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Proof of the averaged null energy condition in a classical curved spacetime using a null-projected quantum inequality

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Abstract

Quantum inequalities are constraints on how negative the weighted average of the renormalized stress-energy tensor of a quantum field can be. A null-projected quantum inequality can be used to prove the averaged null energy condition (ANEC), which would then rule out exotic phenomena such as wormholes and time machines. In this work we derive such an inequality for a massless minimally coupled scalar field, working to first order of the Riemann tensor and its derivatives. We then use this inequality to prove ANEC on achronal geodesics in a curved background that obeys the null convergence condition.

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I. INTRODUCTION

In general relativity it is possible to have exotic spacetimes that allow superluminal travel, closed timelike curves, or wormholes, so long as the appropriate stress-energy tensor $T_{\mu\nu}$ is available. Although general relativity does not provide any restrictions on $T_{\mu\nu}$, quantum field theory does. These constraints are called energy conditions or quantum (energy) inequalities. The simplest energy conditions are bounds on projections of the stress energy tensor at each point in spacetime, but those are easily violated by quantum fields, even by free fields in flat spacetime. But by averaging, we can produce conditions that are not so easily violated. A quantum inequality bounds an average of $T_{\mu\nu}$ over a localized part of a timelike path, and an averaged energy condition bounds the energy along an entire geodesic.

A good technique to rule out exotic phenomena [1] is to use the achronal averaged null energy condition (achronal ANEC), which requires that the null-projected stress-energy tensor cannot be negative when averaged along any complete achronal null geodesic,

$$\int_\gamma T_{ab}\ell^a\ell^b \geq 0\,, \tag{1}$$

where γ is an achronal null geodesic (also called a null line), i.e., no two points of γ can be connected by a timelike path, and ℓ^a is the tangent vector to γ.

Ref. [2] proved ANEC for null geodesics traveling in flat spacetime (though there could be curvature elsewhere) using a quantum inequality. In previous work [3] we studied ANEC in the case of a classical curved background, meaning a spacetime generated by matter that obeys the null energy condition,

$$T_{ab}\ell^a\ell^b \geq 0\,, \tag{2}$$

at all points and for all null vectors ℓ. We conjectured a particular form for a curved-space quantum inequality, and from that we were able to show that that a quantum scalar field in a classical curved background would obey achronal ANEC. Here we complete the proof by demonstrating a curved-space quantum inequality (somewhat different from the one we conjectured before) and using it to prove the same conclusion.

The rest of the paper is structured as follows. In Sec. II we state our assumptions and present the ANEC theorem we will prove. We begin the proof by constructing a parallelogram which can be understood as a congruence of null geodesic segments or of timelike paths, as in Ref. [3]. In Sec. III we present and discuss the general quantum inequality of Fewster and Smith [4]. Secs. IV–VI apply that general inequality to the specific case needed here, using results from our previous application of Fewster and Smith's inequality in Ref. [5]. In Sec. VII we present the proof of the ANEC theorem of Sec. II using the quantum inequality. Finally, Sec. VIII is a summary of our results and discussion of some open problems.

We use the sign convention $(-,-,-)$ in the classification of Misner, Thorne and Wheeler [6]. Latin indices (in small or capital letters) from the beginning of the alphabet will denote all coordinates; those from the middle of the alphabet will denote only spatial coordinates.

II. THE THEOREM

A. Assumptions

We consider a spacetime M containing a null geodesic γ with tangent vector ℓ, and define a "tubular neighborhood" M' around γ, which is composed of a congruence of null geodesics as in Ref. [3].

Then we define Fermi-like coordinates [7] on M' as follows [3]. First pick some point p on the geodesic γ. Let $E_{(u)} = \ell$, and pick a null vector $E_{(v)}$ at p such that $E_{(v)}^a \ell_a = 1$, and two unit spacelike vectors $E_{(x)}$ and $E_{(y)}$ at p, perpendicular to $E_{(u)}$ and $E_{(v)}$ and to each other, giving a pseudo-orthonormal tetrad. Then the point $q = (u, v, x, y)$ in these coordinates is found by traveling unit distance along the geodesic generated by $vE_{(v)} + xE_{(x)} + yE_{(y)}$, parallel transporting $E_{(u)}$, and then unit distance along the geodesic generated by $uE_{(u)}$.

We suppose that the curvature inside the tubular neighborhood M' obeys the null convergence condition, $R_{ab}V^aV^b \geq 0$ for any null vector V. This will be true if the matter generating this curvature obeys the null energy condition, Eq. (2).

We require that in M' the curvature is smooth and obeys the bounds,

$$|R_{abcd}| < R_{\max}\,, \tag{3}$$

and

$$|R_{abcd,\alpha}| < R'_{\max}, \qquad |R_{abcd,\alpha\beta}| < R''_{\max}, \qquad |R_{abcd,\alpha\beta\gamma}| < R'''_{\max}\,, \tag{4}$$

in the coordinate system described above, where the greek indices $\alpha, \beta, \gamma, \ldots$ take values v, x, y but not u, and $R_{\max}, R'_{\max}, R''_{\max}, R'''_{\max}$ are finite numbers but not necessarily small. These bounds need not apply outside M'.

Finally, we consider a quantum scalar field in M. Inside M' it is massless, free, and minimally coupled but outside M' we allow interactions and different curvature couplings. For further details see Sec. II E of Ref. [3].

B. The theorem

Theorem 1. Let (M, g) be a spacetime and γ an achronal null geodesic and suppose that around γ there is a tubular neighborhood M'. We suppose that the curvature is bounded in the sense of Sec. II A and the causal structure of M' is not affected by conditions outside M' [3]. Let T_{ab} be the renormalized expectation value of the stress-energy tensor of a minimally coupled quantum field in some Hadamard state ω.

Then the ANEC integral,

$$A = \int_{-\infty}^{\infty} d\lambda T_{ab}\ell^a\ell^b(\Gamma(\lambda)) \tag{5}$$

cannot converge uniformly to negative values on all geodesics $\Gamma(\lambda)$ in M'.

C. The parallelogram

We will use the (u, v, x, y) coordinates of the Fermi-like coordinate system defined in Sec. II A. Let r be a positive number small enough such that whenever $|v|, |x|, |y| < r$, the point $(0, v, x, y)$ is inside the tubular neighborhood M' defined in Sec. II A. Then the point $(u, v, x, y) \in M'$ for any u. Define the points

$$\Phi(u, v) = (u, v, 0, 0)\,, \tag{6}$$

with v fixed and u varying. Write the ANEC integral

$$A(v) = \int_{-\infty}^{\infty} du\, T_{uu}(\Phi(u, v))\,. \tag{7}$$

As in Ref. [3] we suppose that, contrary to Theorem 1, Eq. (7) converges uniformly to negative values, and show that this leads to a contadiction.

Given any positive number $v_0 < r$ we can find a negative number $-A$ greater than all $A(v)$ with $v \in (-v_0, v_0)$. By uniform continuity, it is then possible to find some number u_1 large enough that

$$\int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u, v)) < -A/2\,, \tag{8}$$

for any $v \in (-v_0, v_0)$ as long as

$$u_+(v) > u_1 \tag{9a}$$

$$u_-(v) < -u_1\,. \tag{9b}$$

We define a sequence of parallelograms in the (u, v) plane, and integrate over each parallelogram in null and timelike directions. The parallelograms have the form

$$v \in (-v_0, v_0) \tag{10a}$$

$$u \in (u_-(v), u_+(v))\,, \tag{10b}$$

where $u_-(v), u_+(v)$ are linear functions of v defined below.,

Let f be a smooth sampling function supported only on $(-1, 1)$ and normalized

$$\int_{-1}^{1} da f(a)^2 = 1\,. \tag{11}$$

Then we can take a weighted integral over the whole parallelogram,

$$\int_{-v_0}^{v_0} dv\, f(v/v_0)^2 \int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u, v)) < -v_0 A/2\,. \tag{12}$$

We choose a velocity V and define the Doppler shift parameter

$$\delta = \sqrt{\frac{1+V}{1-V}}\,. \tag{13}$$

We pick any fixed number α with $0 < \alpha < 1/3$ and let

$$t_0 = \delta^{-\alpha} r\,, \tag{14}$$

and choose

$$v_0 = t_0/(\sqrt{2}\delta)\,. \tag{15}$$

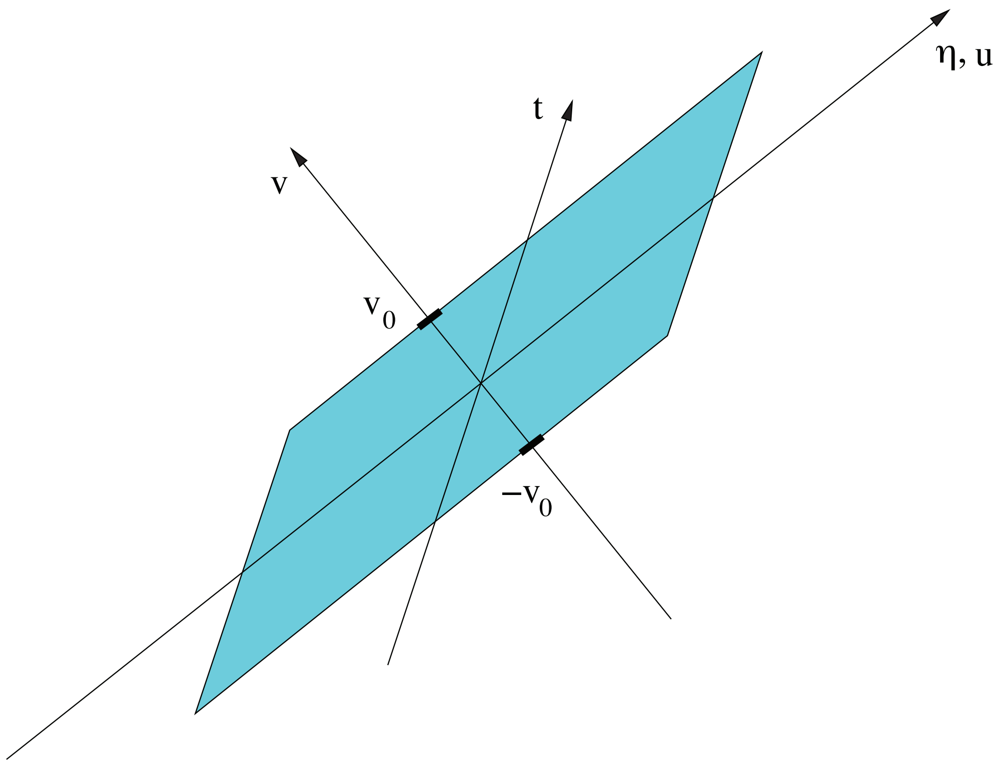

FIG. 1. The parallelogram $\Phi(u,v)$, $v \in (-v_0, v_0)$, $u \in (u_-(v), u_+(v))$, or equivalently $\Phi_V(\eta,t)$, $t \in (-t_0, t_0)$, $\eta \in (-\eta_0, \eta_0)$

Then as $V \to 0$, $\delta \to \infty$ and $t_0, v_0 \to 0$. We define

$$\eta_0 = u_1 + t_0\delta/\sqrt{2} \tag{16a}$$

$$u_\pm(v) = \pm\eta_0 + \delta^2 v\,. \tag{16b}$$

The points

$$\Phi_V(\eta,t) = \Phi\left(\eta + \frac{\delta t}{\sqrt{2}}, \frac{t}{\sqrt{2}\delta}\right)\,, \tag{17}$$

with $|\eta| < \eta_0$ and $|t| < t_0$, are the same parallelogram described above, but parameterized in a different way (see Fig. 1). For constant η the paths are timelike and in flat space parametrized by proper time. In curved spacetime t is approximately the proper time as shown in Ref. [3].

