal reverse reconciliation rate is given by [149]

$$R(V_0) = \frac{1}{2}\log_2 \frac{\eta^2 V_0 + \omega + \eta^3(\omega - V_0)}{(1-\eta)\left[(1-\eta^2)\omega + \eta V_0\right]} + s(\bar{\nu}_2) - s(\omega), \tag{120}$$

where

$$\bar{\nu}_2 = \sqrt{\frac{\omega\left[1 + 4\eta^2 V_0 \omega + \eta^3(1 - 4\omega V_0)\right]}{4\left[\eta^2 V_0 + \omega + \eta^3(\omega - V_0)\right]}}. \tag{121}$$

In particular, we consider the limit of coherent states ($V_0 = 1/2$) and that of largely-thermal states ($V_0 \gg 1$).

VIII. GENERAL METHODOLOGY

Here we discuss how the methodology devised in PLOB for point-to-point protocols (and extended in Ref. [66] to end-to-end protocols) can be applied to any entanglement measure E with suitable properties. For simplicity, here we start considering DV systems. Then we extend the arguments to CV systems via truncation.

A. Main ingredients

Assume that E is an entanglement measure that satisfies the following conditions:

(1) Normalization. For a target state ϕ^n encoding nR_n bits (e.g., ebits or private bits), we have

$$E(\phi^n) \geq nR_n. \tag{122}$$

(2) Continuity. For d-dimensional ρ and σ such that $\|\rho - \sigma\| \leq \varepsilon$, we have the Fannes-type inequality

$$|E(\rho) - E(\sigma)| \leq g(\varepsilon)\log_2 d + h(\varepsilon), \tag{123}$$

where g, h are regular functions going to zero in ε.

(3) Monotonicity. For any trace-preserving LOCC $\bar{\Lambda}$, we may write the data processing inequality

$$E\left[\bar{\Lambda}(\rho)\right] \leq E(\rho)\ . \tag{124}$$

(4) Subadditivity. For any ρ and σ, we may write

$$E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)\ , \tag{125}$$

so that the regularization satisfies

$$E^{\infty}(\rho) := \lim_n n^{-1} E(\rho^{\otimes n}) \leq E(\rho)\ . \tag{126}$$

It is clear that these properties are satisfied by the REE $E_{\rm R}$ [84] with the specific choice

$$g(\varepsilon) = 4\varepsilon,\ h(\varepsilon) = 2H_2(\varepsilon), \tag{127}$$

where H_2 is the binary Shannon entropy. They are also satisfied by the squashed entanglement $E_{\rm sq}$ [150] with

$$g(\varepsilon) = 16\sqrt{\varepsilon},\ h(\varepsilon) = 2H_2(2\sqrt{\varepsilon}). \tag{128}$$

Consider now an arbitrary adaptive protocol $\mathcal{P}$ between two users, Alice and Bob. This protocol may be point-to-point over a quantum channel [77] or an end-to-end protocol along a repeater chain or within a quantum network [66]. After n uses, assume that Alice and Bob's output is ε close to a d-dimensional target state ϕ^n. By applying Eqs. (122) and (123), we derive

$$R_n \leq \frac{E(\rho^n_{\mathbf{ab}}) + g(\varepsilon)\log_2 d + h(\varepsilon)}{n}. \tag{129}$$

Now assume the following property for the target state, which is certainly true for a maximally entangled state and also for a private state.

(5) Exponential size. The effective total dimension of the target state grows at most exponentially

$$d \leq 2^{\alpha_n n}, \tag{130}$$

where $\liminf_n \alpha_n = \alpha$ for constant α. In particular, for a maximally-entangled state $\alpha_n = \alpha = 1$.

Combining Eq. (129) and (130), and taking the limit for large n we derive

$$\lim_n R_n \leq \lim_n \frac{E(\rho^n_{\mathbf{ab}})}{n} + \lim_n \inf g(\varepsilon)\alpha_n + \lim_n \frac{h(\varepsilon)}{n} \quad (131)$$

$$= \lim_n n^{-1} E(\rho^n_{\mathbf{ab}}) + g(\varepsilon)\alpha \ . \quad (132)$$

Then, by taking the limit for small ε, we derive the weak converse bound

$$\lim_n R_n \leq \lim_n n^{-1} E(\rho^n_{\mathbf{ab}}) \ . \quad (133)$$

Finally, consider the optimization over all protocols $\mathcal{P}$ (more precisely, over the equivalent class that satisfies the property in Eq. (130) on the exponential size). This leads to

$$\mathcal{C} := \sup_{\mathcal{P}} \lim_n R_n \leq \sup_{\mathcal{P}} \lim_n n^{-1} E(\rho^n_{\mathbf{ab}}) \ , \quad (134)$$

where $\mathcal{C}$ may be a two-way assisted capacity over a quantum channel, or an end-to-end capacity of a repeater chain or a quantum network.

The next step is to simplify the upper bound in Eq. (134) to a single-letter quantity via a suitable decomposition of the output state. For simplicity, restrict the analysis to point-to-point protocols over a quantum channel (generalization to repeaters and networks goes along the lines of Ref. [66] and requires the introduction of additional tools from network information theory). We know that we have the following powerful tool [77].

(6) Teleportation stretching. Simulating a channel $\mathcal{E}$ with a resource state σ, we may re-organize any point-to-point (generally-adaptive) protocol in a block form so as to decompose its output as

$$\rho^n_{\mathbf{ab}} = \bar{\Lambda}(\sigma^{\otimes n}), \quad (135)$$

for a trace-preserving LOCC $\bar{\Lambda}$.

By replacing Eq. (134) into (135), and exploiting the properties of monotonicity and subadditivity in Eqs. (124) and (125), we achieve and generalize the main insight of PLOB, i.e., the simplification

$$\mathcal{C}(\mathcal{E}) \leq E^{\infty}(\sigma^{\otimes n}) \leq E(\sigma), \quad (136)$$

where $\sigma = \rho_{\mathcal{E}}$ if $\mathcal{E}$ is teleportation covariant.

B. Channel approximations

It is clear that the technique can be extended to bound the capacity of a quantum channel $\mathcal{E}$ which is approximated by another channel $\tilde{\mathcal{E}}$ whose simulation is known and based on some resource state $\tilde{\sigma}$. Consider two DV channels $\mathcal{E}$ and $\tilde{\mathcal{E}}$ with diamond distance

$$||\mathcal{E} - \tilde{\mathcal{E}}||_{\diamond} \leq \delta \ . \quad (137)$$

For the same adaptive protocol $\mathcal{P} = \{\Lambda_0, \ldots, \Lambda_n\}$, consider the output state generated by n transmissions over these two channels, i.e.,

$$\rho^n_{\mathbf{ab}} := \Lambda_n \circ \mathcal{E} \circ \Lambda_{n-1} \cdots \circ \Lambda_1 \circ \mathcal{E}(\rho^0_{\mathbf{ab}}), \quad (138)$$

$$\tilde{\rho}^n_{\mathbf{ab}} := \Lambda_n \circ \tilde{\mathcal{E}} \circ \Lambda_{n-1} \cdots \circ \Lambda_1 \circ \tilde{\mathcal{E}}(\rho^0_{\mathbf{ab}}), \quad (139)$$

where we also have $\tilde{\rho}^n_{\mathbf{ab}} = \bar{\Lambda}(\tilde{\sigma}^{\otimes n})$ by applying teleportation stretching to the simulable channel.

