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Quantum inequality for a scalar field with a background potential

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Abstract

Quantum inequalities are bounds on negative time-averages of the energy density of a quantum field. They can be used to rule out exotic spacetimes in general relativity. We study quantum inequalities for a scalar field with a background potential (i.e., a mass that varies with spacetime position) in Minkowski space. We treat the potential as a perturbation and explicitly calculate the first-order correction to a quantum inequality with an arbitrary sampling function, using general results of Fewster and Smith. For an arbitrary potential, we give bounds on the correction in terms of the maximum values of the potential and its first three derivatives. The techniques we develop here will also be applicable to quantum inequalities in general spacetimes with small curvature, which are necessary to rule out exotic phenomena.

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I. INTRODUCTION

General Relativity relates spacetime curvature to the stress-energy tensor T_{ab}, but does not provide any constraints on what T_{ab} might be. Thus relativity alone allows us to construct any spacetime, including those with exotic features, such as wormholes and time machines. However in the context of quantum field theory, while negative energies are possible, for example in the Casimir effect, there are various constraints on the stress energy tensor. One example is averaged energy conditions that provide bounds on integrals of T_{ab} along an entire geodesic. Another example of bounding the stress-energy tensor is quantum inequalities that bound the total energy when averaging over a time period.

Quantum inequalities were introduced by Ford [1] to avoid the possibility of violating the second law of thermodynamics by sending a flux of negative energy into a black hole. The general form of a quantum inequality is

$$\int_{-\infty}^{\infty} d\tau\, w(\tau) T_{ab}(x(\tau)) V^a V^b > -B\,, \tag{1}$$

where $x(\tau)$ is a timelike path parameterized by proper time τ with tangent vector V, and w is a sampling function. The quantity B is a bound, depending on the function w and the quantum field of interest.

Since the original work of Ford, quantum inequalities have been derived for a wide range of different fields and sampling functions. However, these quantum inequalities apply only to free fields in Minkowski space without boundaries. In other cases, there are difference quantum inequalities [2], in which T_{ab} in Eq. (1) is replaced by the difference between T_{ab} in some state of interest and T_{ab} in a reference state. The bound B may also then depend on the reference state. However, such difference inequalities cannot be used to rule out exotic spacetimes, at least in the case where the exotic matter that supports the spacetime comes from the vacuum state in the presence of the boundaries.

Nevertheless Ref. [3] shows that boundaries do not allow violation of the average null energy condition (ANEC), which states that

$$\int_{-\infty}^{\infty} d\lambda\, T_{ab}(\gamma(\lambda)) V^a V^b \geq 0\,, \tag{2}$$

where the integral is taken on a null geodesic γ, affinely parameterized by λ with tangent vector V. Reference [4] proved that ANEC is sufficient to rule out many exotic spacetimes. The proof made use of quantum inequalities for null contractions of the stress tensor averaged over timelike geodesics [5].

None of this work, however, really addresses the possibility of exotic spacetimes. The quantum inequalities on which it depends apply only in flat spacetime, so they cannot be used to rule out spacetimes with exotic curvature. For that, we need limits on the stress-energy tensor in curved spacetimes. One possible approach is to appeal to the principle of equivalence to say that if the averaging timescale of the quantum inequality is small compared to the curvature radius of the spacetime, then flat-space results should apply approximately [6]. We used such reasoning in Ref. [7] to conjecture that flat-space quantum inequalities apply, even in curved space, with certain corrections, which we hoped were not too large. From this conjecture, we were able to extend the argument of Ref. [3] to curved spacetime. But the truth of our conjecture is not known.

As a first step toward proving the conjecture of Ref. [7], we derive in the present work a quantum inequality in a flat spacetime with a background potential, i.e., a field with a mass depending on spacetime position. This is a simpler system that has many of the important features of quantum fields in curved spacetime. For a scalar field Φ in a background potential, the Lagrangian is

$$L = \frac{1}{2}\left[\partial_\mu \Phi \partial^\mu \Phi - V(x)\Phi^2\right] , \tag{3}$$

the equation of motion is

$$(\Box + V(x))\Phi = 0 , \tag{4}$$

and the classical energy density is

$$T_{00} = \frac{1}{2}\left[(\partial_t \Phi)^2 + (\nabla\Phi)^2 + V(x)\Phi^2\right] . \tag{5}$$

We work only in first order in V but don't otherwise assume that it is small. We can express the maximum values of the background potential and its derivatives as

$$\begin{array}{ll} |V| \le V_{\max} & |V_{,a}| \le V'_{\max} \\ |V_{,ab}| \le V''_{\max} & |V_{,abc}| \le V'''_{\max} , \end{array} \tag{6}$$

where $V_{\max}$, $V'_{\max}$, $V''_{\max}$ and $V'''_{\max}$ are positive numbers, finite but not necessarily small.

Our proof uses a general absolute quantum inequality proven by Fewster and Smith [8], which we discuss in Sec. II. This inequality gives a bound on the renormalized energy density based on the Fourier transform of the point-split energy density operator applied to the Hadamard series. In Sec. III, we discuss this operator, in Sec. IV we compute the Hadamard series, and in Sec. V, we apply the operator. In Sec. VI, we perform the Fourier transform, leading to the final quantum inequality in Sec. VII. We conclude in Sec. VIII with a discussion of future possibilities.

We use metric signature $(+,-,-,-)$. Indices $a, b, c, \ldots$ denote all spacetime coordinates while $i, j, k \ldots$ denote only spatial coordinates.

II. ABSOLUTE QUANTUM ENERGY INEQUALITY

We start by defining the renormalized energy density according to the renormalization procedure of Wald [9]. Let $\langle\phi(x)\phi(x')\rangle$ be the two-point function of the scalar field, and define the Hadamard form

$$H(x,x') = \frac{1}{4\pi^2}\left[\frac{1}{\sigma_+(x,x')} + \sum_{j=0}^{\infty} v_j(x,x')\sigma_+^j(x,x')\ln(\sigma_+(x,x')) + \sum_{j=0}^{\infty} w_j(x,x')\sigma^j(x,x')\right] , \tag{7}$$

where

$$\sigma(x,x') = -\eta_{ab}(x-x')^a(x-x')^b , \tag{8}$$

so that $\sigma(x,x') < 0$ when the separation between x and x' is timelike. By $F(\sigma_+)$, for some function F, we mean the distributional limit

$$F(\sigma_+) = \lim_{\epsilon\to 0^+} F(\sigma_\epsilon) , \tag{9}$$

where

$$\sigma_\epsilon(x,x') = \sigma(x,x') + 2i\epsilon(t-t') + \epsilon^2\,, \tag{10}$$

with t and t' being the time components of the 4-vectors x and x'. In most of the calculation we consider x and x' separated only in time. In that case, we define

$$\tau = t - t'\,, \tag{11}$$

and write $F(\tau_-)$ to mean $\lim_{\epsilon\to 0^+} F(\tau - i\epsilon)$. In general the sums in Eq. (7) do not converge, but we will be concerned only with the first few terms. Following Wald [9], we will choose $w_0 = 0$.

