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Multi-step Fermi normal coordinates

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Abstract

We generalize the concept of Fermi normal coordinates adapted to a geodesic to the case where the tangent space to the manifold at the base point is decomposed into a direct product of an arbitrary number of subspaces, so that we follow several geodesics in turn to find the point with given coordinates. We compute the connection and the metric as integrals of the Riemann tensor. In the case of one subspace (Riemann normal coordinates) or two subspaces, we recover some results previously found by Nesterov, using somewhat different techniques.

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I. INTRODUCTION

The construction of Riemann normal coordinates is well known. For any point p of a Riemannian or Lorentzian manifold $(\mathcal{M}, g)$ and any vector $\mathbf{V}_p$ at p there exists a maximal geodesic $\gamma_{\mathbf{V}}(\lambda)$ with starting point p and initial direction $\mathbf{V}_p$. We define the exponential map $\exp_p$ that takes a subset of T_p, the tangent space to $\mathcal{M}$ at p, into $\mathcal{M}$, such that $\exp_p(\mathbf{V})$ is the point q a unit parameter distance along the geodesic $\gamma_{\mathbf{V}}$ from p.

We can choose an orthonormal tetrad basis $\{\mathbf{E}_{(\alpha)}\}$ at p and then define the coordinates at q by the relation $q = \exp(x^\alpha \mathbf{E}_{(\alpha)})$. Such coordinates are called Riemann normal coordinates.

The Fermi normal coordinate construction [1] is also well known. We start with a timelike geodesic $\gamma_{\mathbf{K}}(\lambda)$ with tangent vector $\mathbf{K}$ at p. (We will consider only geodesics, not arbitrary timelike curves.) Given any vector $\mathbf{V} \in T_p(\mathcal{M})$, we can write $\mathbf{V} = \mathbf{A} + \mathbf{B}$, where $\mathbf{A}$ is in the direction of $\mathbf{K}$ and $\mathbf{B}$ perpendicular to $\mathbf{K}$. We then let $q = \exp_p(A)$ and define a map Fermi_p such that $\mathrm{Fermi}_p(V) = \exp_q(B)$. That is to say, $\mathrm{Fermi}_p(V)$ is found by first moving unit distance along the geodesic $\gamma_{\mathbf{A}}$ from p to q. We parallel transport B from p to q and then move unit distance along the geodesic whose tangent vector at q is B.

We can define an orthonormal basis $\{\mathbf{E}_{(\alpha)}\}$ at p such that $\mathbf{E}_{(0)}$ is parallel to $\mathbf{K}$. Then $\mathbf{A} = x^0 \mathbf{E}_{(0)}$, $\mathbf{B} = x^i \mathbf{E}_{(i)}$, giving the usual construction of Fermi normal coordinates [1].

In this paper, we will generalize this construction to allow an arbitrary number of arbitrary subspaces, rather than just a timelike geodesic and the perpendicular space, and an arbitrary number d of dimensions. In Sec. II we will construct the generalized coordinate system, in Sec. III we compute the connection, and in Sec. IV we compute the metric in the generalized Fermi coordinates. We conclude in Sec. V.

We use the sign convention $(+,+,+)$ in the classification of Misner, Thorne and Wheeler [2].

II. MULTI-STEP FERMI COORDINATES

Consider a d-dimensional Riemannian or Lorentzian manifold $(\mathcal{M}, g)$. We will start our construction by choosing a base point $p \in \mathcal{M}$. We decompose the tangent space T_p into n subspaces, $T_p = A_p^{(1)} \times A_p^{(2)} \times A_p^{(3)} \ldots \times A_p^{(n)}$ so that any $\mathbf{V} \in T_p$ can be uniquely written as $\mathbf{V} = \mathbf{V}_{(1)} + \mathbf{V}_{(2)} + \mathbf{V}_{(3)} + \ldots + \mathbf{V}_{(n)}$. We choose, as a basis for T_p, d linearly independent vectors $\{\mathbf{E}_{(\alpha)}\}$ adapted to the decomposition of T_p so that for each $m = 1 \ldots n$, $\{\mathbf{E}_{(\alpha)} | \alpha \in c_m\}$ is a basis for $A_p^{(m)}$, where $c_1, c_2, \ldots c_n$ is an ordered partition of $\{1 \ldots d\}$. Thus each $V_{(m)} = \sum_{\alpha \in c_m} x^\alpha \mathbf{E}_{(\alpha)}$. The vectors $\{\mathbf{E}_{(\alpha)}\}$ need not be normalized or orthogonal.