Now we change variables in Eq. (12) using the Jacobian

$$\left|\frac{\partial(u,v)}{\partial(\eta,t)}\right| = \frac{1}{\sqrt{2}\delta} \tag{18}$$

to get

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-t_0}^{t_0} dt\, T_{uu}(\Phi_V(\eta,t)) f(t/t_0)^2 < -At_0/2\,. \tag{19}$$

We will show that this upper bound conflicts with a lower bound that we will derive using quantum inequalities on the paths given by fixing η and varying t in $\Phi_V(\eta,t)$.

III. A GENERAL QUANTUM INEQUALITY

Quantum inequalities are bounds on weighted averages along a timelike path of projections of the stress-energy tensor T_{ab}. The general form is

$$\int_{-\infty}^{\infty} dt f(t) T_{ab}(w(t)) V^a V^b \geq -B\,, \tag{20}$$

where $w(t)$ is a timelike path parametrized by t, V is a vector field onto which the stress-energy tensor will be projected, $f(t)$ is a smooth sampling function, and B is some positive number depending on the choice of quantum field, the spacetime, the projection direction V, and the function f. In this paper we will apply the general quantum inequality of Fewster and Smith [4] to the case of $T_{uu}(\Phi_V)$ appearing in Eq. (19).

Following Refs. [4, 8], we define the renormalized stress-energy tensor,

$$\langle T_{ab}^{\rm ren}\rangle \equiv \lim_{x\to x'} T_{ab'}^{\rm split}\left(\langle\phi(x)\phi(x')\rangle - H(x,x')\right) - Qg_{ab} + C_{ab}\,. \tag{21}$$

The quantities appearing in Eq. (21) are defined as follows. The operator $T_{ab'}^{\rm split}$ is the point-split energy density operator,

$$T_{ab'}^{\rm split} = \nabla_a \otimes \nabla_{b'} - g_{ab'} g^{cd'} \nabla_c \otimes \nabla_{d'}\,, \tag{22}$$

which is applied to the difference between the two point function and the Hadamard series,

$$H(x,x') = \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma_+(x,x')} + \sum_{j=0}^{\infty} v_j(x,x')\sigma_+^j(x,x')\ln\left(\frac{\sigma_+(x,x')}{l^2}\right) + \sum_{j=0}^{\infty} w_j(x,x')\sigma^j(x,x')\right]\,, \tag{23}$$

We have introduced a length l so that the argument of the logarithm in Eq. (23) is dimensionless. The possibility of changing this scale creates an ambiguity in the definition of H, but this ambiguity for curved spacetime can be absorbed into the ambiguity involving local curvature terms discussed below [4]. For simplicity of notation, we will work in units where $l = 1$.

In the first term $\Delta^{1/2}$ is the Van Vleck-Morette determinant, and σ is the squared invariant length of the geodesic between x and x', negative for timelike distance. In flat space.

$$\sigma(x,x') = -\eta_{ab}(x-x')^a(x-x')^b\,. \tag{24}$$

By $F(\sigma_+)$, for some function F, we mean the distributional limit

$$F(\sigma_+) = \lim_{\epsilon\to 0^+} F(\sigma_\epsilon)\,, \tag{25}$$

where

$$\sigma_\epsilon(x,x') = \sigma(x,x') + 2i\epsilon(t(x)-t(x')) + \epsilon^2\,. \tag{26}$$

In some parts of the calculation it is possible to assume that the two points have the same spatial coordinates, so we define

$$\tau = t - t'\,, \tag{27}$$

and write

$$F(\sigma_+) = F(\tau_-) = \lim_{\epsilon\to 0} F(\tau_\epsilon)\,, \tag{28}$$

where

$$\tau_\epsilon = \tau - i\epsilon \,. \tag{29}$$

The Hadamard series can be written

$$H(x,x') = \sum_{j=-1}^{\infty} H_j(x,x')\,, \tag{30}$$

where the subscript j shows the power of σ in the term. Following the notation of Ref. [9], we let $H_{(j)}$ denote the sum of all terms from H_{-1} through H_j.

The quantity Q is added "by hand" to ensure that the stress-energy tensor is conserved [8]. But since we will be interested here in projection on a null vector ℓ, Q will not contribute, because $g_{ab}\ell^a\ell^b = 0$.

The term C_{ab} handles the possibility of including local curvature terms with arbitrary coefficients in the definition of the stress-energy tensor. From Ref. [10] we find that these terms include

$$^{(1)}H_{ab} = 2R_{;ab} - 2g_{ab}\Box R - g_{ab}R^2/2 + 2RR_{ab} \tag{31a}$$

$$^{(2)}H_{ab} = R_{;ab} - \Box R_{ab} - g_{ab}\Box R/2 - g_{ab}R^{cd}R_{cd}/2 + 2R^{cd}R_{acbd}\,. \tag{31b}$$

So we must include a term in Eq. (21) given by a linear combination of Eqs. (31a) and (31b). However, we keep only first order in R, and ignore those terms that vanish on null projection, so for our purposes,

$$C_{ab} = a\,{}^{(1)}H_{ab} + b\,{}^{(2)}H_{ab} \approx 2aR_{,ab} - b(R_{,ab} - \Box R_{ab})\,, \tag{32}$$

where a and b are undetermined constants.

From Ref. [4] we have the definition

$$\tilde{H}(x,x') = \frac{1}{2}\left[H(x,x') + H(x',x) + iE(x,x')\right]\,, \tag{33}$$

where iE is the antisymmetric part of the two-point function. We will let E_j be the part of E involving σ^j, define a "remainder term",

$$R_j = E - \sum_{k=-1}^{j} E_k\,, \tag{34}$$

and let

$$\tilde{H}_j(x,x') = \frac{1}{2}\left[H_j(x,x') + H_j(x',x) + iE_j(x,x')\right] \tag{35a}$$

$$\tilde{H}_{(j)}(x,x') = \frac{1}{2}\left[H_{(j)}(x,x') + H_{(j)}(x',x) + iE(x,x')\right]\,. \tag{35b}$$

We will use the Fourier transform convention

$$\hat{f}(k) \text{ or } f^\wedge[k] = \int_{-\infty}^{\infty} dx f(x) e^{ixk}\,. \tag{36}$$

We can now state the quantum inequality of Ref. [4], on a timelike path $w(t)$ with the stress-energy tensor contracted with null vector field ℓ^a

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle \ell^a \ell^b T_{ab}^{\rm ren}\rangle_\omega(w(t)) \geq -\int_0^\infty \frac{d\xi}{\pi}\left[(g\otimes g)(\theta^* T_{ab'}^{\rm split}\ell^a\ell^{b'}\tilde{H}_{(5)})\right]^\wedge(-\xi,\xi) + \int_{-\infty}^{\infty} dt\, g^2(t) C_{ab}\ell^a\ell^b\,, \tag{37}$$

where $g(t)$ is a smooth function with compact support and the operator θ^* denotes the pullback of the function to the path,

$$(\theta^* T_{ab'}^{\rm split}\ell^a\ell^{b'}\tilde{H}_{(5)})(t,t') \equiv (T_{ab'}^{\rm split}\ell^a\ell^{b'}\tilde{H}_{(5)})(w(t),w(t'))\,. \tag{38}$$

The subscript (5) means that we include only terms through $j=5$ in the sums of Eq. (23). However, as we proved in Ref. [9], terms of order $j>1$ make no contribution to Eq. (37). Thus we can write Eq. (37) with $w(t)=\Phi_V(\eta,t)$ for a specific value of η and the stress energy tensor null contracted with vector field ℓ^a pointing only in the u direction with $\ell^u=1$,

$$\int_{-\infty}^{\infty} d\tau\, g(t)^2 \langle T_{uu}^{\rm ren}\rangle(w(t)) \geq -B\,, \tag{39}$$

with

$$B = \int_0^\infty \frac{d\xi}{\pi}\hat{F}(-\xi,\xi) - \int_{-\infty}^{\infty} dt\, g^2(t)(2a+b)R_{,uu}\,, \tag{40}$$

where

$$F(t,t') = g(t)g(t')T_{uu'}^{\rm split}\tilde{H}_{(1)}(w(t),w(t'))\,, \tag{41}$$

$\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (36), and we used the fact that $R_{uu}=0$ according to Ref. [3].

IV. CALCULATION OF $T_{uu'}^{\rm split}\tilde{H}_{(1)}$

We will now evaluate Eq. (40) in the case of interest. In this section we will calculate $T^{\rm split}\tilde{H}_{(1)}$ and thus $F(t,t')$. In Sec. V, we will Fourier transform $F(t,t')$, and in Sec. VI we will find the form of B in terms of limits on the curvature and its derivatives.

To simplify the calculation we will evaluate $T^{\rm split}\tilde{H}_{(1)}$ in a coordinate system (t,x,y,z) where the timelike path $w(t)$ points only in the t direction, the z direction is perpendicular to it, and x and y are the previously defined ones. More specifically t and z are

$$t = \frac{\delta^{-1}u+\delta v}{\sqrt{2}}\,, \qquad z = \frac{\delta^{-1}u-\delta v}{\sqrt{2}}\,, \tag{42}$$

where we extend the definition of t from Sec. II to cover the whole spacetime. The new null coordinates $\tilde{u}$ and $\tilde{v}$ are defined by

$$\tilde{u} = \frac{t+z}{\sqrt{2}}\,, \qquad \tilde{v} = \frac{t-z}{\sqrt{2}}\,, \tag{43}$$

and are connected with u and v,

$$\tilde{u} = \delta^{-1}u\,, \qquad \tilde{v} = \delta v\,. \tag{44}$$

The operator $T^{\rm split}_{uu'}$ can be written

$$T^{\rm split}_{uu'} = \delta^{-2}\partial_{\tilde u}\partial_{\tilde u'}\,. \tag{45}$$

If we define $\zeta = z - z'$ and $\bar u$ as the $\tilde u$ coordinate of $\bar x$, the center point between x and x', we have

$$T^{\rm split}_{uu'} = \frac{1}{2}\delta^{-2}\left(\frac{1}{2}\partial^2_{\bar u} - (\partial^2_\tau + 2\partial_\zeta\partial_\tau + \partial^2_\zeta)\right)\,. \tag{46}$$

A. Derivatives of $\tilde H_{-1}$

For the derivatives of $\tilde H_{-1}$ it is simpler to use Eq. (45). We have

$$\partial_{\tilde u'}\partial_{\tilde u}\tilde H_{-1} = \frac{1}{4\pi^2}\left(\frac{\partial}{\partial x^{\tilde u}}\frac{\partial}{\partial x'^{\tilde u}}\right)\left(\frac{1}{\sigma_+}\right)\,. \tag{47}$$

In flat spacetime it is straightforward to apply the derivatives to $\tilde H_{-1}$. However in curved spacetime, there will be corrections first order in the Riemann tensor to both σ and its derivatives.

