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1 **Non-coaxial soil model with an anisotropic yield criterion**
2 **and its application to the analysis of strip footing problems**

3
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ABSTRACT

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This paper presents numerical applications of a non-coaxial soil model, in which an anisotropic yield criterion is incorporated, to analyze two-dimensional strip-footing problems. Semi-analytical solutions of the bearing capacity for a strip footing that rests on anisotropic, weightless, cohesive-frictional soils are developed based on the slip line method. The degrees of influences of soil anisotropy and non-coaxiality on the bearing capacity of the strip footing are examined. From the viewpoint of strength and stiffness, it is necessary to incorporate both the strength anisotropy and non-coaxiality into numerical simulations and practical designs of geotechnical problems.

KEYWORDS: non-coaxial plasticity, soil anisotropy, numerical simulation, strip footing.

52

1. INTRODUCTION

53 Extensive experimental (e.g., [1-6]) and micromechanics-based (e.g., [7-11]) evidence has
54 demonstrated that non-coaxiality, which refers to the non-coincidence of the principal axes of
55 the stress and plastic strain rate tensors, is an intrinsic characteristic of granular materials.
56 These fundamental insights have guided the development of numerous realistic continuum soil
57 models. Approaches for constitutive modelling can be broadly classified into the
58 phenomenological approach and the multi-scale approach for rate-independent elasto-plastic
59 behaviors of granular materials under a quasi-static loading. The phenomenological approach
60 directly describes the observed phenomena using an approximate and sophisticated
61 mathematical formulation. In recent decades, a number of phenomenological models have been
62 developed that consider the non-coaxial behavior of soils, and examples include the hypo-
63 plastic models [12], the generalized sub-loading surface model [13]; among others ([14-16]).
64 On the other hand, multi-scale approaches have been proposed to describe non-coaxial
65 behavior of soils based on micro-mechanics. The macroscopic mechanical behavior of granular
66 materials is then directly related to the evolution of the internal structure. One popular category
67 within this framework can be classified as elasto-plastic models with fabric tensors (e.g., [17-
68 19]).

69 However, analysis of practical geotechnical problems that consider the non-coaxial plasticity
70 of granular soils is rare. Although phenomenological models have demonstrated their ability to
71 capture many of the most salient features, e.g., dilatancy, soil anisotropy, hardening and strain
72 localization, they often introduce too many parameters without physical meaning and are
73 difficult to be calibrated. Indeed, the mathematical formulations for most of the current models
74 based on phenomenological approaches are complex; hence, it is difficult for those non-coaxial
75 models to be implemented into non-linear numerical codes for the solution of boundary value

76 problems. With respect to the models that use multi-scale approaches, information on the
77 evolution of the internal structure is difficult to define using the laboratory work. These reasons
78 might explain why these non-coaxial constitutive models have not been widely applied to
79 investigate boundary value problems.

80 Many real engineering problems subjected to proportional loading, e.g., tidal waves,
81 earthquakes and footing-penetration, demonstrate obvious principal stress rotations [20-21]. It
82 is accepted that the soil mass underneath a footing, especially in the vicinity of the footing
83 edges, experiences a large amount of stress rotations under loading [22]. Yu and other authors
84 [22, 23] numerically applied non-coaxial constitutive models to investigate shallow
85 foundations. In these researchers' work, the application of non-coaxial models predicted a
86 larger settlement prior to collapse compared with the conventional coaxial models. The
87 conclusions drawn from this study clearly stated that without considering the non-coaxial
88 behavior of soil, a high chance of unsafe design exists in shallow foundations. Nevertheless,
89 work of the above researchers is restricted to soil strength isotropy. The natural characteristic
90 of soils is anisotropic, and recent experimental observations have demonstrated that non-
91 coaxiality is a significant aspect of soil anisotropy (e.g., [4]). As concluded by Tsutsumi and
92 Hashiguchi [24], both the tangent effect (non-coaxiality) and the anisotropy in the yield
93 condition must be incorporated into constitutive equations for a description of the general non-
94 proportional loading behavior of soils. Assuming non-coaxiality in the context of soil isotropy
95 might result in poor predictions of stability and serviceability problems in geotechnical
96 engineering. Hence, it remains a key issue to gain insight into the different aspects that might
97 be introduced into footing problems modeled by non-coaxial plasticity in the context of soil
98 strength anisotropy compared with those that are modeled using coaxial plasticity.

99

100 In this paper, a plane-strain, elastic/perfectly plastic non-coaxial soil model with an anisotropic
101 yield criterion is applied to simulate strip footing problems. The anisotropic yield criterion is
102 generalized from the conventional isotropic Mohr-Coulomb yield criterion to account for the
103 effects of initial strength anisotropy, which is characterized by the variation of internal friction
104 angles (angles of shearing resistance) with the direction of the principal stresses. Based on the
105 slip line method, a semi-analytical solution of the bearing capacity is presented for a strip
106 footing that rests on an anisotropic, weightless, cohesive-frictional soil. Comparison between
107 the numerical predictions and semi-analytical results of the bearing capacity are performed.
108 The influences of degrees of soil anisotropy and non-coaxiality on the bearing capacity of strip
109 footings are also discussed.

110

111 2. A NON-COAXIAL MODEL: DEVELOPMENT AND IMPLEMENTATION

112 The plane strain non-coaxial soil model used in this paper emphasizes on two ingredients: the
113 anisotropic yield function and the non-coaxial plastic flow rule. The signs of the stress (rate)
114 are chosen as positive for compression.

115 2.1 The anisotropic yield criterion

116 Following Booker and Davis [25], the anisotropic yield function in the stress space of
117 $(\frac{\sigma_x - \sigma_y}{2}, \sigma_{xy})$ is a known function of the mean pressure p and the direction of principal stresses
118 Θ . As shown in Fig. 1 and in line with the experimental evidence that the internal friction angle
119 varies with the direction of principal stresses (e.g.,[4]), the yield criterion can be written as
120 follows:

$$121 \quad f(\sigma_x, \sigma_y, \sigma_{xy}) = R + F(p, \Theta) = 0 \quad (1)$$

122 where

123
$$F(p, \Theta) = (p - c \cdot \cot \phi_{\max}) \cdot \sin \phi(\Theta) \quad (2)$$

124
$$\sin \phi(\Theta) = \frac{n \cdot \sin \phi_{\max}}{\sqrt{n^2 \cos^2(2\Theta - 2\beta) + \sin^2(2\Theta - 2\beta)}} \quad (3)$$

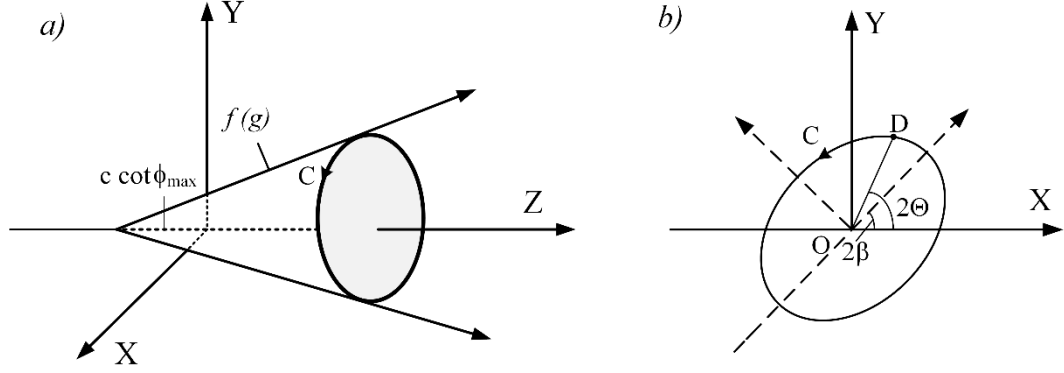
125 and where $R = \frac{1}{2}[(\sigma_x - \sigma_y)^2 + 4\sigma_{xy}^2]^{1/2}$, $p = \frac{1}{2}(\sigma_x + \sigma_y)$, $\tan(2\Theta_p) = 2\sigma_{xy}/(\sigma_x - \sigma_y)$, c denotes

126 cohesion. The expression of Equation (3) is derived by geometric considerations.

127 As indicated in Fig. 1b, the cross-section of the anisotropic yield criterion is assumed to be a
 128 rotational ellipse. The centre of the anisotropic ellipse is assumed to be located at the original
 129 point O , and ϕ_{\max} and ϕ_{\min} are defined as the maximum and minimum peak internal friction
 130 angles, respectively along all possible major principal stress directions. The major and minor
 131 lengths of the ellipse depend on the maximum magnitudes of the peak internal friction angle,
 132 respectively. Two shape parameters n and β , as shown in Equation (3), are added to those
 133 material properties of the conventional isotropic Mohr-Coulomb yield criterion in order to
 134 define the anisotropic yield criterion:

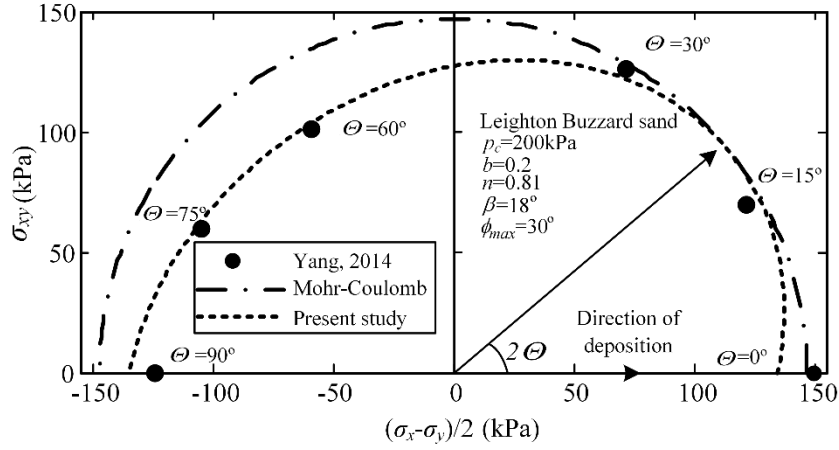
- 135 • $n = \sin \phi_{\min} / \sin \phi_{\max}$, where the range of n is between 0 and 1. In particular, the isotropic Mohr-
 136 Coulomb yield criterion is recovered when $n = 1.0$.
- 137 • β refers to an angle when the major principal stress (corresponding to the case of the
 138 maximum peak internal friction angle) is inclined to the deposition direction; and β ranges
 139 from 0 to $\frac{\pi}{4}$.