The point corresponding to coordinates x^a is then found by starting from p and going along the geodesic whose whose tangent vector is $\mathbf{V}_{(1)}$, parallel transporting the rest of the vectors, then along the geodesic whose tangent vector is $\mathbf{V}_{(2)}$, and so on. An example is shown in Fig. 1. With that general construction of multi-step Fermi coordinates we can define a general Fermi mapping $q = \mathrm{Fermi}_p(\mathbf{V})$ given by

$$\begin{aligned} q_{(0)} &= p \\ q_{(1)} &= \exp_p(\mathbf{V}_{(1)}) \\ q_{(2)} &= \exp_{q_{(1)}}(\mathbf{V}_{(2)}) \\ &\ldots \\ q = q_{(n)} &= \exp_{q_{(n-1)}}(\mathbf{V}_{(n)}) \end{aligned} \tag{1}$$

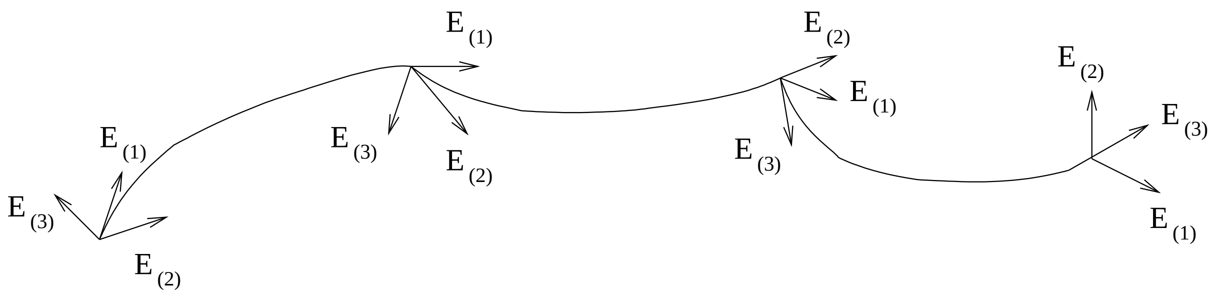

FIG. 1: Construction of 3-step Fermi coordinates in 3 dimensions. We travel first in the direction of $\mathbf{E}_{(1)}$, then $\mathbf{E}_{(2)}$, then $\mathbf{E}_{(3)}$, parallel transporting the triad as we go.

From that general construction we can return to the original Fermi case by choosing $c_1 = \{t\}$ and $c_2 = \{x, y, z\}$. In the Lorentzian case, we could also choose a pseudo-orthonormal tetrad $\mathbf{E}_u, \mathbf{E}_v, \mathbf{E}_x, \mathbf{E}_y$, with $\mathbf{E}_u$ and $\mathbf{E}_v$ null, $\mathbf{E}_u \cdot \mathbf{E}_v = -1$, and other inner products vanishing, and $c_0 = \{u\}$ and $c_1 = \{v, x, y\}$.

For later use we will define

$$\mathbf{V}_{(\leq m)} = \sum_{\alpha \in c_1 \cup c_2 \cup \cdots \cup c_m} x^\alpha \mathbf{E}_{(\alpha)} = \sum_{l=1}^{m} \mathbf{V}_{(l)} \tag{2}$$

$$\mathbf{V}_{(<m)} = \mathbf{V}_{(\leq (m-1))} = \mathbf{V}_{(\leq m))} - \mathbf{V}_{(m)} \tag{3}$$

Then we can write $q_{(m)} = \mathrm{Fermi}_p(\mathbf{V}_{(\leq m)})$.