We are considering a path w whose tangent vector is constant in the coordinate system described in Sec. II A. The length of this path can be written

$$s(x,x') = \int_0^1 d\lambda\sqrt{g_{ab}(w(\lambda))\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}} = \int_0^1 d\lambda\sqrt{g_{ab}(x'')\Delta x^a\Delta x^b}\,. \tag{48}$$

where $\Delta x = x - x'$ and $x'' = x' + \lambda\Delta x$ since $dx^a/d\lambda$ is a constant.

Now σ is the negative squared length of the geodesic connecting x' to x. This geodesic might be slightly different from the path w. However, this deviation results from the connection, which is first order in the curvature (times the coordinate distance from the origin — see Eq. (9) of Ref. [7]). Thus the distance between the two paths is first order, and the difference in the metric is second order in the curvature (see Eqs. (25,27) of Ref. [7]). The difference in length in the same metric due to the different path between the same two points is also second order. All these effects can be neglected, and so we take $\sigma = -s^2$.

Now using Ref. [7] we can write the first-order correction to the metric,

$$g_{ab} = \eta_{ab} + F_{ab} + F_{ba}\,, \tag{49}$$

where F_{ab} is given by Eq. (29) of Ref. [7] because the first step for $x = y = 0$ is in the $\tilde v$ direction and the second in the $\tilde u$ direction. By the symmetries of the Riemann tensor the only non-zero component is

$$F_{\tilde v\tilde v}(x'') = \int_0^1 d\kappa(1-\kappa)R_{\tilde v\tilde u\tilde v\tilde u}(\kappa x''^{\tilde u}, x''^{\tilde v})x''^{\tilde u}x''^{\tilde u}\,, \tag{50}$$

where we took into account the different sign conventions. Putting this in Eq. (48) gives

$$\begin{aligned} s(x,x') &= \int_0^1 d\lambda\sqrt{2\Delta x^{\tilde u}\Delta x^{\tilde v} + 2F_{\tilde v\tilde v}\Delta x^{\tilde v}\Delta x^{\tilde v}} \\ &= \int_0^1 d\lambda\sqrt{2}\left(\sqrt{\Delta x^{\tilde u}\Delta x^{\tilde v}} + \frac{1}{2}F_{\tilde v\tilde v}(\Delta x^{\tilde v})^{3/2}(\Delta x^{\tilde u})^{-1/2}\right)\,. \end{aligned} \tag{51}$$

So to first order in the curvature,

$$\sigma(x,x') = -s(x,x')^2 = -\tau^2 + \zeta^2 - 2\int_0^1 d\lambda F_{\tilde{v}\tilde{v}} \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \,. \tag{52}$$

We define the zeroth order σ,

$$\sigma^{(0)}(x,x') = -\tau^2 + \zeta^2 \,, \tag{53}$$

and the first order,

$$\begin{aligned} \sigma^{(1)}(x,x') &= -2\int_0^1 d\lambda F_{\tilde{v}\tilde{v}} \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \\ &= -2\int_0^1 d\lambda \int_0^1 d\kappa (1-\kappa) R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(\kappa x''^{\tilde{u}}, x''^{\tilde{v}}) x''^{\tilde{u}} x''^{\tilde{u}} \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \\ &= -2\int_0^1 d\lambda \int_0^\ell dy (\ell - y) R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \,, \end{aligned} \tag{54}$$

where we defined $\ell \equiv x''^{\tilde{u}}$ and changed variables to $y = \kappa\ell$. Now to first order,

$$\frac{1}{\sigma_+} = \frac{1}{\sigma^{(0)}} - \frac{\sigma^{(1)}}{(\sigma^{(0)})^2} \,, \tag{55}$$

and the derivatives,

$$\left(\frac{\partial}{\partial x^{\tilde{u}}} \frac{\partial}{\partial x'^{\tilde{u}}}\right) \left(\frac{1}{\sigma_+}\right)\Bigg|_{\zeta=0} = \frac{4}{\tau_-^4} + \frac{12}{\tau_-^6}\sigma^{(1)} - \frac{2\sqrt{2}}{\tau_-^5}\left(\sigma^{(1)}_{,\tilde{u}} - \sigma^{(1)}_{,\tilde{u}'}\right) - \frac{1}{\tau_-^4}\sigma^{(1)}_{,\tilde{u}\tilde{u}'} \,. \tag{56}$$

Now we can take the derivatives of σ ,

$$\begin{aligned} \sigma^{(1)}_{,\tilde{u}} &= -2\int_0^1 d\lambda \lambda \frac{\partial}{\partial \ell} \int_0^\ell dy (\ell - y) R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \\ &= -2\int_0^1 d\lambda \lambda \int_0^\ell dy R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \,. \end{aligned} \tag{57}$$

Similarly,

$$\begin{aligned} \sigma^{(1)}_{,\tilde{u}'} &= -2\int_0^1 d\lambda (1-\lambda) \frac{\partial}{\partial \ell} \int_0^\ell dy (\ell - y) R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \\ &= -2\int_0^1 d\lambda (1-\lambda) \int_0^\ell dy R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \,. \end{aligned} \tag{58}$$

For the two derivatives of $\sigma^{(1)}$,

$$\sigma^{(1)}_{,\tilde{u}\tilde{u}'} = -2\int_0^1 d\lambda (1-\lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''^{\tilde{u}}, x''^{\tilde{v}}) \Delta x^{\tilde{v}} \Delta x^{\tilde{v}} \,. \tag{59}$$

Now we can assume purely temporal separation, so $\Delta x^{\tilde{u}} = \Delta x^{\tilde{v}} = \tau/\sqrt{2}$ and

$$x'' = \frac{1}{\sqrt{2}}(t'' + \bar{z}, t'' - \bar{z}) \,, \tag{60}$$

where $\bar{z} = (z + z')/2$ and $t'' = t' + \lambda\tau$. Then the derivatives of $\tilde{H}_{-1}$ are

$$T^{\text{split}}_{uu'}\tilde{H}_{-1} = \frac{\delta^{-2}}{4\pi^2\tau_-^4}\left(4 - 12\int_0^1 d\lambda \int_0^\ell dy(\ell - y)R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(y, x''^{\tilde{v}})\right.$$
$$-2\sqrt{2}\int_0^1 d\lambda(1 - 2\lambda)\int_0^\ell dy R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(y, x''^{\tilde{v}})\tau$$
$$\left.+\int_0^1 d\lambda(1 - \lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''^{\tilde{u}}, x''^{\tilde{v}})\tau^2\right). \tag{61}$$

Let us define the locations $\bar{x}_\kappa = (\kappa\bar{x}^{\tilde{u}}, \bar{x}^{\tilde{v}})$ and

$$x''_\kappa = \frac{1}{\sqrt{2}}(\kappa(t'' + \bar{z}), t'' - \bar{z})\,. \tag{62}$$

Then Eq. (61) can be written

$$T^{\text{split}}_{uu'}\tilde{H}_{-1} = \frac{\delta^{-2}}{4\pi^2\tau_-^4}\left(4 - \int_0^1 d\lambda\left[12\int_0^1 d\kappa(1 - \kappa)(x''^{\tilde{u}})^2 R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa)\right.\right.$$
$$\left.\left.+2\sqrt{2}(1 - 2\lambda)\int_0^1 d\kappa x''^{\tilde{u}} R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x''_\kappa)\tau - (1 - \lambda)\lambda R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x'')\tau^2\right]\right). \tag{63}$$

The derivatives of $\tilde{H}_{-1}$ can thus be written

$$T^{\text{split}}_{uu'}\tilde{H}_{-1} = \delta^{-2}\left[\frac{1}{\tau_-^4}\left(\frac{1}{\pi^2} + y_1(\bar{t}, \tau)\right) + \frac{1}{\tau_-^3}y_2(\bar{t}, \tau) + \frac{1}{\tau_-^2}y_3(\bar{t}, \tau)\right], \tag{64}$$

where the y_i's are smooth functions of the curvature,

$$y_1(\bar{t}, \tau) = \int_0^1 d\lambda Y_1(t'') \qquad y_2(\bar{t}, \tau) = \int_0^1 d\lambda(1-2\lambda)Y_2(t'') \qquad y_3(\bar{t}, \tau) = \int_0^1 d\lambda(1-\lambda)\lambda Y_3(t'')\,, \tag{65}$$

with

$$Y_1(t'') = -\frac{3}{2\pi^2}\int_0^1 d\kappa(1 - \kappa)(t'' + \bar{z})^2 R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa)\,, \tag{66a}$$

$$Y_2(t'') = \frac{1}{2\pi^2}\int_0^1 d\kappa(t'' + \bar{z})R_{\tilde{v}\tilde{u}\tilde{v}\tilde{u}}(x''_\kappa)\,, \tag{66b}$$

$$Y_3(t'') = -\frac{1}{4\pi^2}R_{\tilde{u}\tilde{v}\tilde{u}\tilde{v}}(x'')\,, \tag{66c}$$

where x'' and x''_κ are defined in terms of t'' by Eqs. (60) and (62).

B. Derivatives with respect to τ and $\bar{u}$

Ref. [5] calculated $\tilde{H}_{(1)}$, but for points separated only in time. Let us use coordinates (T, Z, X, Y) to denote a coordinate system where the coordinates of x and x' differ only in T. Ref. [5] gives

$$\tilde{H}_{(1)}(T, T') = \tilde{H}_{-1}(T, T') + \tilde{H}_0(T, T') + \tilde{H}_1(T, T') + \frac{1}{2}iR_1(T, T')\,, \tag{67}$$

$$\tilde{H}_{(0)}(T, T') = \tilde{H}_{-1}(T, T') + \tilde{H}_0(T, T') + \frac{1}{2}iR_0(T, T')\,, \tag{68}$$

where

$$\tilde{H}_{-1}(T,T') = -\frac{1}{4\pi^2(T-T'-i\epsilon)^2}\,, \tag{69a}$$

$$\tilde{H}_0(T,T') = \frac{1}{48\pi^2}\left[R_{TT}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln\left(-(T-T'-i\epsilon)^2\right)\right], \tag{69b}$$

$$\begin{aligned}\tilde{H}_1(T,T') = \frac{(T-T')^2}{640\pi^2}\Bigg[&\frac{1}{3}R_{TT,TT}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) \\ &-\frac{1}{3}\left(\Box R_{II}(\bar{x}) + \frac{1}{2}R_{,TT}(\bar{x})\right)\ln\left(-(T-T'-i\epsilon)^2\right)\Bigg].\end{aligned} \tag{69c}$$

The order-0 remainder term is

$$\begin{aligned}R_0(T,T') = \frac{1}{32\pi^2}\int d\Omega\Bigg\{&\frac{1}{2}\left[G^{(1)}_{TT}(X'') - G^{(1)}_{RR}(X'')\right] \\ &-\int_0^1 ds\, s^2 G^{(1)}_{TT}(X''_s)\Bigg\}\,\mathrm{sgn}\,(T-T')\,,\end{aligned} \tag{70}$$

where $\int d\Omega$ means to integrate over solid angle with unit 3-vectors $\hat{\Omega}$, the 4-vector $\Omega = (0,\hat{\Omega})$, the subscript R means the radial direction, and we define $X'' = \bar{x} + (1/2)|T-T'|\Omega$, $X''_s = \bar{x} + (s/2)|T-T'|\Omega$, and

$$G^{(1)}_{AB}(X'') = G_{AB}(X'') - G_{AB}(\bar{x}) = \int_0^{|T-T'|/2} dr\, G_{AB,I}(\bar{x}+r\Omega)\Omega^I\,. \tag{71}$$