140 The two shape parameters can be obtained via tests using the hollow cylinder apparatus (HCA).
 141 Experimental investigations from the laboratory [4] can aid in testing the accuracy of the
 142 proposed anisotropic yield criterion, as illustrated in Fig. 2. The non-dimensional parameter b
 143 is the intermediate stress ratio defined as $b = (\sigma_2 - \sigma_3) / (\sigma_1 - \sigma_3)$. For a plane strain condition, $b \approx 0.2$ -
 144 0.4.



145

146 Fig. 1 Anisotropic yield surface in: (a) $(X = \frac{\sigma_x - \sigma_y}{2}, Y = \sigma_{xy}, Z = \frac{\sigma_x + \sigma_y}{2})$ space; (b) $(X = \frac{\sigma_x - \sigma_y}{2},$
 147 $Y = \sigma_{xy})$ space.



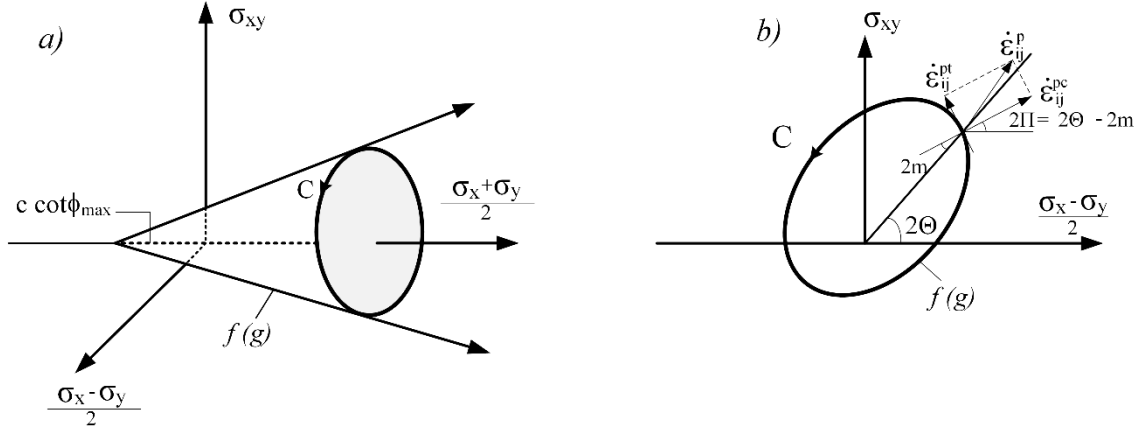
148

149 Fig. 2 Validation of the newly proposed anisotropic yield criterion.

150 **2.2 The non-coaxial plastic flow rule**

151 As indicated in Fig. 3, the general form of the plastic strain rate $\dot{\epsilon}^p$ consists of the conventional
 152 component $\dot{\epsilon}^{pc} = \dot{\lambda} \frac{\partial g}{\partial \sigma}$ and the non-coaxial component $\dot{\epsilon}^{pt} = k \cdot \dot{T}$. The conventional
 153 component is normal to the yield surface derived from the classical plastic potential theory.
 154 The non-coaxial component is tangential to the yield surface induced by the deviatoric stress-
 155 rate component. The general form of the plastic strain rate $\dot{\epsilon}^p$ is shown as follows:

156
$$\dot{\epsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma} + k \cdot \dot{T} \quad \text{if } f = 0 \quad \text{and } \dot{f} = 0 \quad (4)$$



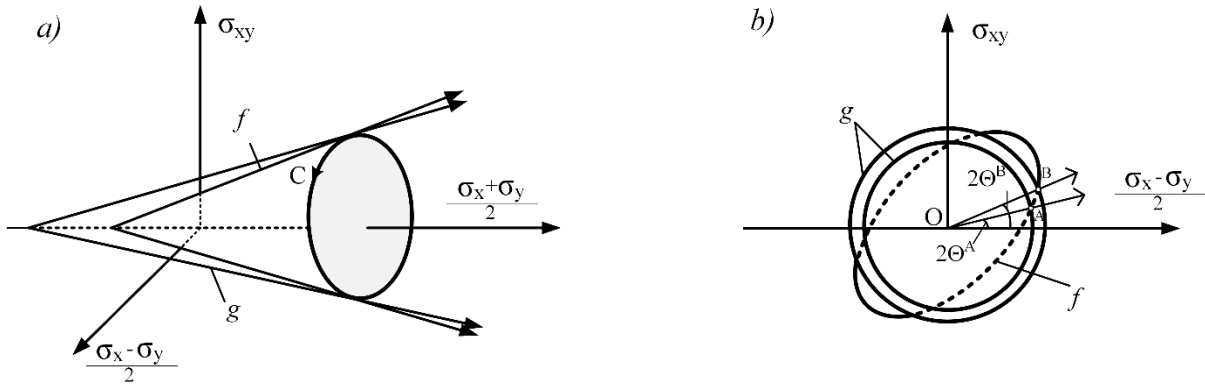
157

158 Fig. 3 Display of the non-coaxial plastic flow rule in: (a) $(X=\frac{\sigma_x-\sigma_y}{2}, Y=\sigma_{xy}, Z=\frac{\sigma_x+\sigma_y}{2})$ space;
 159 (b) $(X=\frac{\sigma_x-\sigma_y}{2}, Y=\sigma_{xy})$ space.

160 where λ denotes a positive scalar, g denotes the plastic potential, f represents the yield surface,
 161 k is a dimensionless scalar (known as the non-coaxial coefficient in this paper), and \dot{T} denotes
 162 the material derivative, which can be displayed in the form of principal stress increments:

163
$$\dot{T} = \frac{1}{k} \cdot N \cdot \sigma \quad (5)$$

164 N is defined in Appendix 1.



165

166 Fig. 4 Illustration of the plastic potential when the non-associativity in the conventional
 167 plastic flow rule is used in the space of: (a) $(X=\frac{\sigma_x-\sigma_y}{2}, Y=\sigma_{xy}, Z=\frac{\sigma_x+\sigma_y}{2})$;
 168 (b) $(X=\frac{\sigma_x-\sigma_y}{2}, Y=\sigma_{xy})$.

169 If $g=f$, then the associativity in the conventional plastic flow rule (abbreviated to asso)
 170 and otherwise, the non-associativity in the conventional plastic flow rule (abbreviated to non-
 171 asso) is used. The plastic potential considers the effect of dilation angle. The dilation angle is

172 assumed to vary with the direction of the principal stresses. As illustrated in Fig. 4, the plastic
 173 potential changes in size corresponding to different stress states (i.e., the plastic potential
 174 surface must pass the current point of the stress state). With this type of assumption, the
 175 conventional component is coaxial with the stress tensor. The form of g is shown with respect
 176 to the non-associativity in the conventional plastic flow rule is written as follows:

$$177 \quad g = R + p \cdot \sin \psi(\Theta) = C \quad (6)$$

178 and

$$179 \quad \sin \psi(\Theta) = \frac{n \cdot \sin \psi_{\max}}{\sqrt{n^2 \cos^2(2\Theta - 2\beta) + \sin^2(2\Theta - 2\beta)}} \quad (7)$$

180 where ψ_{\max} denotes the maximum dilation angle and C denotes a constant.

181 Combining the elastic component in which Hooke's law is used, the general rate equation for
 182 an elasto-plastic relationship can be shown as follows:

$$183 \quad \dot{\sigma} = D^{ep} \dot{\varepsilon} = D^e \left(\dot{\varepsilon} - \dot{\lambda} \frac{\partial g}{\partial \sigma} - N \dot{\sigma} \right) \quad (8)$$

184 where D^{ep} denotes the elasto-plastic stiffness matrix, and D^e denotes the elastic stiffness
 185 matrix. The consistency condition equation for perfect plasticity is written:

$$186 \quad \left(\frac{\partial f}{\partial \sigma} \right)^T \cdot \dot{\sigma} = 0 \quad (9)$$

187 Substituting $\dot{\sigma}$ from Equation (8) into Equation (9), the expression of the scalar multiplier $\dot{\lambda}$
 188 can be obtained as follows:

$$189 \quad \dot{\lambda} = \frac{\overline{D^e} \left(\frac{\partial f}{\partial \sigma} \right)^T \dot{\varepsilon}}{\left(\frac{\partial f}{\partial \sigma} \right)^T \overline{D^e} \frac{\partial g}{\partial \sigma}} \quad (10)$$

190 in which a modified elastic stiffness matrix $\overline{D^e}$ is introduced as follows:

$$191 \quad \overline{D^e} = (I + D^e N)^{-1} D^e \quad (11)$$

192 where I is the identity tensor.

