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Averaged null energy condition and quantum inequalities in curved spacetime

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Abstract

The Averaged Null Energy Condition (ANEC) states that the integral along a complete null geodesic of the projection of the stress-energy tensor onto the tangent vector to the geodesic cannot be negative. ANEC can be used to rule out spacetimes with exotic phenomena, such as closed timelike curves, superluminal travel and wormholes. We prove that ANEC is obeyed by a minimally-coupled, free quantum scalar field on any achronal null geodesic (not two points can be connected with a timelike curve) surrounded by a tubular neighborhood whose curvature is produced by a classical source. To prove ANEC we use a null-projected quantum inequality, which provides constraints on how negative the weighted average of the renormalized stress-energy tensor of a quantum field can be. Starting with a general result of Fewster and Smith, we first derive a timelike projected quantum inequality for a minimally-coupled scalar field on flat spacetime with a background potential. Using that result we proceed to find the bound of a quantum inequality on a geodesic in a spacetime with small curvature, working to first order in the Ricci tensor and its derivatives. The last step is to derive a bound for the null-projected quantum inequality on a general timelike path. Finally we use that result to prove achronal ANEC in spacetimes with small curvature.

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Chapter 1

Introduction

In the context of General Relativity, it is always possible to invent a spacetime with exotic features, such as wormholes, superluminal travel, or the construction of time machines, and then determine what stress-energy tensor is necessary to support the given spacetime according to Einstein's Equations (units: $\hbar = c = 1$)

$$G_{ab} = 8\pi T_{ab}\,. \tag{1.1}$$

However, in quantum field theory, there are restrictions on T_{ab} that could rule out exotic spacetimes. Two examples of these are quantum inequalities and energy conditions.

Quantum inequalities (also called Quantum Energy Inequalities) are bounds on the weighted time averages of the stress-energy tensor. They were first introduced by Ford [17] to prevent the violation of the second law of thermodynamics. The general form of a quantum inequality is

$$\int_{-\infty}^{\infty} d\tau\, f(t) T_{ab}(w(t)) V^a V^b > -B\,, \tag{1.2}$$

where $w(t)$ is a timelike path parameterized by proper time t with tangent vector V, and f is a sampling function. The quantity B is a bound, depending on the function f, the path w and the quantum field of interest.

Since then, they have been derived for a wide range of spacetimes, fields, and weighting functions. We concentrate here on quantum inequalities for minimally coupled scalar fields in curved spacetime. However, quantum inequalities for interacting fields have been derived in 1+1 dimensions [10, 4]. For systems with boundaries, there are difference quantum inequalities [18, 8], in which T_{ab} in Eq. (1.2) is replaced by the difference between T_{ab} in some state of interest and T_{ab} in a reference state. The bound B may also then depend on the reference state. However, such difference inequalities cannot be used to rule out exotic spacetimes, at least in the case where the exotic matter that supports the spacetime comes from the vacuum state in the presence of the boundaries.

Pointwise energy conditions bound the stress-energy tensor at individual spacetime points. One example is the Null Energy Condition (NEC) which requires that the null contracted stress-energy tensor cannot be negative,

$$T_{ab}\ell^a \ell^b \geq 0\,, \tag{1.3}$$

for ℓ^a a null vector. Classically, pointwise energy conditions seem reasonable, but in the quantum context they are violated. Quantum field theory allows arbitrary negative energy densities at individual points, a well known example being the Casimir effect. Even in the simple case of a minimally coupled free scalar field, all known pointwise energy conditions fail and even local averages must admit negative expectation values [7].

On the other hand, averaged energy conditions bound the stress-energy tensor integrated along a complete geodesic; they are weaker and have been proven to hold in a variety of spacetimes. One example is the Averaged Null Energy Condition (ANEC) which bounds the null-projected stress-energy tensor integrated along a null geodesic γ

$$\int_\gamma T_{ab}\ell^a\ell^b d\lambda \geq 0\,. \tag{1.4}$$

To rule out exotic spacetimes such as those with wormholes and closed timelike curves, we would like to prove energy conditions that restrict the stress-energy tensor that might arise from quantum fields and show that the stress-energy necessary to support these spacetimes is impossible. We need a condition which is strong enough to rule out exotic cases, while simultaneously weak enough to be proven correct.

The best possibility for such a condition seems to be the achronal ANEC [21], which requires that γ of Eq. (1.4) is a complete achronal null geodesic i.e., no two points of γ can be connected by a timelike curve. That is to say, we require that the projection of the stress-energy tensor along a null geodesic integrate to a non-negative value, but only for geodesics that are achronal. As far as we know, there is no example of achronal ANEC violation in spacetimes satisfying Einstein's equations with classical matter or free quantum fields as sources.[1] Achronal ANEC is sufficient to rule out many exotic spacetimes [21].

It has been proven that ANEC holds in Minkowski space and Ref. [11] showed that it also holds for geodesics traveling through empty, flat space, even if elsewhere in the spacetime there are boundaries or spacetime curvature, providing that these stay some minimum distance from the geodesic and do not affect the causal structure of the spacetime near the geodesic. This proof made use of quantum inequalities for null contractions of the stress tensor averaged over timelike geodesics [14].

This work however, does not really addresses the possibility of exotic spacetimes. The quantum inequalities on which it depends apply only in flat spacetime, so they cannot be used to rule out spacetimes with exotic curvature. For that, we need limits on the stress-energy tensor in curved spacetimes, the work presented here.

This thesis presents a complete proof of achronal ANEC for minimally coupled scalar fields in spacetimes with curvature in a classical background that obeys NEC, using a null projected quantum inequality. In the first chapter we present a general quantum inequality derived by Fewster and Smith [15] that we use in later chapters to derive a bound. In the second chapter we derive the bound for flat spacetime with a background potential [25], a case similar to the curved spacetime one. In chapter three we present the timelike projected quantum inequality in curved spacetime [26] and discuss the importance of this result. In chapter four we use that result to derive a null projected quantum inequality, which we proceed to use to prove achronal ANEC [23]. Finally, in the Appendix we present a new class of coordinates, called multi-step Fermi coordinates and use them to write the connection and the metric in terms of the curvature [24], results we use throughout this work.

We use the sign convention $(-,-,-)$ in the classification of Misner, Thorne and Wheeler [28]. Indices $a,b,c,\dots$ denote all spacetime coordinates while $i,j,k\dots$ denote only spatial

[1] Except for the case of non-minimally coupled quantum scalar fields.

coordinates. We denote normal derivatives with comma and covariant derivatives with semi-colon.

Chapter 2

Absolute quantum energy inequality

In this chapter we present a general quantum inequality derived by Fewster and Smith [15], result which we are going to use throughout the thesis. First we consider a minimally-coupled scalar field with the usual classical stress-energy tensor,

$$T_{ab} = \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \Phi \nabla_d \Phi + \frac{1}{2} g_{ab} \mu^2 . \tag{2.1}$$

where μ is the mass. Following Ref. [15], we define the renormalized stress-energy tensor

$$\langle T_{ab}^{\text{ren}} \rangle \equiv \lim_{x \to x'} T_{ab'}^{\text{split}} \left(\langle \phi(x) \phi(x') \rangle - H(x, x') \right) - Q g_{ab} + C_{ab} . \tag{2.2}$$

The quantities appear in Eq. (2.2) will be defined below. $T_{ab'}^{\text{split}}$ is the point-split energy density operator,

$$T_{ab'}^{\text{split}} = \nabla_a \otimes \nabla_{b'} - g_{ab'} g^{cd'} \nabla_c \otimes \nabla_{d'} + \frac{1}{2} g_{ab'} \mu^2 . \tag{2.3}$$

The point-split energy density operator acts on the difference between the two-point function and the Hadamard series,

$$\begin{aligned} H(x, x') &= \frac{1}{4\pi^2} \left[\frac{1}{\sigma_+(x, x')} + \sum_{j=0}^{\infty} v_j(x, x') \sigma_+^j(x, x') \ln \left(\frac{\sigma_+(x, x')}{l^2} \right) \right. \\ &\quad \left. + \sum_{j=0}^{\infty} w_j(x, x') \sigma^j(x, x') \right] , \end{aligned} \tag{2.4}$$

where σ is the squared invariant length of the geodesic between x and x', negative for timelike separation. In flat space

$$\sigma(x, x') = -\eta_{ab} (x - x')^a (x - x')^b . \tag{2.5}$$

By $F(\sigma_+)$, for some function F, we mean the distributional limit

$$F(\sigma_+) = \lim_{\epsilon \to 0^+} F(\sigma_\epsilon) , \tag{2.6}$$

where

$$\sigma_\epsilon(x, x') = \sigma(x, x') + 2i\epsilon(t(x) - t(x')) + \epsilon^2 . \tag{2.7}$$

In some parts of the calculation it is possible to assume that the points have only timelike separation, so we define

$$\tau = t - t' \tag{2.8}$$

and write

$$F(\sigma_+) = F(\tau_-) = \lim_{\epsilon\to 0} F(\tau_\epsilon)\,, \tag{2.9}$$

where

$$\tau_\epsilon = \tau - i\epsilon\,. \tag{2.10}$$

We have introduced a length l so that the argument of the logarithm in Eq. (2.4) is dimensionless. The possibility of changing this scale creates an ambiguity in the definition of H, but this ambiguity for curved spacetime can be absorbed into the ambiguity involving local curvature terms discussed below. The ambiguity in the case of a field in flat spacetime with background potential is discussed in Ch. 3. For simplicity of notation, we will assume we are working in units where $l = 1$.

The function Δ is the Van Vleck-Morette determinant bi-scalar, given by

$$\Delta(x, x') = -\frac{\det(\nabla_a \otimes \nabla_{b'}\sigma(x, x')/2)}{\sqrt{-g(x)}\sqrt{-g(x')}}\,. \tag{2.11}$$

The Hadamard coefficients are given by the Hadamard recursion relations, which are the solutions to

$$(\Box + \mu^2)H(x, x') = 0\,. \tag{2.12}$$

The recursion relations for the minimally coupled field in a curved background are [15]

$$(\Box + \mu^2)\Delta^{1/2} + 2v_{0,a}\sigma^{;a} + 4v_0 + v_0\Box\sigma = 0\,, \tag{2.13}$$

$$(\Box + \mu^2)v_j + 2(j+1)v_{j+1,a}\sigma^{;a} - 4j(j+1)v_{j+1} + (j+1)v_{j+1}\Box\sigma = 0\,, \tag{2.14}$$

$$2w_{1,a}\sigma^{;a} + w_1\Box\sigma + 2v_{1,a}\sigma^{;a} - 4v_1 + v_1\Box\sigma = 0\,, \tag{2.15}$$

$$\begin{aligned}&(\Box + \mu^2)w_j + 2(j+1)w_{j+1,a}\sigma^{;a} - 4j(j+1)w_{j+1} + (j+1)w_{j+1}\Box\sigma \\ &+2v_{j+1,a}\sigma^{;a} - 4(2j+1)v_{j+1} + v_{j+1}\Box\sigma = 0\,.\end{aligned} \tag{2.16}$$

All the v_j, and the w_j for $j \geq 1$ are determined by the differential equations discussed above, but w_0 is undetermined. Here we will follow Wald [40] and choose $w_0 = 0$.

From Ref. [15] we have the definition

$$\tilde{H}(x, x') = \frac{1}{2}\left[H(x, x') + H(x', x) + iE(x, x')\right]\,, \tag{2.17}$$

where iE is the antisymmetric part of the two-point function. We can write $H_j(x, x')$, $j = -1, 0, 1, \ldots$, to denote the term in H involving σ^j (with or without $\ln(\sigma_+)$), and $H_{(j)}$ to denote the sum of all terms up through H_j . We will split up $E(x, x')$ into terms labeled E_j that are proportional to σ^j, define a "remainder term"

$$R_j = E - \sum_{k=-1}^{j} E_k\,, \tag{2.18}$$

and let

$$\tilde{H}_j(x, x') = \frac{1}{2}\left[H_j(x, x') + H_j(x', x) + iE_j(x, x'))\right] \tag{2.19a}$$

$$\tilde{H}_{(j)}(x, x') = \frac{1}{2}\left[H_{(j)}(x, x') + H_{(j)}(x', x) + iE(x, x'))\right]\,. \tag{2.19b}$$

The term Q in Eq. (2.2) is the one introduced by Wald to preserve the conservation of the stress-energy tensor. Wald [38] calculated this term in the coincidence limit,

$$Q = \frac{1}{12\pi^2} w_1(x,x)\,. \tag{2.20}$$

The term C_{ab} handles the ambiguities in the definition of the stress-energy tensor T in curved spacetime. We will adopt the axiomatic definition given by Wald [40], but there remains the ambiguity of adding local curvature terms with arbitrary coefficients. From Ref. [3] we find that these terms include

$$^{(1)}H_{ab} \equiv \frac{\delta}{\sqrt{-g}\delta g^{ab}} \int \sqrt{-g} R^2 d^4x = 2R_{;ab} - 2g_{ab}\Box R - \frac{g_{ab}R^2}{2} + 2RR_{ab} \tag{2.21a}$$

$$^{(2)}H_{ab} \equiv \frac{\delta}{\sqrt{-g}\delta g^{ab}} \int \sqrt{-g} R^{ab}R_{ab} = R_{;ab} - \Box R_{ab} - \frac{g_{ab}\Box R}{2} - \frac{g_{ab}R^{cd}R_{cd}}{2} + 2R^{cd}R_{acbd}\,. \tag{2.21b}$$

Thus in Eq. (2.23) we must include a term given by a linear combination of Eqs. (2.21a) and (2.21b) to first order in R,

$$C_{ab} = a\,^{(1)}H_{ab} + b\,^{(2)}H_{ab} \tag{2.22}$$

where a and b are undetermined constants.[1]

A spacetime is globally hyperbolic when it contains a Cauchy surface, a subset of spacetime which is intersected by every causal curve exactly once. Global hyperbolicity requires the existence of unique advanced and retarded Green functions. We define $w(t)$, a timelike path contained in a globally hyperbolic convex [2] normal neighborhood N and for this path we can state the quantum inequality of Ref. [15].

$$\int_{-\infty}^{\infty} dt\, g(t)^2 \langle T^{\text{ren}}_{ab} V^a V^b\rangle_\omega w(t) \geq -\int_0^\infty \frac{d\xi}{\pi} \left[g \otimes g(\theta^* T^{\text{split}}_{ab'} V^a V^{b'} \tilde{H}_{(5)})\right]^\wedge (-\xi,\xi) + \int_{-\infty}^{\infty} dt\, g^2(t)(-Q g_{ab} + C_{ab}) V^a V^b\,. \tag{2.23}$$

The Fourier transform convention we use is

$$\hat{f}(k) \text{ or } f^\wedge[k] = \int_{-\infty}^{\infty} dx f(x) e^{ixk}\,. \tag{2.24}$$

In the inequality we take the transform with respect to both arguments and evaluate at ξ and $-\xi$. The operator θ^* denotes the pullback of the function to the geodesic,

$$(\theta^* T^{\text{split}}_{ab'} \tilde{H}_{(5)})(t,t') \equiv (T^{\text{split}}_{ab'} \tilde{H}_{(5)})(w(t), w(t'))\,, \tag{2.25}$$

[1] There are also ambiguities corresponding to adding multiples of the metric and the Einstein tensor to the stress tensor. The first can be considered renormalization of the cosmological constant and the second renormalization of Newton's constant. We will assume that these renormalization have been performed, and that the cosmological constant is considered part of the gravitational sector, so neither of these affects T_{ab}.

[2] A convex normal neighborhood N is one such that any point $q \in N$ can be connected to any other point $p \in N$ with a unique geodesic totally contained in N. For more detailed discussion of normal neighborhoods and their properties see Ref. [22]

and the subscript (5) means that we include only terms through $j = 5$ in the sums of Eq. (2.4). However, we will prove that terms of order $j > 1$ make no contribution to Eq. (2.23).

We will use the general inequality of Eq. (2.23) to provide a bound for the integral of the renormalized stress-energy tensor in three different cases. In Ch. 3 for the timelike projected $T_{\mu\nu}$ in flat spacetime with a background potential, in Ch. 4 for the energy density in curved spacetime and in Ch. 5 for the null projected stress-energy tensor in spacetimes with curvature.

Chapter 3

Quantum Inequality for a scalar field with a background potential

As a first step toward deriving a bound for the quantum inequality in a spacetime with bounded curvature we first derive a quantum inequality in a flat spacetime with a background potential, i.e., a field with a mass depending on spacetime position. This is a simpler system that has many of the important features of quantum fields in curved spacetime. For a scalar field Φ in a background potential, the Lagrangian is

$$L = \frac{1}{2}\left[\partial_\mu \Phi \partial^\mu \Phi - V(x)\Phi^2\right] , \tag{3.1}$$

the equation of motion is

$$(\Box + V(x))\Phi = 0 , \tag{3.2}$$

and the classical energy density is

$$T_{tt} = \frac{1}{2}\left[(\partial_t \Phi)^2 + (\nabla \Phi)^2 + V(x)\Phi^2\right] . \tag{3.3}$$

So the $T^{\rm split}_{ab'}$ operator of Eq. (2.3) contracted with timelike vectors, for a scalar field with background potential V becomes

$$T^{\rm split}_{tt'} = \frac{1}{2}\left[\sum_{a=0}^{3} \partial_a \partial_{a'} + \frac{V(x)+V(x')}{2}\right] , \tag{3.4}$$

where the potential is analogous to the mass of Eq. (2.3). Since we will take the limit where x and x' coincide the location of evaluation of V does not matter, but the form above will be convenient later. The renormalized stress-energy tensor in this case is

$$\langle T^{\rm ren}_{tt}(x')\rangle \equiv \lim_{x\to x'} T^{\rm split}_{tt'}\left(\langle \phi(x)\phi(x')\rangle - H(x,x')\right) - Q(x') , \tag{3.5}$$

As discussed in Chapter 2 there is an ambiguity in the above procedure. In order to take logarithms, we must divide σ by the square of some length scale l. Changing the scale to some other scale l' decreases H by $\delta H = 2(v_0 + v_1\sigma + \cdots)\ln(l'/l)$. This results in increasing T_{ab} by $\lim_{x\to x'}(\partial_a\partial_b - (1/2)\eta_{ab}\partial^c\partial_c)\delta H$. Using the values for v_0 and v_1 computed below, this becomes $(1/12)(V_{,ab} - \eta_{ab}\Box V)\ln(l'/l)$. Thus we see that the definition of T_{ab} must include arbitrary multiple of $(V_{,ab} - \eta_{ab}\Box V)$. This ambiguity can also be understood as the possibility of including in the Lagrangian density a term of the form $R(x)V(x)$, where R is the scalar curvature. Varying the metric to obtain T_{ab} and then going to flat space yields the above term. The situation is very much analogous to the possible addition of terms of the form R^2 and $R_{ab}R^{ab}$ in the case of a field in curved spacetime, which give rise to the local curvature terms in Eq. (2.22).

Thus we rewrite Eq. (3.5) to include the ambiguous term,

$$\langle T_{tt}^{\text{ren}}(x')\rangle \equiv \lim_{x\to x'} T^{\text{split}}\left(\langle\phi(x)\phi(x')\rangle - H(x,x')\right) - Q(x') + CV_{,ii}\,, \tag{3.6}$$

where C is some constant. Whatever definition of T_{tt} one is trying to use, one can pick an arbitrary scale l and adjust C accordingly.

So the quantum inequality of Eq. (2.23) becomes

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle T_{00}^{\text{ren}}\rangle(t,0) \geq -B\,, \tag{3.7}$$

where

$$B = \int_0^{\infty} \frac{d\xi}{\pi} \hat{F}(-\xi,\xi) + \int_{-\infty}^{\infty} dt\, g^2(t)(Q - CV_{,ii})\,, \tag{3.8}$$

and

$$F(t,t') = g(t)g(t')T^{\text{split}}\tilde{H}_{(5)}((t,0),(t',0))\,, \tag{3.9}$$

$\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (2.24).

We work only in first order in V but don't otherwise assume that it is small. We can express the maximum values of the background potential and its derivatives as

$$\begin{array}{cc} |V| \leq V_{\max} & |V_{,a}| \leq V'_{\max} \\ |V_{,ab}| \leq V''_{\max} & |V_{,abc}| \leq V'''_{\max}\,, \end{array} \tag{3.10}$$

where $V_{\max}$, $V'_{\max}$, $V''_{\max}$ and $V'''_{\max}$ are positive numbers, finite but not necessarily small.

First, we discuss the $T_{tt'}^{\text{split}}$ operator, then we compute the Hadamard series and we apply the operator. After that we perform the Fourier transform, leading to the final quantum inequality.

3.1 General considerations

We will now compute the quantum inequality bound B to first order in the potential V and its derivatives. In the next subsection, we will make some general remarks about terms in $\tilde{H}$ coming from H, which are symmetrical under the exchange of x and x', and show that we need keep terms only through first order, not fifth order as in Eq. (2.23). Then we will simplify the operator $T_{tt'}^{\text{split}}$ defined in Eq. (3.4). In the next section, we will compute, order by order, the terms and $\tilde{H}$. We will then take the Fourier transform to find the quantum inequality bound.

3.1.1 Smooth, symmetrical contributions

Define $\bar{x} = (x - x')/2$, $\bar{t} = (t + t')/2$ and $\tau = t - t'$. Let

$$A(\tau) = \int_{-\infty}^{\infty} d\bar{t}\, F\left(\bar{t} + \frac{\tau}{2}, \bar{t} - \frac{\tau}{2}\right)\,. \tag{3.11}$$

so that $\hat{F}(-\xi,\xi) = \hat{A}(-\xi)$. The presence of the g functions in Eq. (3.9) makes F have compact support in t and t', so A has compact support in τ.

Suppose F contains some term f that is symmetrical in t and t'. Let a be the corresponding term in A according to Eq. (3.11). Then a will be even in τ, so $\hat{a}$ will be even also. If $a \in C^1$, then $\hat{a} \in L^2$, and we can perform the integral of this term separately, giving an inverse Fourier transform,

$$\int_0^\infty \frac{d\xi}{\pi} \hat{f}(-\xi,\xi) = \int_{-\infty}^\infty \frac{d\xi}{2\pi} \hat{a}(\xi) = a(0)\,. \tag{3.12}$$

In particular, if

$$\lim_{t'\to t} f(t,t') = f(t)\,, \tag{3.13}$$

then

$$\int_0^\infty \frac{d\xi}{\pi} \hat{f}(-\xi,\xi) = \int_{-\infty}^\infty dt\, g(t)^2 f(t)\,, \tag{3.14}$$

and if $f(t) = 0$ there is no contribution.

Terms arising from H appear symmetrically in $\tilde{H}$. At orders $j > 1$ they have at least 4 powers of τ, so they vanish in the coincidence limit even when differentiated twice by the operators of $T^{\rm split}_{tt'}$. Thus such terms make no contribution to Eq. (4.4).

3.1.2 Simplification of $T^{\rm split}_{tt'}$

We would like to write the operator $T^{\rm split}_{tt'}$ of Eq. (3.4) in terms of separate derivatives on the center point $\bar{x}$ and the difference between the points. First we separate the derivatives in $T^{\rm split}_{tt'}$ into time and space,

$$\sum_{a=0}^{3} \partial_a \partial_{a'} = \partial_t \partial_{t'} + \nabla_x \cdot \nabla_{x'}\,. \tag{3.15}$$

We can write the spatial derivative with respect to[1] $\bar{x}$ in terms of the derivatives at the endpoints,

$$\nabla^2_{\bar{x}} = \nabla^2_x + 2\nabla_x \cdot \nabla_{x'} + \nabla^2_{x'}\,. \tag{3.16}$$

Then Eqs. (3.4), (3.15),(3.16) give

$$\begin{aligned} T^{\rm split}_{tt'} &= \frac{1}{2}\left[\partial_t\partial_{t'} + \frac{1}{2}\left(\nabla^2_{\bar{x}} - \nabla^2_x - \nabla^2_{x'}\right) + \frac{1}{2}\left(V(x) + V(x')\right)\right] = \\ &= \frac{1}{4}\left[\nabla^2_{\bar{x}} + \Box_x - \partial_t^2 + \Box_{x'} - \partial^2_{t'} + 2\partial_t\partial_{t'} + V(x) + V(x')\right]\,, \end{aligned} \tag{3.17}$$

where $\Box_x$ and $\Box_{x'}$ denote the D'Alembertian operator with respect to x and x'. Then using

$$\partial^2_\tau = \frac{1}{4}\left[\partial^2_t - 2\partial_t\partial_{t'} + \partial^2_{t'}\right]\,, \tag{3.18}$$

we can write

$$T^{\rm split}_{tt'}\tilde{H} = \frac{1}{4}\left[\left(\Box_x + V(x)\right)\tilde{H} + \left(\Box_{x'} + V(x')\right)\tilde{H} + \nabla^2_{\bar{x}}\tilde{H}\right] - \partial^2_\tau \tilde{H}\,. \tag{3.19}$$

[1] When a derivative is with respect to t or t' (or x or x'), we mean to keep the other of these fixed, while when the derivative is with respect to $\bar{t}$ or τ, we mean to keep the other of these fixed. When the derivative is with respect to $\bar{x}$ we mean to keep $x - x'$ fixed.

Consider the first term. The function $H(x, x')$ obeys the equation of motion in x, and so does $E(x, x')$. Thus all of $\tilde{H}(x, x')$ is annihilated by $\Box_x + V(x)$, except for $H(x', x)$,

$$(\Box_x + V(x))\,\tilde{H} = \frac{1}{2}(\Box_x + V)H(x', x)\,. \tag{3.20}$$

The quantities $v_j(x, x')$ are symmetric, so the only sources of asymmetry in H are the functions w_j, which are real but may not be symmetric, and the fact that the imaginary part of $\sigma_+(x, x')$ is antisymmetric. Thus

$$H(x', x) = H(x, x')^* + \frac{1}{4\pi^2}\sum_j (w_j(x', x) - w_j(x, x'))\sigma^j(x, x')\,, \tag{3.21}$$

where $*$ denotes complex conjugation. Since $\Box_x$ is real, if we use Eq. (3.21) in Eq. (3.20), we have $(\Box_x + V)H(x, x')^* = 0$, and we ignore Vw_j, because it is second order in V, leaving

$$(\Box_x + V(x))\,\tilde{H} = \frac{1}{4\pi^2}\Box_x \sum_j (w_j(x', x) - w_j(x, x'))\sigma^j(x, x')\,. \tag{3.22}$$

By the same argument,

$$(\Box_{x'} + V(x'))\,\tilde{H} = \frac{1}{4\pi^2}\Box_{x'} \sum_j (w_j(x, x') - w_j(x', x))\sigma^j(x, x')\,. \tag{3.23}$$

The first two terms in the brackets in Eq. (3.19) are the sum of Eqs. (3.22), (3.23). This sum is smooth, symmetric in x and x', and vanishes in the coincidence limit. Thus according to the analysis of Sec. 3.1.1, it makes no contribution in the Fourier transform of Eq. (4.4), and for our purposes we can take

$$T_{tt'}^{\text{split}}\tilde{H} = \left[\frac{1}{4}\nabla_{\bar{x}}^2 - \partial_\tau^2\right]\tilde{H}\,. \tag{3.24}$$

3.2 Computation of $\tilde{H}$

Examining Eq. (4.15) we see that is sufficient to compute $\tilde{H}$ for purely temporal separation as a function of t, t', and $\mathbf{x}$, the common spatial position of the points. The function $H(t, t')$ is a series of terms with decreasing degree of singularity at coincidence: τ^{-2}, $\ln\tau$, $\tau^2\ln\tau$, etc. For the first term in Eq. (4.15), terms in H that have any positive powers of τ will not contribute by the analysis of Sec. 3.1.1. For the second term we need to keep terms in H up to order τ^2, because the derivatives will reduce the order by 2.

The symmetrical combination $H(t, t') + H(t', t)$, will lead to something whose Fourier transform does not decline rapidly for positive ξ, so that if this alone were put into Eq. (4.4) the integral over ξ would not converge. But each term in $H(t, t') + H(t', t)$ will combine with a term coming from $iE(x, x')$ to give something whose Fourier transform does decline rapidly.

We will work order by order in τ.