When the scalar field is in a Hadamard state, the singularity structure of $\langle\phi(x)\phi(x')\rangle$ is precisely that of $H(x,x')$, so the renormalized two-point function $\langle\phi(x)\phi(x')\rangle - H(x,x')$ is smooth. To this we apply a point-split energy density operator, which is analogous to the classical energy density of Eq. (5),

$$T^{\text{split}} = \frac{1}{2}\left[\sum_{a=0}^{3}\partial_a\partial_{a'} + \frac{V(x)+V(x')}{2}\right]\,, \tag{12}$$

and take the limit where x and x' coincide. In this limit, the location of evaluation of V does not matter, but the form above will be convenient later. Thus we define

$$\langle T^{\text{ren}}_{00}(x')\rangle \equiv \lim_{x\to x'} T^{\text{split}}\left(\langle\phi(x)\phi(x')\rangle - H(x,x')\right) - Q(x')\,, \tag{13}$$

where Q is a term added "by hand" to prevent the failure of conservation of the stress-energy tensor. Wald [10] derived this term for curved spacetime. The calculation for flat space with background potential is essentially the same, giving

$$Q(x) = \frac{1}{12\pi^2}w_1(x,x)\,. \tag{14}$$

Unfortunately, there is an ambiguity in the above procedure. In order to take logarithms, we must divide σ by the square of some length scale l. Changing the scale to some other scale l' decreases H by $\delta H = 2(v_0 + v_1\sigma + \cdots)\ln(l'/l)$. This results in increasing T_{ab} by $\lim_{x\to x'}(\partial_a\partial_b - (1/2)\eta_{ab}\partial^c\partial_c)\delta H$. Using the values for v_0 and v_1 computed below, this becomes $(1/12)(V_{ab} - \eta_{ab}\Box V)\ln(l'/l)$. Thus we see that the definition of T_{ab} must include arbitrary multiple of $(V_{ab} - \eta_{ab}\Box V)$. This ambiguity can also be understood as the possibility of including in the Lagrangian density a term of the form $R(x)V(x)$, where R is the scalar curvature. Varying the metric to obtain T_{ab} and then going to flat space yields the above term. The situation is very much analogous to the possible addition of terms of the form R^2 and $R_{ab}R^{ab}$ in the case of a field in curved spacetime.

Thus we rewrite Eq. (13) to include the ambiguous term,

$$\langle T^{\text{ren}}_{00}(x')\rangle \equiv \lim_{x\to x'} T^{\text{split}}\left(\langle\phi(x)\phi(x')\rangle - H(x,x')\right) - Q(x') + CV_{,ii}\,, \tag{15}$$

where C is some constant. Whenever definition of T_{00} one is trying to use, one can pick an arbitrary scale l and adjust C accordingly.

Now, following Ref. [8] we define

$$\tilde{H}(x,x') = \frac{1}{2}\left[H(x,x') + H(x',x) + iE(x,x')\right]\,, \tag{16}$$

where E is the advanced-minus-retarded Green's function, and thus iE is the antisymmetric part of the two point function. We use the Fourier transform convention

$$\hat{f}(k) \text{ or } [f]^\wedge(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{ixk}\,. \tag{17}$$

We consider the energy density integrated along a geodesic on the t axis with a smooth, positive sampling function $g(t)$. The absolute quantum inequality of Ref. [8] for this case is

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle T_{00}^{\text{ren}} \rangle (t,0) \geq -B\,, \tag{18}$$

where

$$B = \int_0^\infty \frac{d\xi}{\pi} \hat{F}(-\xi,\xi) + \int_{-\infty}^{\infty} dt\, g^2(t)(Q - CV_{,ii})\,, \tag{19}$$

and

$$F(t,t') = g(t)g(t') T^{\text{split}} \tilde{H}_{(5)}((t,0),(t',0))\,, \tag{20}$$

$\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (17), and the subscript (5) means that we include only terms through $j = 5$ in the sums of Eq. (7).

III. GENERAL CONSIDERATIONS

A. Smooth, symmetrical contributions

Let $\bar{x} = \frac{x-x'}{2}$, $\bar{t} = (t+t')/2$ and $\tau = t - t'$. Let

$$A(\tau) = \int_{-\infty}^{\infty} d\bar{t}\, F\left(\bar{t} + \frac{\tau}{2}, \bar{t} - \frac{\tau}{2}\right)\,. \tag{21}$$

Then $\hat{F}(-\xi,\xi) = \hat{A}(-\xi)$.

Suppose F contains some term f that is symmetrical in t and t'. Let a be the corresponding term in A according to Eq. (21). Then a will be even in τ, so $\hat{a}$ will be even also. If $a \in C^1$, then $\hat{a} \in L^2$, and we can perform the integral of this term separately, giving an inverse Fourier transform,

$$\int_0^\infty \frac{d\xi}{\pi} \hat{f}(-\xi,\xi) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \hat{a}(\xi) = a(0)\,. \tag{22}$$

In particular, if

$$\lim_{t\to t'} f(t,t') = f(t)\,, \tag{23}$$

then

$$\int_0^\infty \frac{d\xi}{\pi} \hat{f}(-\xi,\xi) = \int_{-\infty}^{\infty} dt\, g(t)^2 f(t)\,, \tag{24}$$

and if $f(t) = 0$ there is no contribution.

Terms arising from H appear symmetrically in $\tilde{H}$. At orders $j > 1$ they have at least 4 powers of τ, so they vanish in the coincidence limit even when differentiated twice by the operators of T^{split}. Thus such terms make no contribution to Eq. (19).

B. Simplification of $T^{\rm split}$

We would like to write the operator $T^{\rm split}$ in terms of separate derivatives on the center-point $\bar{x}$ and the difference between the points. First we separate the derivatives in $T^{\rm split}$ into time and space,

$$\sum_{a=0}^{3} \partial_a \partial_{a'} = \partial_t \partial_{t'} + \nabla_x \cdot \nabla_{x'} \,. \tag{25}$$