193 The non-coaxial elasto-plastic stress-strain stiffness matrix is shown as follows:

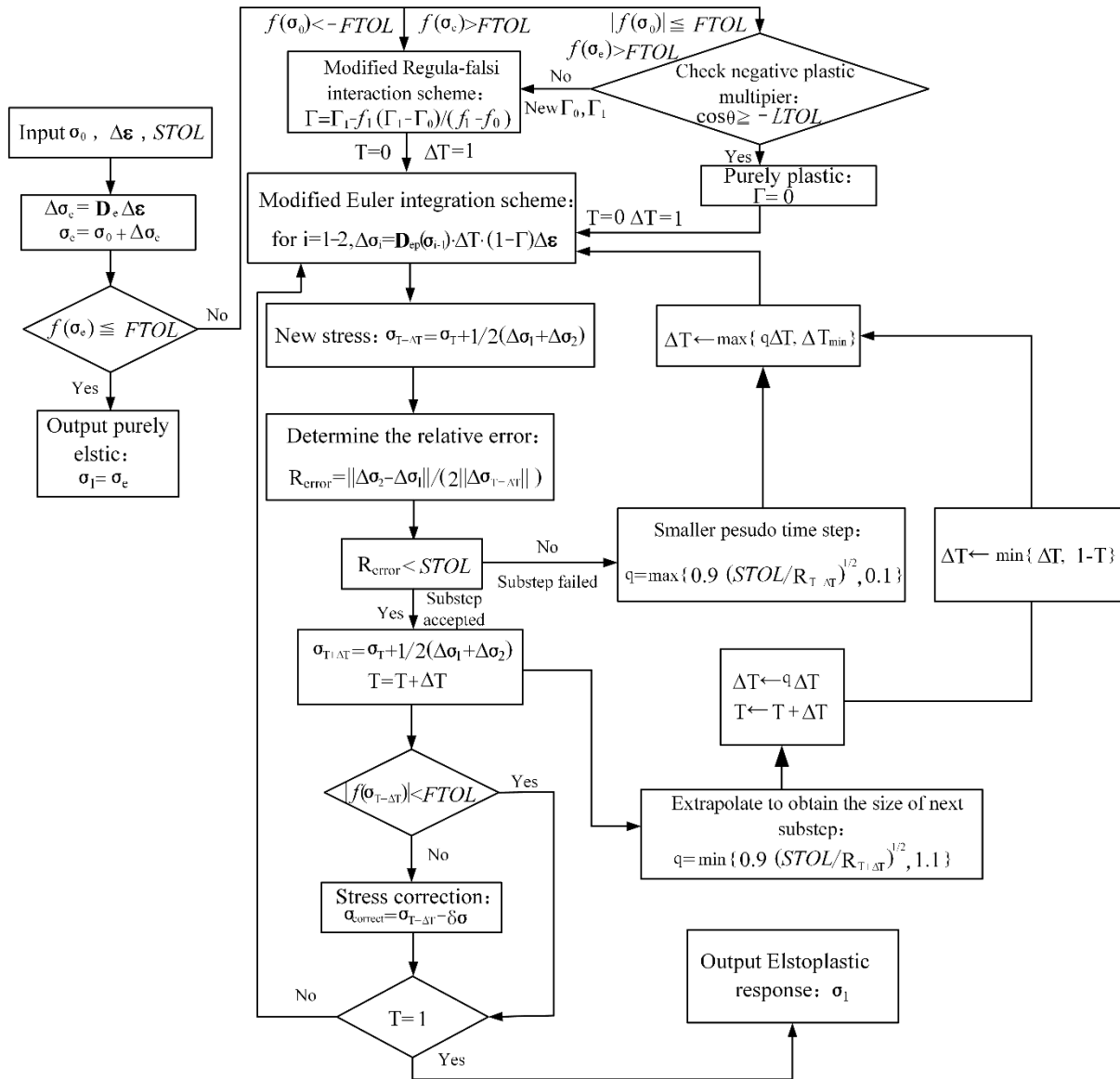
194
$$D^{ep} = \frac{\overline{D^e \frac{\partial g}{\partial \sigma}} \left(\frac{\partial f}{\partial \sigma} \right)^T \overline{D^e}}{\left(\frac{\partial f}{\partial \sigma} \right)^T \overline{D^e \frac{\partial g}{\partial \sigma}}} \quad (12)$$

195 **2.3 Implementation of the proposed model**

196 The developed non-coaxial soil model was implemented in the ABAQUS finite element code
 197 via the user-defined material subroutine (UMAT). A hyperbolic approximation at the tip of the
 198 yield surface is used to eliminate singularity in which the anisotropic yield criterion is modified
 199 as follows [26]:

200
$$f(\sigma_x, \sigma_y, \sigma_{xy}) = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} + a^2 \sin^2 \phi(\Theta) - (p + c \cdot \cot \phi_{\max}) \cdot \sin \phi(\Theta) \quad (13)$$

201 The original anisotropic yield function, i.e., Equations (1) - (3), is recovered if a is set to zero.
 202 As suggested by Abbo [26], the hyperbolic surface closely represents the anisotropic Mohr-
 203 Coulomb yield criterion when $a \leq 0.25c \cdot \cot \phi$. The explicit integration algorithm (an explicit
 204 forward Euler/modified Euler pair) with automatic error controls that returns the stresses to the
 205 yield surface during the integration process is used to perform the numerical implementation
 206 [26]. The modified regula-falsi is used to solve the yield surface intersection problem. The
 207 flowchart for the implementation is displayed in Fig. 5.



208

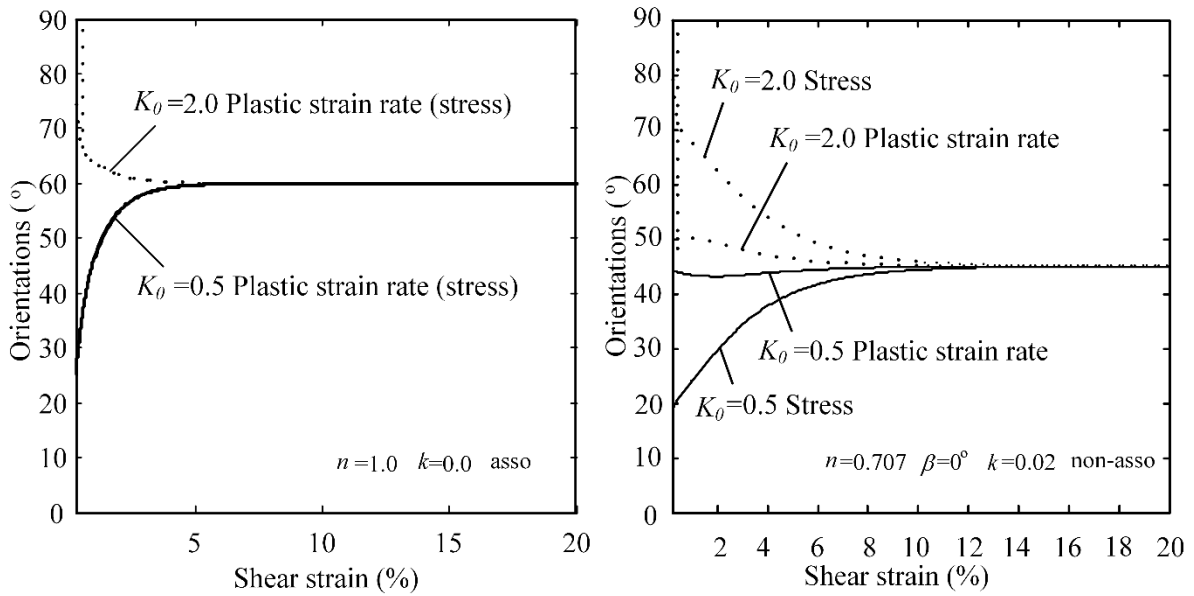
209

Fig. 5 Flowchart of the integration scheme

210 Fig. 6 shows the orientations of the principal stress and plastic strain rate in simple shear tests

211 obtained using the newly proposed non-coaxial soil model. Obviously, the results ideally

212 reproduce non-coaxial behaviors of the principal stress and the principal plastic strain rate.



213

214 Fig. 6 Numerical simulation of simple shear problems in the condition of: (a) associativity
 215 and coaxiality; (2) non-associativity and non-coaxiality.

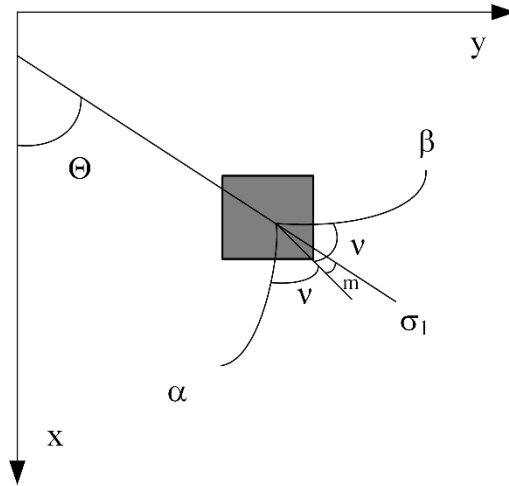
216 Particular attention should be focused on those cases in which severe non-coaxiality or non-
 217 associativity is used in the conventional plastic flow rule. For these situations, negative
 218 eigenvalues might be obtained in the solution of the global finite-element equations. For
 219 example, this scenario is especially prevalent for footing problems in which severe
 220 discontinuity of the stress field occurs in the vicinity of footing corners. Thus, to relax non-
 221 convergence problems in ABAQUS in these situations, the default force residual tolerance
 222 $R_n=0.005$ and the default displacement correction tolerance $C_n=0.01$ are adjusted to larger
 223 numbers (e.g., $R_n=0.01$ and $C_n=0.05$), which might reduce accuracy but within a tolerable range.

224 It should be noted that many findings in the literature have stated that the direction of the major
 225 principal stresses with respect to the x -axis lies on the interval $(0, \pi/2)$ [4, 17]. Consequently,
 226 the anisotropic coefficient β should range within $(0, \pi/4)$. In line with the previous experimental
 227 outcomes and to reduce parametric work, β is chosen as 0 , 22.5° and 45° for discussion in this

228 paper. Following the previous analyses [23], the non-coaxial coefficient k is chosen as 0, 0.02,
 229 and 0.1, to evaluate the effects of non-coaxial plasticity.