An example of a spacetime where these multi-step Fermi coordinates might be used comes from brane-world models. A general brane-world metric with one extra dimension is

$$ds^2 = b(w)[-dt^2 + a(t)(dx^2 + dy^2 + dz^2)] + dw^2. \tag{4}$$

A simpler metric of that form is used, for example, in [3]. In this kind of spacetime it might be useful to introduce three-step Fermi coordinates with: $c_1 = \{w\}$, $c_2 = \{t\}$ and $c_3 = \{x, y, z\}$.

III. CONNECTION

We will parallel transport our orthonormal basis vectors $\mathbf{E}_{(\alpha)}$ along the geodesics that generate the coordinates, and use them as a basis for vectors and tensors throughout the region of $\mathcal{M}$ covered by our coordinates. Components in this basis will be denoted by Greek indices. We will use Latin letters from the beginning of the alphabet to denote indices in the Fermi coordinate basis. Of course at p, there is no difference between these bases.

Latin letters from the middle of the alphabet will denote the subspaces of T_p or equivalently the steps of the Fermi mapping process.

We would like to calculate the covariant derivatives of the basis vectors, $\nabla_\beta \mathbf{E}_{(\alpha)}$. Having done so, we can calculate the covariant derivative of any vector field $\mathbf{V} = V^\beta \mathbf{E}_{(\beta)}$ along a curve $f(\lambda)$ as

$$\frac{DV^\beta}{d\lambda} = \frac{dV^\beta}{d\lambda} + V^\gamma \left(\frac{\partial}{\partial \lambda}\right)^\alpha \nabla_\alpha E^\beta_{(\gamma)} \tag{5}$$

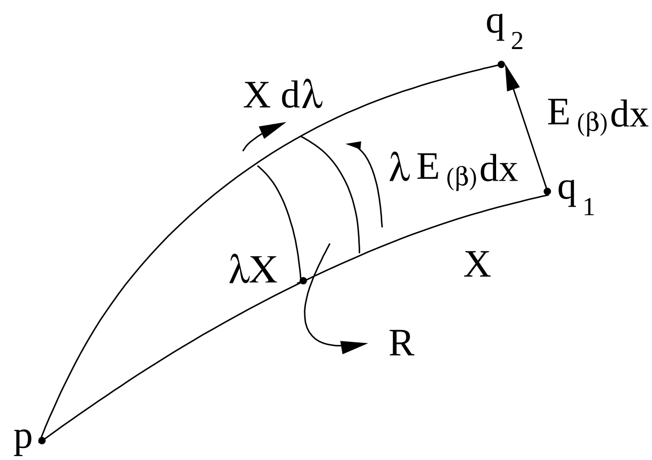

FIG. 2: The covariant derivative $\nabla_\beta \mathbf{E}_{(\alpha)}$ is the change in $\mathbf{E}_{(\alpha)}$ under parallel transport along the path $q_1 \to p \to q_2 \to q_1$, divided by dx. The parallel transport can be decomposed into a series of transports clockwise around trapezoidal regions with sides $\lambda \mathbf{E}_{(\beta)} dx$ and $\mathbf{X} d\lambda$.

To evaluate $\nabla_\beta \mathbf{E}_{(\alpha)}$ at some point $q_1 = \exp_p(\mathbf{X})$, consider an infinitesimally separated point $q_2 = \exp_p(\mathbf{X} + \mathbf{E}_{(\beta)} dx)$. The covariant derivative of $\mathbf{E}_{(\alpha)}$ at q_1 is the difference between $\mathbf{E}_{(\alpha)}(q_2)$ parallel transported to q_1 and the actual $\mathbf{E}_{(\alpha)}(q_1)$, divided by dx. That difference is the same as the change in $\mathbf{E}_{(\alpha)}$ by parallel transport around a loop following the geodesics from q_1 backward to p, the infinitesimally different geodesics forward from p to q_2, and the infinitesimal distance back to q_1. We can write this loop parallel transport as an integral over the Riemann tensor.

Let us first consider the Riemannian case, as shown in Fig. 2. The total parallel transport can be written as the sum of parallel transport around a succession of small trapezoidal regions whose sides are $\lambda \mathbf{E}_{(\beta)}$ and $\mathbf{X} d\lambda$. By using the definition of the Riemann tensor we have

$$\nabla_\beta E^\gamma_{(\alpha)} = \int_0^1 d\lambda \, R^\gamma{}_{\alpha\delta\beta}(\lambda \mathbf{X}) \lambda X^\delta \,. \tag{6}$$

Here R is evaluated at the point $\exp_p(\lambda \mathbf{X})$, which we have denoted merely $\lambda \mathbf{X}$ for compactness.