We can expand the spatial derivative with respect to[1] $\bar{x}$,

$$\nabla_{\bar{x}}^2 = \nabla_x^2 + 2\nabla_x \cdot \nabla_{x'} + \nabla_{x'}^2 \,, \tag{26}$$

and Eqs. (12,25,26) give

$$\begin{aligned} T^{\rm split} &= \frac{1}{2}\left[\partial_t\partial_{t'} + \frac{1}{2}\left(\nabla_{\bar{x}}^2 - \nabla_x^2 - \nabla_{x'}^2\right) + \frac{1}{2}\left(V(x) + V(x')\right)\right] = \\ &= \frac{1}{4}\left[\nabla_{\bar{x}}^2 + \Box_x - \partial_t^2 + \Box_{x'} - \partial_{t'}^2 + 2\partial_t\partial_{t'} + V(x) + V(x')\right] \,, \end{aligned} \tag{27}$$

where $\Box_x$ and $\Box_{x'}$ denote the D'Alembertian operator with respect to x and x'. Then using

$$\partial_\tau^2 = \frac{1}{4}\left[\partial_t^2 - 2\partial_t\partial_{t'} + \partial_{t'}^2\right] \,, \tag{28}$$

we can write

$$T^{\rm split}\tilde{H} = \frac{1}{4}\left[\left(\Box_x + V(x)\right)\tilde{H} + \left(\Box_{x'} + V(x')\right)\tilde{H} + \nabla_{\bar{x}}^2\tilde{H}\right] - \partial_\tau^2\tilde{H} \,. \tag{29}$$

Consider the first term. The function $H(x,x')$ obeys the equation of motion in x, and so does $E(x,x')$. Thus

$$\left(\Box_x + V(x)\right)\tilde{H} = \frac{1}{2}(\Box_x + V)H(x',x) \,. \tag{30}$$

The only asymmetrical part of H comes from the w_j, so

$$H(x',x) = H(x,x') + \frac{1}{4\pi^2}\sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x') \,. \tag{31}$$

Terms involving both V and w_j are second order in V, so we can ignore them, giving

$$\left(\Box_x + V(x)\right)\tilde{H} = \frac{1}{4\pi^2}\Box_x\sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x') \,. \tag{32}$$

Similarly,

$$\left(\Box_{x'} + V(x)\right)\tilde{H} = \frac{1}{4\pi^2}\Box_{x'}\sum_j (w_j(x,x') - w_j(x',x))\sigma^j(x,x') \,. \tag{33}$$

Adding together Eqs. (32,33), we get something which is symmetric in x and x' and vanishes in the coincidence limit. Thus according to the analysis of Sec. III A, it makes no contribution and for our purposes we can take

$$T^{\rm split}\tilde{H} = \left[\frac{1}{4}\nabla_{\bar{x}}^2 - \partial_\tau^2\right]\tilde{H} \,. \tag{34}$$

[1] When a derivative is with respect to x or x', we mean to keep the other of these fixed, while when the derivative is with respect to $\bar{t}$ or τ, we mean to keep the other of these fixed.

IV. COMPUTATION OF $\tilde{H}$

Examining Eq. (34) we see that is sufficient to compute $\tilde{H}$ for purely temporal separation as a function of t, t', and $\mathbf{x}$, the common spatial position of the points. The function $H(t,t')$ is a series of terms with decreasing degree of singularity at coincidence: τ^{-2}, $\ln\tau$, $\tau^2\ln\tau$, etc. For the first term in Eq. (34), terms in H that have any positive powers of τ will not contribute by the analysis of Sec. III A. For the second term we need to keep terms in H up to order τ^2, because the derivatives will reduce the order by 2.

The symmetrical combination $H(t,t') + H(t,t')$, will lead to something whose Fourier transform does not decline rapidly for positive ξ, so that if this alone were put into Eq. (19) the integral over ξ would not converge. But each term in $H(t,t') + H(t,t')$ will combine with a term coming from $iE(x,x')$ to give something whose Fourier transform does decline rapidly.

We will work order by order in $\tau = t - t'$ and write $H_j(t,t')$, $j = -1, 0, 1, \ldots$, to denote the term in H involving τ^{2j} (with or without $\ln\tau$), and $H_{(j)}$ to to denote the sum of all terms up through H_j . We will split up $E(x,x')$ into terms labeled E_j that are proportional to τ^{2j}, define a "remainder term"

$$R_j = E - \sum_{k=-1}^{j} E_k \,, \tag{35}$$

and let

$$\tilde{H}_j(x,x') = \frac{1}{2}\left[H_j(x,x') + H_j(x',x) + iE_j(x,x'))\right] \tag{36a}$$

$$\tilde{H}_{(j)}(x,x') = \frac{1}{2}\left[H_{(j)}(x,x') + H_{(j)}(x',x) + iE(x,x'))\right] . \tag{36b}$$

A. General computation of E

We will need the Green's functions for the background potential, including only first order in V, so we write

$$G = G^{(0)} + G^{(1)} + \cdots . \tag{37}$$

The equation of motion is

$$(\Box + V(x))G(x,x') = \delta^{(4)}(x - x') . \tag{38}$$

Using $\Box G^{(0)}(x,x') = \delta^{(4)}(x,x')$ and keeping only first-order terms we have

$$\Box G^{(1)}(x,x') = -V(x)G^{(0)}(x,x') \,, \tag{39}$$

so

$$G^{(1)}(x,x') = -\int d^4x'' G^{(0)}(x,x'')V(x'')G^{(0)}(x'',x') . \tag{40}$$

For $t > t'' > t'$ we have for the retarded Green's function,

$$G_R^{(0)}(x'',x') = \frac{1}{2\pi}\delta((t''-t')^2 - |\mathbf{x}'' - \mathbf{x}'|^2) = \frac{1}{4\pi}\frac{\delta(t'' - t' - |\mathbf{x}'' - \mathbf{x}'|)}{|\mathbf{x}'' - \mathbf{x}'|} . \tag{41}$$

So we can write

$$G_R^{(1)}(x,x') = -\frac{1}{8\pi^2}\int d^3\mathbf{x}''\int dt''\delta((t-t'')^2-|\mathbf{x}-\mathbf{x}''|^2)\frac{\delta(t''-t'-|\mathbf{x}''-\mathbf{x}'|)}{|\mathbf{x}''-\mathbf{x}'|}V(t'',\mathbf{x}''). \quad (42)$$

Integrating over the second delta function we find $t'' = t' + |\mathbf{x}'' - \mathbf{x}'|$. Again considering purely temporal separation and defining $\mathbf{z}'' = \mathbf{x}'' - \mathbf{x}'$ and $z'' = |\mathbf{z}''|$, we find

$$G_R^{(1)}(t,t') = -\frac{1}{8\pi^2}\int d\Omega\int dz'' z''^2\frac{\delta(\tau^2-2\tau z'')}{z''}V(t'+z'',\mathbf{x}'+z''\hat{\Omega}), \quad (43)$$

where $\int d\Omega$ denotes integration over solid angle, and $\hat{\Omega}$ varies over all unit vectors. We can integrate over z'' to get $z'' = \tau/2$ and

$$G_R^{(1)}(t,t') = -\frac{1}{32\pi^2}\int d\Omega\, V(\bar{t},\mathbf{x}'+\frac{\tau}{2}\hat{\Omega}). \quad (44)$$

If we define a 4-vector $\Omega = (0,\hat{\Omega})$ we can write

$$G_R^{(1)}(t,t') = -\frac{1}{32\pi^2}\int d\Omega\, V(\bar{x}+\frac{\tau}{2}\Omega). \quad (45)$$