230 3. A SEMI-ANALYTICAL SOLUTION: ANISOTROPIC SOIL MASS

231 It is necessary to validate the numerical results with theoretical solutions to ascertain usability
 232 in practical, large-scale applications. To achieve this goal, semi-analytical solutions of the
 233 bearing capacity for a smooth strip footing resting on an anisotropic soil mass are developed
 234 based on the slip line method. For simplicity, a cohesive-frictional, weightless soil is
 235 considered for all analyses. Equations are presented in terms of stress fields, which must be
 236 satisfied in the plastic region of a rigid plastic body, and the magnitude of elastic strains is
 237 disregarded. The rigid plastic body is modeled using the anisotropic Mohr-Coulomb failure
 238 criterion, as shown in Section 2.1. The stress conditions on the boundary are illustrated in Fig.
 239 7, where two families of characteristics can be introduced as (α, β) lines ([25, 27]):



240
 241 Fig. 7 Stress coordinate system and stress characteristics for anisotropic plasticity

242

$$243 \quad \frac{dy}{dx} = \tan(\xi_\alpha) = \tan(\Theta - m - \nu) \quad (14)$$

$$244 \quad \frac{dy}{dx} = \tan(\xi_\beta) = \tan(\Theta - m + \nu) \quad (15)$$

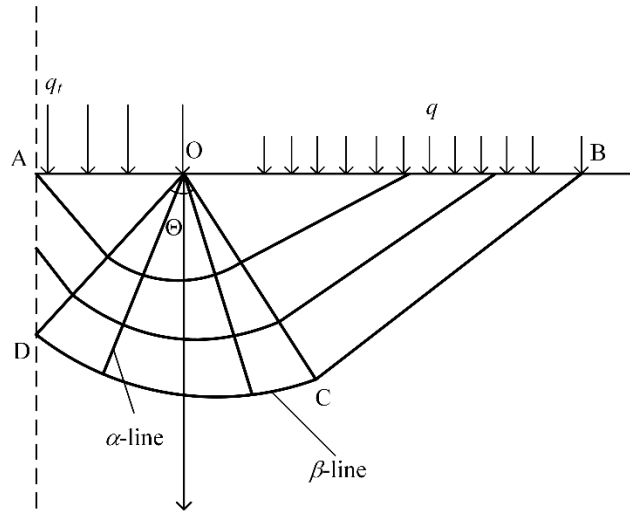
245 And,

246
$$\tan(2m) = \frac{1}{2F} \frac{\partial F}{\partial \Theta} \quad (16)$$

247
$$\cos(2\nu) = \cos(2m) \frac{\partial F}{\partial p} \quad (17)$$

248 where the variable m has a simple geometric interpretation and is introduced purely to ensure
 249 simplicity of the mathematics involved, and F is a function of p and Θ as shown in Equation
 250 (2).

251 The slip line method is illustrated in Fig. 8, where only a symmetrical footing problem is
 252 present. In this figure, AO is the half length of the strip footing, and a surface surcharge of q is
 253 applied on OB . Based on the corollary of Hencky's theory, all α -lines in this field must be
 254 straight lines, and all of these lines must pass through the edge point of the footing at O . The
 255 family of straight α -lines are the characteristics within the region COD that demonstrate an
 256 angle of Θ . By combining the equilibrium equations, if the stresses on the α -lines are integrated
 257 along the β -lines, the solution of vertical pressure at plastic collapse can be stated as follows:



258
 259 Fig. 8 Plastic stress field of strip footing with surcharge on OB .

260
 261
$$q_t = \left(1 + \sqrt{\frac{2}{M}} n \sin \phi_{\max}\right) \left(e^{\int_0^{\pi/2} G(\Theta) d\Theta} \cdot \frac{q\sqrt{M} + \sqrt{M} c \cot \phi_{\max}}{\sqrt{M} - \sqrt{2} n \sin \phi_{\max}}\right) - c \cot \phi_{\max} \quad (18)$$

262 where $G(\theta)$ and the detailed derivation are given in the Appendix 2, and $M =$
 263 $2[(1 - n^2) \sin^2(2\beta) + n^2]$. n and β are shape parameters, as illustrated in Section 2.1.

264

265 The above solution can be further expressed in terms of contributions from the cohesion (c)
 266 and surcharge (q) as follows:

$$267 \quad q_t = N_c c + N_q q \quad (19)$$

268 where

$$269 \quad N_c = (N_q - 1) \cdot \cot \phi_{\max} \quad (20)$$

270 and

$$271 \quad N_q = e^{\int_0^{\pi/2} G(\theta) d\theta} \cdot \left(1 + \sqrt{\frac{2}{M}} n \sin \phi_{\max}\right) \cdot \frac{\sqrt{M}}{\sqrt{M} - \sqrt{2} n \sin \phi_{\max}} \quad (21)$$

272 In a special case in which a smooth strip footing rests on a purely cohesive soil mass without
 273 surface surcharge and the yield criterion is independent of hydrostatic pressure, i.e. $\nu = \frac{\pi}{4}$, the
 274 β -lines are circles. In this case, the yield surface is a cylinder generated by straight lines parallel
 275 to the line corresponding to $\sigma_x = \sigma_y, \tau_{xy} = 0$. The solution becomes much simpler and can
 276 readily be obtained analytically as follows:

$$277 \quad q_t = \pi n c + 2(1 - n)c + 2nc \sqrt{\frac{2}{M}} \quad (22)$$

278 In addition, for a special case of the Tresca model with $\phi = 0^\circ$, the solution can be expressed
 279 in the following well-known form:

$$280 \quad q_t = (2 + \pi) \cdot c \quad (23)$$

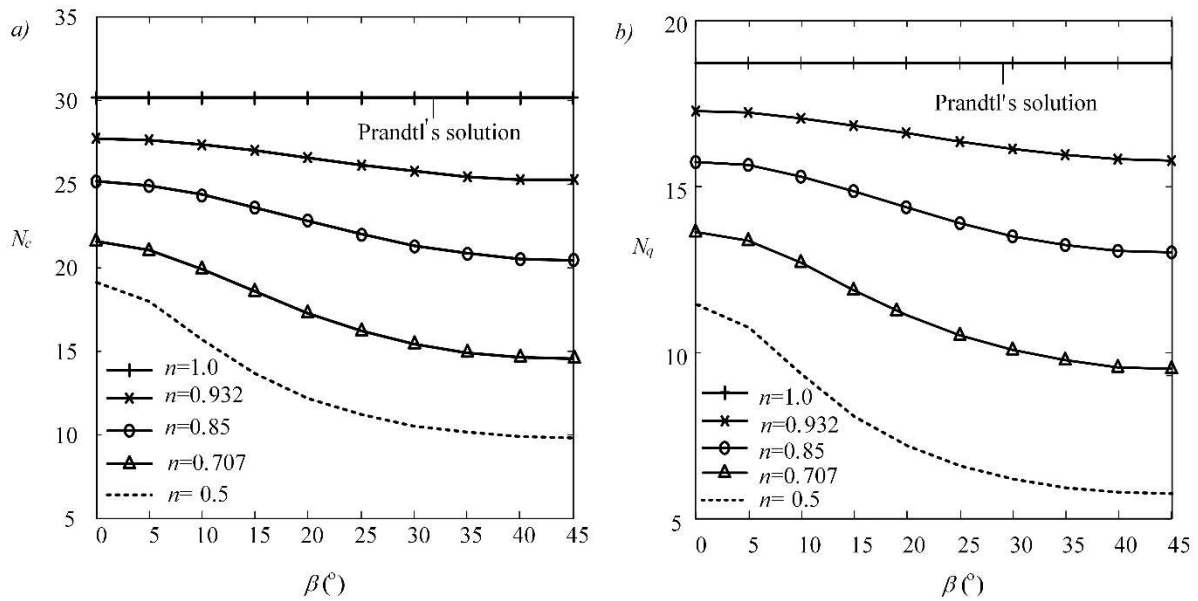
281 which is consistent with Equation (22) when $n=1.0$.

282 As noted by Bishop [28], the stresses in the plastic stress solution have only been demonstrated
 283 to satisfy the yield condition and equilibrium equations in the plastic zone, and these stresses
 284 are referred to as a partial stress field or incomplete solutions. Such incomplete solutions are

285 known as an upper bound solution (as developed in this paper). However, the bearing capacity
286 obtained from the upper bound solution is quite similar to that of the exact solution. This
287 solution has been generally applied to analyze current footing problems (e.g., [28]). In addition,
288 the solutions proposed in this paper assume that an associated flow rule is valid. In this case,
289 the stress and velocity characteristics are coincident such that the determination of a velocity
290 field is not essential, which is the reason why the velocity field is not discussed in this paper.

291

292 In Equation (21), the integration of $G(\theta)$, which is shown in the Appendix 2, is numerically
293 performed in Matlab. A parametric study on the semi-analytical solution is conducted to
294 investigate the influences of the anisotropic coefficients (i.e., shape parameters) n and β on the
295 bearing capacity in terms of N_c and N_q . As shown in Fig. 9, it is evident that when the isotropic
296 Mohr-Coulomb yield criterion is recovered (i.e., $n=1.0$), the bearing capacity obtained from
297 the semi-analytical solution is identical to that obtained from Prandtl's solution. In general, the
298 bearing capacity is lower when the yield surface is anisotropic compared with its isotropic
299 counterpart. The predicted results of the bearing capacity increase with an increase in n but
300 decrease with an increase in β . In addition, further validation can be demonstrated by
301 comparing the semi-analytical solution of the bearing capacity N_c with the results from Cox
302 [29], Spencer [30] and the method of limit analysis (after Chen [31]), as shown in Table 1. If
303 the isotropic yield criterion is recovered when $n=1.0$, the results from those previous methods
304 and the current semi-analytical solution are consistent with various internal friction angles.
305 From the above analysis, it can be concluded that the strength of the soil is reduced when the
306 soil yield anisotropy is considered. Hence, the predicted ultimate bearing capacity is much
307 lower. This situation might result in an unsafe design for strip footing problems if the initial
308 strength anisotropy is ignored.



309

310

Fig. 9 Parametric study of the anisotropic coefficients ($\phi_{max} = 30^\circ$) : (a) N_c ; (b) N_q .