3.2.1 General computation of E

We will need the Green's functions for the background potential, including only first order in V, so we write

$$G = G^{(0)} + G^{(1)} + \cdots\,. \tag{3.25}$$

The equation of motion is

$$(\Box + V(x))G(x,x') = \delta^{(4)}(x-x')\,. \tag{3.26}$$

Using $\Box G^{(0)}(x,x') = \delta^{(4)}(x,x')$ and keeping only first-order terms we have

$$\Box G^{(1)}(x,x') = -V(x)G^{(0)}(x,x')\,, \tag{3.27}$$

so

$$G^{(1)}(x,x') = -\int d^4x'' G^{(0)}(x,x'')V(x'')G^{(0)}(x'',x')\,. \tag{3.28}$$

For $t > t'' > t'$ we have for the retarded Green's function,

$$G_R^{(0)}(x'',x') = \frac{1}{2\pi}\delta((t''-t')^2 - |\mathbf{x}''-\mathbf{x}'|^2) = \frac{1}{4\pi}\frac{\delta(t''-t'-|\mathbf{x}''-\mathbf{x}'|)}{|\mathbf{x}''-\mathbf{x}'|}\,. \tag{3.29}$$

So we can write

$$G_R^{(1)}(x,x') = -\frac{1}{8\pi^2}\int d^3\mathbf{x}''\int dt''\delta((t-t'')^2-|\mathbf{x}-\mathbf{x}''|^2)\frac{\delta(t''-t'-|\mathbf{x}''-\mathbf{x}'|)}{|\mathbf{x}''-\mathbf{x}'|}V(t'',\mathbf{x}'')\,. \tag{3.30}$$

Integrating over the second delta function we find $t'' = t' + |\mathbf{x}''-\mathbf{x}'|$. Again considering purely temporal separation and defining $\zeta'' = \mathbf{x}''-\mathbf{x}'$ and $\zeta'' = |\zeta''|$, we find

$$G_R^{(1)}(t,t') = -\frac{1}{8\pi^2}\int d\Omega\int d\zeta''\zeta''^2\frac{\delta(\tau^2-2\tau\zeta'')}{\zeta''}V(t'+\zeta'',\mathbf{x}'+\zeta''\hat{\Omega})\,, \tag{3.31}$$

where $\int d\Omega$ denotes integration over solid angle, and $\hat{\Omega}$ varies over all unit vectors. We can integrate over ζ'' to get $\zeta'' = \tau/2$ and

$$G_R^{(1)}(t,t') = -\frac{1}{32\pi^2}\int d\Omega\, V(\bar{t},\mathbf{x}'+\frac{\tau}{2}\hat{\Omega})\,. \tag{3.32}$$

If we define a 4-vector $\Omega = (0,\hat{\Omega})$ we can write

$$G_R^{(1)}(t,t') = -\frac{1}{32\pi^2}\int d\Omega\, V(\bar{x}+\frac{\tau}{2}\Omega)\,. \tag{3.33}$$

The advanced Green's functions are the same with t and t' reversed. Since E is the advanced minus the retarded function, we have

$$E^{(1)}(t,t') = \frac{1}{32\pi^2}\int d\Omega\, V(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\,. \tag{3.34}$$

3.2.2 Terms of order τ^{-2}

We now compute the various H_j, $\tilde{H}_j$, and E_j, starting with terms that go as σ^{-1} or τ^{-2}. These terms are exactly what one would have for flat space without potential. Equation (2.4) gives

$$H_{-1}(x,x') = \frac{1}{4\pi^2\sigma_+(x,x')} = -\frac{1}{4\pi^2(\tau_-^2-\zeta^2)}\,, \tag{3.35}$$

where

$$\boldsymbol{\zeta} = \mathbf{x} - \mathbf{x}' \tag{3.36}$$

and

$$\zeta = |\boldsymbol{\zeta}| \,. \tag{3.37}$$

Similarly, the advanced minus retarded Green's function to this order is

$$E_{-1}(x, x') = G_A(x, x') - G_R(x, x') = \frac{\delta(\tau - \zeta) - \delta(\tau + \zeta)}{4\pi\zeta} \,, \tag{3.38}$$

so

$$\tilde{H}_{-1}(t, t') = \lim_{\zeta \to 0} \frac{1}{8\pi^2} \left[-\frac{1}{\tau_+^2 - \zeta^2} - \frac{1}{\tau_-^2 - \zeta^2} + i\pi \frac{\delta(\tau + \zeta) - \delta(\tau - \zeta)}{\zeta} \right] \,, \tag{3.39}$$

where

$$F(\tau_+) = \lim_{\epsilon \to 0} F(\tau + i\epsilon) \,. \tag{3.40}$$

Taking the $\epsilon \to 0$ limit in τ_+ and τ_- gives the formula

$$-\frac{1}{\tau_+^2 - \zeta^2} + \frac{1}{\tau_-^2 - \zeta^2} = -i\pi \frac{\delta(\tau + \zeta) - \delta(\tau - \zeta)}{\zeta} \tag{3.41}$$

so

$$\tilde{H}_{-1}(t, t') = -\frac{1}{4\pi^2 \tau_-^2} = H_{-1}(t, t') \tag{3.42}$$

as discussed in Ref [15].

3.2.3 Terms with no powers of τ

We can find the Hadamard coefficients from the recursion relations, Eq. (2.13), (2.14). To find the zeroth order of the Hadamard series we need only v_0. For flat space, $\sigma_{,a} = -2\eta_{ab}(x'' - x')^b$ and $\Box\sigma = -8$. Putting these in Eq. (2.13) we have

$$(x'' - x')^a v_{0,a} + v_0 = \frac{V(x'')}{4} \,, \tag{3.43}$$

Now let $x'' = x' + \lambda(x - x')$ to integrate along the geodesic going from x' to x. We observe that

$$\frac{dv_0(x'', x')}{d\lambda} = (x - x')^a v_{0,a}(x'', x') \,. \tag{3.44}$$

So Eq. (3.43) gives

$$\lambda \frac{dv_0(x'', x')}{d\lambda} + v_0(x'', x') = \frac{V(x'')}{4} \,, \tag{3.45}$$

or

$$\frac{d(\lambda v_0(x'', x'))}{d\lambda} = \frac{V(x'')}{4} \,, \tag{3.46}$$

from which we immediately find

$$v_0(x, x') = \int_0^1 d\lambda \frac{V(x' + \lambda(x - x'))}{4} \,. \tag{3.47}$$

Now we consider purely temporal separation so the background potential is evaluated at $(t' + \lambda\tau, \mathbf{x})$. We expand V in a Taylor series in τ around 0 with $\bar{t}$ fixed,

$$V(t' + \lambda\tau) = V(\bar{t}) + \tau(\lambda - \frac{1}{2})V_{,t}(\bar{t}) + \frac{\tau^2}{2}(\lambda - \frac{1}{2})^2 V_{,tt}(\bar{t}) + \cdots . \tag{3.48}$$

We are calculating the zeroth order so we keep only the first term of Eq. (3.48), and Eq. (3.47) gives

$$v_0(t, t') = \frac{1}{4}V(\bar{t}) + O(\tau^2) \tag{3.49}$$

and thus

$$H_0(x, x') = \frac{1}{16\pi^2} V(\bar{x}) \ln(-\tau_-^2) , \tag{3.50}$$

and

$$H_0(x, x') + H_0(x', x) = \frac{1}{4\pi^2} V(\bar{x}) \ln|\tau| . \tag{3.51}$$

We can expand V around $\bar{x}$,

$$V(\bar{x} + \frac{\tau}{2}\Omega) = V(\bar{x}) + V^{(1)}(\bar{x} + \frac{\tau}{2}\Omega) , \tag{3.52}$$

where $V^{(1)}$ is the remainder of the Taylor series

$$V^{(1)}(\bar{x} + \frac{\tau}{2}\Omega) = V(\bar{x} + \frac{\tau}{2}\Omega) - V(\bar{x}) = \int_0^{\tau/2} dr\, V_{,i}(\bar{x} + r\Omega)\Omega^i . \tag{3.53}$$

Then from Eq. (3.34),

$$E_0(x, x') = \frac{1}{8\pi} V(\bar{x}) \operatorname{sgn}\tau \tag{3.54a}$$

$$R_0(x, x') = \frac{1}{32\pi^2} \int d\Omega\, V^{(1)}(\bar{x} + \frac{|\tau|}{2}\Omega) \operatorname{sgn}\tau . \tag{3.54b}$$

Using

$$2\ln|\tau| + \pi i \operatorname{sgn}\tau = \ln(-\tau_-^2) , \tag{3.55}$$

we combine Eqs. (3.51), (3.54a) to find

$$\tilde{H}_0(t, t') = \frac{1}{16\pi^2} V(\bar{x}) \ln(-\tau_-^2) . \tag{3.56}$$

Combining all terms through order 0 gives

$$\tilde{H}_{(0)}(t, t') = \tilde{H}_{-1}(t, t') + \tilde{H}_0(t, t') + \frac{1}{2} i R_0(t, t') . \tag{3.57}$$

3.2.4 Terms of order τ^2

Now we compute the terms of order τ^2 in H and E. First we need v_0 at this order, so we use Eq. (3.48) in Eq. (3.47). The $V_{,t}$ term in Eq. (3.48) does not contribute, because it is odd in $\lambda - 1/2$, and the others give

$$v_0(x, x') = \frac{1}{4}V(\bar{x}) + \tau^2 \frac{1}{96} V_{,tt}(\bar{x}) + \cdots . \tag{3.58}$$

Next we need to know v_1, but since v_1 is multiplied by τ^2 in H, we need only the τ-independent term $v_1(x,x)$. From Eq. (2.14),

$$(\Box + V(x))v_0(x,x') + 2\eta^{ab}v_{1,a}(x,x')\sigma_{,b}(x,x') + v_1(x,x')\Box_x\sigma(x,x') = 0\,. \qquad (3.59)$$

We neglect the $V(x)v_0$ term because it is second order in V. At $x = x'$, $\sigma_{,b} = 0$, so

$$v_1(x,x) = \frac{1}{8}\lim_{x'\to x}\Box_x v_0(x,x')\,. \qquad (3.60)$$

Using Eq. (4.62) we find

$$\Box_x v_0(x,x') = \frac{1}{4}\int_0^1 d\lambda \Box_x V(x' + \lambda(x-x')) = \frac{1}{4}\int_0^1 d\lambda\, \lambda^2(\Box V)(x' + \lambda(x-x'))\,, \qquad (3.61)$$

and Eq. (3.60) gives

$$v_1(x,x) = \frac{1}{96}\Box V(\bar{x})\,. \qquad (3.62)$$

We also need to know w_1, but again only at coincidence. Reference [38] gives

$$w_1(x,x) = -\frac{3}{2}v_1(x,x) = -\frac{1}{64}\Box V(x)\,. \qquad (3.63)$$

Combining the second term of Eq. (3.58) with Eqs. (3.62), (3.63) gives

$$H_1(t,t') = \frac{\tau^2}{128\pi^2}\left[\frac{1}{3}V_{,ii}(\bar{x})\ln(-\tau_-^2) + \frac{1}{2}\Box V(\bar{x})\right]\,. \qquad (3.64)$$

Then $H_1(x',x)$ is given by symmetry, so

$$H_1(x,x') + H_1(x',x) = \frac{\tau^2}{64\pi^2}\left[\frac{2}{3}V_{,ii}(\bar{x})\ln|\tau| + \frac{1}{2}\Box V(\bar{x})\right]\,. \qquad (3.65)$$

The calculation of E_1 is similar to that of E_0, but now we have to include more terms in the Taylor expansion of V around $\bar{x}$. So we expand

$$V(\bar{x} + \frac{\tau}{2}\Omega) = V(\bar{x}) + \frac{1}{2}V_{,i}(\bar{x})\Omega^i\tau + \frac{1}{8}V_{,ij}(\bar{x})\Omega^i\Omega^j\tau^2 + V^{(3)}(\bar{x} + \frac{\tau}{2}\Omega)\,, \qquad (3.66)$$

where the remainder of the Taylor series $V^{(3)}$ is

$$V^{(3)}(\bar{x} + \frac{\tau}{2}\Omega) = \frac{1}{2}\int_0^{\tau/2} dr\, V_{,ijk}(\bar{x} + r\Omega)\left(\frac{\tau}{2} - r\right)^2 \Omega^i\Omega^j\Omega^k dr\,. \qquad (3.67)$$

Since $\int d\Omega\,\Omega^i = 0$ and $\int d\Omega\,\Omega^i\Omega^j = (4\pi/3)\delta^{ij}$, Eq. (3.34) gives

$$E_1(x,x') = \frac{1}{192\pi}V_{,ii}(\bar{x})\tau^2\,\mathrm{sgn}\,\tau\,, \qquad (3.68a)$$

$$R_1(x,x') = \frac{1}{32\pi^2}\int d\Omega\, V^{(3)}(\bar{x} + \frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\,. \qquad (3.68b)$$

Again using Eq. (3.55), we combine Eqs. (3.65),(3.68a) to get

$$\tilde{H}_1(x,x') = \frac{\tau^2}{128\pi^2}\left[\frac{1}{3}\ln(-\tau_-^2)V_{,ii} + \frac{1}{2}\Box V(\bar{x})\right]\,. \qquad (3.69)$$

Combining all terms through order 1 gives

$$\tilde{H}_{(1)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \tilde{H}_1(t,t') + \frac{1}{2}iR_1(t,t')\,. \qquad (3.70)$$

3.3 The $T^{\text{split}}_{tt'}\tilde{H}$

Using Eqs. (3.9), (4.15), we need to compute

$$\int_0^\infty \frac{d\xi}{\pi}\hat{F}(-\xi,\xi')\,, \tag{3.71}$$

where

$$F(t,t') = g(t)g(t')\left[\frac{1}{4}\nabla^2_{\bar{x}}\tilde{H}_{(0)}(t,t') - \partial^2_\tau \tilde{H}_{(1)}(t,t')\right]\,. \tag{3.72}$$

Using Eqs. (3.42), (3.54b), (3.56), (3.57), (3.68b, (3.69), (3.70) we can write this

$$F(t,t') = g(t)g(t')\sum_{i=1}^{6} f_i(t,t')\,, \tag{3.73}$$

with

$$f_1 = \frac{3}{2\pi^2\tau_-^4} \tag{3.74a}$$

$$f_2 = \frac{1}{8\pi^2\tau_-^2}V(\bar{x}) \tag{3.74b}$$

$$f_3 = \frac{1}{96\pi^2}V_{,ii}(\bar{x})\ln\left(-\tau_-^2\right) \tag{3.74c}$$

$$f_4 = -\frac{1}{128\pi^2}\left[V_{,tt}(\bar{x}) + V_{,ii}(\bar{x})\right] \tag{3.74d}$$

$$f_5 = \frac{1}{256\pi^2}\int d\Omega\,\nabla^2_{\bar{x}}\left[V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\right] i\,\mathrm{sgn}\,\tau \tag{3.74e}$$

$$f_6 = -\frac{1}{64\pi^2}\int d\Omega\,\partial^2_\tau\left[V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega) i\,\mathrm{sgn}\,\tau\right]\,. \tag{3.74f}$$

3.4 The Fourier transform

We want to calculate the quantum inequality bound B, given by Eq. (3.8). We can write it

$$B = \sum_{i=1}^{8} B_i\,, \tag{3.75}$$

where

$$\begin{aligned} B_i &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt' g(t)g(t')f_i(t,t')e^{i\xi(t'-t)} \\ &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} d\tau\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})f_i(\bar{t},\tau)e^{-i\xi\tau}, i=1\ldots 6 \end{aligned} \tag{3.76a}$$

$$B_7 = \int_{-\infty}^{\infty} dt\, g^2(t)Q(t) = -\frac{1}{768\pi^2}\int_{-\infty}^{\infty} dt\, g^2(t)\Box V(t) \tag{3.76b}$$

$$B_8 = -\int_{-\infty}^{\infty} dt\, g^2(t)CV_{,ii}(t)\,, \tag{3.76c}$$

using Eqs. (2.20), (4.4) and (3.63).

3.4.1 The singular terms

For $i = 1, 2, 3$, f_i consists of a singular function of τ times a function of $\bar{t}$ (or a constant). So we will separate the singular part by writing

$$f_i(\bar{t}, \tau) = g_i(\bar{t})s_i(\tau)\,. \tag{3.77}$$

Then we define

$$G_i(\tau) = \int_{-\infty}^{\infty} d\bar{t}\, g_i(\bar{t})g(\bar{t} - \frac{\tau}{2})g(\bar{t} + \frac{\tau}{2})\,, \tag{3.78}$$

so

$$B_i = \int_0^{\infty} \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau\, G_i(\tau)s_i(\tau)e^{-i\xi\tau}\,. \tag{3.79}$$

This is a Fourier transform of a product, so we can write it as a convolution. The G_i are all real, even functions, and thus their Fourier transforms are also, and we have

$$B_i = \frac{1}{2\pi^2} \int_0^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta\, \hat{G}_i(\xi + \zeta)\hat{s}_i(\zeta)\,. \tag{3.80}$$

Now if we change the order of integrals we can perform another change of variables $\eta = \xi + \zeta$, so we have

$$B_i = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\zeta \int_{\zeta}^{-\infty} d\eta\, \hat{G}_i(\eta)\hat{s}_i(\zeta) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\eta \hat{G}_i(\eta)h_i(\eta)\,, \tag{3.81}$$

where

$$h_i(\eta) = \int_{-\infty}^{\eta} d\zeta\, \hat{s}_i(\zeta)\,. \tag{3.82}$$

The arguments of Ref. [15] show that the integrals over ξ in Eq. (3.79) and η in Eq. (3.82) converge.

We now calculate the Fourier transforms in turn, starting with B_1. We have

$$g_1(\bar{t}) = \frac{3}{2\pi^2} \tag{3.83a}$$

$$s_1(\tau) = \frac{1}{\tau_-^4}\,. \tag{3.83b}$$

The Fourier transform of s_1 is [20]

$$\hat{s_1}(\zeta) = \frac{\pi}{3}\zeta^3\Theta(\zeta)\,, \tag{3.84}$$

so

$$h_1(\eta) = \int_0^{\eta} d\zeta\, \frac{\pi}{3}\zeta^3\Theta(\eta) = \frac{\pi}{12}\eta^4\Theta(\eta)\,. \tag{3.85}$$

From Eq. (3.81) we have

$$B_1 = \frac{1}{24\pi} \int_0^{\infty} d\eta\, \hat{G}_1(\eta)\eta^4\,. \tag{3.86}$$

Using $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$, we get

$$B_1 = \frac{1}{24\pi} \int_0^{\infty} d\eta\, \widehat{G_1''''}(\eta)\,. \tag{3.87}$$

The function G_1 is even, so its Fourier transform is also even and we can extend the integral

$$B_1 = \frac{1}{48\pi}\int_{-\infty}^{\infty} d\eta\, \widehat{G_1''''}(\eta) = \frac{1}{24}G_1''''(0)\,. \tag{3.88}$$

For G_1 we have

$$G_1(\tau) = \frac{3}{2\pi^2}\int d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,, \tag{3.89}$$

and taking the derivatives and integrating by parts gives

$$B_1 = \frac{1}{16\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, g''(\bar{t})^2\,, \tag{3.90}$$

reproducing a result of Ref. [15].

For B_2 we have

$$g_2(\bar{t}) = \frac{1}{8\pi^2}V(\bar{t}) \tag{3.91a}$$

$$s_2(\tau) = \frac{1}{\tau_-^2}\,. \tag{3.91b}$$

This calculation is the same as before except the Fourier transform of s_2 is [20]

$$\hat{s}_2(\zeta) = -2\pi\zeta\Theta(\zeta)\,. \tag{3.92}$$

So we have

$$B_2 = \frac{1}{2}G_2''(0)\,, \tag{3.93}$$

where

$$G_2(\tau) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t})g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,. \tag{3.94}$$

After taking the derivatives

$$B_2 = \frac{1}{32\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t})[g(\bar{t})g''(\bar{t}) - g'(\bar{t})^2]\,. \tag{3.95}$$

For B_3 we have

$$s_3(\tau) = \ln(-\tau_-^2)\,. \tag{3.96}$$

In the appendix, we find the Fourier transform of s_3 as a distribution,

$$\hat{s}_3[f] = 4\pi\int_0^{\infty} dk\, f'(k)\ln|k| - 4\pi\gamma f(0)\,. \tag{3.97}$$

From Eq. (3.82), we can write

$$h_3(\eta) = \int_{-\infty}^{\infty} d\zeta\, \hat{s}_3(\zeta)\Theta(\eta-\zeta)\,, \tag{3.98}$$

which is given by Eq. (3.97) with $f(\zeta) = \Theta(\eta-\zeta)$, so

$$h_3(\eta) = -4\pi\int_0^{\infty} d\zeta\, \delta(\eta-\zeta)\ln|\zeta| - 4\pi\gamma\Theta(\eta) = -4\pi\Theta(\eta)(\ln\eta+\gamma)\,. \tag{3.99}$$

Then Eq. (3.81) gives

$$B_3 = -\frac{2}{\pi}\int_0^\infty d\eta\,\hat{G}_3(\eta)\,(\ln\eta+\gamma) = -\frac{1}{\pi}\int_{-\infty}^\infty d\eta\,\hat{G}_3(\eta)(\ln|\eta|+\gamma)\,, \tag{3.100}$$

since G_3 is even. The integral is just the distribution w of Eq. (B.10) applied to $\hat{G}_3$, which is by definition $\hat{w}[G_3]$, so Eq. (B.11) gives

$$B_3 = -\int_{-\infty}^\infty d\tau\,G_3'(\tau)\ln|\tau|\,\mathrm{sgn}\,\tau\,, \tag{3.101}$$

with

$$G_3(\tau) = \frac{1}{96\pi^2}\int_{-\infty}^\infty dtV_{,ii}(\bar{t})g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,, \tag{3.102}$$

so

$$B_3 = -\frac{1}{96\pi^2}\int_{-\infty}^\infty d\tau\ \ln|\tau|\,\mathrm{sgn}\,\tau\int_{-\infty}^\infty d\bar{t}\,V_{,ii}(\bar{t})g(\bar{t}-\frac{\tau}{2})g'(\bar{t}+\frac{\tau}{2})\,. \tag{3.103}$$

3.4.2 The non-singular terms

For $i=4,5,6$, f_i is not singular at $\tau=0$. We include everything in

$$F_i(\tau) = \int_{-\infty}^\infty d\bar{t}\,f_i(\tau,\bar{t})g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\,, \tag{3.104}$$

so

$$B_i = \int_0^\infty\frac{d\xi}{\pi}\int_{-\infty}^\infty d\tau F_i(\tau)e^{-i\xi\tau} = \int_0^\infty\frac{d\xi}{\pi}\hat{F}_i(-\xi) = \frac{1}{\pi}\int_{-\infty}^\infty d\xi\,\Theta(\xi)\hat{F}_i(-\xi)\,. \tag{3.105}$$

The integral is the distribution Θ applied to $\hat{F}_i(-\xi))$, which is the Fourier transform of Θ applied to $F_i(-\tau)$. The Fourier transform of the Θ function acts on a function f as [20]

$$\Theta[f] = iP\int_{-\infty}^\infty d\tau\left(\frac{1}{\tau}f(\tau)\right)+\pi f(0)\,, \tag{3.106}$$

where P denotes principal value, so

$$B_i = -\frac{i}{\pi}P\int_{-\infty}^\infty d\tau\left(\frac{1}{\tau}F_i(\tau)\right)+F_i(0)\,. \tag{3.107}$$

The first of the non-singular terms is a constant: f_4 does not depend on τ. Thus F_4 is even in τ, and only the second term of Eq. (3.107) contributes, giving

$$B_4 = F_4(0) = -\frac{1}{128\pi^2}\int_{-\infty}^\infty d\bar{t}\,g(\bar{t})^2\left[V_{,tt}(\bar{t})+V_{,ii}(\bar{t})\right]\,. \tag{3.108}$$

The functions f_5 and f_6 are odd in τ and vanish as $\tau\to 0$, so in these cases only the first term in Eq. (3.107) contributes and the principal value symbol is not needed. Equations (3.74e,3.104,3.107) give

$$B_5 = \frac{1}{256\pi^3}\int_{-\infty}^\infty d\tau\frac{1}{\tau}\int_{-\infty}^\infty d\bar{t}\,g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega\,\nabla_{\bar{x}}^2V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{3.109}$$

and Eqs. (3.74f), (3.104) and (3.107) give

$$B_6 = -\frac{1}{64\pi^3}\int_{-\infty}^{\infty} d\tau \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega\, \partial_\tau^2 \left[V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\right]. \tag{3.110}$$

Here we can integrate by parts twice, giving

$$B_6 = -\frac{1}{64\pi^3}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\, \partial_\tau^2 \left[\frac{1}{\tau} g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\right]\int d\Omega\, V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau\,. \tag{3.111}$$

From Eqs. (3.53), (3.67), $V^{(1)}$ goes as τ and $V^{(3)}$ as τ^3 for small τ, so the τ integrals converge.

3.5 The Quantum Inequality

Now we can collect all the terms of B from Eqs. (3.76b), (3.76c), (3.90), (3.95), (3.103), (3.108), (3.109) and (3.111). Since B_7 is made of the same quantities as B_4, we merge these together. We find

$$B = \frac{1}{16\pi^2}\left[I_1 + \frac{1}{2}I_2^V - \frac{1}{6}I_3^V - \frac{1}{8}I_4^V + \frac{1}{16\pi}I_5^V - \frac{1}{4\pi}I_6^V\right] - I_7^V\,, \tag{3.112}$$

where

$$I_1 = \int_{-\infty}^{\infty} dt\, g''(t)^2 \tag{3.113a}$$

$$I_2^V = \int_{-\infty}^{\infty} d\bar{t}\, V(\bar{t})[g(\bar{t})g''(\bar{t}) - g'(\bar{t})^2] \tag{3.113b}$$

$$I_3^V = \int_{-\infty}^{\infty} d\tau \ln|\tau|\,\mathrm{sgn}\,\tau \int_{-\infty}^{\infty} d\bar{t}\, V_{,ii}(\bar{t}) g(\bar{t}-\frac{\tau}{2})g'(\bar{t}+\frac{\tau}{2}) \tag{3.113c}$$

$$I_4^V = \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})^2\left[\frac{7}{6}V_{,tt}(\bar{t}) + \frac{5}{6}V_{,ii}(\bar{t})\right] \tag{3.113d}$$

$$I_5^V = \int_{-\infty}^{\infty} d\tau \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\int d\Omega \nabla_{\bar{x}}^2 V^{(1)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{3.113e}$$

$$I_6^V = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\, \partial_\tau^2\left[\frac{1}{\tau}g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})\right]\int d\Omega\, V^{(3)}(\bar{x}+\frac{|\tau|}{2}\Omega)\,\mathrm{sgn}\,\tau \tag{3.113f}$$

$$I_7^V = C\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})^2 V_{,ii}(\bar{t})\,. \tag{3.113g}$$

In the case with no potential, only I_1 remains, reproducing a result of Fewster and Eveson [9].

In Eq. (3.113c), $\ln|\tau|$ really means $\ln(|\tau|/l)$, where l is the arbitrary length discussed in Ch. 2. The choice of a different length changes Eqs. (3.113c) and (3.113d) in compensating ways so that B is unchanged.

Equations (1.2,3.112,3.113) give a quantum inequality useful when the potential V is known and so the integrals in Eqs. (3.113) can be done. If we only know that V and its derivatives are restricted by the bounds of Eq. (3.10), then we can restrict the magnitude of each term of Eq. (3.112) and add those magnitudes. We start with

$$|I_2^V| \le \int_{-\infty}^{\infty} d\bar{t}|V(\bar{t})||g(\bar{t})g''(\bar{t}) - g'(\bar{t})^2| \le V_{\max}\int_{-\infty}^{\infty} d\bar{t}\left[g(\bar{t})|g''(\bar{t})| + g'(\bar{t})^2\right]. \tag{3.114}$$

The cases of I_3^V, I_4^V, and I_7^V are similar. For I_5^V and I_6^V, it is useful to take explicit forms for the Taylor series remainders. From Eq. (3.53), we see that

$$\left|\int d\Omega\, \nabla_{\bar{x}}^2 V^{(1)}(\bar{x} + \frac{|\tau|}{2}\Omega)\right| \leq \frac{|\tau|}{2}\int d\Omega |\nabla^2 V_{,i}||\Omega^i| \leq \frac{3|\tau|}{2} V'''_{\max} \sum_i \int d\Omega |\Omega^i| = 9\pi|\tau| V'''_{\max}\,. \tag{3.115}$$

Similarly from Eq. (3.67) we have

$$\begin{aligned}\left|\int d\Omega\, V^{(3)}(\bar{x} + \frac{|\tau|}{2}\Omega)\right| &\leq \frac{|\tau|^3}{48}\int d\Omega |V_{,ijk}||\Omega^i\Omega^j\Omega^k| \\ &\leq \frac{|\tau|^3}{48} V'''_{\max} \sum_{ijk}\int d\Omega |\Omega^i\Omega^j\Omega^k| = \frac{2\pi+1}{8}|\tau|^3 V'''_{\max}\,,\end{aligned} \tag{3.116}$$

We can then perform the derivatives in Eq. (3.113f) and take the absolute value of each resulting term separately.

We define

$$J_2 = \int_{-\infty}^{\infty} dt\, \left[g(t)|g''(t)| + g'(t)^2\right] \tag{3.117a}$$

$$J_3 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' |g'(t')| g(t) |\ln|t'-t|| \tag{3.117b}$$

$$J_4 = \int_{-\infty}^{\infty} dt\, g(t)^2 \tag{3.117c}$$

$$J_5 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' g(t) g(t') \tag{3.117d}$$

$$J_6 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' |g'(t')| g(t) |t'-t| \tag{3.117e}$$

$$J_7 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left[g(t)|g''(t')| + g'(t)g'(t')\right](t'-t)^2 \tag{3.117f}$$

and find

$$|I_2^V| \leq V_{\max} J_2 \tag{3.118a}$$

$$|I_3^V| \leq 3V''_{\max} J_3 \tag{3.118b}$$

$$|I_4^V| \leq \frac{11}{3} V''_{\max} J_4 \tag{3.118c}$$

$$|I_5^V| \leq 9\pi V'''_{\max} J_5 \tag{3.118d}$$

$$|I_6^V| \leq \frac{2\pi+1}{16} V'''_{\max}(4J_5 + 4J_6 + J_7) \tag{3.118e}$$

$$|I_7^V| \leq 3|C| V''_{\max} J_4\,. \tag{3.118f}$$

Thus we have

$$\begin{aligned}\int_{\mathbb{R}} dt\, g(t)^2 \langle T_{tt}^{ren}\rangle_\omega(t,0) \geq -\frac{1}{16\pi^2} &\left\{ I_1 + \frac{1}{2}V_{\max} J_2 + V''_{\max}\left[\frac{1}{2}J_3 + \left(\frac{11}{24} + 48\pi^2|C|\right)J_4\right]\right. \\ &\left. + V'''_{\max}\left[\frac{11\pi+1}{16\pi}J_5 + \frac{2\pi+1}{64\pi}(4J_6 + J_7)\right]\right\}\end{aligned} \quad .(3.119)$$

3.5.1 An example for a specific sampling function

An example of the quantum inequality with a specific sampling function g is the following. Consider a Gaussian sampling function

$$g(t) = e^{-t^2/t_0^2}\,, \tag{3.120}$$

where t_0 is a positive number with the dimensions of t. Then the integrals of Eqs. (3.117), calculated numerically, become

$$\begin{aligned} J_1 &= 3.75t_0^{-3} \qquad J_2 = 3.15t_0^{-1} \\ J_3 &= 2.70t_0 \qquad J_4 = 1.25t_0 \\ J_5 &= 3.14t_0^2 \qquad J_6 = 3.57t_0^2 \\ J_7 &= 3.58t_0^2\,, \end{aligned} \tag{3.121}$$

so the right hand side of Eq. (3.119) becomes

$$-\frac{1}{16\pi^2 t_0^3}\left\{3.75 + 3.15V_{\max}t_0^2 + (1.63 + 591.25|C|)V''_{\max}t_0^4 + 2.86V'''_{\max}t_0^5\right\}\,. \tag{3.122}$$

3.6 Discussion of the result

In this chapter we have demonstrated a quantum inequality for a flat spacetime with a background potential, considered as a first-order correction, using a general inequality derived by Fewster and Smith, which we presented in Ch. 2 . We calculated the necessary terms from the Hadamard series and the antisymmetric part of the two-point function to get $\tilde{H}$. Next we Fourier transformed the terms, which are, as expected, free of divergences, to derive a bound for a given background potential. We then calculated the maximum values of these terms to give a bound that applies to any potential whose value and first three derivatives are bounded.