Using our previous "peeling argument", we may evolve Eq. (137) into an error on the output state

$$\|\rho^n_{\mathbf{ab}} - \tilde{\rho}^n_{\mathbf{ab}}\| \leq n\delta \ . \quad (140)$$

Then, assume that $\mathcal{P}$ is optimized for $\mathcal{E}$ so that $\rho^n_{\mathbf{ab}}$ approximates a target state ϕ^n with nR_n bits, i.e., $\|\rho^n_{\mathbf{ab}} - \phi^n\| \leq \varepsilon$. Using the triangle inequality, we write

$$\|\tilde{\rho}^n_{\mathbf{ab}} - \phi^n\| \leq \varepsilon' := \varepsilon + n\delta \ . \quad (141)$$

For small enough ε', this leads to

$$R_n \leq \frac{E(\tilde{\rho}^n_{\mathbf{ab}}) + g(\varepsilon')\log_2 d + h(\varepsilon')}{n}. \quad (142)$$

Using stretching and Eq. (130), we have

$$R_n \leq E(\tilde{\sigma}) + \alpha_n g(\varepsilon') + \frac{h(\varepsilon')}{n}. \quad (143)$$

Note that the upper bound is the same for any $\mathcal{P}$, so that it is also true if we consider the supremum over $\mathcal{P}$. This is therefore an upper bound for the n-use two-way capacity of the channel $\mathcal{E}$. In other words, for n uses and epsilon security ε, we may write the secret key capacity

$$K(\mathcal{E}, n, \varepsilon) \leq E(\tilde{\sigma}) + \alpha_n g(\varepsilon + n\delta) + \frac{h(\varepsilon + n\delta)}{n} \ . \quad (144)$$

It is clear that, in order to be valid, we need to have $n\delta$ small enough. This is not a problem in the case of one-shot capacity ($n = 1$), for which we just have

$$K^{(1)}(\mathcal{E}, \varepsilon) := K(\mathcal{E}, 1, \varepsilon) \leq E(\tilde{\sigma}) + \alpha_1 g(\varepsilon + \delta) + h(\varepsilon + \delta) \ . \quad (145)$$

The problem occurs for large n, where $n\delta$ may explode.

C. Sequences of channels

The previous problem is certainly solved in the case of a sequence of simulable channels converging in diamond norm, i.e., for $\tilde{\mathcal{E}}^{\mu}$ such that

$$\delta_{\mu} := ||\mathcal{E} - \tilde{\mathcal{E}}^{\mu}||_{\diamond} \overset{\mu}{\to} 0 \ . \quad (146)$$

In such a case we may simultaneously write

$$\text{Simulation error:} \quad \|\rho^n_{\mathbf{ab}} - \tilde{\rho}^{\mu,n}_{\mathbf{ab}}\| \leq n\delta_{\mu} \overset{\mu}{\to} 0, \quad (147)$$

$$\text{Epsilon closeness:} \quad \|\rho^n_{\mathbf{ab}} - \phi^n\| \leq \varepsilon, \quad (148)$$

$$\text{Stretching:} \quad \tilde{\rho}^{\mu,n}_{\mathbf{ab}} = \bar{\Lambda}(\tilde{\sigma}^{\mu \otimes n}). \quad (149)$$

Using the triangle inequality, we therefore have

$$\left\|\bar{\Lambda}(\tilde{\sigma}^{\mu\otimes n})-\phi^{n}\right\|\leq\varepsilon_{\mu}:=\varepsilon+n\delta_{\mu}. \tag{150}$$

By applying Eqs. (122)-(125) and Eq. (130), we get

$$R_{n}\leq E(\tilde{\sigma}^{\mu})+\alpha_{n}g(\varepsilon_{\mu})+\frac{h(\varepsilon_{\mu})}{n}. \tag{151}$$

Taking the limit of large μ, this becomes

$$R_{n}\leq\lim_{\mu}E(\tilde{\sigma}^{\mu})+\alpha_{n}g(\varepsilon)+\frac{h(\varepsilon)}{n}\ . \tag{152}$$

Then, for large n and small ε, we find the weak converse

$$\lim_{n}R_{n}\leq\lim_{\mu}E(\tilde{\sigma}^{\mu})\ . \tag{153}$$

By optimizing over $\mathcal{P}$, we get

$$K(\mathcal{E})\leq\lim_{\mu}E(\tilde{\sigma}^{\mu})\ . \tag{154}$$

D. Infinite dimension

The previous approach with sequences of channels is particularly useful for CV systems. As we know from PLOB, we need to use a truncation argument which is then released at the very end. Let us assume Alice and Bob use a trace-preserving truncation LOCC $\mathbb{T}_{d}$ on their output state $\rho_{\mathbf{ab}}^{n,d}=\mathbb{T}_{d}(\rho_{\mathbf{ab}}^{n})$. See Ref. [77, Supplementary Note 1] on how to build this local CV-DV mapping. Also assume that the input alphabet is energy-constrained, so that we have $\mathcal{D}_{N}$ with bounded energy N. This latter condition may also be realized by applying $\mathbb{T}_{d}$ before each transmission. In this case, we will have an energy constraint depending on the truncated dimension, i.e., $N=N(d)$.

Let us consider the energy-constrained diamond distance between the original channel and the sequence of simulable channels $\tilde{\mathcal{E}}^{\mu}$. For any finite N, assume that

$$\delta_{\mu}^{N}:=||\mathcal{E}-\tilde{\mathcal{E}}^{\mu}||_{\diamond N}\overset{\mu}{\to}0\ . \tag{155}$$

For the truncated output $\rho_{\mathbf{ab}}^{n,d}$ (approximating a target state $\phi^{n,d}$ with $nR_{n,d}$ bits) and its simulation $\tilde{\rho}_{\mathbf{ab}}^{\mu,n,d}$ (obtained by replacing $\mathcal{E}$ with $\tilde{\mathcal{E}}^{\mu}$ in the protocol), we may write the following (for any d and associated N)

$$\text{Simulation error:}\quad\left\|\rho_{\mathbf{ab}}^{n,d}-\tilde{\rho}_{\mathbf{ab}}^{\mu,n,d}\right\|\leq n\delta_{\mu}^{N}\overset{\mu}{\to}0, \tag{156}$$

$$\text{Epsilon closeness:}\quad\left\|\rho_{\mathbf{ab}}^{n,d}-\phi^{n,d}\right\|\leq\varepsilon, \tag{157}$$

$$\text{Stretching:}\quad\tilde{\rho}_{\mathbf{ab}}^{\mu,n,d}=\bar{\Lambda}_{d}(\tilde{\sigma}^{\mu\otimes n}). \tag{158}$$

Using the triangle inequality, we therefore have

$$\left\|\bar{\Lambda}_{d}(\tilde{\sigma}^{\mu\otimes n})-\phi^{n,d}\right\|\leq\varepsilon_{\mu}:=\varepsilon+n\delta_{\mu}. \tag{159}$$

Using Eq. (130) and previous reasonings, we get

$$R_{n,d}\leq E(\tilde{\sigma}^{\mu})+\alpha_{n}g(\varepsilon_{\mu})+\frac{h(\varepsilon_{\mu})}{n}. \tag{160}$$

Taking the limit of large μ, this becomes

$$R_{n,d}\leq\liminf_{\mu}E(\tilde{\sigma}^{\mu})+\alpha_{n}g(\varepsilon)+\frac{h(\varepsilon)}{n}\ , \tag{161}$$

where we use the inferior limit to account for the fact that $\tilde{\sigma}^{\mu}$ may be an unbounded sequence of states.