311

Table 1 Variation of N_c with ϕ

| ϕ ($^\circ$) | Bearing capacity N_c | | | | |
|---------------------|------------------------|-------------------|--------------------------------------|--------------------------|-------------------------|
| | Cox (1962) | Spencer (1962) | Limit analysis (after Chen, 1975) | Semi-analytical solution | |
| | | | | $n=1.0$ | $n=0.707 \beta=0^\circ$ |
| 10 | 8.34 | 8.35 | 8.35 | 8.35 | 7.27 |
| 20 | 14.8 | 14.8 | 14.8 | 14.8 | 11.99 |
| 30 | 30.1 | 30.1 | 30.1 | 30.1 | 21.48 |
| 40 | 75.3 | 75.3 | 75.3 | 75.3 | 43.18 |

312

313

4. NUMERICAL RESULTS AND DISCUSSION

314

A strip, rigid and smooth footing is assumed to rest on a weightless granular soil mass. Perfect

315

plasticity is assumed for this case. The flow rule is associativity only for the comparison with

316

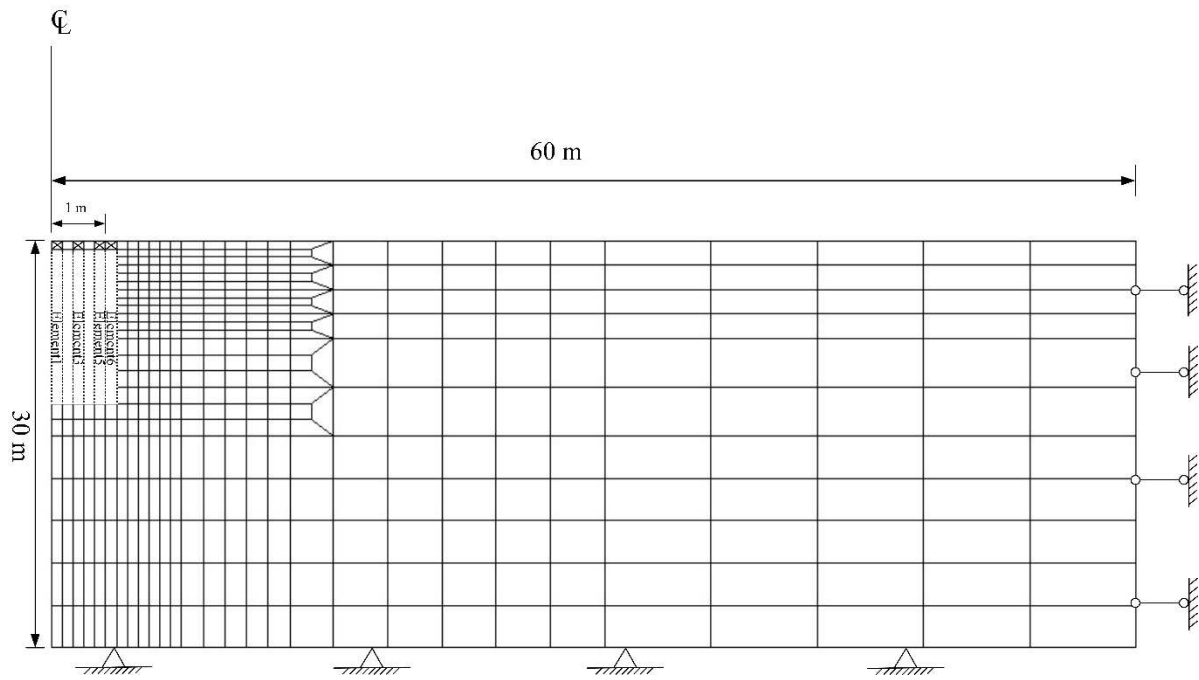
semi-analytical results; otherwise, both associativity and non-associativity in the conventional

317

flow rules are applied for numerical simulations performance to evaluate the effects of the flow

318

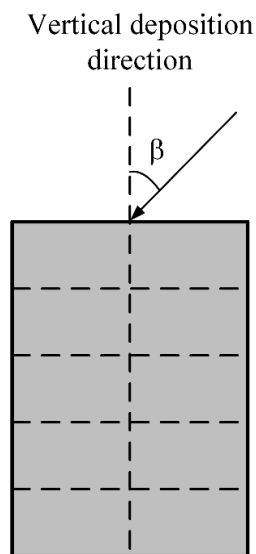
rule.



319

320

Fig. 10 Geometry and finite element discretization of the strip footing



321

322

Fig. 11 Illustration of β

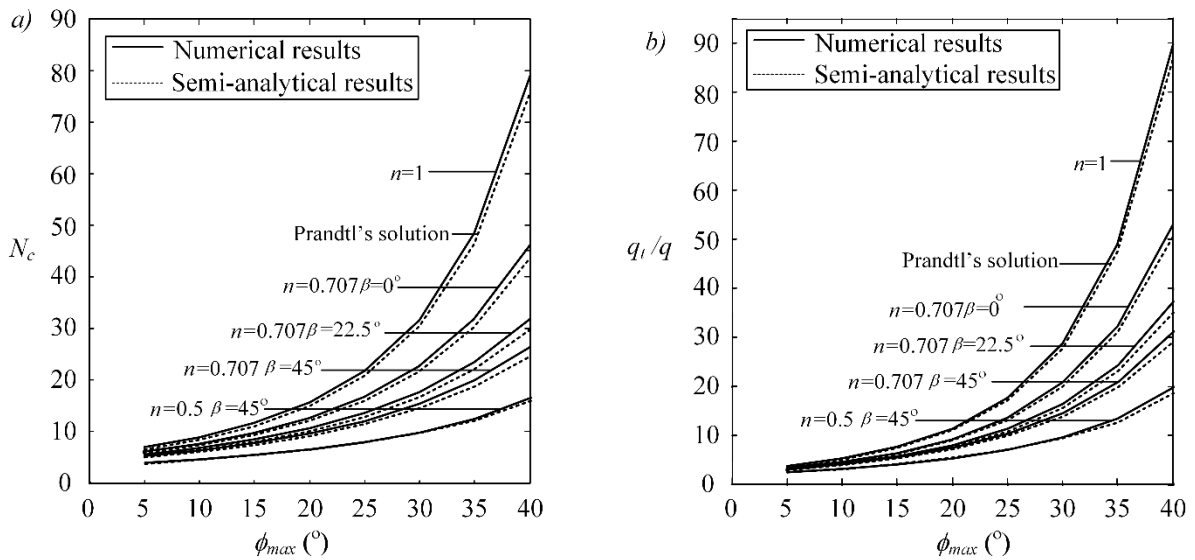
323 The model size for half of the base soil is assumed to be 60 m wide and 30 m deep, with the
 324 half width B of the footing set to 1 m. This negates the impact of the boundary conditions. The
 325 material of the base soil is discretized with first-order 8-node plane-strain reduced elements
 326 (element type CPE8R). The left-hand boundary represents a vertical symmetry axis, whereas
 327 the far-field condition on the right-hand side boundary allows for vertical movement. The
 328 condition on the bottom boundary is fixed in both the vertical and horizontal directions. The

329 nodes immediately underneath the footing are free to move horizontally but are subject to the
330 same amount of vertical downward movement. These nodes are subsequently applied in a
331 gradually increasing, downward vertical displacement to simulate the movement of the footing.
332 Two categories of simulations are performed: the first category has a footing located on a
333 weightless cohesive-frictional soil without a surface surcharge, and the second category
334 involves a footing located on a weightless cohesive-frictional soil with a 100 kPa surface
335 surcharge. The maximum internal friction angle ϕ_{\max} is set as 30° , except in the analysis of the
336 effect of varying ϕ_{\max} (eight values of ϕ_{\max} are applied from 5° to 40° at an interval of 5°) to
337 validate the numerical results with semi-analytical results. Except for the analysis of a purely
338 cohesionless soil, the cohesion c is set as 30 kPa. The typical elastic constants are fixed, i.e.,
339 Young's modulus $E = 10.0 \times 10^4$ kPa and Poisson's ratio $\nu = 0.3$. The shape parameter n
340 defined in the anisotropic yield criterion represents the ratio of the minor axis over the major
341 axis of the ellipse in the deviatoric space, relative to the magnitudes of the peak internal friction
342 angle with the direction of principal stresses. The illustration of another shape parameter β
343 relative to the deposition direction is shown in Fig. 11.

344 4.1 Verification in terms of the bearing capacity

345 The computation of the bearing capacity N_c , which is defined as the ultimate failure pressure
346 normalized by the cohesion as obtained from the semi-analytical solution and numerical
347 simulations with various internal friction angles, is illustrated in Fig. 12 a. The contributions
348 of other bearing capacity factors are not taken into considerations, i.e., $q=0$ kPa. The footing
349 is incrementally displaced immediately before the numerical convergence fails. For
350 computation of the ultimate failure pressure normalized by the surface surcharge (q_i/q), as
351 shown in Fig. 12 b, the cohesion is set as 30 kPa due to convergence problem for small friction
352 angles. A good match of N_c and q_i/q can be observed between the numerical simulations and

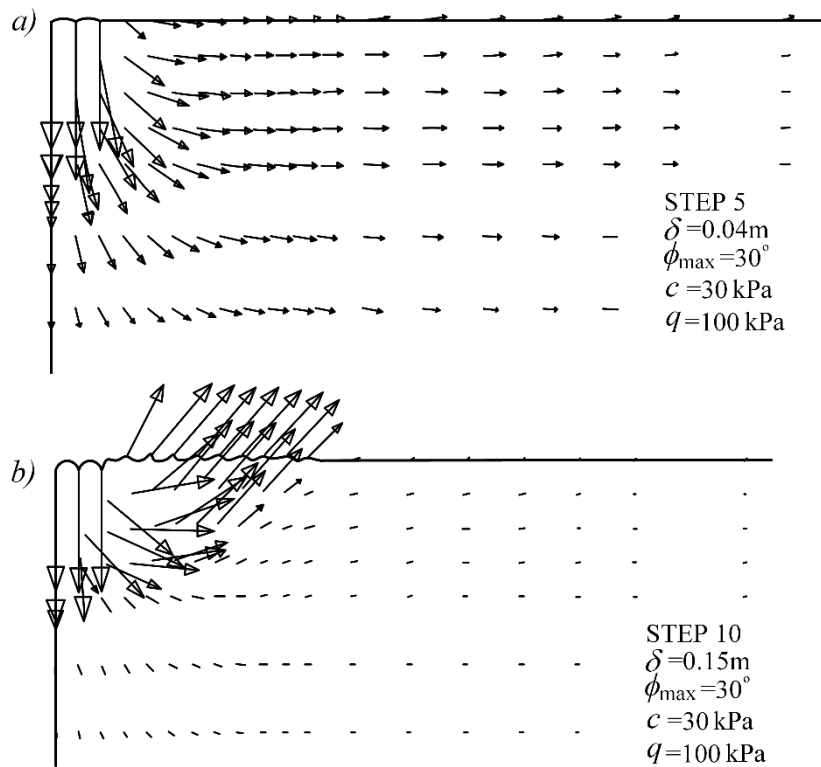
353 semi-analytical calculations for various anisotropic coefficients n and β . Generally, the
 354 numerical results deviate slightly further from the analytical results, but within a tolerable
 355 accuracy. The reasons for this outcome might lie in the presence of elasticity modeled by the
 356 elasto-plastic constitutive model in the numerical simulations, but for the semi-analytical
 357 solutions, the soil mass is modeled as a rigid, plastic body, and the elastic portion is ignored.