To show the meaning of this result, in the last section we presented an example for a specific sampling function. By studying the result we can see the meaning of the right hand side of our quantum inequality. The first term of the bound goes as t_0^{-3}, where t_0 is the sampling time, and agrees with the quantum inequality with no potential [9]. The rest of the terms show the effects of the potential to first order. These corrections will be small, provided that

$$V_{\max}t_0^2 \ll 1 \tag{3.123a}$$

$$V''_{\max}t_0^4 \ll 1 \tag{3.123b}$$

$$V'''_{\max}t_0^5 \ll 1\,. \tag{3.123c}$$

Equation (3.123a) says that the potential is small when its effect over the distance t_0 is considered. Given Eq. (3.123a), Eqs. (3.123b) and (3.123c) say, essentially, that the distance over which V varies is large compared to t_0, so that each additional derivative introduces a factor less than t_0^{-1}. Unlike the flat spacetime case the bound does not go to zero when the sampling time $t_0 \to \infty$ so we cannot obtain the Averaged Weak Energy Condition (AWEC).

Finally, it is interesting to note the relation of this result to the case of a spacetime with bounded curvature. Since the Hadamard coefficients in that case are components of the Riemann tensor and its derivatives, we expect that the bound will be the flat space term plus correction terms that depend on the maximum values of the curvature and its derivatives, just as in our case they depend on the the potential and its derivatives. We will demonstrate that this hypothesis is true in the next chapter.

Chapter 4

Quantum Inequality in spacetimes with small curvature

In this chapter we present a derivation of a timelike-projected quantum inequality in spacetimes with curveture. First we consider a massless, minimally-coupled scalar field with the usual classical stress-energy tensor,

$$T_{ab} = \nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}g^{cd}\nabla_c\Phi\nabla_d\Phi\,. \tag{4.1}$$

Let γ be any timelike geodesic parametrized by proper time t, and let $g(t)$ be any any smooth, positive, compactly-supported sampling function.

Let's construct Fermi normal coordinates [27] in the usual way: We let the vector $e_0(t)$ be the unit tangent to the geodesic γ, and construct a tetrad by choosing arbitrary normalized vectors $e_i(0), i = 1, 2, 3$, orthogonal to $e_0(0)$ and to each other, and define $\{e_i(t)\}$ by parallel transport along γ. The point with coordinates (x^0, x^1, x^2, x^3) is found by traveling unit distance along the geodesic given by $x^i e_i(0)$ from the point $\gamma(0)$.

We work only in first order in the curvature and its derivatives, but don't otherwise assume that it is small. We assume that the components of the Ricci tensor in any Fermi coordinate system, and their derivatives, are bounded,

$$|R_{ab}| \le R_{\max} \qquad |R_{ab,cd}| \le R''_{\max} \qquad |R_{ab,cde}| \le R'''_{\max}\,. \tag{4.2a}$$

These lead to bounds on the Ricci scalar and its derivatives,

$$|R| \le 4R_{\max} \qquad |R_{,cd}| \le 4R''_{\max} \qquad |R_{,cde}| \le 4R'''_{\max}\,, \tag{4.2b}$$

since we are working in four dimensions.

Eqs. (4.2) are intended as universal bounds which hold without regard to the specific choice of Fermi coordinate system above. We will not need a bound on the first derivative. The reason that we bound the Ricci tensor and not the Riemann tensor is that, as we will prove, the additional terms of the quantum inequality do not depend on any other components of the Riemann tensor.

Thus we can write Eq. (2.23) in our case as

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle T_{tt}^{\text{ren}}\rangle(t,0) \ge -B\,, \tag{4.3}$$

where

$$B = \int_0^\infty \frac{d\xi}{\pi} \hat{F}(-\xi,\xi) + \int_{-\infty}^\infty dt\, g^2(t) \left(Q - 2aR_{,ii} - \frac{b}{2}(R_{tt,tt} + R_{ii,tt} - 3R_{tt,ii} + R_{ii,jj}) \right) , \quad (4.4)$$

$$F(t,t') = g(t)g(t')T^{\text{split}} \tilde{H}_{(1)}((t,0),(t',0)) , \quad (4.5)$$

and $\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (2.24).

4.1 Simplification of T^{split}

The $T^{\text{split}}_{tt'}$ operator of Eq. (2.3) for a massless field can be written

$$T^{\text{split}}_{tt'} = \frac{1}{2} \left[\partial_t \partial_{t'} + \sum_{i=1}^{3} \partial_i \partial_{i'} \right] . \quad (4.6)$$

To simplify it, we will define the following operator,

$$\nabla^2_{\bar{x}} = \nabla^2_x + 2 \sum_{i=1}^{3} \partial_i \partial_{i'} + \nabla^2_{x'} , \quad (4.7)$$

which in flat space would be the derivative with respect to the center point. Then Eqs. (4.6) and (4.7) give

$$\begin{aligned} T^{\text{split}}_{tt'} &= \frac{1}{2} \left[\partial_t \partial_{t'} + \frac{1}{2} \left(\nabla^2_{\bar{x}} - \nabla^2_x - \nabla^2_{x'} \right) \right] \\ &= \frac{1}{4} \left[\nabla^2_{\bar{x}} + \Box_x - \partial_t^2 + \Box_{x'} - \partial_{t'}^2 + 2\partial_t \partial_{t'} \right] , \end{aligned} \quad (4.8)$$

where $\Box_x$ and $\Box_{x'}$ denote the D'Alembertian operator with respect to x and x'. Because we are using Fermi coordinates and are on the generating geodesic, the D'Alembertian and Laplacian operators have the same form with respect to Fermi coordinates as they do in flat space. Then using

$$\partial_\tau^2 = \frac{1}{4} \left[\partial_t^2 - 2\partial_t \partial_{t'} + \partial_{t'}^2 \right] , \quad (4.9)$$

we can write

$$T^{\text{split}}_{tt'} \tilde{H} = \frac{1}{4} \left[\Box_x \tilde{H} + \Box_{x'} \tilde{H} + \nabla^2_{\bar{x}} \tilde{H} \right] - \partial_\tau^2 \tilde{H} . \quad (4.10)$$

Consider the first term. The function $H(x,x')$ obeys the equation of motion[1] in x and so does $E(x,x')$. Thus

$$\Box_x \tilde{H} = \frac{1}{2} \Box_x H(x',x) . \quad (4.11)$$

As we discussed in Sec. 3.1.2 we have

$$H(x',x) = H(x,x')^* + \frac{1}{4\pi^2} \sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x') , \quad (4.12)$$

[1] In general the sums in Eq. (2.4) do not converge and we should work only to some finite order in σ. In that case $H(x,x')$ obeys the equation of motion to that order.

Since $\Box_x$ is real, $\Box_x H(x,x')^* = 0$, and we have

$$\Box_x \tilde{H} = \frac{1}{8\pi^2}\Box_x \sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x')\,. \tag{4.13}$$

In the coincidence limit Eq. (4.13) vanishes. There is no $j = 0$ term because $w_0 = 0$. In the $j = 1$ term, we have σ, which vanishes at coincidence unless both derivatives of the $\Box$ are applied to it, in which case w_1 cancel each other, and for $j > 1$, even $\Box_x \sigma^j$ vanishes.

The second term in brackets in Eq. (4.10) gives

$$\Box_{x'} \tilde{H} = \frac{1}{8\pi^2}\Box_{x'} \sum_j (w_j(x,x') - w_j(x',x))\sigma^j(x,x')\,. \tag{4.14}$$

Adding together Eqs. (4.13) and (4.14), we get something which is smooth, symmetric in x and x', and vanishes in the coincidence limit. Following the analysis of Sec. 3.1.2, such a term makes no contribution to Eq. (4.4) and for our purposes we can take

$$T^{\text{split}}\tilde{H} = \left[\frac{1}{4}\nabla^2_{\bar{x}} - \partial^2_\tau\right]\tilde{H}\,. \tag{4.15}$$

4.2 General computation of E

The function E is the advanced minus the retarded Green's function,

$$E(x,x') = G_A(x,x') - G_R(x,x')\,, \tag{4.16}$$

and iE is the imaginary, antisymmetric part of the two-point function. The Green's functions satisfy

$$\Box G(x,x') = \frac{\delta^{(4)}(x-x')}{\sqrt{-g}}\,. \tag{4.17}$$

Following Poisson, et al. [30] and adjusting for different sign and normalization conventions,

$$G(x,x') = \frac{1}{4\pi}\left(2U(x,x')\delta(\sigma) + V(x,x')\Theta(-\sigma)\right)\,, \tag{4.18}$$

where $U(x,x') = \Delta^{1/2}(x,x')$ and $V(x,x')$ are smooth biscalars.

For points y null separated from x', V is called $\check{V}$ [30] and satisfies

$$\check{V}_{,a}\sigma^{;a} + \left[\frac{1}{2}\Box\sigma + 2\right]\check{V} = -\Box U\,, \tag{4.19}$$

with all derivatives with respect to y. Now $\check{V}$ is first order in the curvature, so we will do the rest of the calculation as though we were in flat space. Under this approximation, we will neglect coefficients which depend on the curvature, and also evaluate curvature components at locations that would be relevant if we were in flat space. The distance between these locations and the proper locations is first order in the curvature, so the overall inaccuracy will always be second order in the curvature and its derivatives.

Thus we use $\sigma^{;a} = -2(y-x')^a$ and $\Box\sigma = -8$ in Eq. (4.19) to get

$$(y-x')^a \check{V}_{,a}(y) + \check{V}(y) = \frac{1}{2}\Box U(y)\,. \tag{4.20}$$

Now suppose we want to compute $\check{V}$ at some point x''. We need to integrate along the geodesic going from x' to x''. So let $y = x' + \lambda(x''-x')$ and observe that

$$\frac{d(\lambda\check{V}(y))}{d\lambda} = \lambda\frac{d\check{V}(y)}{d\lambda} + \check{V}(y) = \lambda(x''-x')^a\check{V}_{,a} + \check{V}(y) = (y-x')^a\check{V}_{,a} + \check{V}(y) = \frac{1}{2}\Box U(y)\,, \tag{4.21}$$

so

$$\check{V}(x'',x') = \frac{1}{2}\int_0^1 d\lambda \Box U(y)\,. \tag{4.22}$$

The function V obeys [30]

$$\Box_x V(x,x') = 0\,. \tag{4.23}$$

Let us consider points x and x' on the geodesic γ, which in the flat-space approximation means they are separated only in time, let $\bar{x} = (x+x')/2$, and establish a spherical coordinate system (r,θ,ϕ) with origin at the common spatial position of x and x'. Now $V(x,x')$ can be found in terms of V and its derivatives evaluated at the time $\bar{t}$ (the time component of $\bar{x}$) using Kirchhoff's formula,

$$V(x,x') = \frac{1}{4\pi}\int d\Omega\left[\check{V}(x'',x') + \frac{\tau}{2}\frac{\partial}{\partial r}\check{V}(x'',x') + \frac{\tau}{2}\frac{\partial}{\partial t}\check{V}(x'',x')\right]\,, \tag{4.24}$$

where the derivatives act on the first argument of $\check{V}$, $\int d\Omega$ means to integrate over all spatial unit vectors $\hat{\Omega}$, and we now set

$$x'' = \bar{x} + (\tau/2)\Omega \tag{4.25}$$

with the 4-vector Ω given by $\hat{\Omega}$ with zero time component.

Now define null coordinates $u = t + r$ and $v = t - r$. Then x'' has $u = \tau$, $v = 0$. The derivative $\partial/\partial u$ can be written $(\partial/\partial t + \partial/\partial r)/2$ and so

$$V(x,x') = \frac{1}{4\pi}\int d\Omega\,\frac{d}{du}\left[u\check{V}((u/2)\Omega_1,x')\right]_{u=\tau}\,, \tag{4.26}$$

where Ω_1 is $\hat{\Omega}$ with unit time component. From Eq. (4.22),

$$u\check{V}\big(\frac{u}{2}\Omega,x'\big) = \frac{1}{2}\int_0^u du'(\Box U)((u'/2)\Omega_1,x')\,, \tag{4.27}$$

with the D'Alembertian applied to the first argument, and so

$$V(x,x') = \frac{1}{8\pi}\int d\Omega \Box_{x''}U(x'',x')\,. \tag{4.28}$$

We are only interested in the first order of curvature, so we can expand U, which is just the square root of the Van Vleck determinant, to first order. From Ref. [33],

$$\Delta^{1/2}(x,x') = 1 - \frac{1}{2}\int_0^1 ds(1-s)sR_{ab}(sx+(1-s)x')(x-x')^a(x-x')^b + O(R^2)\,, \tag{4.29}$$

so in the case at hand we can use

$$U(x'', x') = \Delta^{1/2}(x'', x') = 1 - \frac{1}{2}\int_0^1 ds(1-s)sR_{ab}(y)X^aX^b \tag{4.30}$$

where $y = sx'' = (su'', sv'', \theta'', \phi'')$ is a point between 0 and x'', and the tangent vector $X = dy/ds$. We are interested in $\Box_{x''}U(x'', 0)$. To bring the $\Box$ inside the integral, we define $Y = sX = (su'', sv'', 0, 0)$, and

$$\Box U(x'', 0) = -\frac{1}{2}\int_0^1 ds(1-s)s\Box_{x''}[R_{ab}(y)X^aX^b] = -\frac{1}{2}\int_0^1 ds(1-s)s\Box_y[R_{ab}(y)Y^aY^b]\,. \tag{4.31}$$

For the rest of this section, all occurrences of u, v, θ, ϕ, and derivatives with respect to these variables will refer to these components of y or Y.

Now we expand the D'Alembertian in Eq. (4.28), in terms of an angular part,

$$\nabla^2_\Omega = \frac{4}{(v-u)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{4}{(v-u)^2\sin\theta^2}\frac{\partial^2}{\partial\phi^2}\,. \tag{4.32}$$

and a radial and temporal part, which we can write in terms of derivatives in u and v,

$$4\frac{\partial^2}{\partial v\partial u} - \frac{4}{u-v}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\,. \tag{4.33}$$

The angular part vanishes on integration, leaving

$$V(x, x') = -\frac{1}{4\pi}\int d\Omega\int_0^1 dss(1-s)\left[\partial_u\partial_v - \frac{1}{u-v}(\partial_u - \partial_v)\right](R_{ab}(y)Y^aY^b)\,. \tag{4.34}$$

Outside the derivatives, we can take $v = 0$ and change variables to $u = s\tau$, giving

$$\begin{aligned} V(x, x') &= -\frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du(\tau - u)\left[u\partial_u\partial_v - \partial_u + \partial_v\right](R_{ab}(y)Y^aY^b) & (4.35)\\ &= -\frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du(\tau-u)\partial_u[(u\partial_v - 1)(R_{ab}(y)Y^aY^b)]\,. & (4.36)\end{aligned}$$

We can integrate by parts with no surface contribution, giving

$$\begin{aligned} V(x, x') &= \frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du(1-u\partial_v)(R_{ab}(y)Y^aY^b) & (4.37)\\ &= \frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau duu^2\left[-uR_{uu,v}(y) - 2R_{uv}(y) + R_{uu}(y)\right]\,. \end{aligned}$$

Now

$$R_{ab} = G_{ab} - (1/2)g_{ab}G\,, \tag{4.38}$$

where G_{ab} is the Einstein tensor and G its trace. Thus

$$V(x, x') = \frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du\,u^2\left[-uG_{uu,v}(y) - 2G_{uv}(y) + (1/2)G(y) + G_{uu}(y)\right]\,. \tag{4.39}$$

Now define a vector field $Q_a(y) = G_{ab}(y)Y^b$. Then

$$Q_{a;c} = G_{ab;c}(y)Y^b + G_{ab}(y)Y^b{}_{;c}\,. \tag{4.40}$$

We write the covariant derivative only because we are working in null-spherical coordinates, rather than because of spacetime curvature, which we are ignoring because we already have first order quantities.

Since the covariant divergence of G vanishes,

$$g^{ac}Q_{a;c} = g^{ac}G_{ab}(y)Y^b{}_{;c}\,. \tag{4.41}$$

In Cartesian coordinates, $Y^b = y^b$, and $y^b{}_{;c} = \delta^b_c$, which means that (in any coordinate system).

$$g^{ac}Q_{a;c} = G\,. \tag{4.42}$$

Explicit expansion gives

$$g^{ac}Q_{a;c} = 2(Q_{v,u}+Q_{u,v}) - \frac{4}{u-v}(Q_u - Q_v) - \frac{4}{(v-u)^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta Q_\theta) + \frac{1}{\sin\theta^2}Q_{\phi,\phi}\right]\,, \tag{4.43}$$

but the angular terms vanish on integration. Now we expand the derivatives in u and v and set $v = 0$, giving

$$Q_{v,u} = uG_{uv,u} + G_{uv} \tag{4.44a}$$
$$Q_{u,v} = uG_{uu,v} + G_{uv}\,, \tag{4.44b}$$

so

$$\int d\Omega\ (2uG_{uv,u} + 2uG_{uu,v} + 8G_{uv} - 4G_{uu}) = \int d\Omega\, G\,. \tag{4.45}$$

Substituting Eq. (4.45) into Eq. (4.39), we find

$$V(x,x') = \frac{1}{4\pi\tau^3}\int d\Omega \int_0^\tau du\, u^2\left[uG_{uv,u}(y) + 2G_{uv}(y) - G_{uu}(y)\right] \tag{4.46}$$

and integration by parts yields

$$V(x,x') = \frac{1}{4\pi}\int d\Omega\left[G_{uv}(x'') - \frac{1}{\tau^3}\int_0^\tau du\, u^2\left(G_{uv}(y) + G_{uu}(y)\right)\right]\,. \tag{4.47}$$

Now

$$\begin{aligned}\int d\Omega\int_0^\tau du\, u^2\left(G_{uv}(y)+G_{uu}(y)\right) &= \frac{1}{2}\int d\Omega\int_0^\tau du\, u^2\left(G_{tt}(y)+G_{tr}(y)\right)\\ &= \frac{1}{2}\int d\Omega\int_0^\tau du\, u^2\left(G^{tt}(y)-G^{tr}(y)\right)\end{aligned} \tag{4.48}$$

which is 4 times the total flux of G^{ta} crossing inward through the light cone. Since this quantity is conserved, $G^{ta}{}_{;a} = 0$, we can integrate instead over a ball at constant time $\bar{t}$, giving

$$4\int d\Omega\int_0^{\tau/2} dr\, r^2 G^{tt}(\bar{x}+r\Omega) = \frac{\tau^3}{2}\int d\Omega\int_0^1 ds\, s^2 G^{tt}(\bar{x}+s(\tau/2)\Omega) \tag{4.49}$$

so

$$V(x,x') = \frac{1}{8\pi}\int d\Omega\left[\frac{1}{2}\left[G_{tt}(x'') - G_{rr}(x'')\right] - \int_0^1 ds\, s^2 G_{tt}(x''_s)\right]\,, \tag{4.50}$$

where $x''_s = \bar{x} + s(\tau/2)\Omega$, and

$$G_R(x,x') = \Delta^{1/2}(x,x')\frac{\delta(\sigma)}{2\pi} + \frac{1}{32\pi^2}\int d\Omega\Big\{\frac{1}{2}\left[G_{tt}(x'') - G_{rr}(x'')\right] - \int_0^1 ds\, s^2 G_{tt}(x''_s)\Big\} \tag{4.51}$$

$$E(x,x') = \Delta^{1/2}(x,x')\frac{\delta(\tau-|\mathbf{x}-\mathbf{x}'|)-\delta(\tau+|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} + \frac{1}{32\pi^2}\int d\Omega\Big\{\frac{1}{2}\left[G_{tt}(x'') - G_{rr}(x'')\right] - \int_0^1 ds\, s^2 G_{tt}(x''_s)\Big\}\operatorname{sgn}\tau\,. \tag{4.52}$$

4.3 Computation of $\tilde{H}$

We now need to compute $\tilde{H}(x,x')$ and apply $T^{\text{split}}_{tt'}$. First we consider the term in $\tilde{H}(x,x')$ that has no dependence on the curvature. It has the same form as it would in flat space as shown in Ref [15] and Ch. 3

$$\tilde{H}_{-1}(x,x') = H_{-1}(x,x') = \frac{1}{4\pi^2\sigma_+(x,x')}\,. \tag{4.53}$$

In Sec. 4.4, we will apply the fully general $T^{\text{split}}_{tt'}$ from Eq. (4.15) with $\nabla_{\bar{x}}$ defined in Eq. (4.7) to $\tilde{H}_{-1}(x,x')$.

All the remaining terms that we need are first order in the curvature, so for these it is sufficient to take $\nabla_{\bar{x}}$ as the flat-space Laplacian with respect to the center point, $\bar{x}$. For this we only need to compute $\tilde{H}$ at positions given by time coordinates t and t' but the same spatial position.

As we discussed in Sec 3.1.1, we only need to keep terms in $\tilde{H}$ with powers of τ up to τ^2, but we need E exactly. The terms from H alone give a function whose Fourier transform does not decline fast enough for positive ξ for the integral in Eq. (4.4) to converge. Thus we extract the leading order terms from iE and combine these with the terms from H. This combination gives a result that has the appropriate behavior after the Fourier transform.

4.3.1 Terms with no powers of τ

First we want to calculate the zeroth order of the Hadamard series. The Hadamard coefficients are given by Eqs. (2.13,2.14) for a massless field. To find the zeroth order of the Hadamard series we need only $v_0(x,x')$, which we find by integrating Eq. (2.13) along the geodesic from x' to x. Since we are computing a first-order quantity, we can work in flat space by letting $y' = x' + \lambda(x-x')$ and using the first-order formulas $\Box\sigma = -8$ and $\sigma^{;a} = -2(y'-x')^a$. From Eq. (2.13), we have

$$(y'-x')^a v_{0,a} + v_0 = \frac{1}{4}\Box\Delta^{1/2}(y',x')\,, \tag{4.54}$$

and thus

$$v_0(x,x') = \frac{1}{4}\int_0^1 d\lambda(\Box\Delta^{1/2})(x'+\lambda(x-x'),x')\,. \tag{4.55}$$

by the same analysis as Eq. (4.22).

Using the expansion for $\Delta^{1/2}$ from Eq. (4.29) gives

$$v_0(x,x') = -\frac{1}{8}\int_0^1 d\lambda \int_0^1 ds(1-s)s\Box_{y'}[R_{ab}(sy'+(1-s)x')(y'-x')^a(y'-x')^b]$$
$$= -\frac{1}{8}\int_0^1 d\lambda \int_0^1 ds(1-s)s\Big[(\lambda s)^2(\Box R_{ab})(x'+s\lambda(x-x'))(x-x')^a(x-x')^b$$
$$+2\lambda s R_{,b}(x'+s\lambda(x-x'))(x-x')^b + 2R(x'+s\lambda(x-x'))\Big]\,. \qquad (4.56)$$

We can combine the s and λ integrals by defining a new variable $\sigma = s\lambda$

$$\int_0^1 d\lambda \int_0^1 ds(1-s)sf(\lambda s) = \int_0^1 d\lambda \int_0^\lambda d\sigma \left(\frac{\sigma}{\lambda^2}-\frac{\sigma^2}{\lambda^3}\right) f(\sigma) \qquad (4.57)$$
$$= \int_0^1 d\sigma\, f(\sigma)\int_\sigma^1 d\lambda \left(\frac{\sigma}{\lambda^2}-\frac{\sigma^2}{\lambda^3}\right) = \int_0^1 d\sigma\, f(\sigma)\left[-\frac{\sigma}{\lambda}+\frac{\sigma^2}{2\lambda^2}\right]_\sigma^1$$
$$= \frac{1}{2}\int_0^1 d\sigma\, f(\sigma)(1-\sigma)^2\,. \qquad (4.58)$$

Then, changing σ to s, we find

$$v_0(x,x') = -\frac{1}{16}\int_0^1 ds(1-s)^2\Big[s^2(\Box R_{ab})(x'+s(x-x'))(x-x')^a(x-x')^b$$
$$+2sR_{,b}(x'+s(x-x'))(x-x')^b + 2R(x'+s(x-x'))\Big]\,. \qquad (4.59)$$

or when the two points are on the geodesic,

$$v_0(t,t') = -\frac{1}{16}\int_0^1 ds(1-s)^2\Big[s^2(\Box R_{tt})(x'+s\tau)\tau^2 + 4s\eta^{cd}R_{ct,d}(x'+s\tau)\tau + 2R(x'+s\tau)\Big]\,. \qquad (4.60)$$

In the second term we use the contracted Bianchi identity, $\eta^{cd}R_{ct,d} = R_{,t}/2$, giving

$$2\int_0^1 ds(1-s)^2 s\tau R_{,t}(x'+s\tau) = 2\int_0^1 ds(1-s)^2 s\frac{d}{ds}R(x'+s\tau)$$
$$= -2\int_0^1 ds(1-s)(1-3s)R(x'+s\tau)\,, \qquad (4.61)$$

so the final expression for v_0 is

$$v_0(t,t') = -\frac{1}{16}\int_0^1 ds(1-s)\Big[s^2(1-s)\Box R_{tt}(\bar{x}+(s-1/2)\tau)\tau^2 + 4sR(\bar{x}+(s-1/2)\tau)\Big]\,. \qquad (4.62)$$

To calculate H_0 we only need the zeroth order in τ from v_0, so the first term does not contribute. In the second term, we make a Taylor series expansion,

$$R(\bar{x}+(s-1/2)\tau) = R(\bar{x}) + R_{,t}(\bar{x})\tau(s-1/2) + \frac{1}{2}R_{,tt}(\bar{x})\tau^2(s-1/2)^2 + O(\tau^3)\,, \qquad (4.63)$$

but only the first term is relevant here. Thus

$$v_0(t,t') = -\frac{1}{4}\int_0^1 ds(1-s)sR(\bar{x}) = -\frac{1}{24}R(\bar{x})\,. \qquad (4.64)$$

We also need to expand the Van Vleck determinant appearing in the Hadamard series. From Eq. (4.29),

$$\Delta^{1/2}(t,t') = 1 - \frac{1}{12}R_{tt}(\bar{x})\tau^2 - \frac{1}{480}R_{tt,tt}(\bar{x})\tau^4 + O(\tau^6)\,. \tag{4.65}$$

Keeping the first order term from Eq. (4.65) and using Eq. (4.64), we have

$$H_0(x,x') = \frac{1}{48\pi^2}\left[R_{tt}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln(-\tau_-^2)\right]\,. \tag{4.66}$$

Now we can add the $H_0(x',x)$ which is the same except that t and t' interchange

$$H_0(x,x') + H_0(x',x) = \frac{1}{24\pi^2}\left[R_{tt}(\bar{x}) - R(\bar{x})\ln|\tau|\right]\,. \tag{4.67}$$

Next we must include E from Eq. (4.52). We can expand the components of the Einstein tensor around $\bar{x}$,

$$G_{ab}(x'') = G_{ab}(\bar{x}) + G^{(1)}_{ab}(x'')\,, \tag{4.68}$$

where $G^{(1)}_{ab}$ is the remainder of the Taylor series

$$G^{(1)}_{ab}(x'') = G_{ab}(x'') - G_{ab}(\bar{x}) = \int_0^{\tau/2} dr\, G_{ab,i}(\bar{x} + r\Omega)\Omega^i\,. \tag{4.69}$$

To find E_0, we put the first term of Eq. (4.68) into the second term of Eq. (4.52). We use $G_{rr} = G_{ij}\Omega^i\Omega^j$ and $\int d\Omega\,\Omega^i\Omega^j = (4\pi/3)\delta^{ij}$ and find

$$\begin{aligned} E_0(x,x') &= \frac{1}{8\pi}\left\{\frac{1}{2}G_{tt}(\bar{x}) - \frac{1}{6}G_{ii}(\bar{x}) - \int_0^1 ds\, s^2 G_{tt}(\bar{x})\right\}\operatorname{sgn}\tau \\ &= \frac{1}{48\pi}G(\bar{x})\operatorname{sgn}\tau = -\frac{1}{48\pi}R(\bar{x})\operatorname{sgn}\tau\,. \end{aligned} \tag{4.70}$$

For the remainder term R_0, we put the second term of Eq. (4.68) into the second term of Eq. (4.52),

$$R_0(x,x') = \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}\left[G^{(1)}_{tt}(x'') - G^{(1)}_{rr}(x'')\right] - \int_0^1 ds\, s^2 G^{(1)}_{tt}(x''_s)\right\}\operatorname{sgn}\tau\,. \tag{4.71}$$

Using

$$2\ln|\tau| + \pi i \operatorname{sgn}\tau = \ln(-\tau_-^2)\,, \tag{4.72}$$

we combine Eqs. (4.67) and (4.70) to find

$$\tilde{H}_0(t,t') = \frac{1}{48\pi^2}\left[R_{tt}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln(-\tau_-^2)\right]\,. \tag{4.73}$$

Combining all terms through order 0 gives

$$\tilde{H}_{(0)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \frac{1}{2}iR_0(t,t')\,. \tag{4.74}$$

4.3.2 Terms of order τ^2

Now we compute the terms of order τ^2 in H and E. To find v_0 at this order we take Eqs. (4.62) and (4.63) and include terms through second order in τ. The first-order term vanishes, leaving

$$\begin{aligned} v_0(x,x') &= -\frac{1}{24}R(\bar{x}) - \frac{1}{16}\int_0^1 ds(1-s)[s^2(1-s)\Box R_{tt}(\bar{x}) \\ &\qquad\qquad +2s(s-1/2)^2 R_{,tt}(\bar{x})]\tau^2 + \dots \\ &= -\frac{1}{24}R(\bar{x}) - \frac{1}{480}\left(\Box R_{tt}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\tau^2 + \dots . \end{aligned} \tag{4.75}$$

Next we need v_1 but since it is multiplied by τ^2 in H we need only the τ independent term. From Eq. (2.14)

$$\Box v_0 + 2v_{1,a}\sigma^{;a} + v_1\Box\sigma = 0\,, \tag{4.76}$$

At $x = x'$, $\sigma^{;a} = 0$ so

$$v_1(x,x) = \frac{1}{8}\lim_{x\to x'}\Box_x v_0(x,x')\,. \tag{4.77}$$

Using Eq. (4.59) in Eq. (4.77), the only terms that survive in the coincidence limit are those that have no powers of $x - x'$ after differentiation, so

$$v_1(x,x) = -\frac{1}{16}\int_0^1 ds(1-s)^2 s^2 \Box R(\bar{x}) = -\frac{1}{480}\Box R(\bar{x})\,. \tag{4.78}$$

Equations (4.62), (4.75) and (4.78) agree with Ref. [6] if we note that their expansions are around x instead of $\bar{x}$.