Equation (6) reproduces Eq. (13) of Ref. [4]. Note, however, that Eq. (6) is exact and does not require R to be smooth, whereas that of Ref. [4] was given as first order in R and was derived by means of a Taylor series.

We see immediately that the covariant derivative of any $\mathbf{E}_{(\alpha)}$ at $\mathbf{X}$ in the direction of $\mathbf{X}$ vanishes. This happens simply because changes with $d\mathbf{X}$ in the direction of $\mathbf{X}$ correspond to additional parallel transport of $\mathbf{E}_{(\alpha)}$.

Let us now consider the general case where there are n steps, and compute $\nabla_\beta \mathbf{E}_{(\alpha)}$. Since the coordinates are adapted to our construction, the index β must be in some specific set c_m, which is to say that the direction of the covariant derivative, $\mathbf{E}_{(\beta)}$, is part of step m in the Fermi coordinate process. We will write the function that gives that m as $m(\beta)$. Some particular cases are shown in Fig. 3.

If $m = n$ (leftmost in Fig. 3), only the last step is modified. The integration is exactly

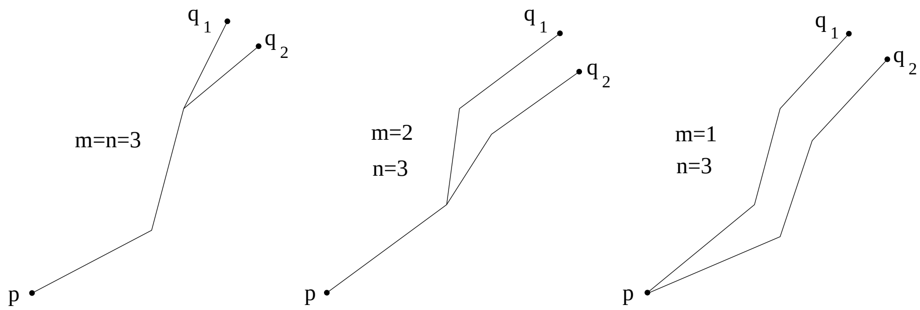

FIG. 3: Original and displaced geodesics for Fermi coordinates with $n = 3$ and $m = 3$, 2, and 1.

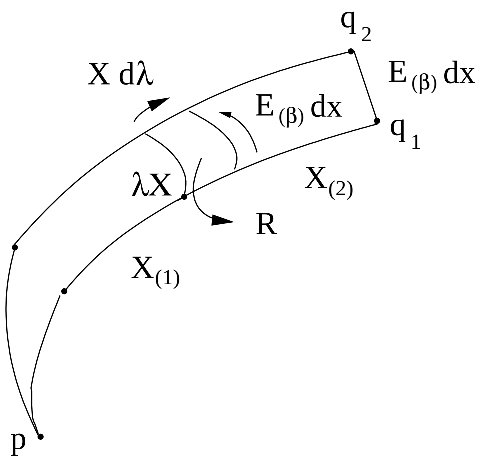

FIG. 4: Part of the calculation of $\nabla_\beta E_{(\alpha)}$ in the case $n = 2$, $m(\beta) = 1$. The geodesic of the first step has been modified, causing the geodesic in the second step to be displaced. The parallel transport integrates the Riemann tensor over a series of rectangular regions between the 2 second-step geodesics.

as shown in Fig. 2, except that it covers only the final geodesic from $\mathbf{X}_{(<n)}$ to $\mathbf{X}_{(n)}$,

$$\nabla_\beta E^\gamma_{(\alpha)} = \int_0^1 d\lambda \, R^\gamma{}_{\alpha\delta\beta}(\mathbf{X}_{(<n)} + \lambda \mathbf{X}_{(n)}) \lambda X^\delta_{(n)}. \tag{7}$$