The advanced Green's functions are the same with t and t' reversed. Since E is the advanced minus the retarded function, we have

$$E^{(1)}(t,t') = \frac{1}{32\pi^2}\int d\Omega\, V(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau. \quad (46)$$

B. Terms of order τ^{-2}

We now compute the various H_j, $\tilde{H}_j$, and E_j, starting with terms that go as σ^{-1} or τ^{-2}. These terms are exactly what one would have for flat space without potential. Equation (7) gives

$$H_{-1}(x,x') = \frac{1}{4\pi^2\sigma_+(x,x')} = -\frac{1}{4\pi^2(\tau_-^2-z^2)}, \quad (47)$$

where

$$\mathbf{z} = \mathbf{x} - \mathbf{x}' \quad (48)$$

and

$$z = |\mathbf{z}|. \quad (49)$$

Similarly, the advanced minus retarded Green's function to this order is

$$E_{-1}(x,x') = G_A(x,x') - G_R(x,x') = \frac{\delta(\tau-z)-\delta(\tau+z)}{4\pi z}, \quad (50)$$

so

$$\tilde{H}_{-1}(t,t') = \lim_{z\to 0}\frac{1}{8\pi^2}\left[-\frac{1}{\tau_+^2-z^2}-\frac{1}{\tau_-^2-z^2}+i\pi\frac{\delta(\tau+z)-\delta(\tau-z)}{z}\right], \quad (51)$$

where

$$F(\tau_+) = \lim_{\epsilon\to 0}F(\tau+i\epsilon). \quad (52)$$

Taking the $\epsilon \to 0$ limit in τ_+ and τ_- gives the formula

$$-\frac{1}{\tau_+^2 - z^2} + \frac{1}{\tau_-^2 - z^2} = -i\pi \frac{\delta(\tau + z) - \delta(\tau - z)}{z} \tag{53}$$

so

$$\tilde{H}_{-1}(t, t') = -\frac{1}{4\pi^2 \tau_-^2} = H_{-1}(t, t') \tag{54}$$

as discussed in Ref [8].

C. Terms with no powers of τ

The Hadamard coefficients are given by the Hadamard recursion relations, which are the solutions to $(\Box + V(x''))H(x'', x') = 0$, giving

$$V(x'') + 2\eta^{ab} v_{0,a}\sigma_{,b} + 4v_0 + v_0 \Box\sigma = 0 \tag{55a}$$

$$(\Box + V(x''))v_j + 2(j+1)\eta^{ab} v_{j+1,a}\sigma_{,b} - 4j(j+1)v_{j+1} + (j+1)v_{j+1}\Box\sigma = 0\,. \tag{55b}$$

In Eqs. (55), σ_j, v_j and their derivatives are functions of x'' and x', and all derivatives act on x''.

To find the zeroth order of the Hadamard series we need only v_0. For flat space, $\sigma_{,a} = -2\eta_{ab}(x'' - x')^b$ and $\Box\sigma = -8$. Putting these in Eq. (55a) we have

$$(x'' - x')^a v_{0,a} + v_0 = \frac{V(x'')}{4}\,, \tag{56}$$

Now let $x'' = x' + \lambda(x - x')$ to integrate along the geodesic going from x' to x. We observe that

$$\frac{dv_0(x'', x')}{d\lambda} = (x - x')^a v_{0,a}(x'', x')\,. \tag{57}$$

So Eq. (56) gives

$$\lambda \frac{dv_0(x'', x')}{d\lambda} + v_0(x'', x') = \frac{V(x'')}{4}\,, \tag{58}$$

or

$$\frac{d(\lambda v_0(x'', x'))}{d\lambda} = \frac{V(x'')}{4}\,, \tag{59}$$

from which we immediately find

$$v_0(x, x') = \int_0^1 d\lambda \frac{V(x' + \lambda(x - x'))}{4}\,. \tag{60}$$

Now we consider purely temporal separation so the background potential is evaluated at $(t' + \lambda\tau, \mathbf{x})$. We expand V in a Taylor series in τ around 0 with $\bar{t}$ fixed,

$$V(t' + \lambda\tau) = V(\bar{t}) + \tau(\lambda - \frac{1}{2})V_{,t}(\bar{t}) + \frac{\tau^2}{2}(\lambda - \frac{1}{2})^2 V_{,tt}(\bar{t}) + \cdots\,. \tag{61}$$

We are calculating the zeroth order so we keep only the first term of Eq. (61), and Eq. (60) gives

$$v_0(t, t') = \frac{1}{4}V(\bar{t}) + O(\tau^2) \tag{62}$$

and thus

$$H_0(x,x') = \frac{1}{16\pi^2} V(\bar{x}) \ln(-\tau_-^2)\,, \tag{63}$$

and

$$H_0(x,x') + H_0(x',x) = \frac{1}{4\pi^2} V(\bar{x}) \ln|\tau|\,. \tag{64}$$

We can expand V around $\bar{x}$,

$$V(\bar{x} + \frac{\tau}{2}\Omega) = V(\bar{x}) + V^{(1)}(\bar{x} + \frac{\tau}{2}\Omega)\,, \tag{65}$$

where $V^{(1)}$ is the remainder of the Taylor series

$$V^{(1)}(\bar{x} + \frac{\tau}{2}\Omega) = V(\bar{x} + \frac{\tau}{2}\Omega) - V(\bar{x}) = \int_0^{\tau/2} dr\, V_{,i}(\bar{x} + r\Omega)\Omega^i\,. \tag{66}$$

Then from Eq. (46),

$$E_0(x,x') = \frac{1}{8\pi} V(\bar{x}) \operatorname{sgn}\tau \tag{67a}$$

$$R_0(x,x') = \frac{1}{32\pi^2} \int d\Omega\, V^{(1)}(\bar{x} + \frac{|\tau|}{2}\Omega) \operatorname{sgn}\tau\,. \tag{67b}$$

Using

$$2\ln|\tau| + \pi i \operatorname{sgn}\tau = \ln(-\tau_-^2)\,, \tag{68}$$

we combine Eqs. (64,67a) to find

$$\tilde{H}_0(t,t') = \frac{1}{16\pi^2} V(\bar{x}) \ln(-\tau_-^2)\,. \tag{69}$$

Combining all terms through order 0 gives

$$\tilde{H}_{(0)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \frac{1}{2} i R_0(t,t')\,. \tag{70}$$

D. Terms of order τ^2

Now we compute the terms of order τ^2 in H and E. First we need v_0 at this order, so we use Eq. (61) in Eq. (60), to get

$$v_0(x,x') = \frac{1}{4} V(\bar{x}) + \tau^2 \frac{1}{96} V_{,tt}(\bar{x}) + \cdots\,. \tag{71}$$