The w_1 at coincidence is given by Ref. [38],

$$w_1(x,x) = -\frac{3}{2}v_1(x,x) = \frac{1}{320}\Box R(\bar{x})\,. \tag{4.79}$$

Combining Eqs. (4.75), (4.78), and (4.79), and the fourth order term from the Van Vleck determinant of Eq. (4.65), and keeping in mind that $\sigma = -\tau^2$ when both points are on the geodesic, we find

$$H_1(x,x') = \frac{1}{640\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln(-\tau_-^2)\right]\tau^2\,. \tag{4.80}$$

Then $H_1(x',x)$ is given by symmetry so

$$H_1(x,x') + H_1(x',x) = \frac{1}{160\pi^2}\left[\frac{1}{6}R_{tt,tt}(\bar{x}) - \frac{1}{4}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln|\tau|\right]\tau^2\,. \tag{4.81}$$

The calculation of E_1 is similar to E_0, but now we have to include more terms to the Taylor expansion,

$$G_{ab}(x'') = G_{ab}(\bar{x}) + \frac{\tau}{2}G_{ab,i}(\bar{x})\Omega^i + \frac{\tau^2}{8}G_{ab,ij}\Omega^i\Omega^j(\bar{x}) + G^{(3)}_{ab}(x'')\,, \tag{4.82}$$

where the remainder of the Taylor series is

$$G^{(3)}_{ab}(x'') = \frac{1}{2}\int_0^{\tau/2} dr G_{ab,ijk}(\bar{x}+r\Omega)\left(\frac{\tau}{2}-r\right)^2\Omega^i\Omega^j\Omega^k\,. \tag{4.83}$$

Now we put Eq. (4.82) into Eq. (4.52), and again use $G_{rr} = G_{ij}\Omega^i\Omega^j$. The first term of Eq. (4.82) gives E_0, which we computed before, and the second term gives nothing, because $\int d\Omega\,\Omega^i = \int d\Omega\,\Omega^i\Omega^j\Omega^k = 0$. Using $\int d\Omega\,\Omega^i\Omega^j = 4\pi/3\delta^{ij}$ and $\int d\Omega\,\Omega^i\Omega^j\Omega^k\Omega^l = (4\pi/15)(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$, the third term gives

$$\begin{aligned} E_1(x,x') &= -\frac{1}{192\pi}\left[\frac{1}{10}G_{ii,jj}(\bar{x}) + \frac{1}{5}G_{ij,ij}(\bar{x}) - \frac{1}{2}G_{tt,ii}(\bar{x}) \right. \\ &\qquad \left. + \int_0^1 ds\, s^4 G_{tt,ii}(\bar{x})\right]\tau^2\,\mathrm{sgn}\,\tau \qquad (4.84) \\ &= -\frac{1}{320\pi}\left[\frac{1}{6}G_{ii,jj}(\bar{x}) + \frac{1}{3}G_{ij,ij}(\bar{x}) - \frac{1}{2}G_{tt,ii}(\bar{x})\right]\tau^2\,\mathrm{sgn}\,\tau\,. \end{aligned}$$

Using the conservation of the Einstein tensor, $0 = \eta^{ab}G_{ia,b} = G_{it,t} - G_{ij,j}$ and $0 = \eta^{ab}G_{ta,b} = G_{tt,t} - G_{it,i}$ we can write

$$G_{ij,ij}(\bar{x}) = G_{tt,tt}(\bar{x})\,. \qquad (4.85)$$

So

$$E_1(x,x') = -\frac{1}{960\pi}\left(\frac{1}{2}G_{ii,jj}(\bar{x}) + G_{tt,tt}(\bar{x}) - \frac{3}{2}G_{tt,ii}(\bar{x})\right)\tau^2\,\mathrm{sgn}\,\tau\,. \qquad (4.86)$$

Now $G_{ab} = R_{ab} - (1/2)R$, so

$$\begin{aligned} G_{ii} &= (3/2)R_{tt} - (1/2)R_{ii} \qquad (4.87a) \\ G_{tt} &= (1/2)R_{tt} + (1/2)R_{ii}\,. \qquad (4.87b) \end{aligned}$$

Putting these in Eq. (4.86) gives

$$\begin{aligned} E_1(x,x') &= -\frac{1}{960\pi}\left(R_{ii,jj}(\bar{x}) + \frac{1}{2}R_{tt,tt}(\bar{x}) + \frac{1}{2}R_{ii,tt}(\bar{x})\right)\tau^2\,\mathrm{sgn}\,\tau] \\ &= -\frac{1}{960\pi}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\tau^2\,\mathrm{sgn}\,\tau]\,. \qquad (4.88) \end{aligned}$$

The fourth term of Eq. (4.82) gives the remainder

$$R_1(x,x') = \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}\left[G^{(3)}_{tt}(x'') - G^{(3)}_{rr}(x'')\right] - \int_0^1 ds\, s^2 G^{(3)}_{tt}(x''_s)\right\}\mathrm{sgn}\,\tau\,. \qquad (4.89)$$

To calculate $\tilde{H}_1$, we combine Eqs. (4.81) and (4.88) and use Eq. (4.72) to get

$$\tilde{H}_1(x,x') = \frac{\tau^2}{640\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln(-\tau_-^2)\right]\,. \qquad (4.90)$$

All terms through order 1 are then given by

$$\tilde{H}_{(1)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \tilde{H}_1(t,t') + \frac{1}{2}iR_1(t,t')\,. \qquad (4.91)$$

4.4 The $T^{\text{split}}_{tt'}\tilde{H}$

We can easily take the derivatives of $\tilde{H}_0$ and $\tilde{H}_1$ using Eq. (4.15), because they are already first order in R. However in the case of the term $\nabla^2_{\bar{x}}\tilde{H}_{-1}$ we have to proceed more carefully. From Eqs. (4.7) and (4.53) we have

$$\begin{aligned} \nabla^2_{\bar{x}}\tilde{H}_{-1} &= \frac{1}{4\pi^2}\sum_{i=1}^{3}\left(\frac{\partial^2}{\partial(x^i)^2} + 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x'^i} + \frac{\partial^2}{\partial(x'^i)^2}\right)\left(\frac{1}{\sigma_+}\right) \\ &= -\frac{1}{4\pi^2\sigma_+^2}\sum_{i=1}^{3}\left(\frac{\partial^2\sigma}{\partial(x^i)^2} + 2\frac{\partial^2\sigma}{\partial x^i\partial x'^i} + \frac{\partial^2\sigma}{\partial(x'^i)^2}\right), \end{aligned} \tag{4.92}$$

where we used $\partial\sigma/\partial x^i = \partial\sigma/\partial x'^i = 0$ when the two points are on the geodesic. From [6], after we shift the Taylor series so that the Riemann tensor is evaluated at $\bar{x}$, we have

$$\frac{\partial^2\sigma}{\partial(x^i)^2} = = -2\eta_{ii} - \frac{2}{3}R_{itit}(\bar{x})\tau^2 - \frac{1}{2}R_{itit,t}(\bar{x})\tau^3 - \frac{1}{5}R_{itit,tt}\tau^4 + O(\tau^5) \tag{4.93a}$$

$$\frac{\partial^2\sigma}{\partial(x'^i)^2} = = -2\eta_{ii} - \frac{2}{3}R_{itit}(\bar{x})\tau^2 + \frac{1}{2}R_{itit,t}(\bar{x})\tau^3 - \frac{1}{5}R_{itit,tt}\tau^4 + O(\tau^5) \tag{4.93b}$$

$$\frac{\partial^2\sigma}{\partial x^i\partial x'^i} = 2\eta_{ii} - \frac{1}{3}R_{itit}(\bar{x})\tau^2 - \frac{7}{40}R_{itit,tt}\tau^4 + O(\tau^5)\,. \tag{4.93c}$$

From Eqs. (4.92) and (4.93), and using $R_{itit} = -R_{tt}$ we have

$$\nabla^2_{\bar{x}}\tilde{H}_{-1} = -\frac{1}{4\pi^2}\left[\frac{2}{\tau_-^2}R_{tt}(\bar{x}) + \frac{3}{4}R_{tt,tt}(\bar{x})\right]. \tag{4.94}$$

From Eqs. (4.5) and (4.15), we need to compute

$$\int_0^\infty \frac{d\xi}{\pi}\hat{F}(-\xi,\xi')\,, \tag{4.95}$$

where

$$F(t,t') = g(t)g(t')\left[\frac{1}{4}\nabla^2_{\bar{x}}\tilde{H}_{(0)}(t,t') - \partial^2_\tau\tilde{H}_{(1)}(t,t')\right]. \tag{4.96}$$

In the first term in brackets it is sufficient to use $\tilde{H}_{(0)}(t,t')$, because higher order terms in H are smooth, even in τ, and vanish at coincidence, and so they do not contribute, as discussed in Sec. 3.1.1. In the second term, two powers of τ are removed by differentiation, so we need $\tilde{H}_{(1)}(t,t')$.

Using Eqs. (4.53), (4.71), (4.73), (4.74), (4.89), (4.90), (4.91) and (4.94) we can combine all terms in F to write

$$F(t,t') = g(t)g(t')\sum_{i=1}^{6} f_i(t,t')\,, \tag{4.97}$$

with

$$f_1 = \frac{3}{2\pi^2\tau_-^4} \tag{4.98a}$$

$$f_2 = \frac{1}{48\pi^2\tau_-^2}[R_{ii}(\bar{x}) - 7R_{tt}(\bar{x})] \tag{4.98b}$$

$$f_3 = \frac{1}{384\pi^2}\left[\frac{1}{5}R_{tt,tt}(\bar{x}) + \frac{1}{5}R_{ii,tt}(\bar{x}) - R_{tt,ii}(\bar{x}) + \frac{3}{5}R_{ii,jj}(\bar{x})\right]\ln(-\tau_-^2) \tag{4.98c}$$

$$f_4 = \frac{1}{320\pi^2}\left[-\frac{43}{3}R_{tt,tt}(\bar{x}) + \frac{7}{6}R_{tt,ii}(\bar{x}) - \frac{1}{2}R_{ii,jj}(\bar{x})\right] \tag{4.98d}$$

$$f_5 = \frac{1}{256\pi^2}\int d\Omega\,\nabla_{\bar{x}}^2\left\{\frac{1}{2}\left[G_{tt}^{(1)}(x'') - G_{rr}^{(1)}(x'')\right] - \int_0^1 dss^2G_{tt}^{(1)}(x_s'')\right\} i\,\mathrm{sgn}\,\tau \tag{4.98e}$$

$$f_6 = -\frac{1}{64\pi^2}\int d\Omega\,\partial_\tau^2\left\{\frac{1}{2}\left[G_{tt}^{(3)}(x'') - G_{rr}^{(3)}(x'')\right] - \int_0^1 dss^2G_{tt}^{(3)}(x_s'')\right\} i\,\mathrm{sgn}\,\tau\,. \tag{4.98f}$$

4.5 The quantum inequality

We want to calculate the quantum inequality bound B, given by Eq. (4.4). We can write it

$$B = \sum_{i=1}^{8} B_i\,, \tag{4.99}$$

where

$$\begin{aligned} B_i &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' g(t)g(t')f_i(t,t')e^{i\xi(t'-t)} \\ &= \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t}-\frac{\tau}{2})g(\bar{t}+\frac{\tau}{2})f_i(\bar{t},\tau)e^{-i\xi\tau}, i = 1\ldots 6 \end{aligned} \tag{4.100a}$$

$$B_7 = \int_{-\infty}^{\infty} dt\, g^2(t)Q(t) = \frac{1}{3840\pi^2}\int_{-\infty}^{\infty} dt\, g^2(t)\Box R(\bar{t}) \tag{4.100b}$$

$$\begin{aligned} B_8 = -\int_{-\infty}^{\infty} dt\, g^2(t)\Big[&2aR_{,ii}(\bar{x}) - \frac{b}{2}(R_{tt,tt}(\bar{x}) + R_{ii,tt}(\bar{x}) - 3R_{tt,ii}(\bar{x}) \\ &+R_{ii,jj}(\bar{x}))\Big] \end{aligned} \tag{4.100c}$$

using Eqs. (2.20), (2.22), (4.4) and (4.79). The first 6 terms have the same τ dependence as the corresponding terms in Ch. 3. So the Fourier transform proceeds in the same way, except that instead of dependence on the potential and its derivatives, we have dependence on the Ricci tensor and its derivatives. After the Fourier transform, we see that B_4 and B_7 have exactly the same form so we merge them in one term. Thus

$$B = \frac{1}{16\pi^2}\left[I_1 + \frac{1}{12}I_2^R - \frac{1}{24}I_3^R + \frac{1}{240}I_4^R + \frac{1}{16\pi}I_5^R - \frac{1}{4\pi}I_6^R\right] - I_7^R\,, \tag{4.101}$$

where

$$I_1 = \int_{-\infty}^{\infty} dt\, g''(t)^2 \tag{4.102a}$$

$$I_2^R = \int_{-\infty}^{\infty} d\bar{t}\,[7R_{tt}(\bar{x}) - R_{ii}(\bar{x})](g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})) \tag{4.102b}$$

$$\begin{aligned} I_3^R &= \int_{-\infty}^{\infty} d\tau\, \ln|\tau|\,\mathrm{sgn}\,\tau \int_{-\infty}^{\infty} d\bar{t}\left[\frac{1}{5}R_{tt,tt}(\bar{x}) + \frac{1}{5}R_{ii,tt}(\bar{x}) - R_{tt,ii}(\bar{x})\right. \\ &\quad \left. + \frac{3}{5}R_{ii,jj}(\bar{x})\right] g(\bar{t} - \frac{\tau}{2})g'(\bar{t} + \frac{\tau}{2}) \end{aligned} \tag{4.102c}$$

$$I_4^R = \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})^2\Big[-171R_{tt,tt}(\bar{x}) - R_{ii,tt}(\bar{x}) + 13R_{tt,ii}(\bar{x}) - 5R_{ii,jj}(\bar{x})\Big] \tag{4.102d}$$

$$\begin{aligned} I_5^R &= \int_{-\infty}^{\infty} d\tau\, \frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t} - \tau/2)g(\bar{t} + \tau/2)\int d\Omega\, \nabla_{\bar{x}}^2\Big\{\frac{1}{2}\left[G_{tt}^{(1)}(x'') - G_{rr}^{(1)}(x'')\right] \\ &\quad - \int_0^1 ds\, s^2\left[G_{tt}^{(1)}(x''_s)\right]\Big\}\,\mathrm{sgn}\,\tau \end{aligned} \tag{4.102e}$$

$$\begin{aligned} I_6^R &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t}\,\partial_\tau^2\left[\frac{1}{\tau}g(\bar{t} - \tau/2)g(\bar{t} + \tau/2)\right]\int d\Omega\left\{\frac{1}{2}\left[G_{tt}^{(3)}(x'') - G_{rr}^{(3)}(x'')\right]\right. \\ &\quad \left. - \int_0^1 ds\, s^2 G_{tt}^{(3)}(x''_s)\right\}\mathrm{sgn}\,\tau \end{aligned} \tag{4.102f}$$

$$\begin{aligned} I_7^R &= \int_{-\infty}^{\infty} dt\, g^2(t)\left[2aR_{,ii}(\bar{x}) - \frac{b}{2}(R_{tt,tt}(\bar{x}) + R_{ii,tt}(\bar{x}) - 3R_{tt,ii}(\bar{x})\right. \\ &\quad \left. + R_{ii,jj}(\bar{x}))\right]. \end{aligned} \tag{4.102g}$$

If we only know that the Ricci tensor and its derivatives are bounded, as in Eqs. (4.2), we can restrict the magnitude of each term of Eq. (4.101). We start with the second term

$$\begin{aligned} |I_2^R| &\le \int_{-\infty}^{\infty} d\bar{t}\ |7R_{tt}(\bar{x}) - R_{ii}(\bar{x})|\,|g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})| \\ &\le 10R_{\max}\int_{-\infty}^{\infty} d\bar{t}[g(\bar{t})|g''(\bar{t})| + g'(\bar{t})^2]\,. \end{aligned} \tag{4.103}$$

Terms I_3^R, I_4^R and I_7^R are similar. For I_5^R and I_6^R, we need bounds on the components of G. From Eq. (4.87b), $|G_{tt}| < 2R_{\max}$. Since Eq. (4.2) holds regardless of rotation, we can bound G_{rr} at any given point by taking the x-axis to point in the radial direction. Then $G_{rr} = G_{xx} = (1/2)[R_{xx} - R_{yy} - R_{zz} + R_{tt}]$ and $|G_{rr}| < 2R_{\max}$. Taking derivatives of G just differentiates the corresponding components of R, which are also bounded. In particular, since there are 3 terms in $\nabla_{\bar{x}}^2$, we have $|\nabla_{\bar{x}}^2 G_{tt}|, |\nabla_{\bar{x}}^2 G_{rr}| < 6R''_{\max}$. Using these results and Eq. (4.69) for the remainder we have

$$\begin{aligned} &\left|\int d\Omega\,\nabla_{\bar{x}}^2\Big\{\frac{1}{2}\left[G_{rr}^{(1)}(x'') - G_{tt}^{(1)}(x'')\right] + \int_0^1 ds\, s^2 G_{tt}^{(1)}(x''_s)\Big\}\right| \\ &\le \frac{|\tau|}{2}\int d\Omega\left\{\frac{1}{2}\left[|\nabla^2 G_{rr,i}(\bar{x})| + |\nabla^2 G_{tt,i}(\bar{x})|\right] + \int_0^1 ds\, s^3|\nabla^2 G_{tt,i}(\bar{x})|\right\}|\Omega^i| \\ &\le R'''_{\max}\frac{15|\tau|}{4}\sum_i\int d\Omega|\Omega^i| = \frac{45\pi}{2}|\tau|R'''_{\max}\,. \end{aligned} \tag{4.104}$$

For I_6^R we use Eq. (4.83) for the remainder

$$\left|\int d\Omega\left\{\frac{1}{2}\left[G^{(3)}_{rr}(x'')-G^{(3)}_{tt}(x'')\right]+\int_0^1 ds\, s^2 G^{(3)}_{tt}(x''_s)\right\}\right|$$
$$\leq \frac{|\tau|^3}{48}\int d\Omega\left\{\frac{1}{2}\left[|G_{rr,ijk}(\bar{x})|+|G_{tt,ijk}(\bar{x})|\right]+\int_0^1 ds\, s^5|G_{tt,ijk}(\bar{x})|\right\}|\Omega^i||\Omega^j||\Omega^k|$$
$$\leq R'''_{\max}\frac{7|\tau|^3}{144}\sum_{i,j,k}\int d\Omega|\Omega^i||\Omega^j||\Omega^k| = \frac{7(2\pi+1)}{24}|\tau|^3 R'''_{\max}\,. \tag{4.105}$$

After we bound all the terms and calculate the derivatives in I_6^R we can define

$$J_2 = \int_{-\infty}^{\infty} dt\left[g(t)|g''(t)|+g'(t)^2\right] \tag{4.106a}$$
$$J_3 = \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt'|g'(t')|g(t)|\ln|t'-t|| \tag{4.106b}$$
$$J_4 = \int_{-\infty}^{\infty} dt\, g(t)^2 \tag{4.106c}$$
$$J_5 = \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt' g(t)g(t') \tag{4.106d}$$
$$J_6 = \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt'|g'(t')|g(t)|t'-t| \tag{4.106e}$$
$$J_7 = \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt'\left[g(t)|g''(t')|+g'(t)g'(t')\right](t'-t)^2 \tag{4.106f}$$

and find

$$|I_2^R| \leq 10R_{\max}J_2 \tag{4.107a}$$
$$|I_3^R| \leq \frac{46}{5}R''_{\max}J_3 \tag{4.107b}$$
$$|I_4^R| \leq 258R''_{\max}J_4 \tag{4.107c}$$
$$|I_5^R| \leq \frac{45\pi}{2}R'''_{\max}J_5 \tag{4.107d}$$
$$|I_6^R| \leq \frac{7(2\pi+1)}{48}R'''_{\max}\left(4J_5+4J_6+J_7\right) \tag{4.107e}$$
$$|I_7^R| \leq (24|a|+11|b|)R''_{\max}J_4\,. \tag{4.107f}$$

Thus the final form of the inequality is

$$\int_{\mathbb{R}} d\tau\, g(t)^2\langle T^{ren}_{tt}\rangle_\omega(t,0) \geq -\frac{1}{16\pi^2}\Bigg\{ I_1+\frac{5}{6}R_{\max}J_2 \tag{4.108}$$
$$+ R''_{\max}\left[\frac{23}{60}J_3+\left(\frac{43}{40}+16\pi^2(24|a|+11|b|)\right)J_4\right]$$
$$+ R'''_{\max}\left[\frac{163\pi+14}{96\pi}J_5+\frac{7(2\pi+1)}{192\pi}(4J_6+J_7)\right]\Bigg\}.$$

Once we have a specific sampling function g, we can compute the integrals of Eqs. (4.106) to get a specific bound. In the case of a Gaussian sampling function,

$$g(t) = e^{-t^2/t_0^2}\,, \tag{4.109}$$

we computed these integrals numerically in Sec. 3.5.1. Using those results, the right hand side of Eq. (4.108) becomes

$$-\frac{1}{16\pi^2 t_0^3}\left\{3.76+2.63R_{\max}t_0^2+[1.71+197.9(24|a|+11|b|)]R''_{\max}t_0^4+6.99R'''_{\max}t_0^5\right\} . \quad (4.110)$$

The leading term is just the flat spacetime bound of Ref. [9] for g given by Eq. (4.109). The possibility of curvature weakens the bound by introducing additional terms, which have the same dependence on t_0 as in Ch. 3, with the Ricci tensor bounds in place of the bounds on the potential.

4.6 Discussion of the result

In this chapter, using the general quantum inequality of Fewster and Smith we presented in Ch. 2, we derived an inequality for a minimally-coupled quantum scalar field on spacetimes with small curvature. We calculated the necessary Hadamard series terms and the Green's function for this problem to the first order in curvature. Combining these terms gives $\tilde{H}$ and taking the Fourier transform gives a bound in terms of the Ricci tensor and its derivatives.

If we know the spacetime explicitly, Eqs. (4.3), (4.101), and (4.102) give an explicit bound on the weighted average of the energy density along the geodesic. This bound depends on integrals of the Ricci tensor and its derivatives combined with the weighting function g.

If we do not know the spacetime explicitly but know that the Ricci tensor and its first 3 derivatives are bounded, Eqs. (4.106) and (4.108) give a quantum inequality depending on the bounds and the weighting function. If we take a Gaussian weighting function, Eq. (4.110) gives a bound depending on the Ricci tensor bounds and the width of the Gaussian, t_0.

As expected, the result shows that the corrections due to curvature are small if the quantities $R_{\max}t_0^2$, $R''_{\max}t_0^4$, and $R'''_{\max}t_0^5$ are all much less than 1. That will be true if the curvature is small when we consider its effect over a distance equal to the characteristic sampling time t_0 (or equivalently if t_0 is much smaller than any curvature radius), and if the scale of variation of the curvature is also small compared to t_0.

In all bounds, there is unfortunately an ambiguity resulting from the unknown coefficients of local curvature terms in the gravitational Lagrangian. This ambiguity is parametrized by the quantities a and b.

Ford and Roman [19] have argued that flat-space quantum inequalities can be applied in curved spacetime, so long as the radius of curvature is small as compared to the sampling time. The present chapter explicitly confirms this claim and calculates the magnitude of the deviation. The curvature must be small not only on the path where the quantum inequality is to be applied but also at any point that is in both the causal future of some point of this path and the causal past of another. All such points are included in the integrals in Eq. (4.102e) and (4.102f).

Is is interesting to consider vacuum spacetimes, i.e., those whose Ricci tensor vanishes. These include, for example, the Schwarzschild and Kerr spacetimes, and those consisting only of gravitational waves. In such spacetimes, the flat-space quantum inequality will hold to first order without modification. There are, of course, second-order corrections. For the Schwarzschild spacetime, for example, these were calculated explicitly by Visser [35, 36, 37].

Chapter 5

Average Null Energy Condition in a classical curved background

In this chapter we present the proof of the achronal ANEC in spacetimes with curvature using a null-projected quantum inequality. It is structured as follows. First we state our assumptions and present the ANEC theorem we will prove. We begin the proof by constructing a parallelogram which can be understood as a congruence of null geodesic segments or of timelike paths. Then we apply the general inequality presented in Ch. 2 to the specific case needed here, using results from Ch. 4. Finally we present the proof of the ANEC theorem using the quantum inequality.

5.1 Assumptions

5.1.1 Congruence of geodesics

As in Ref. [11], we will not be able to rule out ANEC violation on a single geodesic. However, a single geodesic would not lead to an exotic spacetime. It would be necessary to have ANEC violation along a finite congruence of geodesics in order to have a physical effect.

So let us suppose that our spacetime contains a null geodesic γ with tangent vector ℓ and that there is a "tubular neighborhood" M' of γ composed of a congruence of achronal null geodesics, defined as follows. Let p be a point of γ, and let M_p be a normal neighborhood of p. Let v be a null vector at p, linearly independent of ℓ, and let x and y be spacelike vectors perpendicular to v and ℓ. Let q be any point in M_p such that p can be connected to q by a geodesic whose tangent vector is in the span of $\{v, x, y\}$. Let $\gamma(q)$ be the geodesic through q whose tangent vector is the vector $\boldsymbol{\ell}$ parallel transported from p to q. If a neighborhood M' of γ is composed of all geodesics $\gamma(q)$ for some choice of p, M_p, v, x and y, we will say that M' is a tubular neighborhood of γ.

5.1.2 Coordinate system

Given the above construction, we can define Fermi-like coordinates described in Appendix A on M' as follows. Without loss of generality we can take the vector v to be normalized so that $v_a \ell^a = 1$, and x and y to be unit vectors. Then we have a pseudo-orthonormal tetrad at p given by $E_{(u)} = \ell$, $E_{(v)} = v$, $E_{(x)} = x$, and $E_{(y)} = y$. The point $q = (u, v, x, y)$ in these coordinates is found as follows. Let $q^{(1)}$ be found by traveling unit affine parameter from p

along the geodesic generated by $vE_{(v)} + xE_{(x)} + yE_{(y)}$. Then q is found by traveling unit affine parameter from $q^{(1)}$ along the geodesic generated by $uE_{(u)}$. During this process the tetrad is parallel transported. All vectors and tensors will be described using this transported tetrad unless otherwise specified.

The points with u varying but other coordinates fixed form one of the null geodesics of the previous section.

5.1.3 Curvature

We suppose that the curvature inside M' obeys the null convergence condition,

$$R_{ab}V^a V^b \geq 0 \tag{5.1}$$

for any null vector V^a. Equation (5.1) holds whenever the curvature is generated by a "classical background" whose stress tensor obeys the NEC of Eq. (1.3). We will refer to this as a "classical background", but the only way it need be classical is Eq. (1.3).

We would not expect any energy conditions to hold when the curvature is arbitrarily large, because then we would be in the regime of quantum gravity, so we will require that the curvature be bounded. In the coordinate system we described we require

$$|R_{abcd}| < R_{\max} \tag{5.2}$$

and

$$|R_{abcd,\alpha}| < R'_{\max}, \qquad |R_{abcd,\alpha\beta}| < R''_{\max}, \qquad |R_{abcd,\alpha\beta\gamma}| < R'''_{\max} \tag{5.3}$$

in M', where the greek indices $\alpha, \beta, \gamma \cdots = v, x, y$ and $R_{\max}, R'_{\max}, R''_{\max}, R'''_{\max}$ are finite numbers but not necessarily small. Also we assume that the curvature is smooth.