Then, for large n and small ε, we find

$$\lim_{n}R_{n,d}\leq\liminf_{\mu}E(\tilde{\sigma}^{\mu})\ . \tag{162}$$

By optimizing over the protocols $\mathcal{P}$ (i.e., over the equivalent exponential-size class of $\mathcal{P}$), we then get

$$K(\mathcal{E}|N):=\sup_{\mathcal{P}}\lim_{n}R_{n,d}\leq\liminf_{\mu}E(\tilde{\sigma}^{\mu})\ . \tag{163}$$

It is clear that the upper bound does not depend on the constraint N, so that this constraint (and the truncation) can be relaxed. In other words, we have

$$K(\mathcal{E}):=\sup_{N}K(\mathcal{E}|N)\leq\liminf_{\mu}E(\tilde{\sigma}^{\mu})\ . \tag{164}$$

Note that this result holds for an entanglement measure E with the desired properties above, such as the squashed entanglement or the REE. If we specify the result to the REE, then this procedure is an alternate proof of the one given in Sec. VI C. Note that, setting $E=E_{R}$, Eq. (164) becomes

$$\begin{aligned}K(\mathcal{E})&\leq\liminf_{\mu}E_{R}(\tilde{\sigma}^{\mu})=\liminf_{\mu}\inf_{\sigma_{s}}S(\tilde{\sigma}^{\mu}||\sigma_{s})\\&=\inf_{\sigma_{s}^{\mu}}\liminf_{\mu}S(\tilde{\sigma}^{\mu}||\sigma_{s}^{\mu}):=E_{R}(\tilde{\sigma}),\end{aligned} \tag{165}$$

where we use the definition in Eq. (13) with $\tilde{\sigma}:=\lim_{\mu}\tilde{\sigma}^{\mu}$.

IX. LITERATURE ON CHANNEL SIMULATION AND PROTOCOL REDUCTION

Let us here discuss the precursory ideas that were in the literature before the full generalization devised in PLOB. Besides this section, one may also read the Supplementary Notes 8 and 9 in Ref. [77]. A summary of the following discussion is given in Table I, where we make a direct comparison between PLOB and previous approaches and methodologies.

The first insight was introduced in 1996 by BDSW [69]. This was based on the standard teleportation protocol for DV systems and allowed these authors to simulate "generalized depolarizing channels", later known as Pauli channels [1]. The restriction of this original technique to Pauli channels was first shown in Ref. [71] and later re-examined in Ref. [118]. BDSW first recognized that a Pauli channel $\mathcal{P}$ can be simulated by teleporting over its

	BDSW, HHH99 [69, 70]	MH12,W12 [74, 75]	LM15 [76]	GC02,NFC09 [72, 73]	PLOB [77]
Simulated channels	Pauli channels $\mathcal{P}$ [69]. Sub-class of DV channels [70]	All DV channels but probabilistically. If tele-covariant, then deterministically	Tele-covariant DV channels	Gaussian channels	Any channel (DV & CV) LOCC simulable by resource state σ
Amplitude damping	Not simulable	Probabilistically simulable	Not simulable	Not simulable	Simulable
Criterion	N/A	Tele-covariance (for DV)	Tele-covariance (for DV)	N/A	Tele-covariance (for DV & CV)
Simulation error	N/A	Probability of teleportation	N/A	Not controlled	Yes. Controlled for CV channels
Protocol task	QC	QC	QC	QC	Any task (QC, ED, QKD)
Type of reduction	QC $\to$ ED Reduction to ent. distillation	QC $\to$ ED Reduction to ent. distillation	QC $\to$ PPT Reduction to PPT distillation	QC $\to$ ED with Gaussian LOCCs [73]	Adaptive protocol $\to$ block protocol. Task-preserving (any dim, DV/CV)
Type of bound	$Q_1(\mathcal{P}) \leq D_1(\rho_{\mathcal{P}})$ (extended to 2-way CCs) [69, Sec. V]	$Q_2(\mathcal{E}) \leq d^2 D_2(\rho_{\mathcal{E}})$ in finite dim d [74, Theorem 14]	Bounds to DV quantum capacities restricted to PPT-preserving codes	N/A	$Q_2(\mathcal{E}) \leq K(\mathcal{E}) \leq E_R(\sigma)$ $\sigma = \rho_{\mathcal{E}}$ if tele-covariant (any dim, DV/CV)

TABLE I: Comparison between PLOB and previous literature on channel simulation and protocol reduction.

Choi matrix $\rho_{\mathcal{P}}$. See Ref. [69, Section V]. This specific case was later re-considered as a property of mutual *reproducibility* between states and channels [70]. Let us remark that Ref. [70] also explored the possibility to extend channel simulation beyond teleportation by using more general LOCCs. In principle, this allowed them to simulate more channels but still a sub-class of DV channels, due to the specific use of finite-dimensional and non-asymptotic LOCCs (e.g., see Eq. (11) in Ref. [70]).

Similar simulation ideas, but in the setting of quantum computing, were considered in Ref. [151] (see also the more recent Ref. [152]) where a unitary U is stored in its Choi matrix ρ_U. This unitary is then applied to some input state ρ by teleporting such input over ρ_U. This is also known as "quantum gate teleportation". It shows that teleportation is a primitive for quantum computation. Likewise, teleportation can be expressed in terms of primitive quantum computational operations [153]. Quantum gate teleportation is also at the heart of linear-optical quantum computing based on linear optics and probabilistic gates [154] (see also Ref. [80] for a general overview on these applications of teleportation).

Using the teleportation simulation of a Pauli channel $\mathcal{P}$, BDSW first showed how to transform a quantum communication (QC) protocol into an entanglement distillation (ED) protocol over its Choi matrix $\rho_{\mathcal{P}}$. We call this technique "reduction to entanglement distillation". This allowed them to prove the following bound on the one-way quantum capacity

$$Q_1(\mathcal{P}) \leq D_1(\rho_{\mathcal{P}}), \tag{166}$$

where D_1 is the one-way distillability. This result was implicitly extended to two-way CC, so that they also wrote

$$Q_2(\mathcal{P}) \leq D_2(\rho_{\mathcal{P}}). \tag{167}$$

Reduction to entanglement distillation (QC$\to$ED) was originally formulated in an asymptotic fashion, i.e., for large n, which is sufficient to prove Eqs. (166) and (167).

More recently, in 2012, Refs. [74, 75] considered the *probabilistic* simulation of an arbitrary DV quantum channel $\mathcal{E}$ via teleportation. This is done by assuming that Alice and Bob only picks the Bell outcome corresponding to the identity operator, which occurs with

probability $p = d^{-2}$, where d is the dimension of the input system. This version can also be traced back to the probabilistic approach of Ref. [154]. In the presence of a probability of success associated with the simulation, one can derive upper bounds similar to those of BDSW but with a suitable pre-factor. In fact, adopting the probabilistic simulation and BDSW's reduction to entanglement distillation (QC→ED), Ref. [74] showed

$$Q_2(\mathcal{E}) \leq p^{-1} D_2(\rho_{\mathcal{E}}), \tag{168}$$

for an arbitrary DV channel $\mathcal{E}$. Let us remark that Refs. [74, 75] also identified the property of teleportation covariance for DV channels, realizing that these channels can be simulated deterministically, i.e., with an associated success probability $p = 1$.