The order-1 remainder term is

$$\begin{aligned}R_1(T,T') = \frac{1}{32\pi^2}\int d\Omega\Bigg\{&\frac{1}{2}\left[G^{(3)}_{TT}(X'') - G^{(3)}_{RR}(X'')\right] \\ &-\int_0^1 ds\, s^2 G^{(3)}_{TT}(X''_s)\Bigg\}\,\mathrm{sgn}\,(T-T')\,,\end{aligned} \tag{72}$$

where $G^{(3)}_{AB}$ is the remainder after subtracting the second-order Taylor series. We can write

$$G^{(3)}_{AB}(X'') = \frac{1}{2}\int_0^{|T-T'|/2} dr G_{AB,IJK}(\bar{x}+r\Omega)\left(\frac{T-T'}{2} - r\right)^2 \Omega^I\Omega^J\Omega^K\,. \tag{73}$$

When we apply the τ and $\bar{u}$ derivatives from Eq. (46), we can take $(T,Z,X,Y) = (t,z,x,y)$ and calculate $\partial^2_{\bar{u}}\tilde{H}_0$, $\partial^2_\tau\tilde{H}_0$, $\partial^2_\tau\tilde{H}_1$, $\partial^2_{\bar{u}}R_0$, and $\partial^2_\tau R_1$. Applying $\bar{u}$ derivatives to $\tilde{H}_0$ gives

$$\partial^2_{\bar{u}}\tilde{H}_0 = \frac{1}{48\pi^2}\left[R_{tt,\bar{u}\bar{u}}(\bar{x}) - \frac{1}{2}R_{,\bar{u}\bar{u}}\ln\left(-\tau_-^2\right)\right]. \tag{74}$$

For the derivatives with respect to τ we have

$$\partial^2_\tau\tilde{H}_0 = \frac{1}{48\pi^2\tau_-^2}R(\bar{x})\,, \tag{75}$$

and

$$\partial^2_\tau\tilde{H}_1 = \frac{1}{320\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\left(3 + \ln\left(-\tau_-^2\right)\right)\right]. \tag{76}$$

in the $\tau \to 0$ limit.

Applying $\bar{u}$ derivatives to R_0 gives

$$\partial_{\bar{u}}^2 R_0 = \frac{1}{32\pi^2} \int d\Omega \int_0^{|\tau|/2} dr\, \partial_{\bar{u}}^2 \left\{ \frac{1}{2} \left[G_{tt,i}(x''') - G_{rr,i}(x''') \right] - \int_0^1 ds\, s^2 G_{tt,i}(x'''_s) \right\} \Omega^i \operatorname{sgn} \tau \,, \tag{77}$$

where $x''' = \bar{x} + r\Omega$ and $x'''_s = \bar{x} + sr\Omega$.

Now we have to take the second derivative of R_1 with respect to τ, which is $T - T'$ in this case. This appears in three places: the argument of sgn in Eq. (72), the limit of integration in Eq. (73), and the term in parentheses in Eq. (73). When we differentiate the sgn, we get $\delta(\tau)$ and $\delta'(\tau)$. but since $G_{AB}^{(3)} \sim \tau^3$, there are enough powers of τ to cancel the δ or δ', so this gives no contribution. When we differentiate the limit of integration, the term in parentheses in Eq. (73) vanishes immediately. The one remaining possibility gives

$$\partial_\tau^2 R_1 = \frac{1}{128\pi^2} \int d\Omega \int_0^{|\tau|/2} dr \left\{ \frac{1}{2} \left[G_{tt,ijk}(x''') - G_{rr,ijk}(x''') \right] - \int_0^1 ds\, s^2 G_{tt,ijk}(x'''_s) \right\} \Omega^i \Omega^j \Omega^k \operatorname{sgn} \tau \,. \tag{78}$$

C. Derivatives with respect to ζ

To differentiate with respect to ζ, we must consider the possibility that x and x' are not purely temporally separated. We will suppose that the separation is only in the t and z directions and construct new coordinates (T, Z) using a Lorentz transformation that leaves $\bar{x}$ unchanged and maps the interval $(T - T', 0)$ in the new coordinates to (τ, ζ) in the old coordinates. Then

$$T - T' = \operatorname{sgn} \tau \sqrt{\tau^2 - \zeta^2} \,, \tag{79}$$

and the transformation from (T, Z) to (t, z) is given by

$$\Lambda = \frac{1}{\operatorname{sgn} \tau \sqrt{\tau^2 - \zeta^2}} \begin{pmatrix} \tau & \zeta \\ \zeta & \tau \end{pmatrix} . \tag{80}$$

with the x and y coordinates unchanged. Then

$$\begin{pmatrix} \tau \\ \zeta \end{pmatrix} = \Lambda \begin{pmatrix} T - T' \\ 0 \end{pmatrix} . \tag{81}$$

Now let M be some tensor appearing in $\tilde{H}_{(1)}$. The components in the new coordinate system are given in terms of those in the old by

$$M_{ABC\ldots} = \Lambda_A^a \Lambda_B^b \Lambda_C^c \ldots M_{abc\ldots} \tag{82}$$

We would like to differentiate such an object with respect to ζ and then set $\zeta = 0$. The only place ζ can appear is in the Lorentz transformation matrix, where we see

$$\partial_\zeta \Lambda_A^a \Big|_{\zeta=0} = \tau^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{83}$$

and similarly,

$$\partial_\zeta^2 \Lambda^a_A\Big|_{\zeta=0} = \tau^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{84}$$

To simplify notation, we will define P and Q to be the matrices on the right hand sides. Reinstating x and y,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{85}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{86}$$

Now we can write the derivative of $M_{ABC\ldots}$ as

$$\begin{aligned} \partial_\zeta M_{ABC\ldots}\Big|_{\zeta=0} &= \partial_\zeta(\Lambda^a_A \Lambda^b_B \Lambda^c_C \ldots) M_{abc\ldots}\Big|_{\zeta=0} \\ &= \Big[(\partial_\zeta \Lambda^a_A)\delta^b_B \delta^c_C \cdots + \delta^a_A(\partial_\zeta \Lambda^b_B)\delta^c_C \cdots + \ldots\Big] M_{abc\ldots}\Big|_{\zeta=0} \\ &= \frac{1}{\tau_-}(\underbrace{P^a_A \delta^b_B \delta^c_C \cdots + \delta^a_A P^b_B \delta^c_C \cdots + \ldots}_{n}) M_{abc\ldots} = \frac{1}{\tau_-} p^{abc\ldots}_{ABC\ldots} M_{abc\ldots} \end{aligned} \tag{87}$$

where $p^{abc\ldots}_{ABC\ldots}$ is a rank-n matrix of 0's and 1's. With two derivatives, we have

$$\begin{aligned} \partial_\zeta^2 M_{ABC\ldots}\Big|_{\zeta=0} &= \partial_\zeta^2(\Lambda^a_A \Lambda^b_B \Lambda^c_C \ldots) M_{abc\ldots}\Big|_{\zeta=0} \\ &= \Big[(\partial_\zeta^2 \Lambda^a_A)\delta^b_B \delta^c_C \cdots + \delta^a_A(\partial_\zeta^2 \Lambda^b_B)\delta^c_C \cdots + \ldots \\ &\quad + 2(\partial_\zeta \Lambda^a_A)(\partial_\zeta \Lambda^b_B)\delta^c_C \cdots + 2(\partial_\zeta \Lambda^a_A)\delta^b_B(\partial_\zeta \Lambda^c_C) \cdots + \ldots\Big] M_{abc\ldots}\Big|_{\zeta=0} \\ &= \frac{1}{\tau_-^2}(\underbrace{Q^a_A \delta^b_B \delta^c_C \cdots + \delta^a_A Q^b_B \delta^c_C \cdots + \ldots}_{n} \\ &\quad + \underbrace{P^a_A P^b_B \delta^c_C \cdots + P^a_A \delta^b_B P^c_C \cdots + \ldots}_{(n-1)n}) M_{abc\ldots} \\ &= \frac{1}{\tau_-^2} q^{abc\ldots}_{ABC\ldots} M_{abc\ldots} \end{aligned} \tag{88}$$

where $q^{abc\ldots}_{ABC\ldots}$ is a rank-n matrix of nonnegative integers.

There are also places where $T - T'$ appears explicitly in $\tilde{H}_1$. We can differentiate it using Eq. (79),

$$\partial_\zeta(T - T')\Big|_{\zeta=0} = 0 \,, \tag{89a}$$

$$\partial_\zeta^2(T - T')\Big|_{\zeta=0} = -\tau^{-1} . \tag{89b}$$

Now we apply the operators ∂_ζ^2 and $\partial_\tau\partial_\zeta$ to $\tilde{H}_0$, $\tilde{H}_1$, and R_1. First we apply one ζ derivative[1] to Eq. (69b) using Eq. (87),

$$\partial_\tau \left(\partial_\zeta \tilde{H}_0 \Big|_{\zeta=0} \right) = -\frac{1}{48\pi^2\tau_-^2} p_{tt}^{ab} R_{ab}(\bar{x}) \,, \tag{90}$$

and two ζ derivatives using Eqs. (88) and (89a),

$$\partial_\zeta^2 \tilde{H}_0 \Big|_{\zeta=0} = \frac{1}{48\pi^2\tau_-^2} \left[q_{tt}^{ab} R_{ab}(\bar{x}) + R(\bar{x}) \right] \,. \tag{91}$$

Then we apply one ζ derivative to $\tilde{H}_1$,

$$\partial_\tau \left(\partial_\zeta \tilde{H}_1 \Big|_{\zeta=0} \right) = \frac{1}{1920\pi^2} \left[p_{tttt}^{abcd} R_{ab,cd}(\bar{x}) - \left(p_{ii}^{ab} \Box R_{ab}(\bar{x}) + \frac{1}{2} p_{tt}^{ab} R_{,ab}(\bar{x}) \right) \left(\ln(-\tau_-^2) + 2 \right) \right] \,, \tag{92}$$

and two ζ derivatives to $\tilde{H}_1$,

$$\begin{aligned} \partial_\zeta^2 \tilde{H}_1 \Big|_{z=0} = \frac{1}{640\pi^2} \Bigg[& \frac{1}{3} q_{tttt}^{abcd} R_{ab,cd}(\bar{x}) - \frac{2}{3} R_{tt,tt}(\bar{x}) + \Box R(\bar{x}) - \frac{1}{3} \Bigg(q_{ii}^{ab} \Box R_{ab}(\bar{x}) \\ & + \frac{1}{2} q_{tt}^{ab} R_{,ab}(\bar{x}) \Bigg) \ln(-\tau_-^2) + \frac{2}{3} \left(\Box R_{ii}(\bar{x}) + \frac{1}{2} R_{,tt}(\bar{x}) \right) \left(1 + \ln(-\tau_-^2) \right) \Bigg] \,. \end{aligned} \tag{93}$$

Finally we have to apply the derivatives to the remainder R_1. We can apply the ζ derivatives in two places, the Lorentz transformations and $G_{AB}^{(3)}$. Since the three terms are very similar we will apply the derivatives to one of them

$$\partial_\zeta \int d\Omega G_{TT}^{(3)}(\bar{x} + Y) \Big|_{\zeta=0} = \int d\Omega \left(\frac{1}{\tau} p_{tt}^{ab} G_{ab}^{(3)}(x'') + \frac{\partial}{\partial Y^a} G_{tt}^{(3)}(\bar{x} + Y) \partial_\zeta Y^a \Big|_{\zeta=0} \right) \,, \tag{94}$$

where we defined $Y^a \equiv (1/2)|T - T'|\Lambda_I^a \Omega^I$. Then using Eqs. (87) and (89a), we find that that $\partial_\zeta Y^a|_{\zeta=0} = (1/2) p_i^a \Omega^i \operatorname{sgn}\tau$ and taking into account the properties of Taylor expansions,

$$\frac{\partial}{\partial Y^a} G_{tt}^{(3)}(\bar{x} + Y) \Big|_{\zeta=0} = G_{tt,a}^{(2)}(x'') \,, \tag{95}$$

where $G_{ab,c}^{(2)}$ is the remainder of the Taylor expansion of $G_{ab,c}$ after subtracting the first-order Taylor series.