5.1.4 Causal structure

We will also require that conditions outside M' do not affect the causal structure of the spacetime in M' [11][1]

$$J^+(p, M) \cap M' = J^+(p, M') \tag{5.4}$$

for all $p \in M'$. Otherwise the curvature outside M' may be arbitrary.

5.1.5 Quantum field theory

We consider a quantum scalar field in M. We will work entirely inside M', and there we require that the field be massless, free and minimally coupled. Outside M', however, we can allow different curvature coupling, interactions with other fields, and even boundary surfaces with specified boundary conditions.

Because M may not be globally hyperbolic, it is not completely straightforward to specify what we mean by a quantum field theory on M. We will use the same strategy as Ref. [11]. Our results will hold for any quantum field theory on M that reduces to the usual quantum field theory on each globally hyperbolic subspacetime of M. The states of interest will be those that reduce to Hadamard states on each globally hyperbolic subspacetime, and we will refer to any such state as "Hadamard". See Sec. II B of Ref. [11] for further details.

[1]This condition is equivalent to $J^-(p, M) \cap M' = J^-(p, M')$ for all $p \in M'$.

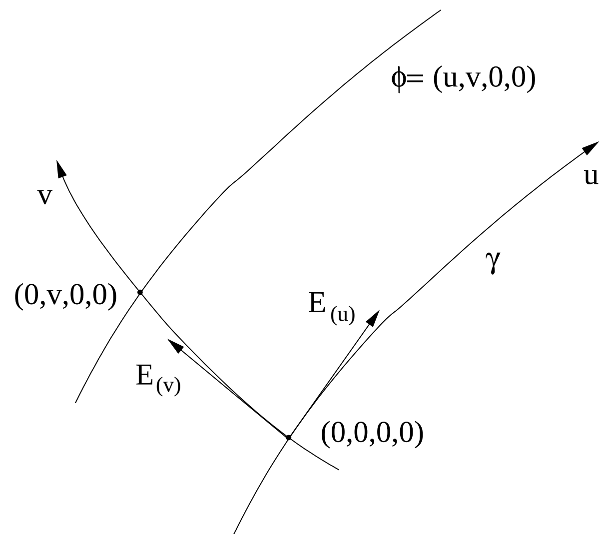

Figure 5.1: Construction of the family of null geodesics Φ using Fermi normal coordinates

5.2 The theorem

5.2.1 Stating the theorem

Theorem 1. Let (M, g) be a (time-oriented) spacetime and let γ be an achronal null geodesic on (M, g), and suppose that γ is surrounded by a tubular neighborhood M' in the sense of Sec. 5.1.1, obeying the null convergence condition, Eq. (1.3), and that we have constructed coordinates by the procedure of Sec. 5.1.2. Suppose that the curvature in this coordinate system is smooth and obeys the bounds of Sec. 5.1.3, that the curvature in the system is localized, i.e., in the distant past and future the spacetime is flat, and that the causal structure of M' is not affected by conditions elsewhere in M, Eq. (5.4).

Let ω be a state of the free minimally coupled quantum scalar field on M' obeying the conditions of Sec. 5.1.5, and let T be the renormalized expectation value of the stress-energy tensor in state ω.

Under these conditions, it is impossible for the ANEC integral,

$$A = \int_{-\infty}^{\infty} d\lambda\, T_{ab}\ell^a\ell^b(\Gamma(\lambda)), \tag{5.5}$$

to converge uniformly to negative values on all geodesics $\Gamma(\lambda)$ *in* M'.

5.2.2 The parallelogram

We will use the (u, v, x, y) coordinates of Sec. 5.1.2. Let r be a positive number small enough such that whenever $|v|, |x|, |y| < r$, the point $(0, v, x, y)$ is inside the normal neighborhood N_p defined in Sec. 5.1.1. Then the point $(u, v, x, y) \in M'$ for any u.

Now consider the points

$$\Phi(u, v) = (u, v, 0, 0)\,. \tag{5.6}$$

With v fixed and u varying, these are null geodesics in M'. (See Fig. 5.1.) Write the ANEC integral

$$A(v) = \int_{-\infty}^{\infty} du\, T_{uu}(\Phi(u,v))\,. \tag{5.7}$$

Suppose that, contrary to Theorem 1, Eq. (5.7) converges uniformly to negative values for all $|v| < r$. We will prove that this leads to a contradiction.

Since the convergence is uniform, $A(v)$ is continuous. Then since $A(v) < 0$ for all $|v| < r$, we can choose a positive number $v_0 < r$ and a negative number $-A$ larger than all $A(v)$ with $v \in (-v_0, v_0)$. Then it is possible to find some number u_1 large enough that

$$\int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u,v)) < -A/2 \tag{5.8}$$

for any $v \in (-v_0, v_0)$ as long as

$$u_+(v) \;>\; u_1 \tag{5.9a}$$
$$u_-(v) \;<\; -u_1\,. \tag{5.9b}$$

As in Ref. [11], we will define a series of parallelograms in the (u,v) plane, and derive a contradiction by integrating over each parallelogram in null and timelike directions. Each parallelogram will have the form

$$v \;\in\; (-v_0, v_0) \tag{5.10a}$$
$$u \;\in\; (u_-(v), u_+(v))\,, \tag{5.10b}$$

where $u_-(v), u_+(v)$ are linear functions of v obeying Eqs. (5.9). On each parallelogram we will construct a weighted integral of Eq. (5.8) as follows. Let $f(a)$ be a smooth function supported only within the interval $(-1,1)$ and normalized

$$\int_{-1}^{1} da f(a)^2 = 1\,. \tag{5.11}$$

Then we can write

$$\int_{-v_0}^{v_0} dv\, f(v/v_0)^2 \int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u,v)) < -v_0 A/2\,. \tag{5.12}$$

We can construct this same parallelogram as follows. First choose a velocity V. Eventually we will take the limit $V \to 1$. Define the Doppler shift parameter

$$\delta = \sqrt{\frac{1+V}{1-V}}\,. \tag{5.13}$$

Let α be some fixed number with $0 < \alpha < 1/3$ and then let

$$t_0 = \delta^{-\alpha} r\,. \tag{5.14}$$

As $V \to 1$, $\delta \to \infty$ and $t_0 \to 0$.

Now define the set of points

$$\Phi_V(\eta, t) = \Phi\left(\eta + \frac{\delta t}{\sqrt{2}}, \frac{t}{\sqrt{2}\delta}\right)\,. \tag{5.15}$$

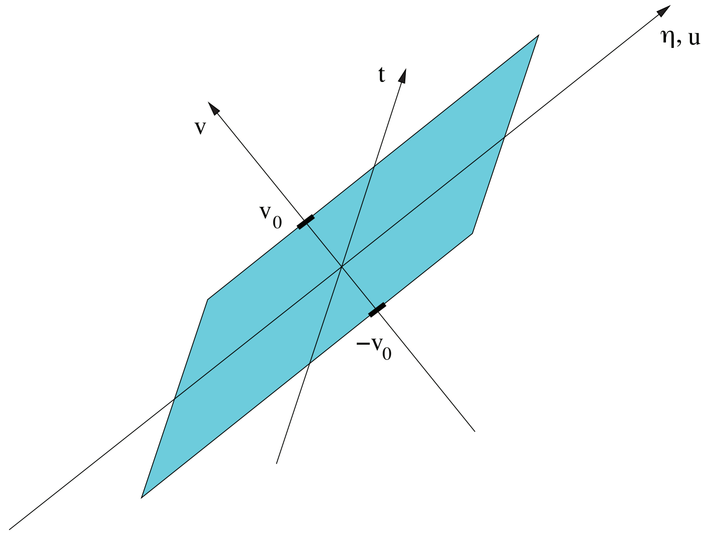

Figure 5.2: The parallelogram $\Phi(u,v)$, $v \in (-v_0, v_0)$, $u \in (u_-(v), u_+(v))$, or equivalently $\Phi_V(\eta, t)$, $t \in (-t_0, t_0)$, $\eta \in (-\eta_0, \eta_0)$

We will be interested in the paths given by $\Phi_V(\eta, t)$ with η fixed and t ranging from $-t_0$ to t_0. In flat space, such paths would be timelike geodesic segments, parameterized by t and moving at velocity V with respect to the original coordinate frame. In our curved spacetime, this is nearly the case, as we will show below. Define

$$\eta_0 = u_1 + t_0\delta/\sqrt{2} \tag{5.16a}$$

$$v_0 = t_0/(\sqrt{2}\delta) \tag{5.16b}$$

$$u_\pm(v) = \pm\eta_0 + \delta^2 v \tag{5.16c}$$

so that $u_\pm$ satisfies Eqs. (5.9). Then the range of points given by Eq. (5.6) with coordinate ranges specified by Eqs. (5.10) is the same as that given by Eq. (5.15) with coordinate ranges

$$-t_0 < t < t_0 \tag{5.17a}$$

$$-\eta_0 < \eta < \eta_0 \tag{5.17b}$$

The parallelogram is shown in Fig. 5.2.

The Jacobian

$$\left|\frac{\partial(u,v)}{\partial(\eta,\tau)}\right| = \frac{1}{\sqrt{2}\delta} \tag{5.18}$$

so Eq. (5.12) becomes

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-t_0}^{t_0} d\tau\, T_{uu}(\Phi_V(\eta,t)) f(t/t_0)^2 < -At_0/2\,. \tag{5.19}$$

We will show that this is impossible.

5.2.3 Transformation of the Riemann tensor

Since we are taking $\delta \to \infty$, components of R with more u's than v's diverge after the transformation. Components of R with fewer u's than v's go to zero and components with equal numbers of u's and v's remain the same. We want the curvature to be bounded by $R_{\max}$ in the primed coordinate system, which will be true if all components of the Riemann tensor with more u's than v's are zero. We will now show that this is the case in our system.

All points of interests are on achronal null geodesics, which thus must be free of conjugate points. Using Eq. (5.1) and proposition 4.4.5 of Ref. [22], each geodesic must violate the "generic condition". That is to say, we must have

$$\ell^c \ell^d \ell_{[a} R_{b]cd[e} \ell_{f]} = 0 \tag{5.20}$$

everywhere in M'.

The only nonvanishing components of the metric in the tetrad basis are $g_{uv} = g_{vu} = -1$ and $g_{xx} = g_{yy} = 1$. The tangent vector $\boldsymbol{\ell}$ has only one nonvanishing component $\ell^u = 1$, while the covector has only one nonvanishing component $\ell_v = -1$. Thus Eq. (5.20) becomes

$$\ell_{[a} R_{b]uu[e} \ell_{f]} = 0\,. \tag{5.21}$$

Let j, k, l, m and n denote indices chosen only from [2] $\{x, y\}$. Choosing $a = m$, $e = n$, and $b = f = v$ we find

$$R_{muun} = 0 \tag{5.22}$$

for all m and n. Thus

$$R_{uu} = 0\,. \tag{5.23}$$

Equation (5.23) also follows immediately from the fact that since R_{uu} cannot be negative, any positive R_{uu} would lead to conjugate points.

If we apply the null convergence condition, Eq. (5.1), to $V = E_{(u)} + \epsilon E_{(m)} + (\epsilon^2/2) E_{(v)}$, where $\epsilon \ll 1$, we get

$$R_{uu} + 2R_{mu}\epsilon + O(\epsilon^2) \geq 0\,. \tag{5.24}$$

Since $R_{uu} = 0$ from Eq. (5.23), in order to have Eq. (5.24) hold for both signs of ϵ, we must have

$$R_{mu} = 0\,. \tag{5.25}$$

Since $R_{mu} = -R_{umvu} + g^{jk} R_{jmku}$,

$$R_{umvu} = g^{jk} R_{jmku}\,. \tag{5.26}$$

Now we use the Bianchi identity,

$$R_{luum;n} + R_{lunu;m} + R_{lumn;u} = 0\,. \tag{5.27}$$

From Eq. (5.22), $R_{luum,n} = 0$. The correction to make the derivatives covariant involves terms of the forms $R_{auum} \nabla_n E_l^{(a)}$ and $R_{laum} \nabla_n E_u^{(a)}$. Because of Eq. (5.22), the only contribution to the first of these comes from $a = v$, which we can transform using Eq. (5.26). For the second, we observe that $0 = \nabla_n (E^{(v)} \cdot E^{(v)}) = 2\nabla_n E^{(v)} \cdot E^{(v)} = 2\nabla_n E_u^{(v)}$, so $a = v$ does not contribute. Furthermore $R_{lumn;u} = R_{lumn,u}$, because the u direction is the single final direction in the coordinate construction of Sec. 5.1.2, and so in this direction the tetrad vectors are just parallel transported. Thus we find

$$\begin{aligned} \frac{dR_{lumn}}{du} &= g^{jk}[R_{jmku} \nabla_n E_l^{(v)} + R_{jlku} \nabla_n E_m^{(v)} - R_{jnku} \nabla_m E_l^{(v)} - R_{jlku} \nabla_m E_n^{(v)}] \\ &\quad + (R_{lkum} + R_{lukm}) \nabla_n E_u^{(k)} + (R_{lknu} + R_{lunk}) \nabla_m E_u^{(k)}\,. \end{aligned} \tag{5.28}$$

[2] This notation applies only in this subsection and not the rest of the thesis

Eq. (5.28) is a first-order differential equation in the pair of independent Riemann tensor components R_{xuxy} and R_{yuxy}. By assumption, the curvature and its derivative vanish in the distant past, and therefore the correct solution to these equations is

$$R_{lumn} = 0\,. \tag{5.29}$$

Eqs. (5.26) and (5.29) then give

$$R_{umvu} = 0\,. \tag{5.30}$$

Combining Eqs. (5.22), (5.29), and (5.30) and their transformations under the usual Riemann tensor symmetries, we conclude that all components of the Riemann tensor with more u's than v's vanish as desired.

5.2.4 Timelike paths

The general quantum inequality of Ch. 2 we will use for this proof is applied on timelike paths. So we are going to show that the paths $\Phi(\eta+\delta t/\sqrt{2}, t/\sqrt{2}\delta)$ are indeed timelike. Differentiating Eq. (5.15), we find the components of the tangent vector $k = d\Phi_V/dt = (1/\sqrt{2}, 1/\sqrt{2})$ in the Fermi coordinate basis. The squared length of p in terms of these components is $g_{ab}k^a k^b$. We showed in Appendix A that $g_{ab} = \eta_{ab} + h_{ab}$, where h_{ab} at some point X is a sum of a small number of terms (6 in the present case of 2-step Fermi coordinates) each of which is a coefficient no greater than 1 times an average of

$$R_{abcd}X^d X^c \tag{5.31}$$

over one of the geodesics used in the construction of the Fermi coordinate system. The summations over d and c in Eq. (5.31) are only over restricted sets of indices depending on the specific term under consideration. From Eqs. (5.15) and (5.16a) the points under consideration satisfy

$$|u| < u_1/\delta + \sqrt{2}\tau_0 \tag{5.32a}$$
$$|v| < \tau_0/\sqrt{2} \tag{5.32b}$$
$$x = y = 0\,. \tag{5.32c}$$

From Eq. (5.14), the first term in Eq. (5.32a) decreases faster than the second, so we find that all components of X are $O(t_0)$. Using the fact that the components of the Riemann tensor are bounded we find

$$h_{ab} = O(R_{\max} t_0^2) \tag{5.33}$$

so

$$g_{ab}k^a k^b = -1 + O(R_{\max} t_0^2)\,. \tag{5.34}$$

Thus for sufficiently large δ, and thus small t_0, k is timelike.

5.2.5 Causal diamond

The quantum inequality of Ch. 2Kontou:2014tha is applied to timelike paths inside a globally hyperbolic region of the spacetime. So this region N must include the timelike path from

$p = \Phi_V(\eta, -t_0)$ to $q = \Phi_V(\eta, t_0)$, and to be globally hyperbolic it must include all points in both the future of p in the past of q, so we can let N be the "double cone" or "causal diamond",

$$N = J^+(p) \cap J^-(q)\,. \tag{5.35}$$

We have shown that the curvature is small everywhere in the tube M', so we must show that $N \subset M'$.

From the previous section, we have that the metric can be written as

$$g_{ab} = \eta_{ab} + h_{ab}\,, \tag{5.36}$$

where h_{ab} consists of terms of the form $R_{abcd}X^dX^c$. The double cone in flat space obeys

$$|x|, |y|, |v| < \tau_0\,, \tag{5.37}$$

so the same is true at zeroth order in the Riemann tensor R. Thus at zeroth order,

$$h_{ab} = O(R_{\max} t_0^2)\,, \tag{5.38}$$

and so at first order in R,

$$|x|, |y|, |v| < \tau_0(1 + O(R_{\max} t_0^2))\,. \tag{5.39}$$

Since $t_0 \ll r$ for large δ, we have

$$|x|, |y|, |v| < r\,. \tag{5.40}$$

and $N \subset M'$ as desired.

5.3 The null-projected quantum inequality

We can write the general quantum inequality of Eq. (2.23) for $w(t) = \Phi_V(\eta, t)$ for a specific value of η and the stress energy tensor contracted with null vector field $\ell^a \equiv u$ as

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle T^{\text{ren}}_{uu} \rangle (w(t)) \geq -B\,, \tag{5.41}$$

where

$$B = \int_0^{\infty} \frac{d\xi}{\pi} \hat{F}(-\xi, \xi) - \int_{-\infty}^{\infty} dt\, g^2(t) \left(2aR_{,uu} + bR_{,uu}\right)\,, \tag{5.42}$$

where we used that $g_{uu} = 0$ so the term Q doesn't contribute at all, $R_{uu} = 0$ according to Sec. 5.2.3 and

$$F(t, t') = g(t) g(t') T^{\text{split}}_{uu'} \tilde{H}_{(1)}(w(t), w(t'))\,, \tag{5.43}$$

$\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (2.24).

5.4 Calculation of $T^{\text{split}}_{uu'} \tilde{H}_{(1)}$

To simplify the calculation we will evaluate the $T\tilde{H}_{(1)}$ in the coordinate system (t, x, y, z) where the timelike path $w(t)$ points only in the t direction, z direction is perpendicular to it and x and y are the previously defined ones. More specifically t and z are

$$t = \frac{\delta^{-1}u + \delta v}{\sqrt{2}}\,, \qquad z = \frac{\delta^{-1}u - \delta v}{\sqrt{2}}\,. \tag{5.44}$$

The new null coordinates $\tilde{u}$ and $\tilde{v}$ are defined by

$$\tilde{u} = \frac{t+z}{\sqrt{2}}\,, \qquad \tilde{v} = \frac{t-z}{\sqrt{2}}\,, \tag{5.45}$$

and are connected with u and v,

$$\tilde{u} = \delta^{-1}u\,, \qquad \tilde{v} = \delta v\,. \tag{5.46}$$

The operator $T^{\rm split}_{uu'}$ can be written as

$$T^{\rm split}_{uu'} = \delta^{-2}\partial_{\tilde{u}}\partial_{\tilde{u}'}\,. \tag{5.47}$$

If we define $\zeta = z - z'$ and $\bar{u}$ as the $\tilde{u}$ coordinate of $\bar{x}$, the center point between x and x' we have

$$T^{\rm split}_{uu'} = \frac{1}{2}\delta^{-2}\left(\frac{1}{2}\partial^2_{\bar{u}} - (\partial^2_\tau + 2\partial_\zeta\partial_\tau + \partial^2_\zeta)\right) \tag{5.48}$$

5.4.1 Derivatives on $\tilde{H}_{-1}$

For the derivatives of $\tilde{H}_{-1}$ it is simpler to use Eq. (5.47). We have

$$\partial_{\tilde{u}'}\partial_{\tilde{u}}\tilde{H}_{-1} \;=\; \frac{1}{4\pi^2}\left(\frac{\partial}{\partial x^{\tilde{u}}}\frac{\partial}{\partial x'^{\tilde{u}}}\right)\left(\frac{1}{\sigma_+}\right)\,. \tag{5.49}$$

In flat spacetime it is straightforward to apply the derivatives to $\tilde{H}_{-1}$. However in curved spacetime, there will be corrections first order in the Riemann tensor to both σ and its derivatives.

We are considering a path w whose tangent vector is constant in the coordinate system described in Sec. 5.1.2. The length of this path can be written

$$s(x,x') = \int_0^1 d\lambda\sqrt{g_{ab}(w(\lambda))\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}} = \int_0^1 d\lambda\sqrt{g_{ab}(x'')\Delta x^a\Delta x^b}\,. \tag{5.50}$$

where $\Delta x = x - x'$ and $x'' = x' + \lambda\Delta x$ since $dx^a/d\lambda$ is a constant.

Now σ is the negative squared length of the geodesic connecting x' to x. This geodesic might be slightly different from the path w. However, the deviation results from the Christoffel symbol Γ^a_{bc}, which is $O(R)$. Thus the distance between the two paths is also $O(R)$, and the difference in the metric between the two paths is thus $O(R^2)$. Similarly, the difference in length in the same metric due to the different path between the same two points is $O(R^2)$. All these effects can be neglected, and so we take $\sigma = -s^2$.

Now using Eq. (A.26) of Appendix A we can write the first-order correction to the metric,

$$g_{ab} = \eta_{ab} + F_{ab} + F_{ba}\,, \tag{5.51}$$

where F_{ab} is given by Eq. (A.28) of the same Appendix because the first step for $x = y = 0$ is in the $\tilde{v}$ direction and the second in the $\tilde{u}$ direction. By the symmetries of the Riemann tensor the only non-zero component is

$$F_{\tilde{v}\tilde{v}}(x'') = \int_0^1 d\kappa(1-\kappa)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(\kappa x''^{\tilde{u}}, x''^{\tilde{v}})x''^{\tilde{u}}x''^{\tilde{u}}\,, \tag{5.52}$$

where we took into account the different sign conventions. Putting this in Eq. (5.50) gives

$$\begin{aligned} s(x,x') &= \int_0^1 d\lambda\sqrt{2\Delta x^{\tilde{u}}\Delta x^{\tilde{v}} + 2F_{\tilde{v}\tilde{v}}\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}} \qquad (5.53)\\ &= \int_0^1 d\lambda\sqrt{2}\left(\sqrt{\Delta x^{\tilde{u}}\Delta x^{\tilde{v}}} + \frac{1}{2}F_{\tilde{v}\tilde{v}}(\Delta x^{\tilde{v}})^{3/2}(\Delta x^{\tilde{u}})^{-1/2}\right). \end{aligned}$$

So to first order in the curvature,

$$\sigma(x,x') = -s(x,x')^2 = -\tau^2 + \zeta^2 - 2\int_0^1 d\lambda F_{\tilde{v}\tilde{v}}\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\,. \qquad (5.54)$$

We define the zeroth order σ,

$$\sigma^{(0)}(x,x') = -\tau^2 + \zeta^2\,, \qquad (5.55)$$

and the first order,

$$\begin{aligned} \sigma^{(1)}(x,x') &= -2\int_0^1 d\lambda F_{\tilde{v}\tilde{v}}\Delta x^{\tilde{v}}\Delta x^{\tilde{v}} \qquad (5.56)\\ &= -2\int_0^1 d\lambda\int_0^1 d\kappa(1-\kappa)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(\kappa x''^{\tilde{u}}, x''^{\tilde{v}})x''^{\tilde{u}}x''^{\tilde{u}}\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\\ &= -2\int_0^1 d\lambda\int_0^\ell dy(\ell - y)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\,, \end{aligned}$$

where we defined $\ell \equiv x''^{\tilde{u}}$ and changed variables to $y = \kappa\ell$. Now to first order,

$$\frac{1}{\sigma_+} = \frac{1}{\sigma^{(0)}} - \frac{\sigma^{(1)}}{(\sigma^{(0)})^2}\,, \qquad (5.57)$$

and the derivatives,

$$\left(\frac{\partial}{\partial x^{\tilde{u}}}\frac{\partial}{\partial x'^{\tilde{u}}}\right)\left(\frac{1}{\sigma_+}\right)\Bigg|_{\zeta=0} = \frac{4}{\tau_-^4} + \frac{12}{\tau_-^6}\sigma^{(1)} - \frac{2\sqrt{2}}{\tau_-^5}\left(\sigma^{(1)}_{,\tilde{u}} - \sigma^{(1)}_{,\tilde{u}'}\right) - \frac{1}{\tau_-^4}\sigma^{(1)}_{,\tilde{u}\tilde{u}'}\,. \qquad (5.58)$$

Now we can take the derivatives of σ ,

$$\begin{aligned} \sigma^{(1)}_{,\tilde{u}} &= -2\int_0^1 d\lambda\lambda\frac{\partial}{\partial\ell}\int_0^\ell dy(\ell - y)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\\ &= -2\int_0^1 d\lambda\lambda\int_0^\ell dyR_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\,. \qquad (5.59) \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma^{(1)}_{,\tilde{u}'} &= -2\int_0^1 d\lambda(1-\lambda)\frac{\partial}{\partial\ell}\int_0^\ell dy(\ell - y)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\\ &= -2\int_0^1 d\lambda(1-\lambda)\int_0^\ell dyR_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\,. \qquad (5.60) \end{aligned}$$

For the two derivatives of $\sigma^{(1)}$,

$$\sigma^{(1)}_{,\tilde{u}\tilde{u}'} = -2\int_0^1 d\lambda(1-\lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''^{\tilde{u}}, x''^{\tilde{v}})\Delta x^{\tilde{v}}\Delta x^{\tilde{v}}\,. \qquad (5.61)$$

Now we can assume purely temporal separation, so $\Delta x^{\tilde{u}} = \Delta x^{\tilde{v}} = \tau/\sqrt{2}$ and

$$x'' = \frac{1}{\sqrt{2}}(t'' + \bar{z}, t'' - \bar{z}) \,, \tag{5.62}$$

where $\bar{z} = (z + z')/2$ and $t'' = t' + \lambda\tau$. Then the derivatives of $\tilde{H}_{-1}$ are

$$\begin{aligned} T^{\rm split}_{uu'}\tilde{H}_{-1} &= \frac{\delta^{-2}}{4\pi^2\tau_-^4}\bigg(4 - 12\int_0^1 d\lambda \int_0^\ell dy(\ell - y)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}}) \\ &\quad -2\sqrt{2}\int_0^1 d\lambda(1-2\lambda)\int_0^\ell dy R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}})\tau \\ &\quad + \int_0^1 d\lambda(1-\lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''^{\tilde{u}}, x''^{\tilde{v}})\tau^2\bigg) \,. \end{aligned} \tag{5.63}$$

Let us define the locations $\bar{x}_\kappa = (\kappa\bar{x}^{\tilde{u}}, \bar{x}^{\tilde{v}})$ and

$$x''_\kappa = \frac{1}{\sqrt{2}}(\kappa(t'' + \bar{z}), t'' - \bar{z}) \,. \tag{5.64}$$

Then Eq. (5.63) can be written

$$\begin{aligned} T^{\rm split}_{uu'}\tilde{H}_{-1} &= \frac{\delta^{-2}}{4\pi^2\tau_-^4}\bigg(4 - \int_0^1 d\lambda\bigg[12\int_0^1 d\kappa(1-\kappa)(x''^{\tilde{u}})^2 R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa) \\ &\quad +2\sqrt{2}(1-2\lambda)\int_0^1 d\kappa x''^{\tilde{u}} R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''_\kappa)\tau - (1-\lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x'')\tau^2\bigg]\bigg) \,. \end{aligned} \tag{5.65}$$

The derivatives of $\tilde{H}_{-1}$ can thus be written

$$T^{\rm split}_{uu'}\tilde{H}_{-1} = \delta^{-2}\left[\frac{1}{\tau_-^4}\left(\frac{1}{\pi^2} + y_1(\bar{t},\tau)\right) + \frac{1}{\tau_-^3}y_2(\bar{t},\tau) + \frac{1}{\tau_-^2}y_3(\bar{t},\tau)\right] , \tag{5.66}$$

where the y_i's are smooth functions of the curvature,

$$y_1(\bar{t},\tau) = \int_0^1 d\lambda Y_1(t'') \qquad y_2(\bar{t},\tau) = \int_0^1 d\lambda(1-2\lambda)Y_2(t'') \qquad y_3(\bar{t},\tau) = \int_0^1 d\lambda(1-\lambda)\lambda Y_3(t'') \,, \tag{5.67}$$

with

$$Y_1(t'') = -\frac{3}{2\pi^2}\int_0^1 d\kappa(1-\kappa)(t''+\bar{z})^2 R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa) \,, \tag{5.68a}$$

$$Y_2(t'') = \frac{1}{2\pi^2}\int_0^1 d\kappa(t''+\bar{z}) R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa) \,, \tag{5.68b}$$

$$Y_3(t'') = -\frac{1}{4\pi^2} R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x'') \,, \tag{5.68c}$$

where x'' and x''_κ are defined in terms of t'' by Eqs. (5.62) and (5.64).