In 2015, Ref. [76] too identified the criterion of teleportation covariance of DV channels and considered the (deterministic) simulation of such channels over their Choi matrices. In particular, Ref. [76] assumed the possibility of more general teleportation protocols as those introduced in Ref. [155]. Because these simulations are non-asymptotic, the class of DV channels is limited and, for instance, it cannot include the amplitude damping channel. Ref. [76] adopted a variation of the BDSW argument to simplify quantum communication. In fact, they showed how to simplify positive-partial transpose (PPT) preserving codes over a teleportation covariant channel into PPT-distillation over copies of its Choi matrix. Thanks to this "reduction to PPT distillation", they were able to write one-shot upper bounds for PPT-preserving code quantum capacities.

In the framework of CV systems, Ref. [72] first studied the simulation of single-mode Gaussian channels by using the BK protocol. Due to the nature of the topics studied in that paper (which is about the impossibility to distill entanglement from Gaussian entangled states with Gaussian LOCCs), no control of the simulation error was considered. The same approach was later followed by Ref. [73]. The latter used the channel simulation to reduce a Gaussian quantum error correcting code into Gaussian entanglement distillation.

Within this general context, PLOB introduced the most general type of channel simulation in a quantum communication scenario, where an LOCC and a resource state are used to simulate an arbitrary quantum channel at any dimension (finite or infinite). See Eqs. (25)-(27). PLOB also established teleportation covariance as a criterion to identify Choi-stretchable (i.e., teleportation-simulable) channels at any dimension. In particular, PLOB extended the technique by developing a rigorous theory of asymptotic channel simulation, which is crucial not only for bosonic channels but also for the deterministic asymptotic simulation of DV channels, such as the amplitude damping channel.

Using channel simulation, PLOB showed how to simplify an arbitrary adaptive protocol implemented over an arbitrary channel at any dimension, finite or infinite (teleportation stretching). Differently from previous approaches (which were about reduction to entanglement distillation), teleportation stretching works by preserving the original communication task. This means that an adaptive protocol of quantum communication (QC), entanglement distribution (ED) or quantum key distribution (QKD), is reduced to a corresponding block protocol with exactly the same original task (QC, ED, or QKD). In particular, the output state is decomposed in terms of a tensor product of resource states as in Eqs. (71) and (77).

The adaptive-to-block reduction of a private communication protocol has been first introduced in PLOB. Most importantly, PLOB has shown how to combine this reduction with the properties of an entanglement measure as the REE. The entire recipe of "REE plus teleportation stretching" has led to the determination of the tightest known upper bound for the secret key capacity (and the other two-way assisted capacities) of a quantum channel at any dimension. See Eqs. (79) and (92).

These techniques developed by PLOB were picked up and exploited in a series of follow-up papers, including WTB [92]. More recently, Ref. [114] introduced a simulation of Gaussian channels based on finite-energy resource states. This was promptly combined with the techniques of PLOB in Ref. [116] to derive sub-optimal approximations of previously-established weak converse bounds for private communication. Finally note that the non-local simulations [156–158] based on deterministic versions of the programmable quantum gate array [159] are clearly not suitable for quantum and private communication where Alice and Bob can only implement LOCCs.

X. STRONG CONVERSE RATES

A. Preliminary comments

At the end of February 2016, four months after the first version of PLOB appeared on the arXiv, the follow-up paper WTB [92] also appeared. An explicit timeline of the contributions is provided in Table II for the sake of clarity. As we can see, the first version of PLOB [77] appeared in October 2015. The first arXiv version of PLOB already contained the most important result for the pure-loss channel (PLOB bound). Full details of the methodology were included in the second arXiv version in December 2015 [100]. All the other two-way capacities and bounds were collected in a twin paper [160] which also appeared in December 2015 and was later merged in the published version of PLOB [77]. In a few words, the main results were all proven in 2015, well before the appearance of WTB. Subsequent arXiv versions of PLOB only added refinements and minor clarifications.

Using the methodology devised in PLOB, WTB studied how the previously-established weak converse bounds for teleportation-covariant channels are also strong converse bounds. In private communication, a weak converse bound means that *perfect* secret keys cannot be

Date:	Manuscripts on the arXiv:	Main contents:
29 Oct 2015	First version of PLOB [77]	Introduces the secret-key capacity of the lossy channel (PLOB bound) $-\log_2(1-\eta)$.
8 Dec 2015	Second version of PLOB [100]	Includes the general methodology: (i) the REE weak converse bound and (ii) its reduction by teleportation stretching to single letter. PLOB bound extended to the thermal-loss channel.
15 Dec 2015	First version of Ref. [160] (merged in published PLOB)	Extends the results to all teleportation-covariant channels, including: Pauli, erasure channels, and bosonic Gaussian channels.
5 Jan 2016	Third version of PLOB and first version of Ref. [66]	Ref. [66] extends methods and results of PLOB to repeater-assisted quantum communications and arbitrary quantum networks.
29 Feb 2016	First version of WTB [92]	Use methods of PLOB to study the strong converse property of the bounds established in PLOB for teleportation-covariant channels.

TABLE II: Timeline of the main results established in the early arXiv versions of PLOB, before the appearance of the follow-up analysis by WTB on the arXiv.

established at rates above the bound. A strong converse bound is a refinement according to which even *imperfect* secret keys (ε-secure with $\varepsilon > 0$) cannot be shared at rates above the bound for many uses. Let us clarify some important points about this paper besides discussing and fixing its technical error.

Even though WTB does not adopt the terminology introduced by PLOB (teleportation stretching, stretchable channels etc.), one can easily check that WTB exploits exactly the methodology previously introduced by PLOB. In fact, WTB combines the following ingredients

- A notion of channel's REE to bound key generation
- Teleportation stretching to simplify adaptive protocols for private communication.

In a few words, WTB adopts the entire reduction idea of PLOB, which is based on using channel's REE on top of teleportation stretching. This is what allows them to write single-letter upper bounds.

To be more precise, WTB first defines "classical pre- and post-processing (CPPP) protocols". These are *non*-adaptive protocols where the remote parties are limited to a single rounds of initial and final LOCCs. In this context, they derive strong converse rates for CPPP-assisted private communication (see Ref. [92, Theorem 13]). To generalize the approach and include adaptive protocols with unlimited two-way CCs (over teleportation-covariant channels), they then employ channel's REE and teleportation stretching. This allows them to write their Theorems 12 and 19, which are the strong converse versions of Ref. [77, Theorem 5] in PLOB.

Indeed, for a teleportation-covariant channel $\mathcal{E}$, WTB wrote the strong converse bound [92, Theorem 19]

$$K(\mathcal{E}) \leq \Phi(\mathcal{E}) + \sqrt{\frac{V(\mathcal{E})}{n}}\varphi^{-1}(\varepsilon) + \mathcal{O}\left(\frac{\log_2 n}{n}\right), \quad (169)$$

for $n \geq 1$ channel uses and security parameter $\varepsilon \in (0,1)$, where $\Phi(\mathcal{E}) = E_R(\rho_{\mathcal{E}})$ is the weak converse bound established in PLOB, and

$$\varphi(a) = \int_{-\infty}^{a} dx\, e^{-x^2/2}/\sqrt{2\pi}. \quad (170)$$

The entropic variance $V(\mathcal{E})$ in Eq. (169) is defined as

$$V(\mathcal{E}) = \begin{cases} \sup_{\sigma_s} V(\rho_{\mathcal{E}}||\sigma_s), & \text{for } 2\varepsilon < 1, \\ \inf_{\sigma_s} V(\rho_{\mathcal{E}}||\sigma_s), & \text{for } 2\varepsilon \geq 1, \end{cases} \quad (171)$$

where $V(\rho||\sigma) = \mathrm{Tr}\left\{\rho[\log_2\rho - \log_2\sigma - S(\rho||\sigma)]^2\right\}$, and the supremum/infimum are taken over the set of separable states σ_s that achieve the minimum in $E_R(\rho_{\mathcal{E}})$.