Next we need to know v_1, but since v_1 is multiplied by τ^2 in H, we need only the τ-independent term $v_1(x,x)$. From Eq. (55b),

$$(\Box + V(x))v_0(x,x') + 2\eta^{ab} v_{1,a}(x,x')\sigma_{,b}(x,x') + v_1(x,x')\Box_x\sigma(x,x') = 0\,. \tag{72}$$

We neglect the $V(x)v_0$ term because it is second order in V. At $x = x'$, $\sigma_{,b} = 0$, so

$$v_1(x,x) = \frac{1}{8} \lim_{x'\to x} \Box_x v_0(x,x')\,. \tag{73}$$

Using Eq. (60) we find

$$\Box_x v_0(x,x') = \frac{1}{4}\int_0^1 d\lambda \Box_x V(x'+\lambda(x-x')) = \frac{1}{4}\int_0^1 d\lambda\, \lambda^2 (\Box V)(x'+\lambda(x-x'))\,, \tag{74}$$

and Eq. (73) gives

$$v_1(x,x) = \frac{1}{96}\Box V(\bar{x})\,. \tag{75}$$

We also need to know w_1, but again only at coincidence. Reference [10] gives

$$w_1(x,x) = -\frac{3}{2}v_1(x,x) = -\frac{1}{64}\Box V(x)\,. \tag{76}$$

Combining the second term of Eq. (71) with Eqs. (75,76) gives

$$H_1(t,t') = \frac{\tau^2}{128\pi^2}\left[\frac{1}{3}V_{,ii}(\bar{x})\ln(-\tau_-^2) + \frac{1}{2}\Box V(\bar{x})\right]\,. \tag{77}$$

Then $H_1(x',x)$ is given by symmetry, so

$$H_1(x,x') + H_1(x',x) = \frac{\tau^2}{64\pi^2}\left[\frac{2}{3}V_{,ii}(\bar{x})\ln|\tau| + \frac{1}{2}\Box V(\bar{x})\right]\,. \tag{78}$$

The calculation of E_1 is similar to that of E_0, but now we have to include more terms in the Taylor expansion of V around $\bar{x}$. So we expand

$$V(\bar{x}+\frac{\tau}{2}\Omega) = V(\bar{x}) + \frac{1}{2}V_{,i}(\bar{x})\Omega^i\tau + \frac{1}{8}V_{,ij}(\bar{x})\Omega^i\Omega^j\tau^2 + V^{(3)}(\bar{x}+\frac{\tau}{2}\Omega)\,, \tag{79}$$

where the remainder of the Taylor series $V^{(3)}$ is

$$V^{(3)}(\bar{x}+\frac{\tau}{2}\Omega) = \frac{1}{2}\int_0^{\tau/2} dr\, V_{,ijk}(\bar{x}+r\Omega)\left(\frac{\tau}{2}-r\right)^2 \Omega^i\Omega^j\Omega^k dr\,. \tag{80}$$

Since $\int d\Omega\,\Omega^i = 0$ and $\int d\Omega\,\Omega^i\Omega^j = (4\pi/3)\delta^{ij}$, Eq. (46) gives

$$E_1(x,x') = \frac{1}{192\pi}V_{,ii}(\bar{x})\tau^2\,\mathrm{sgn}\,\tau\,, \tag{81a}$$

$$R_1(x,x') = \frac{1}{32\pi^2}\int d\Omega\, V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\,. \tag{81b}$$

Again using Eq. (68), we combine Eqs. (78,81a) to get

$$\tilde{H}_1(x,x') = \frac{\tau^2}{128\pi^2}\left[\frac{1}{3}\ln(-\tau_-^2)V_{,ii} + \frac{1}{2}\Box V(\bar{x})\right]\,. \tag{82}$$

Combining all terms through order 1 gives

$$\tilde{H}_{(1)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \tilde{H}_1(t,t') + \frac{1}{2}iR_1(t,t')\,. \tag{83}$$

V. THE $T^{\mathbf{split}}\tilde{H}$

Using Eqs. (20,34), we need to compute

$$\int_0^\infty \frac{d\xi}{\pi}\hat{F}(-\xi,\xi')\,, \tag{84}$$

where

$$F(t,t') = g(t)g(t')\left[\frac{1}{4}\nabla^2_{\bar{x}}\tilde{H}_{(0)}(t,t') - \partial^2_\tau \tilde{H}_{(1)}(t,t')\right]. \tag{85}$$

Using Eqs. (54,67b,69,70,81b,82,83) we can write this

$$F(t,t') = g(t)g(t')\sum_{i=1}^{6} f_i(t,t')\,, \tag{86}$$

with

$$f_1 = \frac{3}{2\pi^2\tau_-^4} \tag{87a}$$

$$f_2 = \frac{1}{8\pi^2\tau_-^2}V(\bar{x}) \tag{87b}$$

$$f_3 = \frac{1}{96\pi^2}V_{,ii}(\bar{x})\ln(-\tau_-^2) \tag{87c}$$

$$f_4 = -\frac{1}{128\pi^2}\left[V_{,tt}(\bar{x}) + V_{,ii}(\bar{x})\right] \tag{87d}$$

$$f_5 = \frac{1}{256\pi^2}\int d\Omega\,\nabla^2_{\bar{x}}\left[V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\right] i\,\mathrm{sgn}\,\tau \tag{87e}$$

$$f_6 = -\frac{1}{64\pi^2}\int d\Omega\,\partial^2_\tau\left[V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)i\,\mathrm{sgn}\,\tau\right]. \tag{87f}$$

VI. THE FOURIER TRANSFORM

We want to calculate the quantum inequality bound B, given by Eq. (19). We can write it

$$B = \sum_{i=1}^{8} B_i\,, \tag{88}$$

where

$$\begin{aligned} B_i &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^\infty dt\int_{-\infty}^\infty dt' g(t)g(t')f_i(t,t')e^{i\xi(t'-t)} \\ &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^\infty d\tau\int_{-\infty}^\infty d\bar{t}\,g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})f_i(\bar{t},\tau)e^{-i\xi\tau} \qquad i=1\ldots 6 \end{aligned} \tag{89a}$$

$$B_7 = \int_{-\infty}^\infty dt\,g^2(t)Q(t) = -\frac{1}{768\pi^2}\int_{-\infty}^\infty dt\,g^2(t)\Box V(t) \tag{89b}$$

$$B_8 = -\int_{-\infty}^\infty dt\,g^2(t)CV_{,ii}(t)\,, \tag{89c}$$

using Eqs. (14,19,76).