358

359 Fig. 12 Bearing capacity factors versus various internal friction angles: (a) N_c ; (b) q_t/q .

360



361

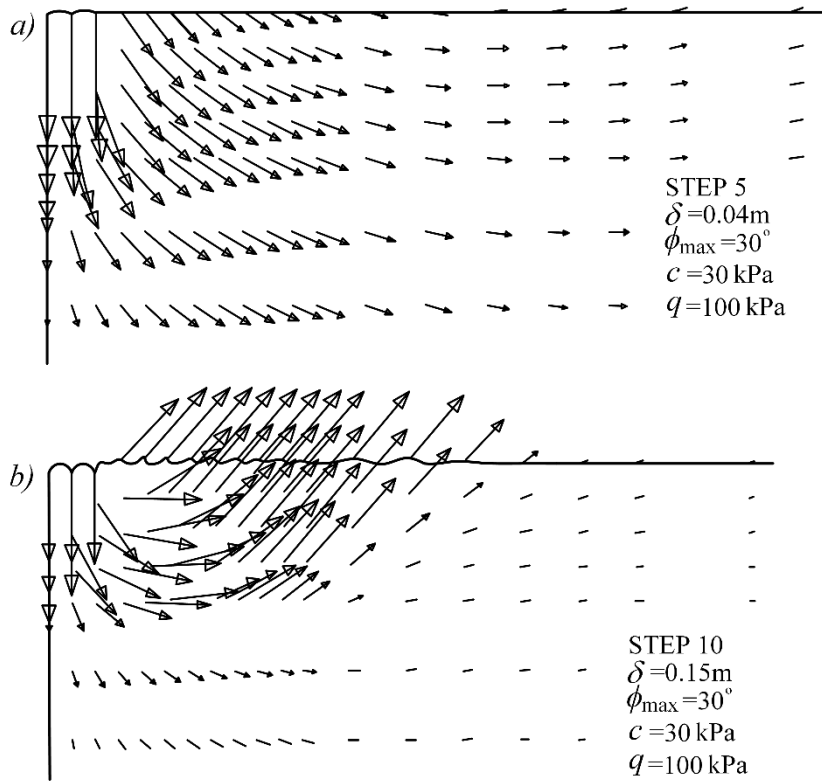
362 Fig. 13 The velocity field for the case of isotropic Mohr-Coulomb yield criterion for different
 363 steps of the computing in ABAQUS: a) fifth step; b) tenth step.

364 4.2 Validation in terms of the velocity field

365 Fig. 13 and Fig. 14 show the velocity fields obtained from the isotropic and anisotropic
 366 soils respectively. The directions of the arrows represent the flows of velocity. The scale of the
 367 magnitude of displacement, which is represented by the length of the arrow, is not identical.
 368 The exact magnitudes of the displacement are not given because they are not focused in the
 369 present study. The pattern of the black arrows visually indicate the β -lines compared with Fig.8.
 370 The velocity zone indicated by the anisotropic Mohr-Coulomb yield criterion (see Fig. 14) is
 371 larger and wider than that indicated by its isotropic counterpart (see Fig. 13). It can be expected
 372 that the failure zone is wider when the yield surface is anisotropic.

373

374

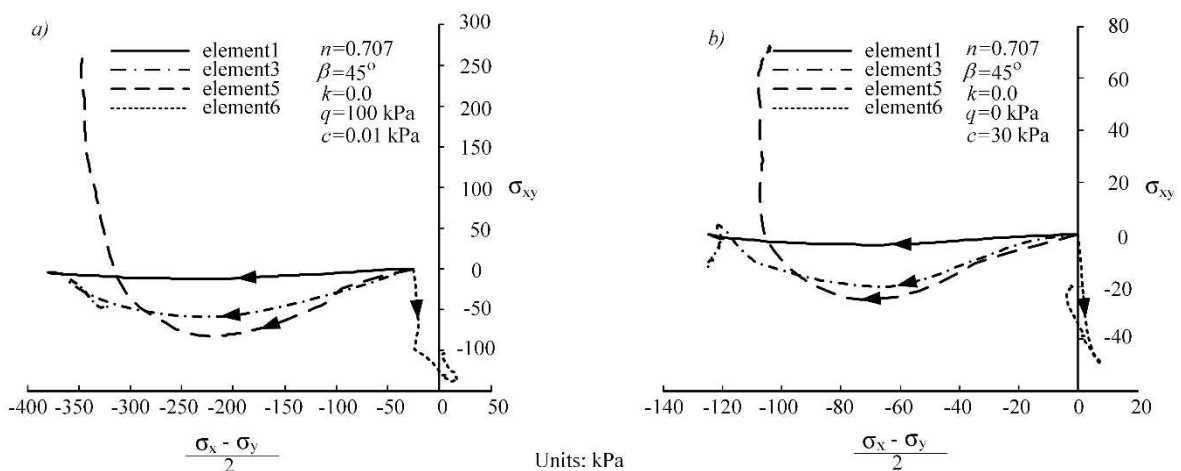


375

376 Fig. 14 The velocity field for the case of anisotropic Mohr-Coulomb yield criterion when
 377 $n=0.707$ and $\beta=45^\circ$, for different steps of the computing in ABAQUS: a) fifth step; b) tenth
 378 step.

379 **4.3 Evidence of principal stress rotations**

380 Four representative soil elements that are underneath and adjacent to the footings are
 381 highlighted in Fig. 10 with a black cross at the top. The stress paths of these representative
 382 elements are shown in Fig. 15. It is visually evident from these figures that these soil elements
 383 experience principal stress rotations.



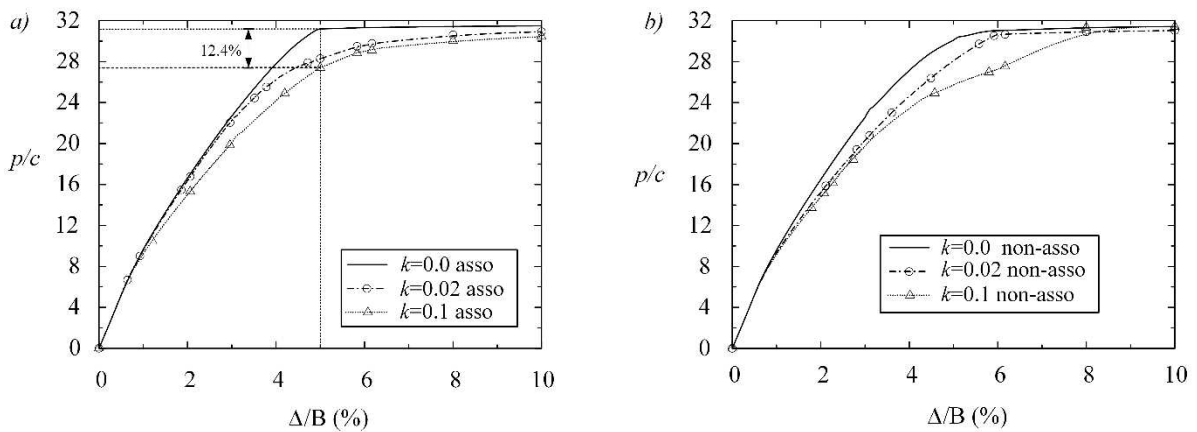
384

385 Fig. 15 Stress paths in the space of $(\frac{\sigma_x - \sigma_y}{2}, \sigma_{xy})$ from the numerical simulations: (a)
 386 computation of N_q ; (b) computation of N_c .

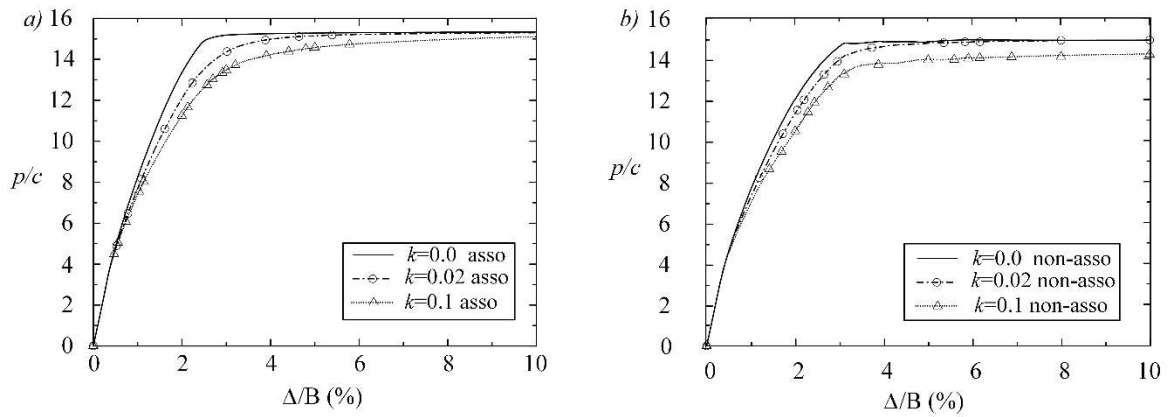
387 **4.4 Influence of the degree of soil anisotropy and non-coaxiality on the bearing capacity**
 388 **due to the contribution of cohesion**

389 As shown in Equation (19), for computation of the bearing capacity N_c , contributions from
 390 other bearing capacity factors are neglected. The soil underneath the footing is assumed to be
 391 purely frictional-cohesive. The maximum internal friction angle is set as $\phi_{max} = 30^\circ$. When a
 392 non-associated condition is used, the dilation angle is set to $\psi_{max} = 20^\circ$ for computational
 393 convenience. The load-displacement curves are presented in Fig. 16 and Fig. 17. The vertical
 394 axis denotes the footing pressure normalized by cohesion (p/c), and the horizontal axis
 395 represents the displacement normalized by the half-width of the footing (Δ/B). The maximum
 396 difference of p/c prior to collapse between the coaxial ($k=0.0$) and non-coaxial predictions
 397 ($k=0.1$) is defined as follows (as illustrated in Fig. 16):

398
$$R_r = \frac{N_c(k=0.0) - N_c(k=0.1)}{N_c(k=0.0)} \quad (24)$$



399
 400 Fig. 16 Load-displacement curves of the bearing capacity N_c when the isotropic Mohr-
 401 Coulomb yield criterion is recovered ($n=1.0$): (a) associativity; (b) non-associativity.