Using a Chebyshev-like bound, one may write the strong converse bound also as [92]

$$K(\mathcal{E}) \leq \Phi(\mathcal{E}) + \sqrt{\frac{V(\mathcal{E})}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n}, \quad (172)$$

where

$$C(\varepsilon) := \log_2 6 + 2\log_2\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \quad (173)$$

As a consequence, the weak-converse bounds for dephasing, erasure and other DV channels established in PLOB

would also be strong converse rates, according to Propositions 22 and 23 stated in WTB [92].

Finally, WTB [92] attempts to generalize the above result to CV systems in Theorem 24. If their Theorem 24 were true, then also the PLOB bounds for Gaussian channels would be strong converse rates. Unfortunately, WTB does *not* rigorously prove its Theorem 24. In fact, following an incorrect interpretation of the BK protocol, WTB assumes that CV teleportation asymptotically induces a perfect quantum channel (i.e., an identity channel) independently from the size of the input alphabet of quantum states. By contrast, we know that the BK teleportation channel *does not* uniformly converge to the identity channel. As a result, the bounds for Gaussian channels stated in Ref. [92, Theorem 24] are technically equal to infinity.

B. Claims and mathematical issues

Let us describe the problem in detail. WTB makes the following equivalent claims on the strong-converse bound.

- **WTB claim ([92, Theorem 24]).** Consider an ε-secure key generation protocol over n uses of a phase-insensitive Gaussian channel $\mathcal{E}$, which may be a thermal-loss channel ($\mathcal{E}_{\eta,\bar{n}}$), a quantum amplifier ($\mathcal{E}_{g,\bar{n}}$) or an additive-noise Gaussian channel ($\mathcal{E}_{\xi}$). For any $\varepsilon \in (0,1)$ and $n \geq 1$, one has the upper bound of Eq. (172) for the secret key rate, where $\Phi(\mathcal{E})$ is the weak-converse bound established in PLOB, and the "unconstrained relative entropy variance" $V(\mathcal{E})$ is respectively given by

$$\begin{aligned} V(\mathcal{E}_{\eta,\bar{n}}) &= \bar{n}(\bar{n}+1)\log_2^2\left[\eta(\bar{n}+1)/\bar{n}\right], \\ V(\mathcal{E}_{g,\bar{n}}) &= \bar{n}(\bar{n}+1)\log_2^2\left[g^{-1}(\bar{n}+1)/\bar{n}\right] \\ V(\mathcal{E}_{\xi}) &= (1-\xi)^2/\ln^2 2\ . \end{aligned} \tag{174}$$

  In particular, for a pure loss channel ($\mathcal{E}_{\eta,0}$) and a quantum-limited amplifier ($\mathcal{E}_{g,0}$), one has

$$K(\mathcal{E}) \leq \Phi(\mathcal{E}) + \frac{C(\varepsilon)}{n}\ . \tag{175}$$

- **WTB claim ([92], with simulation error).** The above claim is obtained starting from a finite simulation energy μ and then taking the limit of $\mu \to \infty$. For any security parameter $\varepsilon \in (0,1)$, number of channel uses $n \geq 1$ and simulation energy μ with "infidelity" $\varepsilon_{\rm TP}(n,\mu)$, one may write the following upper bound for the secret key rate of a phase insensitive Gaussian channel $\mathcal{E}$

$$K(\mathcal{E}) \leq \Phi(\mathcal{E}) + \Delta(n,\mu), \tag{176}$$

  where $\Phi(\mathcal{E})$ is the weak-converse bound established in PLOB. Here $\Delta(n,\mu)$ has the asymptotic expansion

$$\Delta(n,\mu) \simeq \sqrt{\frac{2V(\mathcal{E})+O(\mu^{-1})}{n[1-\varepsilon(n,\mu)]}} + \frac{C[\varepsilon(n,\mu)]}{n} + O(\mu^{-1})\ , \tag{177}$$

  at fixed n and large μ, where

$$\varepsilon(n,\mu) := \min\left\{1, \left[\sqrt{\varepsilon} + \sqrt{\varepsilon_{\rm TP}(n,\mu)}\right]^2\right\}. \tag{178}$$

  For a pure loss channel ($\mathcal{E}_{\eta,0}$) and a quantum-limited amplifier ($\mathcal{E}_{g,0}$), one has Eq. (176), with

$$\Delta(n,\mu) \simeq n^{-1}C[\varepsilon(n,\mu)] + O(\mu^{-1}). \tag{179}$$

In the previous claim, the problem is the infidelity parameter

$$\varepsilon_{\rm TP}(n,\mu) := 1 - F(\rho^n_{\mathbf{ab}}, \rho^{\mu,n}_{\mathbf{ab}}), \tag{180}$$

between the output of the protocol $\rho^n_{\mathbf{ab}}$ and the output of the simulated protocol $\rho^{\mu,n}_{\mathbf{ab}}$ [in WTB denoted with ζ^n_{AB} and $\zeta'_{AB}(n,\mu)$]. In fact, WTB (wrongly) argues that [92]

$$\begin{gathered} \text{"continuous variable teleportation} \\ \text{induces a perfect quantum channel} \\ \text{when infinite energy is available,"} \end{gathered} \tag{181}$$

and then writes [92, Eq. (178)]

$$\limsup_{\mu} \varepsilon_{\rm TP}(n,\mu) = 0, \quad \text{for any } n. \tag{182}$$

The fact that this quantity goes to zero is a crucial step in WTB's proof. In fact, if this is true, then we may write $\lim_\mu \varepsilon(n,\mu) = \varepsilon$ and safely replace this in Eq. (177). By contrast, if Eq. (182) does not hold and we get

$$\limsup_{\mu} \varepsilon_{\rm TP}(n,\mu) = 1, \quad \text{for any } n, \tag{183}$$

then $\lim_\mu \varepsilon(n,\mu) = 1$, and we have $\Delta(n,\mu) = \infty$ both in Eqs. (177) and (179). In this case, one would have proven the trivial upper bound

$$K(\mathcal{E}) \leq \Phi(\mathcal{E}) + \infty. \tag{184}$$

Unfortunately, Eq. (183) is the actual technical result which can be derived following the steps of the proof presented in WTB [92]. This means that WTB proves the trivial bound in Eq. (184), not Eq. (172) or Eq. (175).

C. Technical gap and exploding bound

The first problem is that Eq. (182) is essentially given without any mathematical derivation. To be more precise, it is not proven how the error "$\mathcal{E}^\mu \neq \mathcal{E}$" in the simulation of the Gaussian channel $\mathcal{E}$ (in each single transmission) is propagated into an overall error "$\rho^{\mu,n}_{\mathbf{ab}} \neq \rho^n_{\mathbf{ab}}$" for

the n-use output of the adaptive protocol $\rho^n_{\mathbf{ab}}$, which is exactly what $\varepsilon_{\rm TP}$ is about according to the definition in Eq. (180). For instance, what is the dependence of such an output error with respect to the number n of channel uses? Can this error explode?