Thus Eq. (94) becomes

$$\partial_\zeta \int d\Omega G_{TT}^{(3)}(\bar{x} + Y) \Big|_{\zeta=0} = \int d\Omega \left(\frac{1}{\tau} p_{tt}^{ab} G_{ab}^{(3)}(x'') + \frac{1}{2} G_{tt,a}^{(2)}(x'') p_i^a \Omega^i \operatorname{sgn}\tau \right) \,. \tag{96}$$

---

[1] The Lorentz transformation technique we use here is not quite sufficient to determine the singularity structure of the distribution $\partial_\zeta \tilde{H}_0$ at coincidence. Instead we can use Eq. (47) of Ref. [5] to compute the non-logarithmic term in $\tilde{H}_0$ for arbitrary x and x', which is then $-R_{ab}(\bar{x})(x - x')^a (x - x')^b / (48\pi^2\sigma_+)$. Differentiating this term gives Eq. (90) and explains the presence of τ_- instead of τ in the denominator. The first term of Eq. (91) arises similarly.

Using $G^{(3)}$ from Eq. (73) and

$$G^{(2)}_{ab,c}(x'') = \int_0^{|\tau|/2} dr G_{ab,ijc}(\bar{x}+r\Omega)\left(\frac{|\tau|}{2}-r\right)\Omega^i\Omega^j\,, \tag{97}$$

Eq. (96) becomes

$$\partial_\zeta \int d\Omega G^{(3)}_{TT}(\bar{X}+Y)|_{\zeta=0} = \int d\Omega \int_0^{|\tau|/2} dr\left(\frac{|\tau|}{2}-r\right)\left[p^{ab}_{tt}G_{ab,ijk}(x''')\frac{1}{\tau}\left(\frac{|\tau|}{2}-r\right)\right. \\ \left. +\frac{1}{2}p^a_i G_{tt,ija}(x''')\,\mathrm{sgn}\,\tau\right]\Omega^i\Omega^j\Omega^k\,. \tag{98}$$

We could simplify further by using the explicit values of the p matrices, but our strategy here is to show that all terms are bounded by some constants without computing the constants explicitly, since the actual constant values will not matter to the proof.

Applying the τ derivative gives

$$\partial_\tau\left(\partial_\zeta\int d\Omega G^{(3)}_{TT}(\bar{x}+Y)\Big|_{\zeta=0}\right) = \int d\Omega\int_0^{|\tau|/2} dr\left[\left(\frac{1}{4}-\frac{r^2}{\tau^2}\right)p^{ab}_{tt}G_{ab,ijk}(x''')\right. \\ \left. +\frac{1}{4}p^a_i G_{tt,ija}(x''')\right]\Omega^i\Omega^j\Omega^k\,. \tag{99}$$

We do not have to differentiate $\mathrm{sgn}\,\tau$ here, because the rest of the term is $O(\tau^2)$ and so a term involving $\delta(\tau)$ would not contribute.

The same procedure can be applied to all three terms. Terms involving X''_s will get an extra power of s each time G is differentiated. The final result is

$$\partial\tau\left(\partial_\zeta R_1(T,T')\Big|_{\zeta=0}\right) \tag{100}$$
$$= \frac{1}{32\pi^2}\int d\Omega\int_0^{|\tau|/2} dr\left\{\left(\frac{1}{4}-\frac{r^2}{\tau^2}\right)\left[\frac{1}{2}(p^{ab}_{tt}-p^{ab}_{rr})G_{ab,ijk}(x''')-\int_0^1 ds s^2 p^{ab}_{tt}G_{ab,ijk}(x'''_s)\right]\right. \\ \left. +\frac{1}{4}\left[\frac{p^a_i}{2}(G_{tt,ija}(x''')-G_{rr,ija}(x'''))-\int_0^1 ds s^3 p^a_i G_{tt,ija}(x'''_s)\right]\right\}\Omega^i\Omega^j\Omega^k\,\mathrm{sgn}\,\tau\,.$$

For two ζ derivatives we can apply both on the Lorentz transforms, both on the Einstein tensor or one on each,

$$\partial^2_\zeta\int d\Omega G^{(3)}_{TT}(\bar{x}+Y)\Big|_{\zeta=0} = \int d\Omega\left(\frac{q^{ab}_{tt}}{\tau^2}G^{(3)}_{ab}(x'')+\frac{\partial^2}{\partial Y^a\partial Y^b}G^{(3)}_{tt}(\bar{x}+Y)\partial_\zeta Y^a\partial_\zeta Y^b\Big|_{\zeta=0}\right. \\ +\frac{\partial}{\partial Y^a}G^{(3)}_{tt}(\bar{x}+Y)\partial^2_\zeta Y^a\Big|_{\zeta=0} \\ \left. +2\frac{p^{ab}_{tt}}{\tau}\frac{\partial}{\partial Y^c}G^{(3)}_{ab}(\bar{x}+Y)\partial_\zeta Y^c\Big|_{\zeta=0}\right)\,. \tag{101}$$

Using Eqs. (89b) and (88),

$$\partial^2_\zeta Y^j\Big|_{\zeta=0} = \frac{q^j_i\Omega^i}{2\tau}-\frac{\Omega^j}{2\tau} = \frac{1}{2\tau}h^j_i\Omega^i\,, \tag{102}$$

with $h_i^j \equiv q_i^j - \delta_i^j$, while

$$\partial_\zeta^2 Y^t\Big|_{\zeta=0} = 0\,, \tag{103}$$

since $q_i^t = 0$ and $\Omega^t = 0$. Using properties of the Taylor series as before, we can write

$$\frac{\partial^2}{\partial Y^a \partial Y^b} G_{tt}^{(3)}(\bar{x}+Y) = G_{tt,ab}^{(1)}(x'')\,, \tag{104}$$

so Eq. (101) becomes

$$\begin{aligned}\partial_\zeta^2 \int d\Omega G_{TT}^{(3)}(\bar{x}+Y)\Big|_{\zeta=0} &= \int d\Omega\Bigg[\frac{q_{tt}^{ab}}{\tau^2}G_{ab}^{(3)}(x'') + \frac{1}{4}p_i^a p_j^b G_{tt,ab}^{(1)}(x'')\Omega^i\Omega^j \\ &+\frac{1}{2|\tau|}\left(2p_{tt}^{ab}p_i^c G_{ab,c}^{(2)}(x'') + G_{tt,j}^{(2)}(x'')h_i^j\right)\Omega^i\Bigg]\,. \end{aligned} \tag{105}$$

Using $G^{(1)}$ as in Eq. (71) and $G^{(2)}$ and $G^{(3)}$ from Eqs. (97) and (73) this becomes

$$\begin{aligned}\partial_\zeta^2 \int d\Omega G_{TT}^{(3)}(\bar{x}+Y)\Big|_{\zeta=0} &= \int d\Omega \int_0^{|\tau|/2} dr\Bigg[\left(\frac{1}{2}-\frac{r}{|\tau|}\right)^2 q_{tt}^{ab}G_{ab,ijk}(x''') \\ &+\frac{1}{4}p_i^a p_j^b G_{tt,kab}(x''') + \left(\frac{1}{4}-\frac{r}{2|\tau|}\right)\Big(2p_{tt}^{ab}p_i^c G_{ab,jkc}(x''') \\ &+h_i^l G_{tt,ljk}(x''')\Big)\Bigg]\Omega^i\Omega^j\Omega^k\,. \end{aligned} \tag{106}$$

For all three terms

$$\begin{aligned}&\partial_\zeta^2 R_1(T,T')\Big|_{\zeta=0} \\ &= \frac{1}{32\pi^2}\int d\Omega\int_0^{|\tau|/2} dr\Bigg\{\left(\frac{1}{2}-\frac{r}{|\tau|}\right)^2\left[\frac{1}{2}(q_{tt}^{ab}-q_{rr}^{ab})G_{ab,ijk}(x''') - \int_0^1 ds s^2 q_{tt}^{ab}G_{ab,ijk}(x_s''')\right] \\ &\quad+\frac{1}{4}p_i^a p_j^b\left[\frac{1}{2}(G_{tt,kab}(x''') - G_{rr,kab}(x''')) - \int_0^s ds s^4 G_{tt,kab}(x_s''')\right] \\ &\quad+\left(\frac{1}{4}-\frac{r}{2|\tau|}\right)\Bigg[p_i^c(p_{tt}^{ab}-p_{rr}^{ab})G_{ab,jkc}(x''') + \frac{1}{2}h_i^l(G_{tt,ljk}(x''') - G_{rr,ljk}(x''')) \\ &\quad-\int_0^1 ds s^3(2p_i^c p_{tt}^{ab}G_{ab,jkc}(x_s''') + h_i^l G_{tt,ljk}(x_s'''))\Bigg]\Bigg\}\Omega^i\Omega^j\Omega^k\,\mathrm{sgn}\,\tau\,. \end{aligned} \tag{107}$$

V. THE FOURIER TRANSFORM

Eqs. (64), (74), (75), (76), (77), (78), (90), (91), (92), (93), (100) and (107) include all the $T_{uu'}^{\rm split}\tilde{H}_{(1)}$ terms. To perform the Fourier transform we expand $T_{uu'}^{\rm split}\tilde{H}_{(1)}$ according to

Eqs. (45) and (46) and separate the terms by their τ dependence,

$$
\begin{aligned}
T^{\rm split}_{uu'}\tilde{H}_{(1)} &= \delta^{-2}\Bigg[\partial_{\tilde{u}}\partial_{\tilde{u}'}\tilde{H}_{-1} + \frac{1}{4}\left(\partial^2_{\tilde{u}}\tilde{H}_0 + \frac{1}{2}iR_0\right) \\
&\qquad -\frac{1}{2}(\partial^2_\tau + \partial^2_\zeta + 2\partial_\tau\partial_\zeta)\left(\tilde{H}_0 + \tilde{H}_1 + \frac{1}{2}iR_1\right)\Bigg] \\
&= \delta^{-2}\Bigg[\frac{1}{\tau_-^4}\left(\frac{1}{\pi^2} + y_1(\bar{t},\tau)\right) + \frac{1}{\tau_-^3}y_2(\bar{t},\tau) + \frac{1}{\tau_-^2}(c_1(\bar{t}) + y_3(\bar{t},\tau)) \\
&\qquad + \ln(-\tau_-^2)c_2(\bar{t}) + c_3(\bar{t}) + c_4(\bar{t},\tau)\Bigg],
\end{aligned}
\tag{108}
$$

where c_1, c_2, and c_3 are smooth and have no τ dependence and c_4 is odd, C_1 and bounded. As mentioned in Sec. IV A, the functions y_i depend on τ but are smooth. Explicit expressions for the c_i are given in Appendix A.