5.4.2 Derivatives with respect to τ and $\bar{u}$

In Ch. 4 we calculated $\tilde{H}_{(1)}$, but for points separated only in time. Let us use coordinates (T, Z, X, Y) to denote a coordinate system where the coordinates of x and x' differ only in T. Ref. [26] gives

$$\tilde{H}_{(1)}(T,T') = \tilde{H}_{-1}(T,T') + \tilde{H}_0(T,T') + \tilde{H}_1(T,T') + \frac{1}{2} i R_1(T,T'), \tag{5.69}$$

$$\tilde{H}_{(0)}(T,T') = \tilde{H}_{-1}(T,T') + \tilde{H}_0(T,T') + \frac{1}{2} i R_0(T,T'), \tag{5.70}$$

where

$$\tilde{H}_{-1}(T,T') = -\frac{1}{4\pi^2 (T-T'-i\epsilon)^2}, \tag{5.71a}$$

$$\tilde{H}_0(T,T') = \frac{1}{48\pi^2}\left[R_{TT}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln\left(-(T-T'-i\epsilon)^2\right)\right], \tag{5.71b}$$

$$\tilde{H}_1(T,T') = \frac{(T-T')^2}{640\pi^2}\left[\frac{1}{3}R_{TT,TT}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{II}(\bar{x}) + \frac{1}{2}R_{,TT}(\bar{x})\right)\ln\left(-(T-T'-i\epsilon)^2\right)\right]. \tag{5.71c}$$

The order-0 remainder term is

$$R_0(T,T') = \frac{1}{32\pi^2}\int d\Omega \left\{ \frac{1}{2}\left[G^{(1)}_{TT}(X'') - G^{(1)}_{RR}(X'')\right] - \int_0^1 ds\, s^2 G^{(1)}_{TT}(X''_s)\right\} \operatorname{sgn}(T-T'), \tag{5.72}$$

where $\int d\Omega$ means to integrate over solid angle with unit 3-vectors $\hat{\Omega}$, the 4-vector $\Omega = (0, \hat{\Omega})$, the subscript R means the radial direction, and we define $X'' = \bar{x} + (1/2)|T-T'|\Omega$, $X''_s = \bar{x} + (s/2)|T-T'|\Omega$, and

$$G^{(1)}_{AB}(X'') = G_{AB}(X'') - G_{AB}(\bar{x}) = \int_0^{|T-T'|/2} dr\, G_{AB,I}(\bar{x} + r\Omega)\Omega^I. \tag{5.73}$$

The order-1 remainder term is

$$R_1(T,T') = \frac{1}{32\pi^2}\int d\Omega \left\{ \frac{1}{2}\left[G^{(3)}_{TT}(X'') - G^{(3)}_{RR}(X'')\right] - \int_0^1 ds\, s^2 G^{(3)}_{TT}(X''_s)\right\} \operatorname{sgn}(T-T'), \tag{5.74}$$

where $G^{(3)}_{AB}$ is the remainder after subtracting the second-order Taylor series. We can write

$$G^{(3)}_{AB}(X'') = \frac{1}{2}\int_0^{|T-T'|/2} dr G_{AB,IJK}(\bar{x} + r\Omega)\left(\frac{T-T'}{2} - r\right)^2 \Omega^I\Omega^J\Omega^K. \tag{5.75}$$

When we apply the τ and $\bar{u}$ derivatives from Eq. (5.48), we can take $(T, Z, X, Y) = (t, z, x, y)$ and calculate $\partial^2_{\bar{u}}\tilde{H}_0$, $\partial^2_\tau \tilde{H}_0$, $\partial^2_\tau \tilde{H}_1$, $\partial^2_{\bar{u}} R_0$, and $\partial^2_\tau R_1$. Applying $\bar{u}$ derivatives to $\tilde{H}_0$ gives

$$\partial^2_{\bar{u}}\tilde{H}_0 = \frac{1}{48\pi^2}\left[R_{tt,\bar{u}\bar{u}}(\bar{x}) - \frac{1}{2}R_{,\bar{u}\bar{u}}\ln(-\tau_-^2)\right]. \tag{5.76}$$

For the derivatives with respect to τ we have

$$\partial_\tau^2 \tilde{H}_0 = \frac{1}{48\pi^2\tau_-^2} R(\bar{x})\,, \tag{5.77}$$

and

$$\partial_\tau^2 \tilde{H}_1 = \frac{1}{320\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\left(3+\ln\left(-\tau_-^2\right)\right)\right]. \tag{5.78}$$

in the $\tau \to 0$ limit.

Applying $\bar{u}$ derivatives to R_0 gives

$$\partial_{\bar{u}}^2 R_0 = \frac{1}{32\pi^2}\int d\Omega \int_0^{|\tau|/2} dr\, \partial_{\bar{u}}^2 \left\{ \frac{1}{2}\left[G_{tt,i}(x''') - G_{rr,i}(x''')\right] - \int_0^1 ds\, s^2 G_{tt,i}(x_s''')\right\}\Omega^i \operatorname{sgn}\tau\,, \tag{5.79}$$

where $x''' = \bar{x} + r\Omega$ and $x_s''' = \bar{x} + sr\Omega$.

Now we have to take the second derivative of R_1 with respect to τ, which is $T - T'$ in this case. This appears in three places: the argument of sgn in Eq. (5.74), the limit of integration in Eq. (5.75), and the term in parentheses in Eq. (5.75). When we differentiate the sgn, we get $\delta(\tau)$ and $\delta'(\tau)$. but since $G_{AB}^{(3)} \sim \tau^3$, there are enough powers of τ to cancel the δ or δ', so this gives no contribution. When we differentiate the limit of integration, the term in parentheses in Eq. (5.75) vanishes immediately. The one remaining possibility gives

$$\partial_\tau^2 R_1 = \frac{1}{128\pi^2}\int d\Omega \int_0^{|\tau|/2} dr \left\{ \frac{1}{2}\left[G_{tt,ijk}(x''') - G_{rr,ijk}(x''')\right] - \int_0^1 ds\, s^2 G_{tt,ijk}(x_s''')\right\}\Omega^i\Omega^j\Omega^k \operatorname{sgn}\tau\,. \tag{5.80}$$

5.4.3 Derivatives with respect to ζ

To differentiate with respect to ζ, we must consider the possibility that x and x' are not purely temporally separated. We will suppose that the separation is only in the t and z directions and construct new coordinates (T, Z) using a Lorentz transformation that leaves $\bar{x}$ unchanged and maps the interval $(T - T', 0)$ in the new coordinates to (τ, ζ) in the old coordinates. Then

$$T - T' = \operatorname{sgn}\tau \sqrt{\tau^2 - \zeta^2}\,, \tag{5.81}$$

and the transformation from (T, Z) to (t, z) is given by

$$\Lambda = \frac{1}{\operatorname{sgn}\tau\sqrt{\tau^2-\zeta^2}}\begin{pmatrix} \tau & \zeta \\ \zeta & \tau \end{pmatrix}. \tag{5.82}$$

with the x and y coordinates unchanged. Then

$$\begin{pmatrix} \tau \\ \zeta \end{pmatrix} = \Lambda \begin{pmatrix} T - T' \\ 0 \end{pmatrix}. \tag{5.83}$$

Now let M be some tensor appearing in $\tilde{H}_{(1)}$. The components in the new coordinate system are given in terms of those in the old by

$$M_{ABC\ldots} = \Lambda^a_A \Lambda^b_B \Lambda^c_C \ldots M_{abc\ldots} \tag{5.84}$$

We would like to differentiate such an object with respect to ζ and then set $\zeta = 0$. The only place ζ can appear is in the Lorentz transformation matrix, where we see

$$\partial_\zeta \Lambda^a_A \Big|_{\zeta=0} = \tau^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.85}$$

and similarly,

$$\partial^2_\zeta \Lambda^a_A \Big|_{\zeta=0} = \tau^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.86}$$

To simplify notation, we will define P and Q to be the matrices on the right hand sides. Reinstating x and y,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.87}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.88}$$

Now we can write the derivative of $M_{ABC\ldots}$ as

$$\begin{aligned} \partial_\zeta M_{ABC\ldots}\Big|_{\zeta=0} &= \partial_\zeta(\Lambda^a_A \Lambda^b_B \Lambda^c_C \ldots) M_{abc\ldots}\Big|_{\zeta=0} \\ &= \Big[(\partial_\zeta \Lambda^a_A)\delta^b_B \delta^c_C \cdots + \delta^a_A (\partial_\zeta \Lambda^b_B)\delta^c_C \cdots + \ldots \Big] M_{abc\ldots}\Big|_{\zeta=0} \\ &= \frac{1}{\tau_-}(\underbrace{P^a_A \delta^b_B \delta^c_C \cdots + \delta^a_A P^b_B \delta^c_C \cdots + \ldots}_{n}) M_{abc\ldots} = \frac{1}{\tau_-} p^{abc\ldots}_{ABC\ldots} M_{abc\ldots} \end{aligned} \tag{5.89}$$

where $p^{abc\ldots}_{ABC\ldots}$ is a rank-n matrix of 0's and 1's. With two derivatives, we have

$$\begin{aligned} \partial^2_\zeta M_{ABC\ldots}\Big|_{\zeta=0} &= \partial^2_\zeta(\Lambda^a_A \Lambda^b_B \Lambda^c_C \ldots) M_{abc\ldots}\Big|_{\zeta=0} \\ &= \Big[(\partial^2_\zeta \Lambda^a_A)\delta^b_B \delta^c_C \cdots + \delta^a_A (\partial^2_\zeta \Lambda^b_B)\delta^c_C \cdots + \ldots \\ &\quad + 2(\partial_\zeta \Lambda^a_A)(\partial_\zeta \Lambda^b_B)\delta^c_C \cdots + 2(\partial_\zeta \Lambda^a_A)\delta^b_B(\partial_\zeta \Lambda^c_C) \cdots + \ldots \Big] M_{abc\ldots}\Big|_{\zeta=0} \\ &= \frac{1}{\tau_-^2}(\underbrace{Q^a_A \delta^b_B \delta^c_C \cdots + \delta^a_A Q^b_B \delta^c_C \cdots + \ldots}_{n} \\ &\quad + \underbrace{P^a_A P^b_B \delta^c_C \cdots + P^a_A \delta^b_B P^c_C \cdots + \ldots}_{(n-1)n}) M_{abc\ldots} \\ &= \frac{1}{\tau_-^2} q^{abc\ldots}_{ABC\ldots} M_{abc\ldots} \end{aligned} \tag{5.90}$$

where $q^{abc\cdots}_{ABC\cdots}$ is a rank-n matrix of nonnegative integers.

There are also places where $T - T'$ appears explicitly in $\tilde{H}_1$. We can differentiate it using Eq. (5.81),

$$\partial_\zeta (T - T')\bigg|_{\zeta=0} = 0\,, \tag{5.91a}$$

$$\partial_\zeta^2 (T - T')\bigg|_{\zeta=0} = -\tau^{-1}\,. \tag{5.91b}$$

Now we apply the operators ∂_ζ^2 and $\partial_\tau\partial_\zeta$ to $\tilde{H}_0$, $\tilde{H}_1$, and R_1. First we apply one ζ derivative[3] to Eq. (5.71b) using Eq. (5.89),

$$\partial_\tau \left(\partial_\zeta \tilde{H}_0 \bigg|_{\zeta=0} \right) = -\frac{1}{48\pi^2\tau_-^2} p^{ab}_{tt} R_{ab}(\bar{x})\,, \tag{5.92}$$

and two ζ derivatives using Eqs. (5.90) and (5.91a),

$$\partial_\zeta^2 \tilde{H}_0 \bigg|_{\zeta=0} = \frac{1}{48\pi^2\tau_-^2} \left[q^{ab}_{tt} R_{ab}(\bar{x}) + R(\bar{x}) \right]\,. \tag{5.93}$$

Then we apply one ζ derivative to $\tilde{H}_1$,

$$\partial_\tau \left(\partial_\zeta \tilde{H}_1 \bigg|_{\zeta=0} \right) = \frac{1}{1920\pi^2} \left[p^{abcd}_{tttt} R_{ab,cd}(\bar{x}) - \left(p^{ab}_{ii} \Box R_{ab}(\bar{x}) + \frac{1}{2} p^{ab}_{tt} R_{,ab}(\bar{x}) \right) (\ln(-\tau_-^2) + 2) \right]\,, \tag{5.94}$$

and two ζ derivatives to $\tilde{H}_1$,

$$\begin{aligned} \partial_\zeta^2 \tilde{H}_1 \bigg|_{z=0} = \frac{1}{640\pi^2} \bigg[& \frac{1}{3} q^{abcd}_{tttt} R_{ab,cd}(\bar{x}) - \frac{2}{3} R_{tt,tt}(\bar{x}) + \Box R(\bar{x}) - \frac{1}{3} \bigg(q^{ab}_{ii} \Box R_{ab}(\bar{x}) \\ & + \frac{1}{2} q^{ab}_{tt} R_{,ab}(\bar{x}) \bigg) \ln(-\tau_-^2) + \frac{2}{3} \left(\Box R_{ii}(\bar{x}) + \frac{1}{2} R_{,tt}(\bar{x}) \right) (1 + \ln(-\tau_-^2)) \bigg]\,. \end{aligned} \tag{5.95}$$

Finally we have to apply the derivatives to the remainder R_1. We can apply the ζ derivatives in two places, the Lorentz transformations and $G^{(3)}_{AB}$. Since the three terms are very similar we will apply the derivatives to one of them

$$\partial_\zeta \int d\Omega G^{(3)}_{TT}(\bar{x} + Y) \bigg|_{\zeta=0} = \int d\Omega \left(\frac{1}{\tau} p^{ab}_{tt} G^{(3)}_{ab}(x'') + \frac{\partial}{\partial Y^a} G^{(3)}_{tt}(\bar{x} + Y) \partial_\zeta Y^a \bigg|_{\zeta=0} \right)\,, \tag{5.96}$$

where we defined $Y^a \equiv (1/2)|T - T'|\Lambda^a_I \Omega^I$. Then using Eqs. (5.89) and (5.91a), we find that that $\partial_\zeta Y^a|_{\zeta=0} = (1/2) p^a_i \Omega^i \operatorname{sgn}\tau$ and taking into account the properties of Taylor expansions,

$$\frac{\partial}{\partial Y^a} G^{(3)}_{tt}(\bar{x} + Y) \bigg|_{\zeta=0} = G^{(2)}_{tt,a}(x'')\,, \tag{5.97}$$

[3] The Lorentz transformation technique we use here is not quite sufficient to determine the singularity structure of the distribution $\partial_\zeta \tilde{H}_0$ at coincidence. Instead we can use Eq. (4.29) to compute the non-logarithmic term in $\tilde{H}_0$ for arbitrary x and x', which is then $-R_{ab}(\bar{x})(x - x')^a(x - x')^b/(48\pi^2\sigma_+)$. Differentiating this term gives Eq. (5.92) and explains the presence of τ_- instead of τ in the denominator. The first term of Eq. (5.93) arises similarly.

where $G^{(2)}_{ab,c}$ is the remainder of the Taylor expansion of $G_{ab,c}$ after subtracting the first-order Taylor series.

Thus Eq. (5.96) becomes

$$\partial_\zeta \int d\Omega G^{(3)}_{TT}(\bar{x}+Y)\Big|_{\zeta=0} = \int d\Omega \left(\frac{1}{\tau} p^{ab}_{tt} G^{(3)}_{ab}(x'') + \frac{1}{2} G^{(2)}_{tt,a}(x'') p^a_i \Omega^i \operatorname{sgn}\tau\right) . \tag{5.98}$$

Using $G^{(3)}$ from Eq. (5.75) and

$$G^{(2)}_{ab,c}(x'') = \int_0^{|\tau|/2} dr G_{ab,ijc}(\bar{x}+r\Omega)\left(\frac{|\tau|}{2}-r\right)\Omega^i\Omega^j , \tag{5.99}$$

Eq. (5.98) becomes

$$\begin{aligned}\partial_\zeta \int d\Omega G^{(3)}_{TT}(\bar{X}+Y)|_{\zeta=0} = \int d\Omega \int_0^{|\tau|/2} dr \left(\frac{|\tau|}{2}-r\right) \quad & \left[p^{ab}_{tt} G_{ab,ijk}(x''')\frac{1}{\tau}\left(\frac{|\tau|}{2}-r\right)\right. \\ & \left.+\frac{1}{2} p^a_i G_{tt,ija}(x''') \operatorname{sgn}\tau\right]\Omega^i\Omega^j\Omega^k .\end{aligned} \tag{5.100}$$

We could simplify further by using the explicit values of the p matrices, but our strategy here is to show that all terms are bounded by some constants without computing the constants explicitly, since the actual constant values will not matter to the proof.

Applying the τ derivative gives

$$\begin{aligned}\partial_\tau\left(\partial_\zeta \int d\Omega G^{(3)}_{TT}(\bar{x}+Y)\Big|_{\zeta=0}\right) = \int d\Omega \int_0^{|\tau|/2} dr \Bigg[\quad & \left(\frac{1}{4}-\frac{r^2}{\tau^2}\right) p^{ab}_{tt} G_{ab,ijk}(x''') \\ & +\frac{1}{4} p^a_i G_{tt,ija}(x''')\Bigg]\Omega^i\Omega^j\Omega^k .\end{aligned} \tag{5.101}$$

We do not have to differentiate $\operatorname{sgn}\tau$ here, because the rest of the term is $O(\tau^2)$ and so a term involving $\delta(\tau)$ would not contribute.

The same procedure can be applied to all three terms. Terms involving X''_s will get an extra power of s each time G is differentiated. The final result is

$$\begin{aligned}&\partial\tau\left(\partial_\zeta R_1(T,T')\Big|_{\zeta=0}\right) \\ &= \frac{1}{32\pi^2}\int d\Omega \int_0^{|\tau|/2} dr\Bigg\{\left(\frac{1}{4}-\frac{r^2}{\tau^2}\right)\left[\frac{1}{2}(p^{ab}_{tt}-p^{ab}_{rr})G_{ab,ijk}(x''') - \int_0^1 dss^2 p^{ab}_{tt} G_{ab,ijk}(x'''_s)\right] \\ &\quad +\frac{1}{4}\left[\frac{p^a_i}{2}(G_{tt,ija}(x''')-G_{rr,ija}(x''')) - \int_0^1 dss^3 p^a_i G_{tt,ija}(x'''_s)\right]\Bigg\}\Omega^i\Omega^j\Omega^k \operatorname{sgn}\tau .\end{aligned} \tag{5.102}$$

For two ζ derivatives we can apply both on the Lorentz transforms, both on the Einstein tensor or one on each,

$$\begin{aligned}\partial^2_\zeta \int d\Omega G^{(3)}_{TT}(\bar{x}+Y)\Big|_{\zeta=0} = & \int d\Omega\Bigg(\frac{q^{ab}_{tt}}{\tau^2} G^{(3)}_{ab}(x'') + \frac{\partial^2}{\partial Y^a \partial Y^b} G^{(3)}_{tt}(\bar{x}+Y)\partial_\zeta Y^a \partial_\zeta Y^b\Big|_{\zeta=0} \\ & +\frac{\partial}{\partial Y^a} G^{(3)}_{tt}(\bar{x}+Y)\partial^2_\zeta Y^a\Big|_{\zeta=0} \\ & +2\frac{p^{ab}_{tt}}{\tau}\frac{\partial}{\partial Y^c} G^{(3)}_{ab}(\bar{x}+Y)\partial_\zeta Y^c\Big|_{\zeta=0}\Bigg) .\end{aligned} \tag{5.103}$$

Using Eqs. (5.91b) and (5.90),

$$\partial_\zeta^2 Y^j\Big|_{\zeta=0} = \frac{q_i^j\Omega^i}{2\tau} - \frac{\Omega^j}{2\tau} = \frac{1}{2\tau}h_i^j\Omega^i\,, \tag{5.104}$$

with $h_i^j \equiv q_i^j - \delta_i^j$, while

$$\partial_\zeta^2 Y^t\Big|_{\zeta=0} = 0\,, \tag{5.105}$$

since $q_i^t = 0$ and $\Omega^t = 0$. Using properties of the Taylor series as before, we can write

$$\frac{\partial^2}{\partial Y^a \partial Y^b} G_{tt}^{(3)}(\bar{x}+Y) = G_{tt,ab}^{(1)}(x'')\,, \tag{5.106}$$

so Eq. (5.103) becomes

$$\begin{aligned}\partial_\zeta^2 \int d\Omega G_{TT}^{(3)}(\bar{x}+Y)\Big|_{\zeta=0} &= \int d\Omega\Big[\frac{q_{tt}^{ab}}{\tau^2}G_{ab}^{(3)}(x'') + \frac{1}{4}p_i^a p_j^b G_{tt,ab}^{(1)}(x'')\Omega^i\Omega^j \\ &\quad + \frac{1}{2|\tau|}\Big(2p_{tt}^{ab}p_i^c G_{ab,c}^{(2)}(x'') + G_{tt,j}^{(2)}(x'')h_i^j\Big)\Omega^i\Big]\,. \end{aligned} \tag{5.107}$$

Using $G^{(1)}$ as in Eq. (5.73) and $G^{(2)}$ and $G^{(3)}$ from Eqs. (5.99) and (5.75) this becomes

$$\begin{aligned}\partial_\zeta^2 \int d\Omega G_{TT}^{(3)}(\bar{x}+Y)\Big|_{\zeta=0} &= \int d\Omega \int_0^{|\tau|/2} dr\Big[\Big(\frac{1}{2} - \frac{r}{|\tau|}\Big)^2 q_{tt}^{ab}G_{ab,ijk}(x''') \\ &\quad + \frac{1}{4}p_i^a p_j^b G_{tt,kab}(x''') + \Big(\frac{1}{4} - \frac{r}{2|\tau|}\Big)\Big(2p_{tt}^{ab}p_i^c G_{ab,jkc}(x''') \\ &\quad + h_i^l G_{tt,ljk}(x''')\Big)\Big]\Omega^i\Omega^j\Omega^k\,. \end{aligned} \tag{5.108}$$

For all three terms

$$\begin{aligned}&\partial_\zeta^2 R_1(T,T')\Big|_{\zeta=0} \\ &= \frac{1}{32\pi^2}\int d\Omega \int_0^{|\tau|/2} dr\Big\{\Big(\frac{1}{2} - \frac{r}{|\tau|}\Big)^2\Big[\frac{1}{2}(q_{tt}^{ab} - q_{rr}^{ab})G_{ab,ijk}(x''') - \int_0^1 ds s^2 q_{tt}^{ab}G_{ab,ijk}(x_s''')\Big] \\ &\quad + \frac{1}{4}p_i^a p_j^b\Big[\frac{1}{2}(G_{tt,kab}(x''') - G_{rr,kab}(x''')) - \int_0^s ds s^4 G_{tt,kab}(x_s''')\Big] \\ &\quad + \Big(\frac{1}{4} - \frac{r}{2|\tau|}\Big)\Big[p_i^c(p_{tt}^{ab} - p_{rr}^{ab})G_{ab,jkc}(x''') + \frac{1}{2}h_i^l(G_{tt,ljk}(x''') - G_{rr,ljk}(x''')) \\ &\quad - \int_0^1 ds s^3(2p_i^c p_{tt}^{ab}G_{ab,jkc}(x_s''') + h_i^l G_{tt,ljk}(x_s'''))\Big]\Big\}\Omega^i\Omega^j\Omega^k \operatorname{sgn}\tau\,. \end{aligned} \tag{5.109}$$

5.5 The Fourier transform

Eqs. (5.66), (5.76), (5.77), (5.78), (5.79), (5.80), (5.92), (5.93), (5.94), (5.95), (5.102) and (5.109) include all the $T_{uu'}^{\rm split}\tilde{H}_{(1)}$ terms. To perform the Fourier transform we expand $T_{uu'}^{\rm split}\tilde{H}_{(1)}$

according to Eqs. (5.47) and (5.48) and separate the terms by their τ dependence,

$$
\begin{aligned}
T^{\rm split}_{uu'}\tilde{H}_{(1)} &= \delta^{-2}\Bigg[\partial_{\tilde{u}}\partial_{\tilde{u}'}\tilde{H}_{-1}+\frac{1}{4}\Bigg(\partial^2_{\tilde{u}}\tilde{H}_0+\frac{1}{2}iR_0\Bigg) \\
&\qquad -\frac{1}{2}(\partial_\tau^2+\partial_\zeta^2+2\partial_\tau\partial_\zeta)\left(\tilde{H}_0+\tilde{H}_1+\frac{1}{2}iR_1\right)\Bigg] \\
&= \delta^{-2}\Bigg[\frac{1}{\tau_-^4}\left(\frac{1}{\pi^2}+y_1(\bar{t},\tau)\right)+\frac{1}{\tau_-^3}y_2(\bar{t},\tau)+\frac{1}{\tau_-^2}(c_1(\bar{t})+y_3(\bar{t},\tau)) \\
&\qquad +\ln(-\tau_-^2)c_2(\bar{t})+c_3(\bar{t})+c_4(\bar{t},\tau)\Bigg], \qquad (5.110)
\end{aligned}
$$

where c_1, c_2, and c_3 are smooth and have no τ dependence and c_4 is odd, C_1 and bounded. As mentioned in Sec. 5.4.1, the functions y_i depend on τ but are smooth. Explicit expressions for the c_i are

$$
\begin{aligned}
c_1 &= \frac{1}{48\pi^2}\Bigg(-R(\bar{x})+\left(p^{ab}_{tt}-\frac{q^{ab}_{tt}}{2}\right)R_{ab}(\bar{x})\Bigg) \qquad (5.111a)\\
c_2 &= \frac{1}{1920\pi^2}\Bigg(-5R_{,\tilde{u}\tilde{u}}(\bar{x})+\left(p^{ab}_{ii}+\frac{q^{ab}_{ii}}{2}\right)\Box R_{ab}(\bar{x})+\frac{1}{2}\left(p^{ab}_{tt}+\frac{q^{ab}_{tt}}{2}\right)R_{,ab}(\bar{x})\Bigg) \qquad (5.111b)\\
c_3 &= \frac{1}{960\pi^2}\Bigg(5R_{tt,\tilde{u}\tilde{u}}(\bar{x})-\frac{1}{2}\left(p^{abcd}_{tttt}+\frac{q^{abcd}_{tttt}}{2}\right)R_{ab,cd}(\bar{x})+\Box R_{ii}(\bar{x})+\frac{1}{2}R_{,tt}(\bar{x}) \\
&\qquad +p^{ab}_{ii}\Box R_{ab}(\bar{x})+\frac{1}{2}p^{ab}_{tt}R_{,ab}(\bar{x})\Bigg) \qquad (5.111c)\\
c_4 &= \frac{1}{256\pi^2}\int_0^{|\tau|/2}dr\int d\Omega\Bigg(\partial^2_{\tilde{u}}\Bigg\{\frac{1}{2}\left[G_{tt,i}(x''')-G_{rr,i}(x''')\right]-\int_0^1 ds\, s^2G_{tt,i}(x'''_s)\Bigg\} \\
&\quad -\Bigg\{\frac{1}{4}\left[G_{tt,ijk}(x''')-G_{rr,ijk}(x''')\right]-\frac{1}{2}\int_0^1 ds\, s^2G_{tt,ijk}(x'''_s) \\
&\quad +\left(1-\frac{4r^2}{\tau^2}\right)\left[\frac{1}{2}(p^{ab}_{tt}-p^{ab}_{rr})G_{ab,ijk}(x''')-\int_0^1 ds s^2p^{ab}_{tt}G_{ab,ijk}(x'''_s)\right] \\
&\quad +\frac{p^a_i}{2}(G_{tt,jka}(x''')-G_{rr,jka}(x'''))-\int_0^1 ds s^3p^a_iG_{tt,jka}(x'''_s) \\
&\quad +2\left(\frac{1}{2}-\frac{r}{|\tau|}\right)^2\left[\frac{1}{2}(q^{ab}_{tt}-q^{ab}_{rr})G_{ab,ijk}(x''')-\int_0^1 ds s^2q^{ab}_{tt}G_{ab,ijk}(x'''_s)\right] \\
&\quad +\frac{1}{2}p^a_ip^b_j\left[\frac{1}{2}(G_{tt,kab}(x''')-G_{rr,kab}(x'''))-\int_0^s ds s^4G_{tt,kab}(x'''_s)\right] \\
&\quad +\left(\frac{1}{2}-\frac{r}{|\tau|}\right)\Bigg[p^c_i(p^{ab}_{tt}-p^{ab}_{rr})G_{ab,jkc}(x''')+\frac{1}{2}h^l_i(G_{tt,ljk}(x''')-G_{rr,ljk}(x''')) \\
&\quad -\int_0^1 ds s^3(2p^c_ip^{ab}_{tt}G_{ab,jkc}(x'''_s)+h^l_iG_{tt,ljk}(x'''_s))\Bigg\}\Omega^j\Omega^k\Bigg)\Omega^i\,\mathrm{sgn}\,\tau\,. \qquad (5.111d)
\end{aligned}
$$

We now put the terms of Eq. (5.110) into Eq. (5.42), and Fourier transform them, following the procedure of Sec. 3.4 , to obtain the bound B in the form

$$
B=\delta^{-2}\sum_{i=0}^{6}B_i\,. \qquad (5.112)
$$

The first term in Eq. (5.110) is $1/(\pi^2\tau_-^4)$, and we proceed exactly as Sec. 3.4, except for the different numerical coefficient, to obtain

$$B_0 = \frac{1}{24\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, g''(\bar{t})^2 \,. \tag{5.113}$$

Putting only Eq. (5.113) into Eq. (5.112) gives the result for flat space. Fewster and Eveson [9] found a result of the same form, but they considered T_{tt} instead of T_{uu}, so the multiplying constant is different. Fewster and Roman [14] found the result for null projection. Where we have 1/24, they had $(v\cdot\ell)^2/12$, where v is the unit tangent vector to the path of integration. Here $v\cdot\ell = \ell^t = 1/(\delta\sqrt{2})$, from Eq. (5.44), so the results agree.