Said in other words, the fundamental gap in WTB's proof is the absence of a peeling argument [77] as the one discussed in Sec. V B [See Eq. (75)], which shows how the simulation error on the channel $\|\mathcal{E}-\mathcal{E}^{\mu}\|_{\diamond N}$ propagates through the adaptive protocol and is transformed into a corresponding simulation error on the n-use output state $\|\rho^n_{\mathbf{ab}}-\rho^{n,\mu}_{\mathbf{ab}}\|$. This is a crucial technique for the simplification of an adaptive protocol [77], which is based on a suitable combination of triangle inequality and data processing (monotonicity) of the relative entropy.

Because of the absence of any peeling argument able to quantify the infidelity $\varepsilon_{\rm TP}(n,\mu)$ in terms of the channel simulation error "$\mathcal{E}^{\mu}\neq\mathcal{E}$", we may safely say that WTB's proof does not really apply to adaptive protocols. As further discussed in Ref. [111], peeling arguments could be formulated assuming various topologies of convergences (strong, uniform, or bounded-uniform) but none of these formulations can be found in WTB, where the peculiar convergence properties of the BK protocol are clearly not known and completely ignored.

The second problem is that Eq. (182) is not even proven for a single use ($n=1$), and the reasoning followed in WTB leads exactly to the opposite result of Eq. (183). In fact, let us assume a single use ($n=1$) of a trivial adaptive protocol ($\Lambda_1=\mathcal{I}$), so that

$$\rho^1_{\mathbf{ab}}=\mathcal{E}(\rho^0_{\mathbf{ab}}),\ \rho^{1,\mu}_{\mathbf{ab}}=\mathcal{E}^{\mu}(\rho^0_{\mathbf{ab}}), \tag{185}$$

where $\rho^0_{\mathbf{ab}}$ is the initial state of the registers, and the channels are meant to be applied to the input system a_1, i.e., $\mathcal{E}=\mathcal{I}_{\mathbf{a}}\otimes\mathcal{E}_{a_1}\otimes\mathcal{I}_{\mathbf{b}}$ and $\mathcal{E}^{\mu}=\mathcal{I}_{\mathbf{a}}\otimes\mathcal{E}^{\mu}_{a_1}\otimes\mathcal{I}_{\mathbf{b}}$. Then, we may write their infidelity as

$$\begin{aligned}\varepsilon_{\rm TP}(1,\mu)&=1-F(\rho^1_{\mathbf{ab}},\rho^{\mu,1}_{\mathbf{ab}}) &(186)\\ &=1-F[\mathcal{E}(\rho^0_{\mathbf{ab}}),\mathcal{E}^{\mu}(\rho^0_{\mathbf{ab}})] &(187)\\ &\geq 1-F[\rho^0_{\mathbf{ab}},\mathcal{I}^{\mu}_{a_1}(\rho^0_{\mathbf{ab}})], &(188)\end{aligned}$$

where we exploit the monotonicity of the fidelity under the maps $\mathcal{E}=\mathcal{E}\circ\mathcal{I}$ and $\mathcal{E}^{\mu}=\mathcal{E}\circ\mathcal{I}^{\mu}$.

Now the "proof idea" in WTB is based on the statement in (181), which is unfortunately not sufficient to send $\varepsilon_{\rm TP}(1,\mu)$ to zero in the limit of large μ. In fact, this statement fails if, at the input, we consider asymptotic states whose energy $\tilde{\mu}$ "competes" with the one μ of the resource state. It is clear that these states need to be included among all possible inputs, because we are studying *unconstrained* quantum and private capacities, for which the input alphabet is energy-unbounded.

Therefore, as possible input, assume that Alice sends part a_1 of a TMSV state $\Phi^{\tilde{\mu}}_{aa_1}$ with energy $\tilde{\mu}$. This means that we may decompose $\rho^0_{\mathbf{ab}}=\rho^0_{\mathbf{a}}\otimes\Phi^{\tilde{\mu}}_{aa_1}\otimes\rho^0_{\mathbf{b}}$, and write

$$\varepsilon_{\rm TP}(1,\mu)\geq 1-F[\Phi^{\tilde{\mu}}_{aa_1},\mathcal{I}_a\otimes\mathcal{I}^{\mu}_{a_1}(\Phi^{\tilde{\mu}}_{aa_1})], \tag{189}$$

by using the multiplicativity of the fidelity. Now, from Secs. IV B and IV C, we know that, depending on the order of the limits, we may write the two opposite results

$$\lim_{\tilde{\mu}}\lim_{\mu}\varepsilon_{\rm TP}(1,\mu)\geq 0 \tag{190}$$

and

$$\lim_{\mu}\lim_{\tilde{\mu}}\varepsilon_{\rm TP}(1,\mu)=1. \tag{191}$$

In WTB there is no consideration of the unboundedness of the input alphabet, and the authors just consider $\limsup_{\mu}\varepsilon_{\rm TP}$. This generic limit does not imply any specific order of the limits between the simulation energy μ and the input energy $\tilde{\mu}$ of the alphabet. Therefore, for an unbounded alphabet, one must have

$$\limsup_{\mu}\ \varepsilon_{\rm TP}=\max\{\lim_{\mu}\lim_{\tilde{\mu}}\varepsilon_{\rm TP},\lim_{\tilde{\mu}}\lim_{\mu}\varepsilon_{\rm TP}\}\ . \tag{192}$$

It is clear that this leads to

$$\limsup_{\mu}\ \varepsilon_{\rm TP}(1,\mu)=1\ . \tag{193}$$

By extending the reasoning to arbitrary n (via the peeling argument), one obtains Eq. (183) and, therefore, the explosion of the bound as in Eq. (184).

Here it is important to remark that the ambiguity in Eq. (192) is not addressed or noted in any part of WTB. In WTB there is no discussion related to uniform convergence, associated with the first order of the limits in Eq. (192), or strong convergence, associated with the second order of the limits in Eq. (192). The only discussion or motivation we can find is the statement reported in (181) which suggests the (wrong) assumption of uniform convergence, so that the BK teleportation channel $\mathcal{I}^{\mu}$ would "induce" the identity channel $\mathcal{I}$ ("perfect quantum channel") when $\mu\to\infty$ (i.e., for infinite energy). We know that this is not true, because

$$\lim_{\mu\to\infty}\|\mathcal{I}^{\mu}-\mathcal{I}\|_{\diamond}=2\ . \tag{194}$$

Let us stress that, in an infinite-dimensional Hilbert space, considering "an arbitrary protocol" [92] does not necessarily mean that the protocol is energy-bounded. Among the key-generation protocols to be considered in the derivation of upper bounds for the (unconstrained) secret key capacity of bosonic channels, one clearly needs to include protocols based on asymptotic input states. The energy of the input state can diverge in each single use of the protocol or as a monotonic function in the number n of the uses. For instance, one may just use TMSV states with increasing energy as $\tilde{\mu}\simeq O(n)$ or any other scaling in n. For any simulation energy μ, we can always consider such a diverging sequence, so that $\varepsilon_{\rm TP}(n,\mu)\overset{n}{\to}1$. In other words, we generally have

$$\limsup_{n,\mu}\ \varepsilon_{\rm TP}(n,\mu)=1\ . \tag{195}$$

As a consequence, the joint limit for large μ and n in the term $\Delta(n,\mu)$ of Eq. (176) is not defined, and we can easily get the explosion $\Delta(n,\mu)\to\infty$.