We now put the terms of Eq. (108) into Eq. (40), and Fourier transform them, following the procedure of Sec. IV of Ref. [9], to obtain the bound B in the form

$$
B = \delta^{-2}\sum_{i=0}^{6} B_i\,. \tag{109}
$$

The first term in Eq. (108) is $1/(\pi^2\tau_-^4)$, and we proceed exactly as Ref. [9], except for the different numerical coefficient, to obtain

$$
B_0 = \frac{1}{24\pi^2}\int_{-\infty}^{\infty} d\bar{t}\, g''(\bar{t})^2\,. \tag{110}
$$

Putting only Eq. (110) into Eq. (109) gives the result for flat space. Fewster and Eveson [11] found a result of the same form, but they considered T_{tt} instead of T_{uu}, so the multiplying constant is different. Fewster and Roman [12] found the result for null projection. Where we have $1/24$, they had $(v\cdot\ell)^2/12$, where v is the unit tangent vector to the path of integration. Here $v\cdot\ell = \ell^t = 1/(\delta\sqrt{2})$, from Eq. (42), so the results agree.

The remaining τ_-^{-4} term requires more attention, because of the τ dependence in y_1. We write

$$
B_1 = \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} d\tau G_1(\tau)\frac{1}{\tau_-^4}e^{-i\xi\tau}\,, \tag{111}
$$

with

$$
G_1(\tau) = \int_{-\infty}^{\infty} d\bar{t}(y_1(\bar{t},\tau))g\left(\bar{t}-\frac{\tau}{2}\right)g\left(\bar{t}+\frac{\tau}{2}\right)\,. \tag{112}
$$

Then [9]

$$
B_1 = \frac{1}{24}G_1''''(0)\,. \tag{113}
$$

Applying the τ derivatives to G_1 gives

$$
\begin{aligned}
G_1''''(\tau)\Big|_{\tau=0} &= \int_{-\infty}^{\infty} d\bar{t}\Bigg[\frac{d^4}{d\tau^4}y_1(\bar{t},\tau)\Big|_{\tau=0}g(\bar{t})^2 + 3\frac{d^2}{d\tau^2}y_1(\bar{t},\tau)\Big|_{\tau=0}(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) \\
&\qquad + \frac{1}{8}y_1(\bar{t})(g''''(\bar{t})g(\bar{t}) - 4g'''(\bar{t})g(\bar{t}) + 3g''(\bar{t})^2)\Bigg],
\end{aligned}
\tag{114}
$$

where the terms with an odd number of derivatives of the product of the sampling functions vanish after taking $\tau = 0$.

Now y_1 depends on τ and $\bar{t}$ only through $t'' = \bar{t} + (\lambda - 1/2)\tau$, so using Eq. (65), we can write

$$\frac{d}{d\tau} y_1(\bar{t}, \tau) = \frac{d}{d\tau} \int_0^1 d\lambda Y_1(t'') = \frac{d}{d\bar{t}} \int_0^1 d\lambda (\lambda - 1/2) Y_1(t'') \,. \tag{115}$$

Then we integrate by parts and put all the derivatives on the sampling functions g,

$$\begin{aligned} B_1 = \frac{1}{24} \int_{-\infty}^{\infty} d\bar{t} \Bigg[& 2 \int_0^1 d\lambda \left(\lambda - \frac{1}{2}\right)^4 Y_1(\bar{t})(3g''(\bar{t})^2 + 4g'(\bar{t})g'''(\bar{t}) + g(\bar{t})g''''(\bar{t})) \\ & +3 \int_0^1 d\lambda \left(\lambda - \frac{1}{2}\right)^2 Y_1(\bar{t})(g''''(\bar{t})g(\bar{t}) - g''(\bar{t})^2) \\ & + \frac{1}{8} y_1(\bar{t})(g''''(\bar{t})g(\bar{t}) - 4g'''(\bar{t})g'(\bar{t}) + 3g''(\bar{t})^2) \Bigg] \,. \end{aligned} \tag{116}$$

Since we set $\tau = 0$, Y_1 has no λ dependence and we can perform the integral. The result is

$$B_1 = \frac{1}{120} \int_{-\infty}^{\infty} d\bar{t}\, Y_1(\bar{t})(g''(\bar{t})^2 - 2g'''(\bar{t})g'(\bar{t}) + 2g''''(\bar{t})g(\bar{t})) \,. \tag{117}$$

For the term proportional to τ_-^{-3}, we have

$$B_2 = \int_0^{\infty} \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau G_2(\tau) \frac{1}{\tau_-^3} e^{-i\xi\tau} \,. \tag{118}$$

where

$$G_2(\tau) = \int_{-\infty}^{\infty} d\bar{t} y_2(\bar{t}, \tau) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right) \,. \tag{119}$$

We calculate this Fourier transform in Appendix B and the result is

$$B_2 = \frac{1}{6} G_2'''(0) \,. \tag{120}$$

Applying the derivatives to G_2 gives

$$G_2'''(\tau)\Big|_{\tau=0} = \int_{-\infty}^{\infty} d\bar{t} \left[\frac{d^3}{d\tau^3} y_2(\bar{t}, \tau)\Big|_{\tau=0} g(\bar{t})^2 + \frac{3}{2} \frac{d}{d\tau} y_2(\bar{t}, \tau)\Big|_{\tau=0} (g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) \right] . \tag{121}$$

Again the only dependence of y_2 on τ is in the form of t'' so we can integrate by parts

$$\begin{aligned} B_2 = -\frac{1}{3} \int_{-\infty}^{\infty} d\bar{t} \int_0^1 d\lambda \Bigg[& 2 \left(\lambda - \frac{1}{2}\right)^4 Y_2(\bar{t})(3g'(\bar{t})g''(\bar{t}) + g(\bar{t})g'''(\bar{t})) \\ & + \frac{3}{2} \left(\lambda - \frac{1}{2}\right)^2 Y_2(\bar{t})(g'''(\bar{t})g(\bar{t}) - g''(\bar{t})g'(\bar{t})) \Bigg] , \end{aligned} \tag{122}$$

and perform the λ integrals

$$B_2 = \frac{1}{60} \int_{-\infty}^{\infty} d\bar{t}\, Y_2(\bar{t})(g'(\bar{t})g''(\bar{t}) - 3g'''(\bar{t})g(\bar{t})) \,. \tag{123}$$

For the term proportional to τ_-^{-2}, we have

$$B_3 = \int_0^\infty \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau G_3(\tau) \frac{1}{\tau_-^2} e^{-i\xi\tau} \,. \tag{124}$$

where

$$G_3(\tau) = \int_{-\infty}^{\infty} d\bar{t} (c_1(\bar{t}) + y_3(\bar{t},\tau)) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right) \,. \tag{125}$$

Ref. [9] calculated this Fourier transform, but the Fourier transform of $1/\tau_-^2$ given by Ref. [13] was cited with the wrong sign in Eq. (105) of Ref. [9]. The correct result is

$$B_3 = \frac{1}{2} G_3''(0) \,. \tag{126}$$

Applying the derivatives to G_3 gives

$$G_3''(\tau)\bigg|_{\tau=0} = \int_{-\infty}^{\infty} d\bar{t} \bigg[\frac{d^2}{d\tau^2} y_3(\bar{t},\tau)\bigg|_{\tau=0} g(\bar{t})^2 + \frac{1}{2}(c_1(\bar{t}) + y_3(\bar{t}))(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) \bigg] \,. \tag{127}$$

As before, we integrate by parts

$$\begin{aligned} B_3 = \frac{1}{2} \int_{-\infty}^{\infty} d\bar{t} \int_0^1 d\lambda \bigg[& 2\left(\lambda - \frac{1}{2}\right)^2 (1-\lambda)\lambda Y_3(\bar{t})(g'(\bar{t})^2 + g(\bar{t})g''(\bar{t})) \\ & + \frac{1}{2}(c_1(\bar{t}) + (1-\lambda)\lambda Y_3(\bar{t}))(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) \bigg] \,. \end{aligned} \tag{128}$$

Integrating in λ gives

$$B_3 = \frac{1}{4} \int_{-\infty}^{\infty} d\bar{t} \left[c_1(\bar{t})(g''(\bar{t})g(\bar{t}) - g'(\bar{t})^2) + \frac{1}{15} Y_3(\bar{t})(3g''(\bar{t})g(\bar{t}) - 2g'(\bar{t})^2) \right] \,. \tag{129}$$

The three remaining terms have Fourier transforms given in Ref. [9], so we find[2]

$$B_4 = -\int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau g'\left(\bar{t} + \frac{\tau}{2}\right) g\left(\bar{t} - \frac{\tau}{2}\right) \ln|\tau| c_2(\bar{t}) \operatorname{sgn}\tau \tag{130a}$$

$$B_5 = \int_{-\infty}^{\infty} d\bar{t} g(\bar{t})^2 (c_3(\bar{t}) + c_5(\bar{t})) \tag{130b}$$

$$B_6 = \frac{1}{\pi} \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau \frac{1}{\tau} g\left(\bar{t} + \frac{\tau}{2}\right) g\left(\bar{t} - \frac{\tau}{2}\right) c_4(\bar{t},\tau) \,, \tag{130c}$$

where we added

$$c_5(\bar{t}) = -(2a+b) R_{,\tilde{u}\tilde{u}}(\bar{t}) \,, \tag{131}$$

which is the local curvature term from Eq. (40).

The bound is now given by Eqs. (109), (110), (117), (123), (129), (130).