A. The singular terms

For $i = 1, 2, 3$, f_i consists of a singular function of τ times a function of $\bar{t}$ (or a constant). So we will separate the singular part by writing

$$f_i(\bar{t}, \tau) = g_i(\bar{t}) s_i(\tau)\,. \tag{90}$$

Then we define

$$G_i(\tau) = \int_{-\infty}^{\infty} d\bar{t}\, g_i(\bar{t}) g(\bar{t} - \frac{\tau}{2}) g(\bar{t} + \frac{\tau}{2})\,, \tag{91}$$

so

$$B_i = \int_0^{\infty} \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau\, G_i(\tau) s_i(\tau) e^{-i\xi\tau}\,. \tag{92}$$

This is a Fourier transform of a product, so we can write it as a convolution. The G_i are all real, even functions, and thus their Fourier transforms are also, and we have

$$B_i = \frac{1}{2\pi^2} \int_0^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta\, \hat{G}_i(\xi + \zeta) \hat{s}_i(\zeta)\,. \tag{93}$$

Now if we change the order of integrals we can perform another change of variables $\eta = \xi + \zeta$, so we have

$$B_i = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\zeta \int_{\zeta}^{-\infty} d\eta\, \hat{G}_i(\eta) \hat{s}_i(\zeta) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\eta \hat{G}_i(\eta) h_i(\eta)\,, \tag{94}$$

where

$$h_i(\eta) = \int_{-\infty}^{\eta} d\zeta\, \hat{s}_i(\zeta)\,. \tag{95}$$

The arguments of Ref. [8] show that the integrals over ξ in Eq. (92) and η in Eq. (95) converge.

We now calculate the Fourier transforms in turn, starting with B_1. We have

$$g_1(\bar{t}) = \frac{3}{2\pi^2} \tag{96a}$$

$$s_1(\tau) = \frac{1}{\tau_-^4}\,. \tag{96b}$$

The Fourier transform of s_1 is [11]

$$\hat{s_1}(\zeta) = \frac{\pi}{3} \zeta^3 \Theta(\zeta)\,, \tag{97}$$

so

$$h_1(\eta) = \int_0^{\eta} d\zeta\, \frac{\pi}{3} \zeta^3 \Theta(\eta) = \frac{\pi}{12} \eta^4 \Theta(\eta)\,. \tag{98}$$

From Eq. (94) we have

$$B_1 = \frac{1}{24\pi} \int_0^{\infty} d\eta\, \hat{G}_1(\eta) \eta^4\,. \tag{99}$$

Using $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$, we get

$$B_1 = \frac{1}{24\pi} \int_0^{\infty} d\eta\, \widehat{G_1''''}(\eta)\,. \tag{100}$$

The function G_1 is even, so its Fourier transform is also even and we can extend the integral

$$B_1 = \frac{1}{48\pi}\int_{-\infty}^{\infty} d\eta\, \widehat{G_1''''}(\eta) = \frac{1}{24} G_1''''(0)\,. \tag{101}$$

For G_1 we have

$$G_1(\tau) = \frac{3}{2\pi^2}\int d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,, \tag{102}$$

and taking the derivatives and integrating by parts gives

$$B_1 = \frac{1}{16\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, g''(\bar{t})^2\,, \tag{103}$$

reproducing a result of Ref. [8].

For B_2 we have

$$g_2(\bar{t}) \;=\; \frac{1}{8\pi^2} V(\bar{t}) \tag{104a}$$

$$s_2(\tau) \;=\; \frac{1}{\tau_-^2}\,. \tag{104b}$$

This calculation is the same as before except the Fourier transform of s_2 is [11]

$$\hat{s_2}(\zeta) = 2\pi\zeta\Theta(\zeta)\,. \tag{105}$$

So we have

$$B_2 = -\frac{1}{2} G_2''(0)\,, \tag{106}$$

where

$$G_2(\tau) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t}) g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,. \tag{107}$$

After taking the derivatives

$$B_2 = -\frac{1}{32\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t})[g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})]\,. \tag{108}$$

For B_3 we have

$$s_3(\tau) \;=\; \ln(-\tau_-^2)\,. \tag{109}$$

In the appendix, we find the Fourier transform of s_3 as a distribution,

$$\hat{s}_3[f] = 4\pi\int_0^{\infty} dk\, f'(k)\ln|k| - 4\pi\gamma f(0)\,. \tag{110}$$

From Eq. (95), we can write

$$h_3(\eta) = \int_{-\infty}^{\infty} d\zeta\, \hat{s_3}(\zeta)\Theta(\eta-\zeta)\,, \tag{111}$$

which is given by Eq. (110) with $f(\zeta) = \Theta(\eta - \zeta)$, so

$$h_3(\eta) = -4\pi \int_0^\infty d\zeta\, \delta(\eta - \zeta) \ln|\zeta| - 4\pi\gamma\Theta(\eta) = -4\pi\Theta(\eta)(\ln\eta + \gamma)\,. \tag{112}$$

Then Eq. (94) gives

$$B_3 = -\frac{2}{\pi} \int_0^\infty d\eta\, \hat{G}_3(\eta)\,(\ln\eta + \gamma) = -\frac{1}{\pi} \int_{-\infty}^\infty d\eta\, \hat{G}_3(\eta)(\ln|\eta| + \gamma)\,, \tag{113}$$

since G_3 is even. The integral is just the distribution w of Eq. (A12) applied to $\hat{G}_3$, which is by definition $\hat{w}[G_3]$, so Eq. (A13) gives

$$B_3 = -\int_{-\infty}^\infty d\tau\, G_3'(\tau) \ln|\tau| \operatorname{sgn}\tau\,, \tag{114}$$

with

$$G_3(\tau) = \frac{1}{96\pi^2} \int_{-\infty}^\infty dt V_{,ii}(\bar{t}) g(\bar{t} - \frac{\tau}{2}) g(\bar{t} + \frac{\tau}{2})\,, \tag{115}$$

so

$$B_3 = -\frac{1}{48\pi^2} \int_{-\infty}^\infty d\tau\, \ln|\tau| \operatorname{sgn}\tau \int_{-\infty}^\infty d\bar{t}\, V_{,ii}(\bar{t}) g(\bar{t} - \frac{\tau}{2}) g'(\bar{t} + \frac{\tau}{2})\,. \tag{116}$$

B. The non-singular terms

For $i = 4, 5, 6$, f_i is not singular at $\tau = 0$. We include everything in

$$F_i(\tau) = \int_{-\infty}^\infty d\bar{t}\, f_i(\tau, \bar{t}) g(\bar{t} - \frac{\tau}{2}) g(\bar{t} + \frac{\tau}{2})\,, \tag{117}$$

so

$$B_i = \int_0^\infty \frac{d\xi}{\pi} \int_{-\infty}^\infty d\tau F_i(\tau) e^{-i\xi\tau} = \int_0^\infty \frac{d\xi}{\pi} \hat{F}_i(-\xi) = \frac{1}{\pi} \int_{-\infty}^\infty d\xi\, \Theta(\xi) \hat{F}_i(-\xi)\,. \tag{118}$$