D. Fixing the mathematical issues

Let us know rigorously prove the WTB claim. We modify the derivation under the assumption of an energy constraint on the input alphabet, which leads us to write an energy-constrained version of Eq. (176). Only at the very end, the enforced constraint can be relaxed once we have proven that the upper bound does not depend on it. The key step is to prove a rigorous version of Eq. (182) that removes the issue associated with the asymptotic states (unbounded alphabet). The present proof is based on the bounded-uniform convergence of the BK protocol. See Ref. [111] for other proofs that are based on other topologies of convergence (e.g., strong convergence).

Following PLOB, let us restrict Alice's and Bob's registers to a finite-energy alphabet $\mathcal{D}_N$ as in Eq. (61) where N is maximum mean number of photons. We then consider the energy-constrained diamond distance $\|\cdot\|_{\diamond N}$ between a Gaussian channel $\mathcal{E}$ and its simulation $\mathcal{E}^{\mu}$ which defines a simulation error $\delta(\mu, N)$ as in Eq. (63). For any fixed energy, we may now state that the simulation is asymptotically perfect, i.e., $\delta(\mu, N) \overset{\mu}{\to} 0$ as in Eq. (64).

The next step is to propagate this error to the output state as done in PLOB and explained in Sec. V B. For any energy constraint N (bounded alphabet) and finite-energy simulation μ of the Gaussian channel, we may bound the trace distance between the actual output $\rho_{\mathbf{ab}}^{n}$ and the simulated output $\rho_{\mathbf{ab}}^{\mu,n}$ as in Eq. (72). In other words, we may use our peeling argument and write

$$\delta(n, \mu, N) := \|\rho_{\mathbf{ab}}^{n} - \rho_{\mathbf{ab}}^{\mu,n}\| \leq n\delta(\mu, N). \tag{196}$$

Using the Fuchs-van der Graaf relation in Eq. (59), we may now correctly write the infidelity as

$$\varepsilon_{\text{TP}}(n, \mu, N) := 1 - F\left(\rho_{\mathbf{ab}}^{n}, \rho_{\mathbf{ab}}^{\mu,n}\right) \leq \frac{n\delta(\mu, N)}{2} \ . \tag{197}$$

Using the triangle inequality for the trace distance $d(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$, one finds that Eq. (178) has to be changed into the following

$$\varepsilon(n, \mu, N) := \min\left\{1, \left[\sqrt{\varepsilon} + \sqrt{\varepsilon_{\text{TP}}(n, \mu, N)}\right]^2\right\}. \tag{198}$$

As a result, for any n and N, we may derive the energy-constrained version of Eq. (176) which reads

$$K(\mathcal{E}|N) \leq \Phi(\mathcal{E}) + \Delta(n, \mu, N), \tag{199}$$

where $K(\mathcal{E}|N)$ is the reduced key rate associated with the use of an energy-constrained alphabet, $\Phi(\mathcal{E})$ is the weak converse in PLOB, and Δ has the asymptotic expansion

$$\Delta(n, \mu, N) \simeq \sqrt{\frac{2V(\mathcal{E}) + O(\mu^{-1})}{n[1 - \varepsilon(n, \mu, N)]}} + \frac{C[\varepsilon(n, \mu, N)]}{n} + O(\mu^{-1}) \ , \tag{200}$$

with a more simplified expression for the pure-loss channel and the quantum-limited amplifier.

Now, for any number of channel uses n and any energy constraint N for the input alphabet, we may safely take the limit in μ. In these conditions, $\delta(\mu, N) \overset{\mu}{\to} 0$ implies

$$\limsup_{\mu} \varepsilon(n, \mu, N) = \varepsilon \ , \tag{201}$$

so that we may write the upper bound

$$K(\mathcal{E}|N) \leq \Phi(\mathcal{E}) + \Delta, \quad \Delta = \sqrt{\frac{2V(\mathcal{E})}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n} \ . \tag{202}$$

Here we note that the bound $\Phi(\mathcal{E}) + \Delta$ does not depend on N. We can therefore relax the energy constraint by extending the inequality to the supremum

$$\begin{aligned} K(\mathcal{E}) &:= \sup_{N} K(\mathcal{E}|N) \\ &\leq \Phi(\mathcal{E}) + \sqrt{\frac{2V(\mathcal{E})}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n} \ , \end{aligned} \tag{203}$$

which rigorously proves the WTB claim (strong converse) for noisy Gaussian channels. It is easy to see that we may similarly show the claim in Eq. (175) for the pure-loss channel and the quantum-limited amplifier.

E. Further details and technical issues

There are other technical issues in WTB's treatment of bosonic Gaussian channels that can be automatically fixed by correctly applying the tools in PLOB. One of these issues is related with the treatment of the Bell detection which is energy-unbounded for CV systems (being a projection onto infinitely-squeezed displaced TMSV states). To be completely rigorous, this measurement needs to be treated as a sequence of Gaussian measurements with increasing energy. These measurements are quasi-projections onto finite-squeezed displaced TMSV states $D(\alpha)\Phi^{\mu}D(-\alpha)$, with $D(\alpha)$ being the displacement operator with amplitude α [5].

Thus, more precisely, the asymptotic simulation of a bosonic Gaussian channel $\mathcal{E}$ must also involve a sequence of LOCCs $\mathcal{T}_{\mu}$ (including the finite-squeezing Bell measurements) which means that the simulating channel $\mathcal{E}^{\mu}$ should be modified into the following form [77]

$$\mathcal{E}^{\mu}(\rho) = \mathcal{T}_{\mu}(\rho \otimes \rho_{\mathcal{E}}^{\mu}), \tag{204}$$

where $\rho_{\mathcal{E}}^{\mu}$ is the usual quasi-Choi matrix. As a consequence, the teleportation stretching of a protocol over a bosonic Gaussian channel $\mathcal{E}$ involves a sequence of LOCCs $\bar{\Lambda}_{\mu}$, so that it takes the form [77]

$$||\rho_{\mathbf{ab}}^{n} - \bar{\Lambda}_{\mu}(\rho_{\mathcal{E}}^{\mu\otimes n})|| \overset{\mu}{\to} 0, \tag{205}$$

where we need to consider the simultaneous infinite-energy limit in both the Choi-sequence $\rho_{\mathcal{E}}^{\mu}$ and the

LOCCs $\bar{\Lambda}_\mu$. For the sake of simplicity, we have omitted this further technical detail in this review, but this aspect has been fully accounted in PLOB. Unfortunately, no approximation of the CV Bell detection has been discussed or considered in WTB. Furthermore, this problem also affects all the derivations presented in Ref. [104].

XI. CONCLUSION

In this manuscript we have reviewed recent results and techniques in the field of quantum and private communications. We have started with the definition of adaptive protocols which are based on LOs assisted by two-way CCs. Optimizing over these adaptive LOCCs, one can define the various two-way assisted capacities (Q_2, D_2, P_2 and K) associated with a quantum channel. In particular, the secret key capacity of the channel is defined starting from the notion of private state which is a suitable cryptographic generalization of a maximally-entangled state. Following PLOB, we have then introduced the relative entropy of entanglement as a general weak converse upper bound for the secret key capacity for quantum channels of any dimension (finite or infinite). The first rigorous proof of this result was presented back in 2015 [100] and exploits a truncation argument for the case of CV channels. In this regard, we have also demystified some unfounded claims made in recent literature.