402

403 Fig. 17 Load-displacement curves when $n=0.707$ and $\beta=45^\circ$: (a) associativity; (b) non-
 404 associativity.

405 As shown in Fig. 16, when the shape parameter $n=1.0$, i.e., the isotropic Mohr-Coulomb yield
 406 criterion, is recovered, the ultimate failure is reached at a normalised displacement Δ/B around
 407 5%-6%. The settlement prior to collapse is larger when the non-coaxial coefficient is not equal
 408 to zero. The settlement increases with an increase in the non-coaxial coefficient k . It can be
 409 concluded that the soil is softened when non-coaxial plasticity is present. However, the ultimate
 410 bearing capacity N_c is not significantly affected and tend to be identical when approaching a
 411 large displacement for various values of k . As illustrated in Fig. 17, for the anisotropic case,
 412 the ultimate failure is reached at around $\Delta/B = 2.3\%$ and $\Delta/B = 3\%$ for the associativity and
 413 non-associativity in the conventional flow rules, respectively. When compared Fig.16 and
 414 Fig.17, the soil strength anisotropy exhibits a significant impact on the strength of the soil mass.
 415 The results show that the exclusion of initial soil strength anisotropy tends to delay the onset
 416 of the ultimate bearing capacity.

417

418 The parametric study is presented in Table 2. The results indicate that the onset of the ultimate
 419 bearing capacity and the maximum difference R_r depend on the degree of initial strength
 420 anisotropy and non-coaxiality. Conclusions can be drawn that the influence of the use of
 421 associativity/non-associativity is insignificant. For particular cases (e.g., Test5), the results

422 from non-coaxial modelling for $k=0.1$ match closely with those from coaxial modeling for
 423 $k=0.0$. The most influenced case occurs for $n=0.85$ and $\beta=45^\circ$ (Test2). The stiffness of soil mass
 424 prior to failure is definitely influenced by the degree of non-coaxiality, whereas the effects of
 425 non-coaxiality on the bearing capacity are influenced by soil yield anisotropy, but the ultimate
 426 bearing capacity is not significantly affected.

427 Table 2 Maximum difference R_r for the computation of N_c .

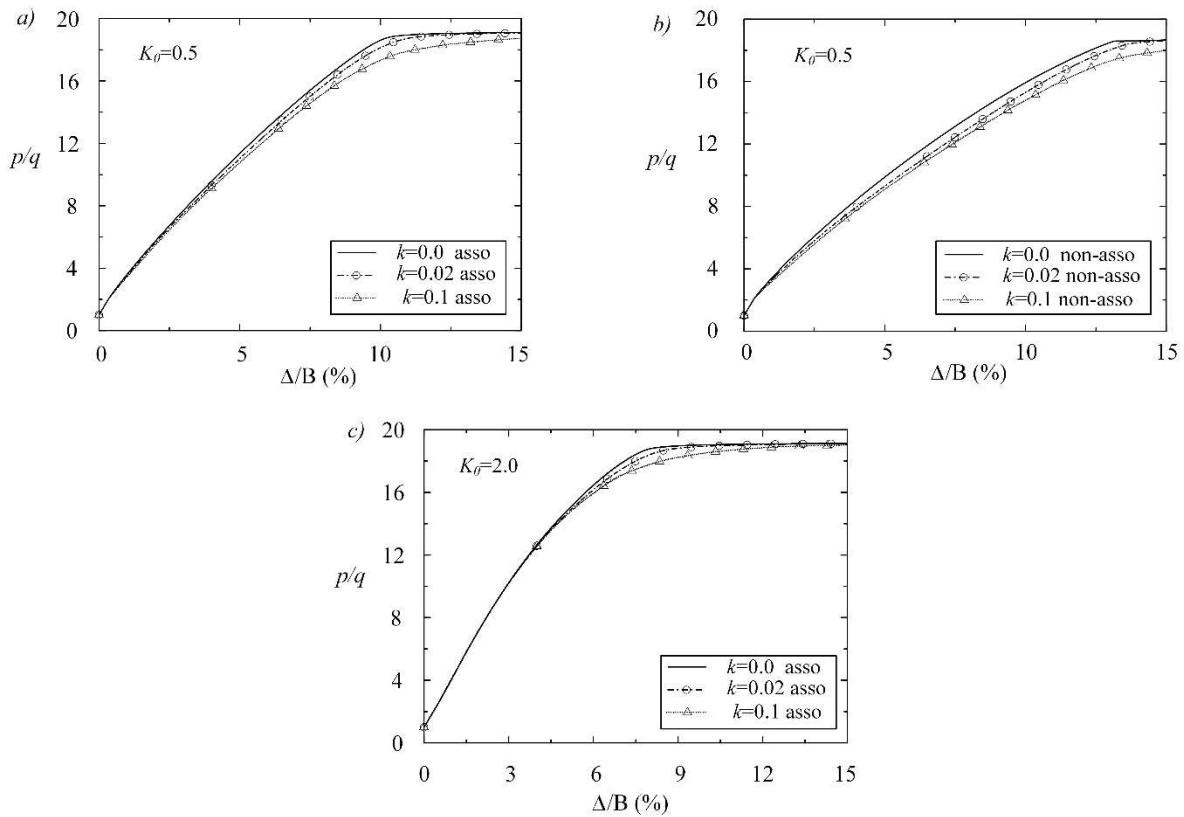
| | n | $\beta(^{\circ})$ | Asso/Non-asso | Δ/B (%) | $R_r(\%)$ |
|-------|-------|-------------------|---------------|----------------|-------------|
| Test1 | 1.0 | N/A | Asso | 5 | 12.4 |
| | | | Non-asso | 6 | 13.1 |
| Test2 | 0.85 | 45 | Asso | 3.4 | 13.5 |
| Test3 | 0.707 | 45 | Asso | 2.3 | 10.9 |
| | | | Non-asso | 3 | 10.0 |
| Test4 | 0.707 | 22.5 | Nsso | 3 | 6.8 |
| | | | Non-asso | 2.8 | 6.8 |
| Test5 | 0.707 | 0 | Asso | 4.3 | 4.6 |
| | | | Non-asso | 5.3 | 4.9 |

428

429 **4.5 Influence of soil anisotropy and non-coaxiality on the bearing capacity due to the**
 430 **contribution of surface surcharge**

431 A uniform surface surcharge of 100 kPa is applied for computation of bearing capacity N_q . The
 432 cohesion is set to $c=0.01$ for convergence convenience. The maximum internal friction angle
 433 is set as $\phi_{max} = 30^\circ$. When a non-associativity in the conventional flow rule is used, the
 434 dilation angle is set as $\psi_{max} = 20^\circ$ for computational convenience. The coefficient of earth
 435 pressure at rest, i.e. K_0 , are assumed as 0.5 and 2.0. Both associativity/non-associativity in the
 436 conventional flow rules are used in this instance. The vertical axis denotes the footing pressure
 437 normalized by the surface surcharge (p/q), and the horizontal axis represents the displacement
 438 normalized by the half-width of the footing (Δ/B). The maximum difference of p/q prior to
 439 collapse between the coaxial ($k=0.0$) and non-coaxial predictions ($k=0.1$) is defined as follows:

$$R_s = \frac{N_q(k=0.0) - N_q(k=0.1)}{N_q(k=0.0)} \quad (25)$$

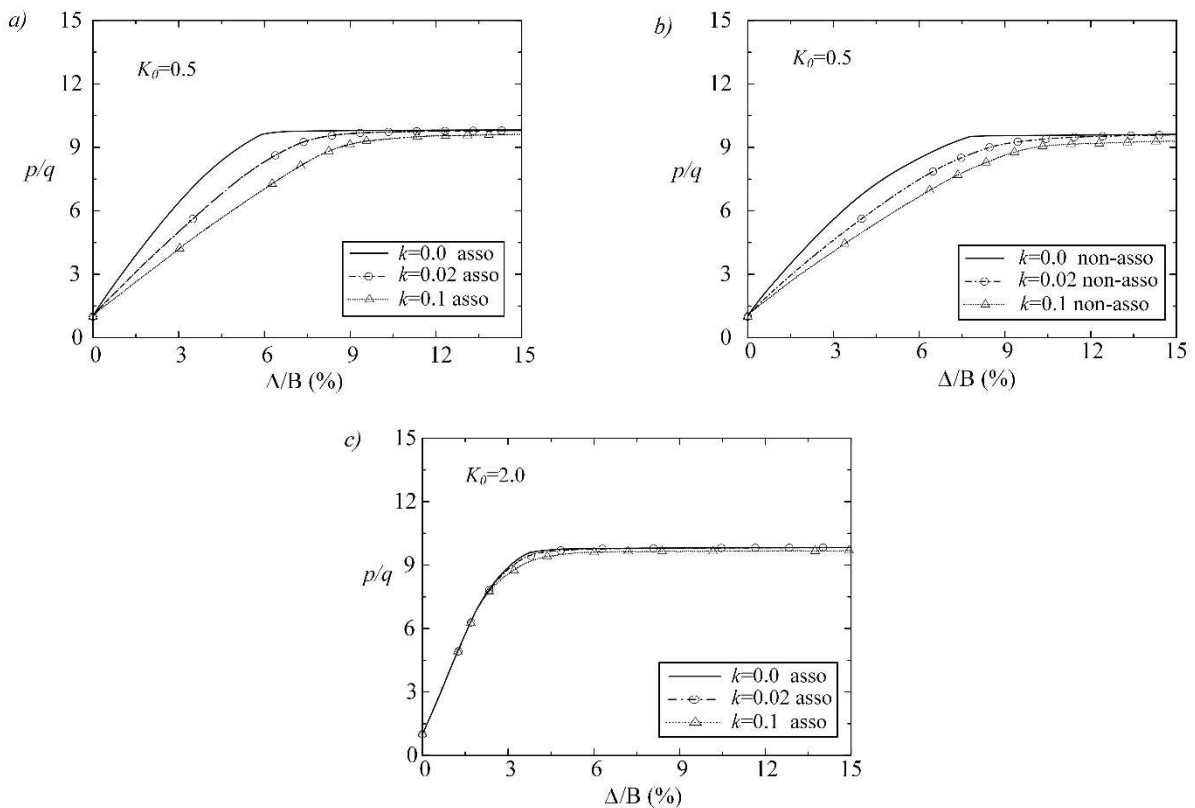


441

442 Fig. 18 Load-displacement curves of the bearing capacity N_q when the isotropic Mohr-
 443 Coulomb yield criterion is recovered ($n=1.0$): (a) lateral stress ratio $K_0=0.5$ and associativity;
 444 (b) lateral stress ratio $K_0=0.5$ and non-associativity; (c) lateral stress ratio $K_0=2.0$ and
 445 associativity.