The integral is the distribution Θ applied to $\hat{F}_i(-\xi))$, which is the Fourier transform of Θ applied to $F_i(-\tau)$. The Fourier transform of the Θ function acts on a function f as [11]

$$\Theta[f] = iP \int_{-\infty}^\infty d\tau \left(\frac{1}{\tau} f(\tau)\right) + \pi f(0)\,, \tag{119}$$

where P denotes principal value, so

$$B_i = -\frac{i}{\pi} P \int_{-\infty}^\infty d\tau \left(\frac{1}{\tau} F_i(\tau)\right) + F_i(0)\,. \tag{120}$$

The first of the non-singular terms is a constant: f_4 does not depend on τ. Thus F_4 is even in τ, and only the second term of Eq. (120) contributes, giving

$$B_4 = F_4(0) = -\frac{1}{128\pi^2} \int_{-\infty}^\infty d\bar{t}\, g(\bar{t})^2 \left[V_{,tt}(\bar{t}) + V_{,ii}(\bar{t})\right]\,. \tag{121}$$

The functions f_5 and f_6 are odd in τ, so in these cases only the first term in Eq. (120) contributes. Equations (87e,117,120) give

$$B_5 = \frac{1}{256\pi^3}\int_{-\infty}^{\infty} d\tau \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega\, \nabla^2_{\bar{x}} V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{122}$$

and Eqs. (87f,117,120) give

$$B_6 = -\frac{1}{64\pi^3}\int_{-\infty}^{\infty} d\tau \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega\, \partial^2_\tau \left[V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\right]. \tag{123}$$

Here we can integrate by parts twice, giving

$$B_6 = -\frac{1}{64\pi^3}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\, \partial^2_\tau \left[\frac{1}{\tau} g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\right]\int d\Omega\, V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\,. \tag{124}$$

VII. THE QUANTUM INEQUALITY

Now we have can collect all the terms of B from Eqs. (89b,89c,103,108,116,121,122,124). Since B_7 is made of the same quantities as B_4, we merge these together. We find

$$B = \frac{1}{16\pi^2}\left[I_1 - \frac{1}{2}I_2 - \frac{1}{3}I_3 - \frac{1}{8}I_4 + \frac{1}{16\pi}I_5 - \frac{1}{4\pi}I_6\right] - I_7\,, \tag{125}$$

where

$$I_1 = \int_{-\infty}^{\infty} dt\, g''(t)^2 \tag{126a}$$

$$I_2 = \int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t})[g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})] \tag{126b}$$

$$I_3 = \int_{-\infty}^{\infty} d\tau \ln|\tau|\,\mathrm{sgn}\,\tau \int_{-\infty}^{\infty} d\bar{t}\, V_{,ii}(\bar{t})g(\bar{t}-\frac{\tau}{2})g'(\bar{t}+\frac{\tau}{2}) \tag{126c}$$

$$I_4 = \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})^2\left[\frac{7}{6}V_{,tt}(\bar{t}) + \frac{5}{6}V_{,ii}(\bar{t})\right] \tag{126d}$$

$$I_5 = \int_{-\infty}^{\infty} d\tau \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega\, \nabla^2_{\bar{x}} V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{126e}$$

$$I_6 = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\, \partial^2_\tau\left[\frac{1}{\tau} g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\right]\int d\Omega\, V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{126f}$$

$$I_7 = C\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})^2 V_{,ii}(\bar{t})\,. \tag{126g}$$

In Eq. (126c), $\ln|\tau|$ really means $\ln(|\tau|/l)$, where l is the arbitrary length discussed in Sec. II. The choice of a different length changes Eqs. (126c,126d) in compensating ways so that B is unchanged.

Equations (1,125,126) give a quantum inequality useful when the potential V is known and so the integrals in Eqs. (126) can be done. If we only know that V and its derivatives are restricted by the bounds of Eq. (6), then we can restrict the magnitude of each term of Eq. (125) and add those magnitudes. We start with

$$|I_2| \leq \int_{-\infty}^{\infty} d\bar{t}|V(\bar{t})|[g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})] \leq V_{\max} \int_{-\infty}^{\infty} d\bar{t}\left[g(\bar{t})|g''(\bar{t})| + g'(\bar{t})^2\right] . \tag{127}$$

The cases of I_3, I_4, and I_7 are similar. For I_5 and I_6, it is useful to take explicit forms for the Taylor series remainders. From Eq. (66), we see that

$$\left|\int d\Omega\, \nabla^2_{\bar{x}} V^{(1)}(\bar{x} + \frac{|\tau|}{2}\Omega)\right| \leq \frac{|\tau|}{2}\int d\Omega|\nabla^2 V_{,i}||\Omega^i| \leq \frac{3|\tau|}{2}V'''_{\max}\sum_i \int d\Omega|\Omega^i| = 9\pi|\tau|V'''_{\max} . \tag{128}$$

Similarly from Eq. (80) we have

$$\begin{aligned}\left|\int d\Omega\, V^{(3)}(\bar{x} + \frac{|\tau|}{2}\Omega)\right| &\leq \frac{|\tau|^3}{48}\int d\Omega|V_{,ijk}||\Omega^i\Omega^j\Omega^k| \qquad (129)\\ &\leq \frac{|\tau|^3}{48}V'''_{\max}\sum_{ijk}\int d\Omega|\Omega^i\Omega^j\Omega^k| = \frac{2\pi+1}{8}|\tau|^3 V'''_{\max} ,\end{aligned}$$

We can then perform the derivatives in Eq. (126f) and take the absolute value of each resulting term separately.

We define

$$J_2 = \int_{-\infty}^{\infty} dt\left[g(t)|g''(t)| + g'(t)^2\right] \tag{130a}$$

$$J_3 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'|g'(t')|g(t)|\ln|t'-t|| \tag{130b}$$

$$J_4 = \int_{-\infty}^{\infty} dt\, g(t)^2 \tag{130c}$$

$$J_5 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' g(t)g(t') \tag{130d}$$

$$J_6 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'|g'(t')|g(t)|t'-t| \tag{130e}$$

$$J_7 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left[g(t)|g''(t')| + g'(t)g'(t')\right](t'-t)^2 \tag{130f}$$

and find

$$|I_2| \leq V_{\max} J_2 \tag{131a}$$

$$|I_3| \leq 3V''_{\max} J_3 \tag{131b}$$

$$|I_4| \leq \frac{11}{3}V''_{\max} J_4 \tag{131c}$$

$$|I_5| \leq 9\pi V'''_{\max} J_5 \tag{131d}$$

$$|I_6| \leq \frac{2\pi+1}{16}V'''_{\max}(4J_5 + 4J_6 + J_7) \tag{131e}$$

$$|I_7| \leq 3|C|V''_{\max} J_4 . \tag{131f}$$

Thus we have

$$\int_{\mathbb{R}} d\tau\, g(t)^2 \langle T_{00}^{ren} \rangle_\omega(t,0) \geq -\frac{1}{16\pi^2} \Bigg\{ I_1 + \frac{1}{2} V_{\max} J_2 + V''_{\max} \left[J_3 + \left(\frac{11}{24} + 48\pi^2 |C| \right) J_4 \right] + V'''_{\max} \left[\frac{11\pi+1}{16\pi} J_5 + \frac{2\pi+1}{64\pi} (4J_6 + J_7) \right] \Bigg\} . \qquad (132)$$