[2] Equation (130a) corrects an error of a factor of 2 between Eqs. (114) and (116) of Ref. [9]

VI. THE INEQUALITY

We would like to bound the correction terms B_1 through B_6 using bounds on the curvature and its derivatives. Using Eq. (3) in Eq. (66a), we find

$$|Y_1(\bar{t})| < \frac{3}{2\pi^2}|\bar{x}^{\tilde{u}}|^2 R_{\max} \,. \tag{132}$$

We can use Eq. (132) in Eq. (117) to get a bound on $|B_1|$. But will not be interested in specific numerical factors, only the form of the quantities that appear in our bounds. So we will write

$$|B_1| \leq J_1^{(3)}[g]|\bar{x}^{\tilde{u}}|^2 R_{\max} \,, \tag{133}$$

where $J_1^{(3)}[g]$ is an integral of some combination of the sampling function and its derivatives appearing in Eq. (117). We will need many similar functionals $J_n^{(k)}[g]$, which are listed in Appendix A. The number in the parenthesis shows the dimension of the integral,

$$J_n^{(k)}[g] \sim \frac{1}{[L]^k} \,. \tag{134}$$

Similar analyses apply to B_2 and B_3 and the results are

$$|B_2| \leq J_2^{(2)}[g]|\bar{x}^{\tilde{u}}| R_{\max} \tag{135a}$$

$$|B_3| \leq J_3^{(1)}[g] R_{\max} \,. \tag{135b}$$

Among the rest of the terms in B there are some components of the form $R_{abcd,\tilde{u}}$ which diverge after boosting to the null geodesic, as shown in Ref. [3]. However we can show that these derivatives are not a problem since we can integrate them by parts. Suppose we have a term of the form

$$B_n = \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_n(\tau,\bar{t}) R_{abcd,\tilde{u}}(\bar{x}) \,, \tag{136}$$

where $L_n(\tau,\bar{t})$ is a function that contains the sampling function g and its derivatives. The $\tilde{u}$ derivative on the Riemann tensor can be written

$$R_{abcd,\tilde{u}} = R_{abcd,t} - R_{abcd,\tilde{v}} \,. \tag{137}$$

The term can be reorganized the following way by grouping the terms with t and $\tilde{v}, x, y$ derivatives

$$B_n = \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_n(\tau,\bar{t}) (A_n^{abcd} R_{abcd,t}(\bar{x}) + A_n^{abcd\alpha} R_{abcd,\alpha}(\bar{x})) \,, \tag{138}$$

where $A_n^{abcd\ldots}$ are arrays with constant components and the subscript n denotes the term they come from. Here the greek indices $\alpha, \beta, \cdots = \tilde{v}, x, y$. The term with one derivative on α can be bounded, while the term with one derivative on t can be integrated by parts,

$$B_n = -\int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau (L_n'(\bar{t},\tau) A_n^{abcd} R_{abcd}(\bar{x}) + L_n(\bar{t},\tau) A_n^{abcd\alpha} R_{abcd,\alpha}(\bar{x})) \,. \tag{139}$$

where the primes denote derivatives with respect to $\bar{t}$. The sampling function is C_0^{∞} so $L'(\tau,\bar{t})$ is still smooth and the boundary terms vanish. Now it is possible to bound this term,

$$|B_n| \leq \int_{\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau (|L_n'(\bar{t},\tau)| a_n^{(0)} R_{\max} + |L_n(\bar{t},\tau)| a_n^{(1)} R'_{\max}) \,, \tag{140}$$

where we defined

$$a_n^{(m)} = \sum_{abcd \underbrace{\alpha\beta\ldots}_{m}} \left| A_n^{abcd \overbrace{\alpha\beta\ldots}^{m}} \right| . \tag{141}$$

The same method can be applied with more than one $\tilde{u}$ derivative.

Now we apply this method to the integrals B_4, B_5 and B_6 of Eq. (130). We start with B_4, which has the form

$$\begin{aligned} B_4 = \int_{-\infty}^{\infty} d\tau \ln|\tau| \operatorname{sgn}\tau \int_{-\infty}^{\infty} d\bar{t} L_4(\bar{t},\tau) \Big(& A_4^{abcd} R_{abcd,tt}(\bar{x}) + A_4^{abcd\alpha} R_{abcd,\alpha t}(\bar{x}) \\ & + A_4^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \,, \end{aligned} \tag{142}$$

where

$$L_4(\bar{t},\tau) = g(\bar{t}+\tau/2) g'(\bar{t}-\tau/2) \,. \tag{143}$$

After integration by parts

$$\begin{aligned} B_4 = \int_{-\infty}^{\infty} d\tau \ln|\tau| \operatorname{sgn}\tau \int_{-\infty}^{\infty} d\bar{t} \Big(& L_4''(\bar{t},\tau) A_4^{abcd} R_{abcd}(\bar{x}) - L_4'(\bar{t},\tau) A_4^{abcd\alpha} R_{abcd,\alpha}(\bar{x}) \\ & + L_4(\bar{t},\tau) A_8^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \,. \end{aligned} \tag{144}$$

Taking the bound gives

$$|B_4| \leq \sum_{m=0}^{2} J_4^{(1-m)}[g] R_{\max}^{(m)} \,. \tag{145}$$

Reorganizing B_5 based on the number of t derivatives gives

$$\begin{aligned} B_5 &= \int_{-\infty}^{\infty} d\bar{t} L_5(\bar{t}) \Big(A_5^{abcd} R_{abcd,tt}(\bar{x}) + A_5^{abcd\alpha} R_{abcd,\alpha t}(\bar{x}) + A_5^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \\ &= \int_{-\infty}^{\infty} d\bar{t} \Big(L_5''(\bar{t}) A_5^{abcd} R_{abcd}(\bar{x}) - L_5'(\bar{t}) A_5^{abcd\alpha} R_{abcd,\alpha}(\bar{x}) + L_5(\bar{t}) A_5^{abcd\alpha\beta} R_{abcd,\alpha\beta}(\bar{x}) \Big) \,, \end{aligned} \tag{146}$$

where

$$L_5(\bar{t}) = g(\bar{t})^2 \,, \tag{147}$$

and the bound is

$$|B_5| \leq \sum_{m=0}^{2} J_5^{(1-m)}[g] R_{\max}^{(m)} \,. \tag{148}$$

Finally the remainder term is

$$B_6 = \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau L_6(\bar{t},\tau) \int d\Omega \int_0^1 d\lambda \Big\{ A_6^{abcd}(\lambda,\Omega) R_{abcd,ttt}(\lambda\Omega)$$
$$+A_6^{abcd\alpha}(\lambda,\Omega) R_{abcd,\alpha tt}(\lambda\Omega) + A_6^{abcd\alpha\beta}(\lambda,\Omega) R_{abcd,\alpha\beta t}(\lambda\Omega)$$
$$+A_6^{abcd\alpha\beta\gamma}(\lambda,\Omega) R_{abcd,\alpha\beta\gamma}(\lambda\Omega) \Big\} \operatorname{sgn}\tau\,, \tag{149}$$

where we changed variables to $\lambda = r/\tau$ and now arrays $A_6^{abcd\dots}$ have components that depend on λ and Ω, and

$$L_6(\bar{t},\tau) = g(\bar{t} - \tau/2) g(\bar{t} + \tau/2)\,. \tag{150}$$

After integration by parts

$$B_6 = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t} \int d\Omega \int_0^1 d\lambda \Big\{ L_6'''(\tau,\bar{t}) A_6^{abcd}(\lambda,\Omega) R_{abcd}(\lambda\Omega)$$
$$+L_6(\tau,\bar{t})'' A_6^{abcd\alpha}(\lambda,\Omega) R_{abcd,\alpha}(\lambda\Omega) + L_6(\tau,\bar{t})' A_6^{abcd\alpha\beta}(\lambda,\Omega) R_{ab,\alpha\beta}(\lambda\Omega)$$
$$+L_6(\tau,\bar{t}) A_6^{abcd\alpha\beta\gamma}(\lambda,\Omega) R_{ab,\alpha\beta\gamma}(\lambda\Omega) \Big\} \operatorname{sgn}\tau\,. \tag{151}$$

We define constants $a_6^{(m)}$

$$a_6^{(m)} = \sum_{abcd\,\underbrace{\alpha\beta\dots}_{m}} \left| \int d\Omega \int_0^1 d\lambda A_6^{abcd\,\overbrace{\alpha\beta\dots}^{m}}(\lambda,\Omega) \right|\,, \tag{152}$$

and now we can take the bound

$$|B_6| \le \sum_{m=0}^{3} J_6^{(1-m)}[g] R_{\max}^{(m)}\,. \tag{153}$$

Putting everything together gives

$$B \le \delta^{-2} \left(B_0 + \sum_{n=1}^{3} J_n^{(4-n)}[g] |\bar{x}^{\tilde{u}}|^{3-n} R_{\max} + \sum_{n=4}^{6} \sum_{m=0}^{3} J_n^{(1-m)}[g] R_{\max}^{(m)} \right)\,. \tag{154}$$

We can change the argument of the sampling function, writing $g(t) = f(t/t_0)$, where f is defined in Sec. II and normalized according to Eq. (11), so Eq. (154) becomes

$$\int dt T_{uu}(w(t)) g(t)^2 \ge -\frac{\delta^{-2}}{t_0^3} \Big\{ \frac{1}{24\pi^2 t_0} \int_{-t_0}^{t_0} dt f''(t/t_0)^2 + \sum_{n=1}^{3} J_n^{(4-n)}[f] |\bar{x}^{\tilde{u}}|^{3-n} R_{\max} t_0^{n-1}$$
$$+ \sum_{n=4}^{6} \sum_{m=0}^{3} J_n^{(1-m)}[f] R_{\max}^{(m)} t_0^{m+2} \Big\}\,, \tag{155}$$

where we used $J_n^{(k)}[g] = t_0^{-k} J_n^{(k)}[f]$. We can simplify the inequality by defining

$$F = \int f''(\alpha)^2 d\alpha = \frac{1}{t_0} \int f''(t/t_0)^2 dt\,, \tag{156}$$

$$F^{(m)} = \sum_{n=4}^{6} J_n^{(1-m)}[f]\,, \tag{157}$$

and

$$\bar{F}^{(n)} = J_n^{(4-n)}[f]\,. \tag{158}$$

Then Eq. (155) becomes

$$\int dt T_{uu}(w(t))g(t)^2 \geq \tag{159}$$
$$-\frac{\delta^{-2}}{t_0^3}\left\{\frac{1}{24\pi^2}F + \sum_{m=0}^{3} F^{(m)} R_{\max}^{(m)} t_0^{m+2} + \sum_{n=1}^{3} |\bar{x}^{\tilde{u}}|^{3-n} \bar{F}^{(n)} R_{\max} t_0^{n-1}\right\}.$$

We will use this result to prove the achronal ANEC.

VII. THE PROOF OF THE THEOREM

We use Eq. (159) with $w(t) = \Phi_V(\eta, t)$ and integrate in η to get

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-t_0}^{t_0} T_{uu}(\Phi_V(\eta, t)) f(t/t_0)^2 \geq \tag{160}$$
$$-\frac{\eta_0}{\delta^2 t_0^3}\left\{\frac{1}{24\pi^2}F + \sum_{m=0}^{3} F^{(m)} R_{\max}^{(m)} t_0^{m+2} + \sum_{n=1}^{3} |\bar{x}^{\tilde{u}}|^{3-n} \bar{F}^{(n)} R_{\max} t_0^{n-1}\right\}.$$

As $\delta \to \infty$, $t_0 \to 0$ but $F^{(m)}$, $\bar{F}^{(n)}$, $R_{\max}$, and $R_{\max}^{(m)}$ are constant. Now $\bar{x}^{\tilde{u}} = \bar{x}^u/\delta$, and using Eqs. (10), (15), (16), $|\bar{x}^u| < u_1 + \sqrt{2}\delta t_0$. Thus as $\delta \to \infty$, $\bar{x}^{\tilde{u}} \to 0$. Therefore only the first term in braces in Eq. (160) survives, so the bound goes to zero as

$$\frac{\eta_0}{\delta^2 t_0^3} \sim \delta^{2\alpha-1}\,. \tag{161}$$

Equation (160) is a lower bound. It says that its left-hand side can be no more negative than the bound, which declines as $\delta^{2\alpha-1}$. But Eq. (19) gives an upper bound on the same quantity, saying that it must be more negative than $-At_0/2$, which goes to zero as $t_0 \sim \delta^{-\alpha}$. Since $\alpha < 1/3$, the lower bound goes to zero more rapidly, and therefore for sufficiently large δ, the lower bound will be closer to zero than the upper bound, and the two inequalities cannot be satisfied at the same time. This contradiction proves Theorem 1.