We have then presented the most general kind of simulation for a quantum channel in a quantum or private communication scenario. This must be based on LOs performed by the two remote users. In fact, it is generally defined as an LOCC applied to a resource state, and this formulation may also be asymptotic, i.e., involving the limit of sequence of states, which is particularly relevant for CV channels. Such a general LOCC simulation, first considered in PLOB [77], has a number of precursory ideas based on teleportation that have been developed in the last 20 years or so [69–76].

In the context of channel simulation, we have discussed the important criterion of teleportation covariance, which is a way to determine if a quantum channel is Choi-stretchable, i.e., simulable by teleportation over its Choi matrix. This criterion was first identified for DV channels [74–76] and then extended to channels of any dimension [77]. Most importantly, we have fully clarified how to handle the asymptotic (and optimal) simulation of bosonic Gaussian channels, for which the simulation error must be carefully controlled and correctly defined in terms of energy-constrained diamond distance.

The tool of channel simulation is at the core of the most powerful techniques of protocol reduction. This was first shown in the teleportation-based approach of BDSW [69] with the formulation of a protocol reduction into entanglement distillation that was later picked up by several other works [73, 74, 76]. More recently, PLOB showed how channel simulation (standard or asymptotic) is even more powerful and can be used to reduce any adaptive protocol into a block protocol, while preserving the original quantum task. This method of teleportation stretching has been already widely exploited in recent literature, not only in the area of quantum/private communication, but also in those of quantum channel discrimination and quantum metrology (e.g., see Refs. [81, 82]).

With all the ingredients in our hands, we have discussed how their combination leads to the computation of single-letter upper bounds for the two-way capacities of quantum channels at any dimension (finite or infinite). Some of these upper bounds coincide with corresponding lower bounds, and fully establish the two-way capacities of fundamental quantum channels, such as the lossy channel. In order to fully clarify the procedure, we have separately discussed the results involving standard non-asymptotic simulations from those that require asymptotic simulations (important for bosonic Gaussian channels). While this recipe was designed in PLOB [77] for the relative entropy of entanglement, here we also discuss its full generality and applicability to other entanglement measures, including the squashed entanglement.

There are a number of questions still open. What are the two-way capacities ($Q_2 = D_2$ and $P_2 = K$) of the depolarizing channel? Same question for the amplitude damping channel. In the CV setting, the two-way capacities are still to be determined for all the "noisy" single-mode phase insensitive Gaussian channels, where the environment is not just the vacuum. The most notable case is the thermal-loss channel for its importance in QKD. From this point of view, this paper has also faced another crucial question, what is the maximum excess noise that is tolerable in QKD? Our study shows that the gap between upper and lower bound is still too large. Perhaps these questions may be closed by following the approach recently put forward in Ref. [161] where port-based teleportation [80, 162–164] is adopted as more general tool for channel simulation and protocol stretching.

In conclusion, we have also re-considered the derivations of the follow-up work WTB [92], which aimed at proving the strong converse property of the previous upper bounds established in PLOB. Because of a problem associated with the unboundedness of the alphabet in the teleportation of CV channels, the treatment of bosonic Gaussian channels was affected by a technical issue that we fix in this paper. Furthermore, we also fill a fundamental gap in the proof of WTB which was not properly designed for adaptive protocols, due to the absence of a crucial peeling argument [77]. In this way, we provide a complete and rigorous proof of the claims presented in WTB in relation to the strong converse bounds for private communication over Gaussian channels. Further validations can also be found in Ref. [111].

Let us conclude by saying that, despite the lack of technical rigor in treating the simulation of bosonic Gaussian channels, we think that it is fair to attribute to WTB [92] the derivation of their strong converse bounds. By contrast, let us stress that WTB *did not* play any role in the derivation of the previous weak converse bounds (and

two-way assisted capacities) for the same channels, because these results were already and rigorously established in PLOB [77], which also laid down the main methodology.

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classical communication cost is at most linear in m [98], which implies [95] that the shield size d_S increases at most exponentially in $m < n$.

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APPENDIX A: FIDELITY LIMITS IN THE BK TELEPORTATION PROTOCOL

To explicitly show Eq. (51), recall that a TMSV state $\Phi_{Aa}^{\tilde{\mu}}$ is a bipartite Gaussian state with zero mean value and CM of the form

$$\mathbf{V} = \mathbf{V}_q \oplus \mathbf{Z}\mathbf{V}_q\mathbf{Z}, \tag{A1}$$

where $\mathbf{Z} := \mathrm{diag}(1,-1)$ and $\mathbf{V}_q$ is explicitly given by

$$\mathbf{V}_q(\tilde{\mu}) = \begin{pmatrix} \tilde{\mu} & \sqrt{\tilde{\mu}^2-1/4} \\ \sqrt{\tilde{\mu}^2-1/4} & \tilde{\mu} \end{pmatrix}. \tag{A2}$$

Now assume that we apply the BK protocol to mode a of the input state $\Phi_{Aa}^{\tilde{\mu}}$ by using a TMSV state $\Phi_{a'B}^{\mu}$ as a resource. The ideal CV Bell detection on modes a and a', and the CC of the outcome realizes the BK channel $\mathcal{I}^{\mu}$ from mode a to mode B. This is locally (i.e., pointwise) equivalent to an additive-noise Gaussian channel with added noise [114, 116] $\xi = 2\mu - \sqrt{4\mu^2-1}$. When applied to $\Phi_{Aa}^{\tilde{\mu}}$, we get the output $\Phi_{Aa}^{\mu,\tilde{\mu}} := \mathcal{I}_A \otimes \mathcal{I}_a^{\mu}(\Phi_{Aa}^{\tilde{\mu}})$ whose CM $\mathbf{V}^{\mu,\tilde{\mu}}$ has the form in Eq. (A1) with

$$\mathbf{V}_q^{\mu,\tilde{\mu}} = \begin{pmatrix} \tilde{\mu} & \sqrt{\tilde{\mu}^2-1/4} \\ \sqrt{\tilde{\mu}^2-1/4} & \tilde{\mu}+\xi \end{pmatrix}. \tag{A3}$$

Using the formula for the quantum fidelity between arbitrary multimode Gaussian states [125], we find

$$\begin{aligned} F(\mu,\tilde{\mu}) &:= F\left(\Phi_{Aa}^{\mu,\tilde{\mu}}, \Phi_{Aa}^{\tilde{\mu}}\right) \\ &= \frac{1}{\sqrt[4]{1-4\tilde{\mu}\left[\sqrt{4\mu^2-1}+\tilde{\mu}-2\mu\left(1+2\tilde{\mu}\xi\right)\right]}}. \end{aligned} \tag{A4}$$

We can easily check the asymptotic expansions

$$F(\mu,\tilde{\mu}) \simeq 1 - O(\mu^{-1}), \quad \text{for large } \mu, \tag{A5}$$

$$F(\mu,\tilde{\mu}) \simeq O(\tilde{\mu}^{-1}), \quad \text{for large } \tilde{\mu}, \tag{A6}$$

which imply the opposite limits in Eqs. (44) and (51), when $\mathcal{E} = \mathcal{I}$.