The remaining τ_-^{-4} term requires more attention, because of the τ dependence in y_1. We write

$$B_1 = \int_0^{\infty}\frac{d\xi}{\pi}\int_{-\infty}^{\infty} d\tau G_1(\tau)\frac{1}{\tau_-^4}e^{-i\xi\tau}\,, \tag{5.114}$$

with

$$G_1(\tau) = \int_{-\infty}^{\infty} d\bar{t}(y_1(\bar{t},\tau))g\left(\bar{t}-\frac{\tau}{2}\right)g\left(\bar{t}+\frac{\tau}{2}\right)\,. \tag{5.115}$$

Then

$$B_1 = \frac{1}{24}G_1''''(0)\,. \tag{5.116}$$

Applying the τ derivatives to G_1 gives

$$\begin{aligned} G_1''''(\tau)\Big|_{\tau=0} = \int_{-\infty}^{\infty} d\bar{t}\Big[&\frac{d^4}{d\tau^4}y_1(\bar{t},\tau)\Big|_{\tau=0} g(\bar{t})^2 + 3\frac{d^2}{d\tau^2}y_1(\bar{t},\tau)\Big|_{\tau=0}(g''(\bar{t})g(\bar{t})-g'(\bar{t})^2) \\ &+\frac{1}{8}y_1(\bar{t})(g''''(\bar{t})g(\bar{t})-4g'''(\bar{t})g(\bar{t})+3g''(\bar{t})^2)\Big]\,, \end{aligned} \tag{5.117}$$

where the terms with an odd number of derivatives of the product of the sampling functions vanish after taking $\tau=0$.

Now y_1 depends on τ and $\bar{t}$ only through $t''=\bar{t}+(\lambda-1/2)\tau$, so using Eq. (5.67), we can write

$$\frac{d}{d\tau}y_1(\bar{t},\tau) = \frac{d}{d\tau}\int_0^1 d\lambda Y_1(t'') = \frac{d}{d\bar{t}}\int_0^1 d\lambda(\lambda-1/2)Y_1(t'')\,. \tag{5.118}$$

Then we integrate by parts and put all the derivatives on the sampling functions g,

$$\begin{aligned} B_1 = \frac{1}{24}\int_{-\infty}^{\infty} d\bar{t}\Big[& 2\int_0^1 d\lambda\left(\lambda-\frac{1}{2}\right)^4 Y_1(\bar{t})(3g''(\bar{t})^2+4g'(\bar{t})g'''(\bar{t})+g(\bar{t})g''''(\bar{t})) \\ &+3\int_0^1 d\lambda\left(\lambda-\frac{1}{2}\right)^2 Y_1(\bar{t})(g''''(\bar{t})g(\bar{t})-g''(\bar{t})^2) \\ &+\frac{1}{8}y_1(\bar{t})(g''''(\bar{t})g(\bar{t})-4g'''(\bar{t})g'(\bar{t})+3g''(\bar{t})^2)\Big]\,. \end{aligned} \tag{5.119}$$

Since we set $\tau=0$, Y_1 has no λ dependence and we can perform the integral. The result is

$$B_1 = \frac{1}{120}\int_{-\infty}^{\infty} d\bar{t}\, Y_1(\bar{t})(g''(\bar{t})^2-2g'''(\bar{t})g'(\bar{t})+2g''''(\bar{t})g(\bar{t}))\,. \tag{5.120}$$

For the term proportional to τ_-^{-3}, we have

$$B_2 = \int_0^\infty \frac{d\xi}{\pi} \int_{-\infty}^\infty d\tau G_2(\tau) \frac{1}{\tau_-^3} e^{-i\xi\tau} . \tag{5.121}$$

where

$$G_2(\tau) = \int_{-\infty}^\infty d\bar{t} y_2(\bar{t},\tau) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right) . \tag{5.122}$$

We calculate this Fourier transform in Appendix B.2 and the result is

$$B_2 = \frac{1}{6} G_2'''(0) . \tag{5.123}$$

Applying the derivatives to G_2 gives

$$G_2'''(\tau)\bigg|_{\tau=0} = \int_{-\infty}^\infty d\bar{t}\left[\frac{d^3}{d\tau^3} y_2(\bar{t},\tau)\bigg|_{\tau=0} g(\bar{t})^2 + \frac{3}{2}\frac{d}{d\tau} y_2(\bar{t},\tau)\bigg|_{\tau=0} (g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2)\right] . \tag{5.124}$$

Again the only dependence of y_2 on τ is in the form of t'' so we can integrate by parts

$$B_2 = -\frac{1}{3}\int_{-\infty}^\infty d\bar{t} \int_0^1 d\lambda \Bigg[2\left(\lambda - \frac{1}{2}\right)^4 Y_2(\bar{t})(3g'(\bar{t})g''(\bar{t}) + g(\bar{t})g'''(\bar{t})) + \frac{3}{2}\left(\lambda - \frac{1}{2}\right)^2 Y_2(\bar{t})(g'''(\bar{t})g(\bar{t}) - g''(\bar{t})g'(\bar{t}))\Bigg] , \tag{5.125}$$

and perform the λ integrals

$$B_2 = \frac{1}{60}\int_{-\infty}^\infty d\bar{t}\, Y_2(\bar{t})(g'(\bar{t})g''(\bar{t}) - 3g'''(\bar{t})g(\bar{t})) . \tag{5.126}$$

For the term proportional to τ_-^{-2}, we have

$$B_3 = \int_0^\infty \frac{d\xi}{\pi} \int_{-\infty}^\infty d\tau G_3(\tau) \frac{1}{\tau_-^2} e^{-i\xi\tau} . \tag{5.127}$$

where

$$G_3(\tau) = \int_{-\infty}^\infty d\bar{t}(c_1(\bar{t}) + y_3(\bar{t},\tau)) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right) . \tag{5.128}$$

The result from Sec. 3.4 is

$$B_3 = \frac{1}{2} G_3''(0) . \tag{5.129}$$

Applying the derivatives to G_3 gives

$$G_3''(\tau)\bigg|_{\tau=0} = \int_{-\infty}^\infty d\bar{t}\left[\frac{d^2}{d\tau^2} y_3(\bar{t},\tau)\bigg|_{\tau=0} g(\bar{t})^2 + \frac{1}{2}(c_1(\bar{t}) + y_3(\bar{t}))(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2)\right] . \tag{5.130}$$

As before, we integrate by parts

$$B_3 = \frac{1}{2}\int_{-\infty}^\infty d\bar{t} \int_0^1 d\lambda \Bigg[2\left(\lambda - \frac{1}{2}\right)^2 (1-\lambda)\lambda Y_3(\bar{t})(g'(\bar{t})^2 + g(\bar{t})g''(\bar{t})) + \frac{1}{2}(c_1(\bar{t}) + (1-\lambda)\lambda Y_3(\bar{t}))(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2)\Bigg] . \tag{5.131}$$

Integrating in λ gives

$$B_3 = \frac{1}{4}\int_{-\infty}^{\infty} d\bar{t}\left[c_1(\bar{t})(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) + \frac{1}{15}Y_3(\bar{t})(3g''(\bar{t})g(\bar{t}) - 2g'(\bar{t})^2)\right]. \tag{5.132}$$

The three remaining terms have Fourier transforms given in Sec. 3.4

$$B_4 = -\int_{-\infty}^{\infty} d\bar{t}\int_{-\infty}^{\infty} d\tau g'\left(\bar{t}+\frac{\tau}{2}\right) g\left(\bar{t}-\frac{\tau}{2}\right)\ln|\tau| c_2(\bar{t})\,\mathrm{sgn}\,\tau \tag{5.133a}$$

$$B_5 = \int_{-\infty}^{\infty} d\bar{t} g(\bar{t})^2(c_3(\bar{t}) + c_5(\bar{t})) \tag{5.133b}$$

$$B_6 = \frac{1}{\pi}\int_{-\infty}^{\infty} d\bar{t}\int_{-\infty}^{\infty} d\tau\frac{1}{\tau} g\left(\bar{t}+\frac{\tau}{2}\right) g\left(\bar{t}-\frac{\tau}{2}\right) c_4(\bar{t},\tau)\,, \tag{5.133c}$$

where we added

$$c_5(\bar{t}) = -(2a+b)R_{,\tilde{u}\tilde{u}}(\bar{t})\,, \tag{5.134}$$

which is the local curvature term from Eq. (5.42).

The bound is now given by Eqs. (5.112), (5.113), (5.120), (5.126), (5.132), (5.133).

5.6 The inequality

We would like to bound the correction terms B_1 through B_6 using bounds on the curvature and its derivatives. Using Eq. (5.2) in Eq. (5.68a), we find

$$|Y_1(\bar{t})| < \frac{3}{2\pi^2}|\bar{x}^{\tilde{u}}|^2 R_{\max}\,. \tag{5.135}$$

We can use Eq. (5.135) in Eq. (5.120) to get a bound on $|B_1|$. But will not be interested in specific numerical factors, only the form of the quantities that appear in our bounds. So we will write

$$|B_1| \le J_1^{(3)}[g]|\bar{x}^{\tilde{u}}|^2 R_{\max}\,, \tag{5.136}$$

where $J_1^{(3)}[g]$ is an integral of some combination of the sampling function and its derivatives appearing in Eq. (5.120). We will need many similar functionals $J_n^{(k)}[g]$, which are listed at the end of the section. The number in the parenthesis shows the dimension of the integral,

$$J_n^{(k)}[g] \sim \frac{1}{[L]^k}\,. \tag{5.137}$$

Similar analyses apply to B_2 and B_3 and the results are

$$|B_2| \le J_2^{(2)}[g]|\bar{x}^{\tilde{u}}| R_{\max} \tag{5.138a}$$

$$|B_3| \le J_3^{(1)}[g] R_{\max}\,. \tag{5.138b}$$

Among the rest of the terms in B there are some components of the form $R_{abcd,\tilde{u}}$ which diverge after boosting to the null geodesic, as shown in Ref. [23]. However we can show that these derivatives are not a problem since we can integrate them by parts. Suppose we have a term of the form

$$B_n = \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_n(\tau,\bar{t}) R_{abcd,\tilde{u}}(\bar{x})\,, \tag{5.139}$$

where $L_n(\tau,\bar{t})$ is a function that contains the sampling function g and its derivatives. The $\tilde{u}$ derivative on the Riemann tensor can be written

$$R_{abcd,\tilde{u}} = R_{abcd,t} - R_{abcd,\tilde{v}}\,. \tag{5.140}$$

The term can be reorganized the following way by grouping the terms with t and $\tilde{v}, x, y$ derivatives

$$B_n = \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_n(\tau,\bar{t})(A_n^{abcd} R_{abcd,t}(\bar{x}) + A_n^{abcd\alpha} R_{abcd,\alpha}(\bar{x}))\,, \tag{5.141}$$

where $A_n^{abcd\ldots}$ are arrays with constant components and the subscript n denotes the term they come from. Here the greek indices $\alpha, \beta, \cdots = \tilde{v}, x, y$. The term with one derivative on α can be bounded while the term with one derivative on t can be integrated by parts,

$$B_n = -\int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau (L_n'(\bar{t},\tau) A_n^{abcd} R_{abcd}(\bar{x}) + L_n(\bar{t},\tau) A_n^{abcd\alpha} R_{abcd,\alpha}(\bar{x}))\,. \tag{5.142}$$

where the primes denote derivatives with respect to $\bar{t}$. The sampling function is C_0^∞ so $L'(\tau,\bar{t})$ is still smooth and the boundary terms vanish. Now it is possible to bound this term,

$$|B_n| \leq \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau (|L_n'(\bar{t},\tau)| a_n^{(0)} R_{\max} + |L_n(\bar{t},\tau)| a_n^{(1)} R'_{\max})\,, \tag{5.143}$$

where we defined

$$a_n^{(m)} = \sum_{abcd\,\underbrace{\alpha\beta\ldots}_{m}} \left| A_n^{abcd\,\overbrace{\alpha\beta\ldots}^{m}} \right|\,. \tag{5.144}$$

The same method can be applied with more than one $\tilde{u}$ derivative.

Now we apply this method to the integrals B_4, B_5 and B_6 of Eq. (5.133). We start with B_4, which has the form

$$\begin{aligned} B_4 \;&=\; \int_{-\infty}^{\infty} d\tau \ln|\tau| \operatorname{sgn}\tau \int_{-\infty}^{\infty} d\bar{t} L_4(\bar{t},\tau)\Big(A_4^{abcd} R_{abcd,tt}(\bar{x}) + A_4^{abcd\alpha} R_{abcd,\alpha t}(\bar{x}) \\ &\qquad + A_4^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x})\Big)\,, \end{aligned} \tag{5.145}$$

where

$$L_4(\bar{t},\tau) = g(\bar{t}+\tau/2) g'(\bar{t}-\tau/2)\,. \tag{5.146}$$

After integration by parts

$$\begin{aligned} B_4 \;&=\; \int_{-\infty}^{\infty} d\tau \ln|\tau| \operatorname{sgn}\tau \int_{-\infty}^{\infty} d\bar{t} \Big(L_4''(\bar{t},\tau) A_4^{abcd} R_{abcd}(\bar{x}) - L_4'(\bar{t},\tau) A_4^{abcd\alpha} R_{abcd,\alpha}(\bar{x}) \\ &\qquad + L_4(\bar{t},\tau) A_8^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x})\Big)\,. \end{aligned} \tag{5.147}$$

Taking the bound gives

$$|B_4| \leq \sum_{m=0}^{2} J_4^{(1-m)}[g] R_{\max}^{(m)} \,. \tag{5.148}$$

Reorganizing B_5 based on the number of t derivatives gives

$$\begin{aligned} B_5 &= \int_{-\infty}^{\infty} d\bar{t} L_5(\bar{t}) \Big(A_5^{abcd} R_{abcd,tt}(\bar{x}) + A_5^{abcd\alpha} R_{abcd,\alpha t}(\bar{x}) + A_5^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \qquad (5.149) \\ &= \int_{-\infty}^{\infty} d\bar{t} \Big(L_5''(\bar{t}) A_5^{abcd} R_{abcd}(\bar{x}) - L_5'(\bar{t}) A_5^{abcd\alpha} R_{abcd,\alpha}(\bar{x}) + L_5(\bar{t}) A_5^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \,, \end{aligned}$$

where

$$L_5(\bar{t}) = g(\bar{t})^2 \,, \tag{5.150}$$

and the bound is

$$|B_5| \leq \sum_{m=0}^{2} J_5^{(1-m)}[g] R_{\max}^{(m)} \,. \tag{5.151}$$

Finally the remainder term is

$$\begin{aligned} B_6 &= \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_6(\bar{t},\tau) \int d\Omega \int_0^1 d\lambda \Big\{ A_6^{abcd}(\lambda,\Omega) R_{abcd,ttt}(\lambda\Omega) \\ &\quad + A_6^{abcd\alpha}(\lambda,\Omega) R_{abcd,\alpha tt}(\lambda\Omega) + A_6^{abcd\alpha\beta}(\lambda,\Omega) R_{abcd,\alpha\beta t}(\lambda\Omega) \\ &\quad + A_6^{abcd\alpha\beta\gamma}(\lambda,\Omega) R_{abcd,\alpha\beta\gamma}(\lambda\Omega) \Big\} \operatorname{sgn}\tau \,, \end{aligned} \tag{5.152}$$

where we changed variables to $\lambda = r/\tau$ and now arrays $A_6^{abcd\ldots}$ have components that depend on λ and Ω, and

$$L_6(\bar{t},\tau) = g(\bar{t} - \tau/2) g(\bar{t} + \tau/2) \,. \tag{5.153}$$

After integration by parts

$$\begin{aligned} B_6 &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t} \int d\Omega \int_0^1 d\lambda \Big\{ L_6'''(\tau,\bar{t}) A_6^{abcd}(\lambda,\Omega) R_{abcd}(\lambda\Omega) \\ &\quad + L_6(\tau,\bar{t})'' A_6^{abcd\alpha}(\lambda,\Omega) R_{abcd,\alpha}(\lambda\Omega) + L_6(\tau,\bar{t})' A_6^{abcd\alpha\beta}(\lambda,\Omega) R_{ab,\alpha\beta}(\lambda\Omega) \\ &\quad + L_6(\tau,\bar{t}) A_6^{abcd\alpha\beta\gamma}(\lambda,\Omega) R_{ab,\alpha\beta\gamma}(\lambda\Omega) \Big\} \operatorname{sgn}\tau \,. \end{aligned} \tag{5.154}$$

We define constants $a_6^{(m)}$

$$a_6^{(m)} = \sum_{abcd\,\underbrace{\alpha\beta\ldots}_{m}} \left| \int d\Omega \int_0^1 d\lambda A_6^{abcd\,\overbrace{\alpha\beta\ldots}^{m}}(\lambda,\Omega) \right| \,, \tag{5.155}$$

and now we can take the bound

$$|B_6| \leq \sum_{m=0}^{3} J_6^{(1-m)}[g] R_{\max}^{(m)} \,. \tag{5.156}$$

Putting everything together gives

$$B \le \delta^{-2}\left(B_0 + \sum_{n=1}^{3} J_n^{(4-n)}[g]|\bar{x}^{\tilde{u}}|^{3-n} R_{\max} + \sum_{n=4}^{6}\sum_{m=0}^{3} J_n^{(1-m)}[g] R_{\max}^{(m)}\right). \tag{5.157}$$

The functionals $J_n^{(k)}[g]$ are

$$J_1^{(3)}[g] = \int_{-\infty}^{\infty} dt(a_{11}|g''''(t)|g(t) + a_{12}|g'''(t)g'(t)| + a_{13}g''(t)^2) \tag{5.158a}$$

$$J_2^{(2)}[g] = \int_{-\infty}^{\infty} dt(a_{21}|g'''(t)|g(t) + a_{22}|g''(t)g'(t)|) \tag{5.158b}$$

$$J_3^{(1)}[g] = \int_{-\infty}^{\infty} dt(a_{31}|g''(t)|g(t) + a_{32}g'(t)^2) \tag{5.158c}$$

$$J_4^{(1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \,|\ln|t-t'||\,(a_{41}|g'''(t')|g(t) + a_{42}|g''(t)g'(t')|) \tag{5.158d}$$

$$J_4^{(0)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \,|\ln|t-t'||\,(a_{43}|g''(t')|g(t) + a_{44}|g'(t)g'(t')|) \tag{5.158e}$$

$$J_4^{(-1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \,|\ln|t-t'||\, a_{45}|g'(t')|g(t) \tag{5.158f}$$

$$J_5^{(1)}[g] = \int_{-\infty}^{\infty} dt(a_{51}|g''(t)|g(t) + a_{52}g'(t)^2) \tag{5.158g}$$

$$J_5^{(0)}[g] = \int_{-\infty}^{\infty} dt\, a_{53}|g'(t)|g(t) \tag{5.158h}$$

$$J_5^{(-1)}[g] = \int_{-\infty}^{\infty} dt\, a_{54}g(t)^2 \tag{5.158i}$$

$$J_6^{(1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'(a_{61}|g'''(t)|g(t') + a_{62}|g''(t)g'(t')|) \tag{5.158j}$$

$$J_6^{(0)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'(a_{63}|g''(t)|g(t') + a_{64}|g'(t)g'(t')|) \tag{5.158k}$$

$$J_6^{(-1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, a_{65}|g'(t)|g(t') \tag{5.158l}$$

$$J_6^{(-2)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, a_{66}g(t)g(t'), \tag{5.158m}$$

where a_{nk} are positive constants that may depend on $a_n^{(m)}$.

We can change the argument of the sampling function, writing $g(t) = f(t/t_0)$, where f is defined in Sec. 5.2 and normalized according to Eq. (5.11), so Eq. (5.157) becomes

$$\int dt T_{uu}(w(t))g(t)^2 \ge -\frac{\delta^{-2}}{t_0^3}\left\{\frac{1}{24\pi^2 t_0}\int_{-t_0}^{t_0} dt f''(t/t_0)^2 + \sum_{n=1}^{3} J_n^{(4-n)}[f]|\bar{x}^{\tilde{u}}|^{3-n} R_{\max} t_0^{n-1} + \sum_{n=4}^{6}\sum_{m=0}^{3} J_n^{(1-m)}[f] R_{\max}^{(m)} t_0^{m+2}\right\}, \tag{5.159}$$

where we used $J_n^{(k)}[g] = t_0^{-k} J_n^{(k)}[f]$. We can simplify the inequality by defining

$$F = \int f''(\alpha)^2 d\alpha = \frac{1}{t_0} \int f''(t/t_0)^2 dt\,, \tag{5.160}$$

$$F^{(m)} = \sum_{n=4}^{6} J_n^{(1-m)}[f]\,, \tag{5.161}$$

and

$$\bar{F}^{(n)} = J_n^{(4-n)}[f]\,. \tag{5.162}$$

Then Eq. (5.159) becomes

$$\int dt T_{uu}(w(t))g(t)^2 \geq \tag{5.163}$$
$$-\frac{\delta^{-2}}{t_0^3}\left\{\frac{1}{24\pi^2}F + \sum_{m=0}^{3} F^{(m)} R_{\max}^{(m)} t_0^{m+2} + \sum_{n=1}^{3} |\bar{x}^{\tilde{u}}|^{3-n} \bar{F}^{(n)} R_{\max} t_0^{n-1}\right\}.$$

We will use this result to prove the achronal ANEC.

5.7 The proof of the theorem

We use Eq. (5.163) with $w(t) = \Phi_V(\eta, t)$ and integrate in η to get

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-t_0}^{t_0} T_{uu}(\Phi_V(\eta,t)) f(t/t_0)^2 \geq \tag{5.164}$$
$$-\frac{\eta_0}{\delta^2 t_0^3}\left\{\frac{1}{24\pi^2}F + \sum_{m=0}^{3} F^{(m)} R_{\max}^{(m)} t_0^{m+2} + \sum_{n=1}^{3} |\bar{x}^{\tilde{u}}|^{3-n} \bar{F}^{(n)} R_{\max} t_0^{n-1}\right\}.$$

As $\delta \to \infty$, $t_0 \to 0$ but $F^{(m)}$, $\bar{F}^{(n)}$, $R_{\max}$, and $R_{\max}^{(m)}$ are constant. Now $\bar{x}^{\tilde{u}} = \bar{x}^u/\delta$, and using Eqs. (5.10), (5.16), $|\bar{x}^u| < u_1 + \sqrt{2}\delta t_0$. Thus as $\delta \to \infty$, $\bar{x}^{\tilde{u}} \to 0$. Therefore only the first term in braces in Eq. (5.164) survives, so the bound goes to zero as

$$\frac{\eta_0}{\delta^2 t_0^3} \sim \delta^{2\alpha - 1}\,. \tag{5.165}$$

Equation (5.164) is a lower bound. It says that its left-hand side can be no more negative than the bound, which declines as $\delta^{2\alpha-1}$. But Eq. (5.19) gives an upper bound on the same quantity, saying that it must be more negative than $-At_0/2$, which goes to zero as $t_0 \sim \delta^{-\alpha}$. Since $\alpha < 1/3$, the lower bound goes to zero more rapidly, and therefore for sufficiently large δ, the lower bound will be closer to zero than the upper bound, and the two inequalities cannot be satisfied at the same time. This contradiction proves Theorem 1.

The ambiguous local curvature terms do not contribute in the limit $\eta_0 \to \infty$ because they are total derivatives proportional to

$$\int_{-\eta_0}^{\eta_0} d\eta R_{,uu}(\bar{x})) = 0\,. \tag{5.166}$$

Chapter 6

Conclusions

In this thesis we presented the derivation of quantum inequalities and a proof of the averaged null energy condition in curved spacetimes. Using a general quantum inequality derived by Fewster and Smith [15], we first derived a bound for a quantum inequality in flat spacetime with a background potential, a case with similarities to the curved spacetime case,

$$\int_{\mathbb{R}} dt\, g(t)^2 \langle T_{tt}^{ren}\rangle_\omega(t,0) \geq -\frac{1}{16\pi^2}\Bigg\{ \quad I_1 + \frac{1}{2}V_{\max}J_2 + V''_{\max}\left[\frac{1}{2}J_3 + \left(\frac{11}{24}+48\pi^2|C|\right)J_4\right] + V'''_{\max}\left[\frac{11\pi+1}{16\pi}J_5 + \frac{2\pi+1}{64\pi}(4J_6+J_7)\right]\Bigg\}, \tag{6.1}$$

where the J_i integrals are given in Eq. (3.117). We then calculated the bound for a timelike projected quantum inequality in curved spacetime,

$$\int_{\mathbb{R}} dt\, g(t)^2 \langle T_{tt}^{ren}\rangle_\omega(t,0) \geq -\frac{1}{16\pi^2}\Bigg\{ \quad I_1 + \frac{5}{6}R_{\max}J_2 \tag{6.2}$$
$$+ \; R''_{\max}\left[\frac{23}{60}J_3 + \left(\frac{43}{40}+16\pi^2(24|a|+11|b|)\right)J_4\right]$$
$$+ \; R'''_{\max}\left[\frac{163\pi+14}{96\pi}J_5 + \frac{7(2\pi+1)}{192\pi}(4J_6+J_7)\right]\Bigg\},$$

where the J_i's in these case are given in Eq. (4.106). The importance and application of these results, for example in the special case of vacuum spacetimes, is discussed in Chapters 3 and 4. Next we presented the derivation of a null projected quantum inequality in curved spacetime

$$\int_{-\infty}^{\infty} dt\, g(t)^2 \langle T_{uu}^{\text{ren}}\rangle(w(t)) \;\; \geq \;\; \delta^{-2}\left(B_0 + \sum_{n=1}^{3} J_n^{(4-n)}[g]|\bar{x}^{\tilde{u}}|^{3-n}R_{\max} + \sum_{n=4}^{6}\sum_{m=0}^{3} J_n^{(1-m)}[g]R_{\max}^{(m)}\right) \tag{6.3}$$
$$,$$

where $J_n^{(m)}$ integrals are presented in Eq. (5.158). Finally we used this result to prove achronal ANEC in spacetimes with curvature,

$$\int_\gamma T_{ab}\ell^a\ell^b d\lambda \geq 0\,. \tag{6.4}$$

As discussed in the introduction, Ref. [21] showed that to have an exotic spacetime there would have to be violation of ANEC on achronal geodesics, generated by a state of quantum fields in that same spacetime. The result discussed above concerns integrals of the stress-energy tensor of a quantum field in a background spacetime; we have so far not been concerned about the back-reaction of the stress-energy tensor on the spacetime curvature. Thus we have shown that no spacetime that obeys NEC can be perturbed by a minimally-coupled quantum scalar field into one which violates achronal ANEC. This analysis is correct in the case where the quantum field under consideration produces only a small perturbation of the spacetime. Thus no such perturbation of a classical spacetime would allow wormholes, superluminal travel, or construction of time machines.

What possibilities remain for the generation of such exotic phenomena?

Could it be that a single effect both violates ANEC and produces the curvature that allows ANEC to be violated? The following heuristic argument casts doubt on this possibility. Suppose ANEC violation and NEC violation have the same source. We will say that they are produced by an exotic stress-energy tensor T_{exotic}. This T_{exotic} gives rise to an exotic Einstein curvature tensor,

$$G_{\text{exotic}} = 8\pi l_{\text{Planck}}^2 T_{\text{exotic}} \,. \tag{6.5}$$

It is G_{exotic} that permits T_{exotic} to arise from the quantum field. Without G_{exotic}, the spacetime would obey the null convergence condition, and so, since T_{exotic} violates ANEC, it would have to vanish. A reasonable conjecture is that as $G_{\text{exotic}} \to 0$, $T_{\text{exotic}} \to 0$ at least linearly.[1] We can then write schematically

$$|T_{\text{exotic}}| \lesssim l^{-2} |G_{\text{exotic}}| \,, \tag{6.6}$$

where l is a constant length obeying $l \gg l_{\text{Planck}}$. The parameter l, needed on dimensional grounds, might be the wavelength of some excited modes of the quantum field. Equation (6.6) is schematic because we have not said anything about the places at which these tensors should be compared, or in what coordinate system they should be measured. Combining Eqs. (6.5) and (6.6), we find

$$|T_{\text{exotic}}| \lesssim (l_{\text{Planck}}/l)^2 |T_{\text{exotic}}| \,, \tag{6.7}$$

which is impossible since $l \gg l_{\text{Planck}}$.

Given the assumptions of this work, it appears that the only remaining possibility for self-consistent achronal ANEC violation using minimally coupled free fields is to have first a quantum field that violates NEC but obeys ANEC, and then a second quantum field (or a second, weaker effect produced by the same field) that violates ANEC when propagating in the spacetime generated by the first field. The stress-energy tensor of the second field would be a small correction to that of the first, but this correction might lead to ANEC violation on geodesics that were achronal (and thus obeyed ANEC only marginally), taking into account only the first field.

There is also the possibility of different fields. We have not studied higher-spin fields, but these typically obey the same energy conditions as minimally-coupled scalars.

If one considers quantum scalar fields with non-minimal curvature coupling, the situation is rather different. Even classical non-minimally coupled scalar fields can violate ANEC [1, 2], with large enough (Planck-scale) field values. However, as the field values increase toward such levels, the effective Newton's constant first diverges and then becomes negative. Recently an even stronger result has been proven; the effective Newton's constant has to change sign between the two asymptotic regions on different ends of the wormhole [5]. Such situations may not be physically realizable. If one excludes such field values, some restrictions are known,

[1]Not, for example, changing discontinuously for infinitesimal but nonzero G_{exotic} or going as $G_{\text{exotic}}^{1/2}$.

but there are no quantum inequalities of the usual sort [12, 13], and there are general [34] and specific [31, 32] cases where conformally coupled quantum scalar fields violate ANEC in curved space. It may be possible to control such situations by considering only cases where a spacetime is produced self-consistently by fields propagating in that spacetime, but the status of this "self-consistent achronal ANEC" for non-minimally coupled scalar fields outside the large-field region is not known.

Appendix A

Multi-step Fermi coordinates

In this Appendix, we generalize the Fermi coordinate construction to allow an arbitrary number of arbitrary subspaces (and an arbitrary number of dimensions d), rather than just a timelike geodesic and the perpendicular space. First, we construct the generalized coordinate system, then we compute the connection, and finally we compute the metric in the generalized Fermi coordinates. Results shown in this Appendix are used throughout the main body of the thesis. Here Greek indices $\alpha, \beta \ldots$ refer to tetrad components while latin $a, b, \ldots$ to coordinate basis. This notation is not used in the rest of the thesis.