A. An example for a specific sampling function

An example of the quantum inequality with a specific sampling function g is the following. Consider a Gaussian sampling function

$$g(t) = e^{-t^2/t_0^2} , \qquad (133)$$

where t_0 is a positive number with the dimensions of t. Then the integrals of Eqs. (130), calculated numerically, become

$$\begin{aligned} J_1 &= 3.75 t_0^{-3} \quad J_2 = 3.15 t_0^{-1} \\ J_3 &= 2.70 t_0 \quad J_4 = 1.25 t_0 \\ J_5 &= 3.14 t_0^2 \quad J_6 = 3.57 t_0^2 \\ J_7 &= 3.58 t_0^2 , \end{aligned} \qquad (134)$$

so the right hand side of Eq. (132) becomes

$$-\frac{1}{16\pi^2 t_0^3} \left\{ 3.75 + 3.15 V_{\max} t_0^2 + (3.26 + 591.25|C|) V''_{\max} t_0^4 + 2.86 V'''_{\max} t_0^5 \right\} . \qquad (135)$$

VIII. CONCLUSION

In this work we have demonstrated a quantum inequality for a flat spacetime with a background potential, considered as a first-order correction, using a general inequality presented by Fewster and Smith [8]. We calculated the necessary terms from the Hadamard series and the antisymmetric part of the two-point function to get $\tilde{H}$. Next we Fourier transformed the terms, which are, as expected, free of divergences, to derive a bound for a given background potential. We then calculated the maximum values of these terms to give a bound that applies to any potential whose value and first three derivatives are bounded.

To show the meaning of this result, in the last section we presented an example for a specific sampling function. By studying the result we can see the meaning of the right hand side of our quantum inequality. The first term of the bound goes as t_0^{-3}, where t_0 is the sampling time, and agrees with the quantum inequality with no potential [12]. The rest of the terms show the effects of the potential to first order. These corrections will be small, provided that

$$V_{\max} t_0^2 \ll 1 \qquad (136a)$$

$$V''_{\max} t_0^4 \ll 1 \qquad (136b)$$

$$V'''_{\max} t_0^5 \ll 1 . \qquad (136c)$$

Equation (136a) says that the potential is small when its effect over the distance t_0 is considered. Given Eq. (136a), Eqs. (136b,136c) say, essentially, that the distance over which V varies is large compared to t_0, so that each additional derivative introduces a factor less than t_0^{-1}.

Finally, it is interesting to note the relation of the current work to the case of a spacetime with bounded curvature. Since the Hadamard coefficients in that case are components of the Riemann tensor and its derivatives, we expect that the bound will be the flat space term plus correction terms that depend on the maximum values of the curvature and its derivatives, just as in our case they depend on the the potential and its derivatives. We intend to analyze that case in future work.

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Appendix A: Fourier transforms of some distributions involving logarithms

In this appendix will compute the Fourier transforms of the distributions given by

$$u(\tau) = \ln|\tau| \tag{A1}$$

$$v(\tau) = \ln(-\tau_-^2)\,. \tag{A2}$$

We write u as a distributional limit,

$$u = \lim_{\epsilon\to 0^+} u_\epsilon\,, \tag{A3}$$

where

$$u_\epsilon(\tau) = \ln|\tau| e^{-\epsilon|\tau|}\,, \tag{A4}$$

so its Fourier transform is

$$\hat{u}_\epsilon(k) = \int_{-\infty}^{\infty} d\tau \ln|\tau| e^{-\epsilon|\tau|} e^{ik\tau} = 2\,\mathrm{Re}\int_0^\infty d\tau \ln\tau\, e^{(ik-\epsilon)\tau} = -2\,\mathrm{Re}\,\frac{\gamma+\ln(\epsilon-ik)}{\epsilon-ik}\,. \tag{A5}$$

Thus the action of $\hat{u}$ on a test function f is

$$\hat{u}[f] = -2\lim_{\epsilon\to 0^+}\mathrm{Re}\int_{-\infty}^{\infty} dk\frac{\gamma+\ln(\epsilon-ik)}{\epsilon-ik} f(k)\,. \tag{A6}$$

The term involving γ is

$$-2\gamma\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} dk\frac{\epsilon}{k^2+\epsilon^2} f(k) = -2\pi\gamma f(0)\,. \tag{A7}$$

In the other term we integrate by parts,

$$-2\lim_{\epsilon\to 0^+}\mathrm{Re}\int_{-\infty}^{\infty}dk\frac{\ln(\epsilon-ik)}{\epsilon-ik}f(k) \;=\; -\lim_{\epsilon\to 0^+}\mathrm{Im}\int_{-\infty}^{\infty}dk\,f'(k)[\ln(\epsilon-ik)]^2 \tag{A8}$$

$$= -\mathrm{Im}\int_{-\infty}^{\infty}dk\,f'(k)[\ln|k|-i(\pi/2)\,\mathrm{sgn}\,k]^2 \tag{A9}$$

$$= \pi\int_{-\infty}^{\infty}dk\,f'(k)\ln|k|\,\mathrm{sgn}\,k\,, \tag{A10}$$

and thus

$$\hat{u}[f]=\pi\int_{-\infty}^{\infty}dk\,f'(k)\ln|k|\,\mathrm{sgn}\,k-2\pi\gamma f(0)\,. \tag{A11}$$

Since the Fourier transform of the constant γ is just $2\pi\gamma\delta(k)$, the transform of

$$w(\tau)=\ln|\tau|+\gamma \tag{A12}$$

is just

$$\hat{w}[f]=\pi\int_{-\infty}^{\infty}dk\,f'(k)\ln|k|\,\mathrm{sgn}\,k\,. \tag{A13}$$

Now

$$v(\tau)=\lim_{\epsilon\to 0}\ln(-(\tau-i\epsilon)^2)=2\ln|\tau|+\pi i\,\mathrm{sgn}\,\tau\,. \tag{A14}$$

The Fourier transform of sgn acts on f as [11]

$$2iP\int_{-\infty}^{\infty}dk\,\frac{f(k)}{k}=-2i\int_{-\infty}^{\infty}dk\,f'(k)\ln|k|\,, \tag{A15}$$

Putting Eqs. (A11,A15) in Eq. (A14) gives

$$\hat{v}[f]=4\pi\int_{0}^{\infty}dk\,f'(k)\ln|k|-4\pi\gamma f(0)\,. \tag{A16}$$

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