If $m < n$, then we are modifying some intermediate step, and the path followed at later steps is displaced parallel to itself. In that case we get an integral over rectangular rather than trapezoidal regions, as shown in Fig. 4. For general m there is a contribution for each step $j \geq m$. The $j = m$ contribution integrates over trapezoids that grow with λ, while the $j > m$ contributions integrate over rectangles with fixed width dx. We can write the complete result

$$\nabla_\beta E^\gamma_{(\alpha)} = \sum_{j=m}^{n} \int_0^1 d\lambda \, a_{jm}(\lambda) R^\gamma{}_{\alpha\delta\beta}(\mathbf{X}_{(<j)} + \lambda \mathbf{X}_{(j)}) X^\delta_{(j)} \tag{8}$$

where $m = m(\beta)$ and

$$a_{jm}(\lambda) = \begin{cases} 1 & j \neq m \\ \lambda & j = m \, . \end{cases} \tag{9}$$

Equation (8) is exact and includes Eqs. (6,7) as special cases.

Consider the case where c_1 consists only of one index. If $m > 1$, there is no $j = 1$ term in Eq. (8). If $m = 1$, then β is the single index in c_1, and $X^\delta_{(1)} = 0$ unless $\delta = \beta$, so the $j = 1$ term in vanishes because $R_{\alpha\gamma\delta\beta}$ is antisymmetric under $\delta \leftrightarrow \beta$. Thus there is never a $j = 1$ contribution to Eq. (8) when there is only one index in c_1.

Now suppose X lies on the first generating geodesic, so $X_{(j)} = 0$ for $j > 1$. Then all $j > 1$ terms vanish in Eq. (8). So if c_1 consists only of one index, all Christoffel symbols vanish at X. This is well known in the case of the usual Fermi coordinates.

IV. METRIC

Now we would like to compute the metric g at some point $\mathbf{X}$. Specifically, we would like to compute the metric component g_{ab} in our generalized Fermi coordinates.

We will start by considering the vectors $\mathbf{Z}_{(a)} = \partial/\partial x^a$. These are the basis vectors of the Fermi coordinate basis for the tangent space, so the metric is given by $g_{ab} = \mathbf{Z}_{(a)} \cdot \mathbf{Z}_{(b)}$. Thus if we compute the orthonormal basis components $Z^\alpha_{(a)}$ we can write $g_{ab} = \eta_{\alpha\beta} Z^\alpha_{(a)} Z^\beta_{(b)}$.

Again we will start with the case of Riemann normal coordinates. Let $W(t, s)$ be the point $\exp_p s(\mathbf{X} + t\mathbf{E}_{(a)})$. Define $\mathbf{Y} = \partial W/\partial t$ and $\mathbf{V} = \partial W/\partial s$. Then $\mathbf{Y}(X) = \mathbf{Z}_{(a)}$ and $V^\beta = X^\beta + t\delta^\beta_a$. The components of $\mathbf{Z}_{(a)}$ at $\mathbf{X}$ can be calculated by integration,

$$Z^\beta_{(a)}(\mathbf{X}) = Y^\beta(\mathbf{X}) = \int_0^1 ds \frac{\partial Y^\beta(s\mathbf{X})}{\partial s}. \tag{10}$$

Because the orthonormal basis is parallel transported we can write

$$\frac{d}{ds} Y^\beta = \frac{DY^\beta}{ds}. \tag{11}$$

By construction, the Lie derivative $L_\mathbf{V}\mathbf{Y} = 0$ and thus [5, Ch. 4]

$$\frac{D\mathbf{Y}}{ds} = \frac{D\mathbf{V}}{dt} \tag{12}$$

From Eq. (5) we have

$$\frac{DV^\beta}{dt} = \frac{dV^\beta}{dt} + V^\gamma Y^\alpha \nabla_\alpha E^\beta_{(\gamma)} = \delta^\beta_a + s\delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2). \tag{13}$$

where we have retained δ^α_a instead of writing $\nabla_a E^\beta_{(\gamma)}$ to make it clear that the covariant derivative is with respect to the orthonormal basis.