A.1 Multi-step Fermi coordinates

Consider a d-dimensional Riemannian or Lorentzian manifold $(\mathcal{M}, g)$. We will start our construction by choosing a base point $p \in \mathcal{M}$. We decompose the tangent space T_p into n subspaces, $T_p = A_p^{(1)} \times A_p^{(2)} \times A_p^{(3)} \ldots \times A_p^{(n)}$ so that any $V \in T_p$ can be uniquely written as $V = V_{(1)} + V_{(2)} + V_{(3)} + \ldots + V_{(n)}$. We choose, as a basis for T_p, d linearly independent vectors $\{E_{(\alpha)}\}$ adapted to the decomposition of T_p so that for each $m = 1 \ldots n$, $\{E_{(\alpha)} | \alpha \in c_m\}$ is a basis for $A_p^{(m)}$, where $c_1, c_2, \ldots c_n$ is an ordered partition of $\{1 \ldots d\}$. Thus each $V_{(m)} = \sum_{\alpha \in c_m} x^\alpha E_{(\alpha)}$. The vectors $\{E_{(\alpha)}\}$ need not be normalized or orthogonal.

The point corresponding to coordinates x^a is then found by starting from p and going along the geodesic whose whose tangent vector is $V_{(1)}$, parallel transporting the rest of the vectors, then along the geodesic whose tangent vector is $V_{(2)}$, and so on. An example is shown in Fig. A.1. With that general construction of multi-step Fermi coordinates we can define a

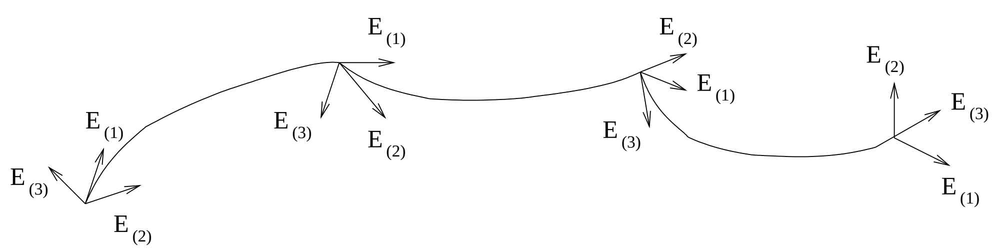

Figure A.1: Construction of 3-step Fermi coordinates in 3 dimensions. We travel first in the direction of $E_{(1)}$, then $E_{(2)}$, then $E_{(3)}$, parallel transporting the triad as we go.

general Fermi mapping $q = \text{Fermi}_p(V)$ given by

$$\begin{aligned} q_{(0)} &= p \\ q_{(1)} &= \exp_p(V_{(1)}) \\ q_{(2)} &= \exp_{q_{(1)}}(V_{(2)}) \\ &\dots \\ q = q_{(n)} &= \exp_{q_{(n-1)}}(V_{(n)}) \end{aligned} \tag{A.1}$$

From that general construction we can return to the original Fermi case by choosing $c_1 = \{t\}$ and $c_2 = \{x, y, z\}$. In the Lorentzian case, we could also choose a pseudo-orthonormal tetrad E_u, E_v, E_x, E_y, with E_u and E_v null, $E_u \cdot E_v = -1$, and other inner products vanishing, and $c_0 = \{u\}$ and $c_1 = \{v, x, y\}$.

For later use we will define

$$V_{(\leq m)} = \sum_{\alpha \in c_1 \cup c_2 \cup \cdots \cup c_m} x^\alpha E_{(\alpha)} = \sum_{l=1}^{m} V_{(l)} \tag{A.2}$$

$$V_{(<m)} = V_{(\leq(m-1))} = V_{(\leq m))} - V_{(m)} \tag{A.3}$$

Then we can write $q_{(m)} = \text{Fermi}_p(V_{(\leq m)})$.

A.2 Connection

We will parallel transport our orthonormal basis vectors $E_{(\alpha)}$ along the geodesics that generate the coordinates, and use them as a basis for vectors and tensors throughout the region of $\mathcal{M}$ covered by our coordinates. Components in this basis will be denoted by Greek indices. We will use Latin letters from the beginning of the alphabet to denote indices in the Fermi coordinate basis. Of course at p, there is no difference between these bases.

Latin letters from the middle of the alphabet will denote the subspaces of T_p or equivalently the steps of the Fermi mapping process.

We would like to calculate the covariant derivatives of the basis vectors, $\nabla_\beta E_{(\alpha)}$, which are connected with connection one-forms, see for example Ref. [39] by

$$\omega_{\beta\alpha\delta} = \eta_{\gamma\delta} \nabla_\beta E^\gamma_{(\alpha)} \,, \tag{A.4}$$

because we are using a orthonormal tetrad basis.

We can then calculate the covariant derivative of any vector field $V = V^\beta E_{(\beta)}$ along a curve $f(\lambda)$ as

$$\frac{DV^\beta}{d\lambda} = \frac{dV^\beta}{d\lambda} + V^\gamma \left(\frac{\partial}{\partial\lambda}\right)^\alpha \nabla_\alpha E^\beta_{(\gamma)} \tag{A.5}$$

To evaluate $\nabla_\beta E_{(\alpha)}$ at some point $q_1 = \exp_p(X)$, consider an infinitesimally separated point $q_2 = \exp_p(X + E_{(\eta)} dx)$. The covariant derivative of $E_{(\alpha)}$ at q_1 is the difference between $E_{(\alpha)}(q_2)$ parallel transported to q_1 and the actual $E_{(\alpha)}(q_1)$, divided by dx. That difference is the same as the change in $E_{(\alpha)}$ by parallel transport around a loop following the geodesics from q_1 backward to p, the infinitesimally different geodesics forward from p to q_2, and the infinitesimal distance back to q_1. We can write this loop parallel transport as an integral over the Riemann tensor.

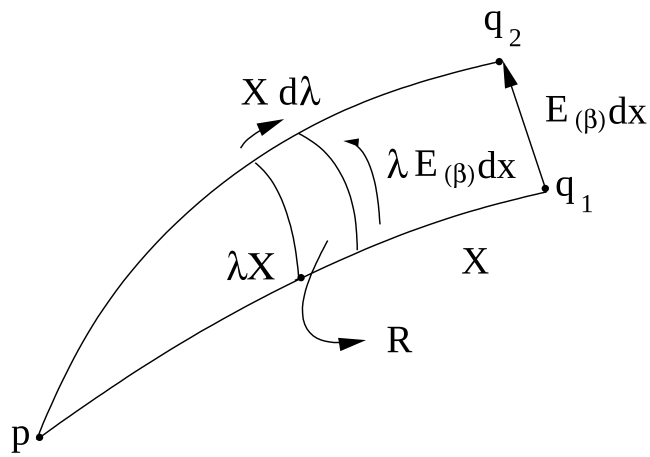

Figure A.2: The covariant derivative $\nabla_\beta E_{(\alpha)}$ is the change in $E_{(\alpha)}$ under parallel transport along the path $q_1 \to p \to q_2 \to q_1$, divided by dx. The parallel transport can be decomposed into a series of transports clockwise around trapezoidal regions with sides $\lambda E_{(\beta)} dx$ and $X d\lambda$.

Let us first consider the Riemannian case, as shown in Fig. A.2. The total parallel transport can be written as the sum of parallel transport around a succession of small trapezoidal regions whose sides are $\lambda E_{(\beta)}$ and $X d\lambda$. By using the definition of the Riemann tensor we have

$$\nabla_\beta E^\gamma_{(\alpha)} = \int_0^1 d\lambda\, R^\gamma{}_{\alpha\delta\beta}(\lambda X)\lambda X^\delta\,. \tag{A.6}$$

Here R is evaluated at the point $\exp_p(\lambda X)$, which we have denoted merely λX for compactness.

Equation (A.6) reproduces Eq. (13) of Ref. [29]. Note, however, that Eq. (A.6) is exact and does not require R to be smooth, whereas that of Ref. [29] was given as first order in R and was derived by means of a Taylor series.

We see immediately that the covariant derivative of any $E_{(\alpha)}$ at X in the direction of X vanishes. This happens simply because changes with dX in the direction of X correspond to additional parallel transport of $E_{(\alpha)}$.

Let us now consider the general case where there are n steps, and compute $\nabla_\beta E_{(\alpha)}$. Since the coordinates are adapted to our construction, the index β must be in some specific set c_m, which is to say that the direction of the covariant derivative, $E_{(\beta)}$, is part of step m in the Fermi coordinate process. We will write the function that gives that m as $m(\beta)$. Some particular cases are shown in Fig. A.3.

If $m = n$ (leftmost in Fig. A.3), only the last step is modified. The integration is exactly as shown in Fig. A.2, except that it covers only the final geodesic from $X_{(<n)}$ to $X_{(n)}$,

$$\nabla_\beta E^\gamma_{(\alpha)} = \int_0^1 d\lambda\, R^\gamma{}_{\alpha\delta\beta}(X_{(<n)} + \lambda X_{(n)})\lambda X^\delta_{(n)}. \tag{A.7}$$

If $m < n$, then we are modifying some intermediate step, and the path followed at later steps is displaced parallel to itself. In that case we get an integral over rectangular rather than trapezoidal regions, as shown in Fig. A.4. For general m there is a contribution for each step $j \geq m$. The $j = m$ contribution integrates over trapezoids that grow with λ, while the $j > m$

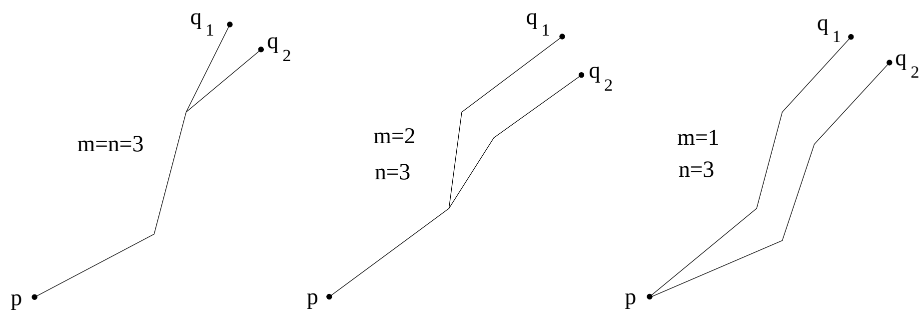

Figure A.3: Original and displaced geodesics for Fermi coordinates with $n = 3$ and $m = 3$, 2, and 1.

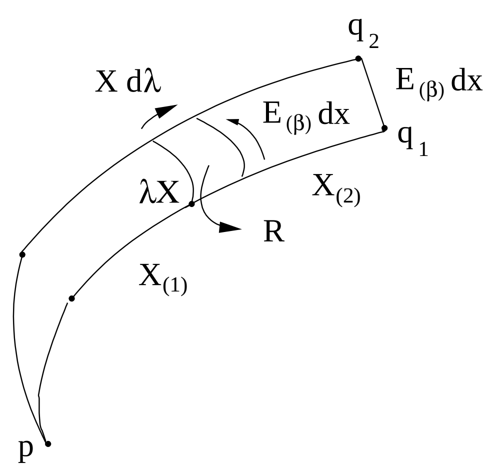

Figure A.4: Part of the calculation of $\nabla_\beta E_{(\alpha)}$ in the case $n = 2$, $m(\beta) = 1$. The geodesic of the first step has been modified, causing the geodesic in the second step to be displaced. The parallel transport integrates the Riemann tensor over a series of rectangular regions between the 2 second-step geodesics.

contributions integrate over rectangles with fixed width dx. We can write the complete result

$$\nabla_\eta E^\gamma_{(\alpha)} = \sum_{j=m}^{n} \int_0^1 d\lambda\, a_{jm}(\lambda) R^\gamma{}_{\alpha\delta\beta}(X_{(<j)} + \lambda X_{(j)}) X^\delta_{(j)} \tag{A.8}$$

where $m = m(\beta)$ and

$$a_{jm}(\lambda) = \begin{cases} 1 & j \neq m \\ \lambda & j = m\,. \end{cases} \tag{A.9}$$

Equation (A.8) is exact and includes Eqs. (A.6,A.7) as special cases.

Consider the case where c_1 consists only of one index. If $m > 1$, there is no $j = 1$ term in Eq. (A.8). If $m = 1$, then β is the single index in c_1, and $X^\delta_{(1)} = 0$ unless $\delta = \beta$, so the $j = 1$ term in vanishes because $R_{\alpha\gamma\delta\beta}$ is antisymmetric under $\delta \leftrightarrow \beta$. Thus there is never a $j = 1$ contribution to Eq. (A.8) when there is only one index in c_1.

Now suppose X lies on the first generating geodesic, so $X_{(j)} = 0$ for $j > 1$. Then all $j > 1$ terms vanish in Eq. (A.8). So if c_1 consists only of one index, all Christoffel symbols vanish at X. This is well known in the case of the usual Fermi coordinates.

A.3 Metric

Now we would like to compute the metric g at some point X. Specifically, we would like to compute the metric component g_{ab} in our generalized Fermi coordinates.

We will start by considering the vectors $Z_{(a)} = \partial/\partial x^a$. These are the basis vectors of the Fermi coordinate basis for the tangent space, so the metric is given by $g_{ab} = Z_{(a)} \cdot Z_{(b)}$. Thus if we compute the orthonormal basis components $Z^\alpha_{(a)}$ we can write $g_{ab} = \eta_{\alpha\beta} Z^\alpha_{(a)} Z^\beta_{(b)}$.

Again we will start with the case of Riemann normal coordinates. Let $W(t,s)$ be the point $\exp_p s(X + tE_{(a)})$. Define $Y = \partial W/\partial t$ and $V = \partial W/\partial s$. Then $Y(X) = Z_{(a)}$ and $V^\beta = X^\beta + t\delta^\beta_a$. The components of $Z_{(a)}$ at X can be calculated by integration,

$$Z^\beta_{(a)}(X) = Y^\beta(X) = \int_0^1 ds \frac{\partial Y^\beta(sX)}{\partial s}. \tag{A.10}$$

Because the orthonormal basis is parallel transported we can write

$$\frac{d}{ds} Y^\beta = \frac{DY^\beta}{ds}. \tag{A.11}$$

By construction, the Lie derivative $L_V Y = 0$ and thus [22, Ch. 4]

$$\frac{DY}{ds} = \frac{DV}{dt} \tag{A.12}$$

From Eq. (A.5) we have

$$\frac{DV^\beta}{dt} = \frac{dV^\beta}{dt} + V^\gamma Y^\alpha \nabla_\alpha E^\beta_{(\gamma)} = \delta^\beta_a + s\delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2). \tag{A.13}$$

where we have retained δ^α_a instead of writing $\nabla_a E^\beta_{(\gamma)}$ to make it clear that the covariant derivative is with respect to the orthonormal basis.

From Eq. (A.6) we have

$$\nabla_\alpha E^\beta_{(\gamma)}(sX) = \int_0^1 d\lambda\,\lambda R^\beta{}_{\gamma\delta\alpha}(\lambda sX)sX^\delta = \frac{1}{s}\int_0^s d\lambda\,\lambda R^\beta{}_{\gamma\delta\alpha}(\lambda X)X^\delta \tag{A.14}$$

Taking $t=0$, V is just X. Combining Eqs. (A.10-A.14), we find

$$\begin{aligned} Z^\beta_{(a)}(X) &= \int_0^1 ds\left[\delta^\beta_a + \delta^\alpha_a \int_0^s d\lambda\,\lambda R^\beta{}_{\gamma\delta\alpha}(\lambda X)X^\delta X^\gamma\right] + O(R^2) \\ &= \delta^\beta_a + \delta^\alpha_a \int_0^1 d\lambda\,\lambda(1-\lambda)R^\beta{}_{\gamma\delta\alpha}(\lambda X)X^\delta X^\gamma + O(R^2). \end{aligned} \tag{A.15}$$

From Eq. (A.15), the metric is given by

$$g_{ab} = \eta_{ab} + 2\delta^\alpha_a \delta^E_b ta \int_0^1 d\lambda\,\lambda(1-\lambda)R_{\alpha\gamma\delta\beta}(\lambda X)X^\delta X^\gamma + O(R^2). \tag{A.16}$$

Equation (A.16) reproduces Eq. (14) of Ref. [29][1].

Next let us consider the case where there are n steps in our procedure. We will define a set of functions W_j as

$$W_j(s) = \mathrm{Fermi}_p(X_{(<j)} + sX_{(j)}). \tag{A.17}$$

The path $W_j(s), j = 1\ldots n, s = 0\ldots 1$ traces the geodesics generating the Fermi coordinates for the point X. Now consider $Z_{(a)} = \partial/\partial x^a$. Let $m = m(a)$, so $Z_{(a)}(p) \in A^{(m)}_p$. Then let

$$W_j(s,t) = \mathrm{Fermi}_p \begin{cases} sX_{(j)} & j < m \\ X_{(<j)} + s(X_{(j)} + tE_{(a)}) & j = m \\ X_{(<j)} + tE_{(a)} + sX_{(j)} & j > m \end{cases} \tag{A.18}$$

Let $Y = \partial W/\partial t$ and $V = \partial W/\partial s$ as before. To find $Z_{(a)}$ we now must integrate over a multi-step path from p,

$$Z^\beta_{(a)}(X) = \sum_{j=1}^n \int_0^1 ds \frac{\partial Y^\beta(W_j(s))}{\partial s}. \tag{A.19}$$

The generalized version of Eq. (A.13) is

$$\frac{DV^\beta(W_j(s,t))}{dt} = \frac{dV^\beta}{dt} + V^\gamma Y^\alpha \nabla_\alpha E^\beta_{(\gamma)} = \begin{cases} 0 & j < m \\ \delta^\beta_a + s\delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2) & j = m \\ \delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2) & j > m. \end{cases} \tag{A.20}$$

Now

$$\nabla_\alpha E^\beta_{(\gamma)}(W_j(s)) = \sum_{k=m}^j \frac{1}{s_{kj}(s)} \int_0^{s_{kj}(s)} d\lambda\, a_{km}(\lambda) R^\beta{}_{\gamma\delta\alpha}(X_{(<k)} + \lambda X_{(k)})X^\delta_{(k)} \tag{A.21}$$

[1]Ref. [29] uses the same sign convention for $R^\alpha{}_{\beta\gamma\delta}$ as the present thesis, but the opposite convention for g_{ab} and consequently also for $R_{\alpha\beta\gamma\delta}$.

where

$$s_{kj}(s) = \begin{cases} 1 & k \neq j \\ s & k = j. \end{cases} \tag{A.22}$$

The $k = j$ term is analogous to Eq. (A.14), while the others have no dependence on s.

Combining Eqs. (A.11,A.12,A.19–A.21) we get

$$Z^{\beta}_{(a)}(X) = \delta^{\beta}_{a} + F^{\beta}_{a} + O(R^2) \tag{A.23}$$

where

$$\begin{aligned} F^{\beta}_{\alpha} &= \sum_{j=m}^{n} \sum_{k=m}^{j} \int_0^1 ds \int_0^{s_{kj}(s)} d\lambda\, a_{km}(\lambda) R^{\beta}{}_{\gamma\delta\alpha}(X_{(<k)} + \lambda X_{(k)}) X^{\delta}_{(k)} X^{\gamma}_{(j)} \\ &= \sum_{j=m}^{n} \sum_{k=m}^{j} \int_0^1 d\lambda\, a_{km}(\lambda) b_{kj}(\lambda) R^{\beta}{}_{\gamma\delta\alpha}(X_{(<k)} + \lambda X_{(k)}) X^{\delta}_{(k)} X^{\gamma}_{(j)} \end{aligned} \tag{A.24}$$

where $m = m(\alpha)$ and

$$b_{kj}(\lambda) = \begin{cases} 1 & k \neq j \\ 1 - \lambda & k = j \end{cases}. \tag{A.25}$$

Thus the metric is

$$g_{ab} = \eta_{\alpha\beta} Z^{\alpha}_{(a)} Z^{\beta}_{(b)} = \eta_{ab} + F_{ab} + F_{ba} + O(R^2) \tag{A.26}$$

where

$$F_{\alpha\beta} = \sum_{j=m}^{n} \sum_{k=m}^{j} \int_0^1 d\lambda\, a_{km}(\lambda) b_{kj}(\lambda) R_{\alpha\gamma\delta\beta}(X_{(<k)} + \lambda X_{(k)}) X^{\delta}_{(k)} X^{\gamma}_{(j)} \tag{A.27}$$

where $m = m(\beta)$.

Florides and Synge [16] construct coordinates by taking geodesics perpendicular to an embedded submanifold. Our construction for $n = 2$ is of this kind in the case where all basis vectors with indices in c_1 lie tangent to the surface generated by all first-step geodesics. This will be so if $R_{\alpha\gamma\delta\beta} = 0$ everywhere on this surface whenever $\gamma, \delta, \beta \in c_1$ and $\alpha \in c_2$. In that case, Eq. (A.27) agrees with Theorem I of Ref. [16].

Now, consider again the case where c_1 contains only one index. As discussed with respect to Eq. (A.8), if $\beta \in c_1$, there is no nonvanishing $k = 1$ term in Eq. (A.27). Thus $g_{ab} = \eta_{ab}$ everywhere on the first generating geodesic. This is also well known in the usual Fermi case.

Now suppose c_1 consists only of one index and furthermore $n = 2$. The only possible term in Eq. (A.27) is then $j = k = 2$, so

$$F_{\alpha\beta} = \int_0^1 d\lambda\, a_{2m}(\lambda)(1 - \lambda) R_{\alpha\gamma\delta\beta}(X_{(1)} + \lambda X_{(2)}) X^{\delta}_{(2)} X^{\gamma}_{(2)}. \tag{A.28}$$

where $m = m(\beta)$. Equation Eq. (A.28) is equivalent to Eq. (28) in Ref. [29] in the case where the generating curve of the Fermi coordinates is a geodesic.

A.4 Regularity of the coordinates

Riemann normal coordinates cannot in general be defined over the entirety of a manifold, because there might be points conjugate to the base point p. At such a point, some infinitesimal change to the coordinates would yield no change to the resulting point, and the metric would be singular.

Similar considerations apply to Fermi normal coordinates [27]. No trouble can occur in the first step, because that consists merely of traveling down a geodesic. Along the geodesic, and thus by continuity in some neighborhood surrounding the geodesic, Fermi normal coordinates are regular. If we attempt to extend beyond this neighborhood, we may find points that are conjugate to the generating geodesic. In such places the metric will become singular.

The situation here is more complicated. The metric will be singular whenever an infinitesimal change in coordinates fails to yield a change in the location of the resulting point. But when there are more than two steps, the result can no longer be described in terms of conjugate points. Nevertheless, it is easy to see that one if one chooses a sufficiently small neighborhood around p, all multi-step Fermi coordinates will be well defined, since any such coordinates approach Riemann normal coordinates when all coordinate values are sufficiently small.

In case c_1 contains only one index, multi-step Fermi coordinates will be well defined in a neighborhood of the initial geodesic, because when all coordinate values except for the first are small, the multi-step Fermi coordinates approach the regular Fermi coordinates. There is no particular advantage to having a single index in any later c_m.

One can get a simple condition sufficient for the existence of multi-step Fermi coordinates in a small region by looking at Eqs. (A.26) and (A.27). As long as $|F_{ab}| \ll 1$, the metric g_{ab} cannot degenerate. Thus the coordinates will be well defined if [29]

$$|R_{\alpha\gamma\beta\delta}|(X^{\epsilon})^2 \ll 1 \tag{A.29}$$

throughout the region of interest, for all $\alpha, \gamma, \beta, \delta, \epsilon$. In the case where there is only one index in c_1, there is no contribution to F_{ab} from $X_{(1)}$. Then it is sufficient for Eq. (A.29) to hold for $\epsilon > 1$. Thus if the first step is one-dimensional, it can be arbitrarily long [27], as discussed above.

Appendix B

Fourier transforms of some distributions

B.1 Fourier transforms of some distributions involving logarithms

In this appendix will compute the Fourier transforms of the distributions given by

$$u(\tau) = \ln|\tau| \tag{B.1}$$
$$v(\tau) = \ln(-\tau_-^2)\,. \tag{B.2}$$

We write u as a distributional limit,

$$u = \lim_{\epsilon\to 0^+} u_\epsilon\,, \tag{B.3}$$

where

$$u_\epsilon(\tau) = \ln|\tau| e^{-\epsilon|\tau|}\,, \tag{B.4}$$

so its Fourier transform is

$$\hat{u}_\epsilon(k) = \int_{-\infty}^{\infty} d\tau \ln|\tau| e^{-\epsilon|\tau|} e^{ik\tau} = 2\,\mathrm{Re}\int_0^\infty d\tau \ln\tau\, e^{(ik-\epsilon)\tau} = -2\,\mathrm{Re}\,\frac{\gamma + \ln(\epsilon - ik)}{\epsilon - ik}\,. \tag{B.5}$$

Thus the action of $\hat{u}$ on a test function f is

$$\hat{u}[f] = -2 \lim_{\epsilon\to 0^+} \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{\gamma + \ln(\epsilon - ik)}{\epsilon - ik} f(k)\,. \tag{B.6}$$

The term involving γ is

$$-2\gamma \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dk \frac{\epsilon}{k^2+\epsilon^2} f(k) = -2\pi\gamma f(0)\,. \tag{B.7}$$

In the other term we integrate by parts,

$$\begin{aligned} -2 \lim_{\epsilon\to 0^+} \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{\ln(\epsilon - ik)}{\epsilon - ik} f(k) &= -\lim_{\epsilon\to 0^+} \mathrm{Im}\int_{-\infty}^{\infty} dk\, f'(k)[\ln(\epsilon - ik)]^2 \\ &= -\mathrm{Im}\int_{-\infty}^{\infty} dk\, f'(k)[\ln|k| - i(\pi/2)\,\mathrm{sgn}\,k]^2 \\ &= \pi\int_{-\infty}^{\infty} dk\, f'(k)\ln|k|\,\mathrm{sgn}\,k\,, \end{aligned} \tag{B.8}$$

and thus

$$\hat{u}[f] = \pi \int_{-\infty}^{\infty} dk\, f'(k) \ln|k| \operatorname{sgn} k - 2\pi\gamma f(0)\,. \tag{B.9}$$

Since the Fourier transform of the constant γ is just $2\pi\gamma\delta(k)$, the transform of

$$w(\tau) = \ln|\tau| + \gamma \tag{B.10}$$

is just

$$\hat{w}[f] = \pi \int_{-\infty}^{\infty} dk\, f'(k) \ln|k| \operatorname{sgn} k\,. \tag{B.11}$$

Now

$$v(\tau) = \lim_{\epsilon\to 0} \ln(-(\tau - i\epsilon)^2) = 2\ln|\tau| + \pi i \operatorname{sgn}\tau\,. \tag{B.12}$$

The Fourier transform of sgn acts on f as [20]

$$2iP \int_{-\infty}^{\infty} dk\, \frac{f(k)}{k} = -2i \int_{-\infty}^{\infty} dk\, f'(k) \ln|k|\,, \tag{B.13}$$

Putting Eqs. (B.9,B.13) in Eq. (B.12) gives

$$\hat{v}[f] = 4\pi \int_{0}^{\infty} dk\, f'(k) \ln|k| - 4\pi\gamma f(0)\,. \tag{B.14}$$

B.2 Fourier transform of distribution τ_-^{-3}

We follow the procedure of Sec. 3.4 to calculate

$$B_2 = \int_0^\infty \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau G_2(\tau) s_2(\tau) e^{-i\xi\tau}\,. \tag{B.15}$$

where

$$G_2(\tau) = \int_{-\infty}^{\infty} d\bar{t} y_2(\bar{t},\tau) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right)\,. \tag{B.16}$$

and

$$s_2(\tau) = \frac{1}{\tau_-^3}\,. \tag{B.17}$$

This is the Fourier transform of a product so we can write it as a convolution. The function s_2 is real and odd, so its Fourier transform is imaginary, but G_2 is also real and odd, thus the Fourier transform of their product is real. We have

$$B_2 = \frac{1}{2\pi^2} \int_0^\infty d\xi \int_{-\infty}^{\infty} d\zeta \hat{G}_2(-\xi - \zeta) \hat{s}_2(\zeta)\,. \tag{B.18}$$

We can change the order of integrals and change variables to $\eta = -\xi - \zeta$ which gives

$$\begin{aligned} B_2 &= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\zeta \int_{\zeta}^{\infty} d\eta\, \hat{G}_2(\eta) \hat{s}_2(\zeta) \\ &= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\eta\, \hat{G}_2(\eta) \int_{-\infty}^{\eta} d\zeta \hat{s}_2(\zeta)\,. \end{aligned} \tag{B.19}$$

The Fourier transform of s_2 is [20]

$$\hat{s}_2(\zeta) = -i\pi\zeta^2\Theta(\zeta)\,, \tag{B.20}$$

and

$$\int_0^\eta d\zeta(-i\pi\zeta^2) = -\frac{i\pi}{3}\eta^3\Theta(\eta)\,. \tag{B.21}$$

From Eq. (B.19) we have

$$B_2 = -\frac{i}{6\pi}\int_0^\infty d\eta\,\hat{G}_2(\eta)\eta^3\,. \tag{B.22}$$

Using $\widehat{f'}(\xi) = -i\xi\hat{f}(\xi)$, we get

$$B_2 = \frac{1}{6\pi}\int_0^\infty d\eta\,\widehat{G_2'''}(\eta)\,. \tag{B.23}$$

The function G_2 is odd but with three derivatives it becomes even, so we can extend the intergal

$$B_2 = \frac{1}{12\pi}\int_{-\infty}^\infty d\eta\,\widehat{G_2'''}(\eta) = \frac{1}{6}G_2'''(0)\,. \tag{B.24}$$

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