The ambiguous local curvature terms do not contribute in the limit $\eta_0 \to \infty$ because they are total derivatives proportional to

$$\int_{-\eta_0}^{\eta_0} d\eta R_{,uu}(\bar{x})) = 0\,. \tag{162}$$

VIII. CONCLUSIONS

This work completes the proof of ANEC in curved spacetime for a minimally coupled, free scalar field, on achronal geodesics traveling through a spacetime that obeys NEC. The techniques are similar to those of Ref. [3], but that paper required an unproven conjecture. Here, we use the general absolute quantum inquality of Fewster and Smith [4] to derive a null projected quantum inequality, slightly different from our previous conjecture, and use that inequality, Eq. (155), to prove achronal ANEC. Equation (155) has the form of the flat-space null-projected quantum inequality of Fewster and Roman [12], plus correction terms which vanish as one considers more and more highly boosted timelike paths with smaller and smaller total proper time in the limiting process above.

The result of this paper concerns integrals of the stress-energy tensor of a quantum field in a background spacetime; we have so far not been concerned about the back-reaction of the stress-energy tensor on the spacetime curvature. This analysis is correct in the case where the quantum field under consideration produces only a small perturbation of the spacetime. Thus we have shown that no spacetime that obeys NEC can be perturbed by a minimally-coupled quantum scalar field into one which violates achronal ANEC. Thus no such perturbation of a classical spacetime would allow wormholes, superluminal travel, or construction of time machines[3] [1].

What possibilities remain for the generation of such exotic phenomena? One is that the quantum field is not just a perturbation but generates enough NEC violation to permit itself to violate ANEC also. We argued against this idea on dimensional grounds in Ref. [3]. Another is that there is a field that violates NEC but obeys ANEC, and a second field, propagating in the background generated by the first, that violates ANEC. This three-step process seems unlikely but is open to future investigation.

There is also the possibility of different fields. We have not studied higher-spin fields, but these typically obey the same energy conditions as minimally-coupled scalars. Of more interest is the possibility of a non-minimally coupled scalar field. Such fields can produce ANEC violations even classically [16, 17] with large enough (Planck-scale) field values. However these situations seem unphysical since the effective Newton's constant becomes negative as the field value increases. In the case of a wormhole [18], the effective Newton's constant must be negative not only inside the wormhole but in one of the asymptotic regions. If one disallows Planck-scale field values, there are restrictions on non-minimally coupled classical [19] and quantum [20] fields, but these restrictions are not in the form of the usual quantum inequalities. Whether there is a self-consistent achronal ANEC for non-minimally coupled scalar fields remains an open question.

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[3] More specifically, ANEC rules out compactly generated causality violation [14]. A causality violating region is compactly generated if the generators of the Cauchy horizon followed into the past enter and remain within some compact set. But Ori [15] argues that one can ensure that closed causal curves develop even without compact generation, by arranging the formation of a null hypersurface spanned by closed null geodesics at the future boundary of the domain of dependence.

Appendix A: Explicit expressions

Here are the explicit expressions of the functions c_i used in Eq. (108):

$$c_1 = \frac{1}{48\pi^2}\left(- R(\bar{x}) + \left(p^{ab}_{tt} - \frac{q^{ab}_{tt}}{2}\right) R_{ab}(\bar{x})\right) \tag{A1a}$$

$$c_2 = \frac{1}{1920\pi^2}\left(- 5R_{,\tilde{u}\tilde{u}}(\bar{x}) + \left(p^{ab}_{ii} + \frac{q^{ab}_{ii}}{2}\right) \Box R_{ab}(\bar{x}) + \frac{1}{2}\left(p^{ab}_{tt} + \frac{q^{ab}_{tt}}{2}\right) R_{,ab}(\bar{x})\right) \tag{A1b}$$

$$c_3 = \frac{1}{960\pi^2}\Bigg(5R_{tt,\tilde{u}\tilde{u}}(\bar{x}) - \frac{1}{2}\left(p^{abcd}_{tttt} + \frac{q^{abcd}_{tttt}}{2}\right) R_{ab,cd}(\bar{x}) + \Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})$$
$$+p^{ab}_{ii}\Box R_{ab}(\bar{x}) + \frac{1}{2}p^{ab}_{tt}R_{,ab}(\bar{x})\Bigg) \tag{A1c}$$

$$c_4 = \frac{1}{256\pi^2}\int_0^{|\tau|/2} dr \int d\Omega \Bigg(\partial^2_{\tilde{u}}\left\{\frac{1}{2}\left[G_{tt,i}(x''') - G_{rr,i}(x''')\right] - \int_0^1 ds\, s^2 G_{tt,i}(x'''_s)\right\}$$
$$-\Bigg\{\frac{1}{4}\left[G_{tt,ijk}(x''') - G_{rr,ijk}(x''')\right] - \frac{1}{2}\int_0^1 ds\, s^2 G_{tt,ijk}(x'''_s)$$
$$+\left(1 - \frac{4r^2}{\tau^2}\right)\left[\frac{1}{2}(p^{ab}_{tt} - p^{ab}_{rr})G_{ab,ijk}(x''') - \int_0^1 ds s^2 p^{ab}_{tt} G_{ab,ijk}(x'''_s)\right]$$
$$+\frac{p^a_i}{2}(G_{tt,jka}(x''') - G_{rr,jka}(x''')) - \int_0^1 ds s^3 p^a_i G_{tt,jka}(x'''_s)$$
$$+2\left(\frac{1}{2} - \frac{r}{|\tau|}\right)^2\left[\frac{1}{2}(q^{ab}_{tt} - q^{ab}_{rr})G_{ab,ijk}(x''') - \int_0^1 ds s^2 q^{ab}_{tt} G_{ab,ijk}(x'''_s)\right]$$
$$+\frac{1}{2}p^a_i p^b_j\left[\frac{1}{2}(G_{tt,kab}(x''') - G_{rr,kab}(x''')) - \int_0^s ds s^4 G_{tt,kab}(x'''_s)\right]$$
$$+\left(\frac{1}{2} - \frac{r}{|\tau|}\right)\Bigg[p^c_i(p^{ab}_{tt} - p^{ab}_{rr})G_{ab,jkc}(x''') + \frac{1}{2}h^l_i(G_{tt,ljk}(x''') - G_{rr,ljk}(x'''))$$
$$-\int_0^1 ds s^3 (2p^c_i p^{ab}_{tt} G_{ab,jkc}(x'''_s) + h^l_i G_{tt,ljk}(x'''_s))\Bigg\}\Omega^j\Omega^k\Bigg)\Omega^i \,\mathrm{sgn}\,\tau\,. \tag{A1d}$$

And here are the integrals of the sampling function:

$$J^{(3)}_1[g] = \int_{-\infty}^{\infty} dt (a_{11}|g''''(t)|g(t) + a_{12}|g'''(t)g'(t)| + a_{13}g''(t)^2) \tag{A2a}$$

$$J^{(2)}_2[g] = \int_{-\infty}^{\infty} dt (a_{21}|g'''(t)|g(t) + a_{22}|g''(t)g'(t)|) \tag{A2b}$$

$$J^{(1)}_3[g] = \int_{-\infty}^{\infty} dt (a_{31}|g''(t)|g(t) + a_{32}g'(t)^2) \tag{A2c}$$

$$J^{(1)}_4[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left|\ln|t-t'|\right| (a_{41}|g'''(t')|g(t) + a_{42}|g''(t)g'(t')|) \tag{A2d}$$

$$J^{(0)}_4[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left|\ln|t-t'|\right| (a_{43}|g''(t')|g(t) + a_{44}|g'(t)g'(t')|) \tag{A2e}$$

$$J^{(-1)}_4[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left|\ln|t-t'|\right| a_{45}|g'(t')|g(t) \tag{A2f}$$

$$J_5^{(1)}[g] = \int_{-\infty}^{\infty} dt(a_{51}|g''(t)|g(t) + a_{52}g'(t)^2) \tag{A2g}$$

$$J_5^{(0)}[g] = \int_{-\infty}^{\infty} dt\, a_{53}|g'(t)|g(t) \tag{A2h}$$

$$J_5^{(-1)}[g] = \int_{-\infty}^{\infty} dt\, a_{54}g(t)^2 \tag{A2i}$$

$$J_6^{(1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'(a_{61}|g'''(t)|g(t') + a_{62}|g''(t)g'(t')|) \tag{A2j}$$

$$J_6^{(0)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'(a_{63}|g''(t)|g(t') + a_{64}|g'(t)g'(t')|) \tag{A2k}$$

$$J_6^{(-1)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, a_{65}|g'(t)|g(t') \tag{A2l}$$

$$J_6^{(-2)}[g] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, a_{66}g(t)g(t')\,, \tag{A2m}$$

where a_{nk} are positive constants that may depend on $a_n^{(m)}$.

Appendix B: Fourier transform

We follow the procedure of Ref. [9] to calculate

$$B_2 = \int_0^{\infty} \frac{d\xi}{\pi} \int_{-\infty}^{\infty} d\tau G_2(\tau) s_2(\tau) e^{-i\xi\tau}\,. \tag{B1}$$

where

$$G_2(\tau) = \int_{-\infty}^{\infty} d\bar{t} y_2(\bar{t}, \tau) g\left(\bar{t} - \frac{\tau}{2}\right) g\left(\bar{t} + \frac{\tau}{2}\right)\,. \tag{B2}$$

and

$$s_2(\tau) = \frac{1}{\tau_-^3}\,. \tag{B3}$$

This is the Fourier transform of a product so we can write it as a convolution. The function s_2 is real and odd, so its Fourier transform is imaginary, but G_2 is also real and odd, thus the Fourier transform of their product is real. We have

$$B_2 = \frac{1}{2\pi^2} \int_0^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \hat{G}_2(-\xi - \zeta) \hat{s}_2(\zeta)\,. \tag{B4}$$

We can change the order of integrals and change variables to $\eta = -\xi - \zeta$ which gives

$$\begin{aligned} B_2 &= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\zeta \int_{\zeta}^{\infty} d\eta\, \hat{G}_2(\eta) \hat{s}_2(\zeta) \\ &= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\eta\, \hat{G}_2(\eta) \int_{-\infty}^{\eta} d\zeta\, \hat{s}_2(\zeta)\,. \end{aligned} \tag{B5}$$

The Fourier transform of s_2 is [13]

$$\hat{s}_2(\zeta) = -i\pi\zeta^2\Theta(\zeta)\,, \tag{B6}$$

and

$$\int_0^\eta d\zeta(-i\pi\zeta^2) = -\frac{i\pi}{3}\eta^3\Theta(\eta)\,. \tag{B7}$$

From Eq. (B5) we have

$$B_2 = -\frac{i}{6\pi}\int_0^\infty d\eta\,\hat{G}_2(\eta)\eta^3\,. \tag{B8}$$

Using $\widehat{f'}(\xi) = -i\xi\hat{f}(\xi)$, we get

$$B_2 = \frac{1}{6\pi}\int_0^\infty d\eta\,\widehat{G_2'''}(\eta)\,. \tag{B9}$$

The function G_2 is odd but with three derivatives it becomes even, so we can extend the intergal

$$B_2 = \frac{1}{12\pi}\int_{-\infty}^\infty d\eta\,\widehat{G_2'''}(\eta) = \frac{1}{6}G_2'''(0)\,. \tag{B10}$$

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