From Eq. (6) we have

$$\nabla_\alpha E^\beta_{(\gamma)}(s\mathbf{X}) = \int_0^1 d\lambda\, \lambda R^\beta{}_{\gamma\delta\alpha}(\lambda s\mathbf{X}) s X^\delta = \frac{1}{s}\int_0^s d\lambda\, \lambda R^\beta{}_{\gamma\delta\alpha}(\lambda\mathbf{X}) X^\delta \tag{14}$$

Taking $t = 0$, $\mathbf{V}$ is just $\mathbf{X}$. Combining Eqs. (10-14), we find

$$\begin{aligned} Z^\beta_{(a)}(\mathbf{X}) &= \int_0^1 ds \left[\delta^\beta_a + \delta^\alpha_a \int_0^s d\lambda\, \lambda R^\beta{}_{\gamma\delta\alpha}(\lambda\mathbf{X}) X^\delta X^\gamma\right] + O(R^2) \\ &= \delta^\beta_a + \delta^\alpha_a \int_0^1 d\lambda\, \lambda(1-\lambda) R^\beta{}_{\gamma\delta\alpha}(\lambda\mathbf{X}) X^\delta X^\gamma + O(R^2). \end{aligned} \tag{15}$$

From Eq. (15), the metric is given by

$$g_{ab} = \eta_{ab} + 2\delta^\alpha_a \delta^\beta_b \int_0^1 d\lambda\, \lambda(1-\lambda) R_{\alpha\gamma\delta\beta}(\lambda \mathbf{X}) X^\delta X^\gamma + O(R^2). \tag{16}$$

Equation (16) reproduces Eq. (14) of Ref. [4][1].

Next let us consider the case where there are n steps in our procedure. We will define a set of functions W_j as

$$W_j(s) = \mathrm{Fermi}_p(\mathbf{X}_{(<j)} + s\mathbf{X}_{(j)}). \tag{17}$$

The path $W_j(s), j = 1 \ldots n, s = 0 \ldots 1$ traces the geodesics generating the Fermi coordinates for the point X. Now consider $\mathbf{Z}_{(a)} = \partial/\partial x^a$. Let $m = m(a)$, so $\mathbf{Z}_{(a)}(p) \in A_p^{(m)}$. Then let

$$W_j(s,t) = \mathrm{Fermi}_p \begin{cases} s\mathbf{X}_{(j)} & j < m \\ \mathbf{X}_{(<j)} + s(\mathbf{X}_{(j)} + t\mathbf{E}_{(a)}) & j = m \\ \mathbf{X}_{(<j)} + t\mathbf{E}_{(a)} + s\mathbf{X}_{(j)} & j > m \end{cases} \tag{18}$$

Let $\mathbf{Y} = \partial W/\partial t$ and $\mathbf{V} = \partial W/\partial s$ as before. To find $\mathbf{Z}_{(a)}$ we now must integrate over a multi-step path from p,

$$Z^\beta_{(a)}(\mathbf{X}) = \sum_{j=1}^n \int_0^1 ds \frac{\partial Y^\beta(W_j(s))}{\partial s}. \tag{19}$$

The generalized version of Eq. (13) is

$$\frac{DV^\beta(W_j(s,t))}{dt} = \frac{dV^\beta}{dt} + V^\gamma Y^\alpha \nabla_\alpha E^\beta_{(\gamma)} = \begin{cases} 0 & j < m \\ \delta^\beta_a + s\delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2) & j = m \\ \delta^\alpha_a V^\gamma \nabla_\alpha E^\beta_{(\gamma)} + O(R^2) & j > m. \end{cases} \tag{20}$$

Now

$$\nabla_\alpha E^\beta_{(\gamma)}(W_j(s)) = \sum_{k=m}^{j} \frac{1}{s_{kj}(s)} \int_0^{s_{kj}(s)} d\lambda\, a_{km}(\lambda) R^\beta{}_{\gamma\delta\alpha}(\mathbf{X}_{(<k)} + \lambda\mathbf{X}_{(k)}) X^\delta_{(k)} \tag{21}$$

where

$$s_{kj}(s) = \begin{cases} 1 & k \neq j \\ s & k = j. \end{cases} \tag{22}$$

The $k = j$ term is analogous to Eq. (14), while the others have no dependence on s.

Combining Eqs. (11,12,19–21) we get

$$Z^\beta_{(a)}(\mathbf{X}) = \delta^\beta_a + F^\beta_a + O(R^2) \tag{23}$$

where

$$\begin{aligned} F^\beta_\alpha &= \sum_{j=m}^n \sum_{k=m}^j \int_0^1 ds \int_0^{s_{kj}(s)} d\lambda\, a_{km}(\lambda) R^\beta{}_{\gamma\delta\alpha}(\mathbf{X}_{(<k)} + \lambda\mathbf{X}_{(k)}) X^\delta_{(k)} X^\gamma_{(j)} \\ &= \sum_{j=m}^n \sum_{k=m}^j \int_0^1 d\lambda\, a_{km}(\lambda) b_{kj}(\lambda) R^\beta{}_{\gamma\delta\alpha}(\mathbf{X}_{(<k)} + \lambda\mathbf{X}_{(k)}) X^\delta_{(k)} X^\gamma_{(j)} \end{aligned} \tag{24}$$

[1] Ref. [4] uses the same sign convention for $R^\alpha{}_{\beta\gamma\delta}$ as the present paper, but the opposite convention for g_{ab} and consequently also for $R_{\alpha\beta\gamma\delta}$.

where $m = m(\alpha)$ and

$$b_{kj}(\lambda) = \begin{cases} 1 & k \neq j \\ 1-\lambda & k = j \end{cases} \tag{25}$$

So the metric is

$$g_{ab} = \eta_{\alpha\beta} Z^{\alpha}_{(a)} Z^{\beta}_{(b)} = \eta_{ab} + F_{ab} + F_{ba} + \mathcal{O}(R^2) \tag{26}$$

where

$$F_{\alpha\beta} = \sum_{j=m}^{n} \sum_{k=m}^{j} \int_0^1 d\lambda\, a_{km}(\lambda) b_{kj}(\lambda) R_{\alpha\gamma\delta\beta}(\mathbf{X}_{(>k)} + \lambda \mathbf{X}_{(k)}) X^{\delta}_{(k)} X^{\gamma}_{(j)} \tag{27}$$

where $m = m(\beta)$.

Once again consider the case where c_1 contains only one index. As discussed with respect to Eq. (8), if $\beta \in c_1$, there is no nonvanishing $k = 1$ term in Eq. (27). Thus $g_{ab} = \eta_{ab}$ at points on the first generating geodesic. This is also well known in the usual Fermi case.

Now suppose c_1 consists only of one index and furthermore $n = 2$. The only possible term in Eq. (27) is then $j = k = 2$, so

$$F_{\alpha\beta} = \int_0^1 d\lambda\, a_{2m}(\lambda)(1-\lambda) R_{\alpha\gamma\delta\beta}(\mathbf{X}_{(1)} + \lambda \mathbf{X}_{(2)}) X^{\delta}_{(2)} X^{\gamma}_{(2)}. \tag{28}$$

where $m = m(\beta)$. Equation Eq. (28) is equivalent to Eq. (28) in Ref. [4] in the case where the generating curve of the Fermi coordinates is a geodesic.

Now we are in a position to discuss the region of the manifold over which the multi-step Fermi coordinates are well defined. Assuming there are no singularities or edges in the manifold, the only problem would be if the metric g_{ab} is degenerate, which in turn can happen only if F_{ab} is of order 1. Thus the Fermi coordinates will be well defined providing that [4]

$$|R_{\alpha\gamma\beta\delta}| (X^{\epsilon})^2 \ll 1 \tag{29}$$

throughout the region of interest, for all $\alpha, \gamma, \beta, \delta, \epsilon$.

In the case where there is only one index in c_1, there is no contribution to F_{ab} from $X_{(1)}$. Then it is sufficient for Eq. (29) to hold for $\epsilon > 1$. In other words, if the first step is one-dimensional, it can be arbitrarily long [1].

V. CONCLUSION

We have generalized the usual Fermi normal coordinates in the case where the generating curve is a geodesic to allow for any number of steps and for a subspace of any dimension at each step. We have derived the connection (exactly) and the metric (to first order in the curvature) as integrals over the Riemann tensor. Our results reproduce several formulas previously derived by Nesterov [4] with a more geometric approach in a more general setting, and without reference to any derivative of R.

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