446 Fig. 18 and Fig. 19 show the pressure-displacement curves obtained from the isotropic and
 447 anisotropic modelling, respectively. Non-coaxial modelling affects the settlement prior to
 448 collapse, which indicates that the soil is softened. In this scenario, the results indicate that the
 449 maximum difference R_s , which can account for the influence of non-coaxiality on the stiffness
 450 of soil mass prior to failure, depends on the lateral stress ratio, degree of initial strength
 451 anisotropy and flow rule. The ultimate value of the bearing capacity N_q is rarely affected by
 452 the introduction of non-coaxial plasticity. However, the ultimate bearing capacity is reached
 453 with $\frac{\Delta}{B} = 6\%$ when the yield strength anisotropy is applied (Fig. 19 a), which is nearly half of
 454 that with $\frac{\Delta}{B} = 10.5\%$ when its isotropic counterpart is applied (Fig. 18 a). Generally, a similar

455 phenomenon is obtained as compared with that due to the contribution of cohesion. However,
 456 the influences of non-coaxiality and initial strength anisotropy are pronounced when compared
 457 with those of the bearing capacity due to the contribution of cohesion, as shown in Figure 19.
 458 A parametric study with respect to different values of lateral stress ratio, anisotropic
 459 coefficients (n and β), non-coaxial coefficient (k) and flow rules is shown in Table 3. For a
 460 value of K_0 of 2.0, few differences exist between the coaxial and non-coaxial predictions, for
 461 which the minimum of R_s can be 4.6%. The maximum difference R_s increases with a decrease
 462 in the value of n . However, when comparing Tests 8, 9 and 10, the maximum difference R_s
 463 sharply decreases with smaller values of β . The value drops from a maximum of $R_s=28.3\%$ to
 464 a minimum of $R_s=3.4\%$ for such a scenario. Hence, the effects of two shape parameters from
 465 the anisotropic yield criterion on R_s are highly evident.



466

467 Fig. 19 Load-displacement curves of the bearing capacity N_q when $n=0.707$ and $\beta=45^\circ$: (a)
 468 lateral stress ratio $K_0=0.5$ and associativity; (b) lateral stress ratio $K_0=0.5$ and non-
 469 associativity; (c) lateral stress ratio $K_0=2.0$ and associativity.

470

471

Table 3 Maximum difference R_s for the computation of N_q

| | n | $\beta(^{\circ})$ | Asso/Non-asso | $\frac{\Delta}{B}(\%)$ | K_0 | $R_s(\%)$ |
|--------|-------|-------------------|---------------|------------------------|-------|-------------|
| Test6 | 1.0 | N/A | Nsso | 10.5 | 0.5 | 7.0 |
| | | | Non-asso | 12.5 | 0.5 | 6.6 |
| | | | Asso | 8 | 2.0 | 6.1 |
| Test7 | 0.85 | 45 | Asso | 7.6 | 0.5 | 20.0 |
| Test8 | 0.707 | 45 | Asso | 6 | 0.5 | 28.3 |
| | | | Non-asso | 7.6 | 0.5 | 24.4 |
| | | | Asso | 4.2 | 2.0 | 4.6 |
| Test9 | 0.707 | 22.5 | Asso | 6.5 | 0.5 | 6.5 |
| Test10 | 0.707 | 0 | Asso | 10 | 0.5 | 3.4 |

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5. CONCLUSIONS

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In this paper, a plane-strain elastic-perfectly-plastic non-coaxial soil model with an anisotropic yield criterion was applied to investigate smooth strip footing problems. Semi-analytical solutions of the bearing capacity for a smooth strip footing resting on an anisotropic, weightless, cohesive-frictional soil were developed based on the slip line method. Influences of the degree of soil anisotropy and non-coaxiality, on the bearing capacity of footing problems, were discussed. Based on the above analyses, the following conclusions can be drawn:

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- The soil mass adjacent to the footing edge exhibited severe principal stress rotations.
- Without considering the non-coaxial plasticity (i.e., $k=0.0$), the numerical results were similar to the semi-analytical solutions, highlighting the capability of the numerical procedures and validation of the proposed model.
- The ultimate bearing capacity was much lower if soil yield anisotropy was involved. The exclusion of initial soil strength anisotropy tended to delay the onset of the ultimate bearing capacity N_c and N_q .
- Non-coaxial modelling affected the settlement prior to collapse, which indicated that the soil was softened. The degree of non-coaxial effects depended on the initial stress state,

489 the degree of initial strength anisotropy, and the flow rule. The ultimate bearing capacity
 490 was rarely affected by the inclusion of non-coaxial plasticity. It is necessary to consider
 491 both initial strength anisotropy and non-coaxiality when analyzing strip footing problems.

492

493

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 498 Nos. 2682015CX092 and 2682016CX084).

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APPENDIX 1

501 The matrix \mathbf{N} can be written as follows:

$$502 \quad \mathbf{N} = \begin{bmatrix} a & -a & b \\ -a & a & -b \\ c & -c & d \end{bmatrix} \quad \text{A.1}$$

503 The expressions for a , b , c and d are listed below:

$$504 \quad a = k \cdot H \cdot \left[-\frac{\sigma_{xy}}{4\sigma_{xy}^2 + (\sigma_x - \sigma_y)^2} \right] \quad \text{A.2}$$

$$505 \quad b = k \cdot H \cdot \left[\frac{\sigma_x - \sigma_y}{4\sigma_{xy}^2 + (\sigma_x - \sigma_y)^2} \right] \quad \text{A.3}$$

$$506 \quad c = k \cdot I \cdot \left[-\frac{\sigma_{xy}}{4\sigma_{xy}^2 + (\sigma_x - \sigma_y)^2} \right] \quad \text{A.4}$$

$$507 \quad d = k \cdot I \cdot \left[\frac{\sigma_x - \sigma_y}{4\sigma_{xy}^2 + (\sigma_x - \sigma_y)^2} \right] \quad \text{A.5}$$

508 where

$$509 \quad H = -2 \sin(2\Theta + 2m) \cdot (1 + m_\Theta) \quad \text{A.6}$$

510
$$I = 2 \cos(2\Theta + 2m) \cdot (1 + m_{\Theta}) \quad \text{A.7}$$

511 For a rotational ellipse anisotropic yield criterion, the definition of m_{Θ} is written as follows:

512
$$m_{\Theta} = \frac{4(1 - n^2) \cdot \cos(4\Theta - 4\beta) \cdot C - 4D^2}{C^2} \quad \text{A.8}$$

513 where

514
$$C = 2(n^2 - 1) \cos^2(2\Theta - 2\beta) + 2 \quad \text{A.9}$$

515
$$D = (1 - n^2) \sin(4\Theta - 4\beta) \quad \text{A.10}$$

516 **APPENDIX 2**

517 **1. Governing equations of stresses**

518 The two characteristics lines, i.e., α -lines and β -lines, are integrals of Equation (14) and
 519 Equation (15), respectively. Hence, the canonical form of the equilibrium equation can be
 520 written as follows:

521
$$\sin[2(m - \nu)] \frac{\partial p}{\partial \alpha} + 2F \frac{\partial \Theta}{\partial \alpha} + \gamma \cos(2m) \left[\sin(2\nu) \frac{\partial x}{\partial \alpha} + \cos(2\nu) \frac{\partial y}{\partial \alpha} \right] = 0 \quad \text{A.11}$$

522
$$\sin[2(m + \nu)] \frac{\partial p}{\partial \beta} + 2F \frac{\partial \Theta}{\partial \beta} + \gamma \cos(2m) \left[-\sin(2\nu) \frac{\partial x}{\partial \beta} + \cos(2\nu) \frac{\partial y}{\partial \beta} \right] = 0 \quad \text{A.12}$$

523 For a cohesive-frictional soil with no self-weight, γ is neglected. Then Equation A.11 and A.12
 524 are reduced to the definitions shown below:

525
$$\sin[2(m - \nu)] \frac{\partial p}{\partial \alpha} + 2F \frac{\partial \Theta}{\partial \alpha} \quad \text{A.13}$$

526
$$\sin[2(m + \nu)] \frac{\partial p}{\partial \alpha} + 2F \frac{\partial \Theta}{\partial \alpha} \quad \text{A.14}$$

527 Which are hyperbolic if the characteristics defined in Equation (14) and Equation (15) are real
 528 and distinct.

529 Recalling the anisotropic yield criterion in Section 2.1, the variation of the stress state in an
 530 anisotropic plastic region can be shown as follows:

531
$$dp + (p + c \cot \phi_{max}) \frac{2 \sin \phi(\Theta)}{\sin 2(m - \nu)} d\Theta = 0 \quad \text{A.15}$$

532
$$dp + (p + c \cot \phi_{max}) \frac{2 \sin \phi(\Theta)}{\sin 2(m + \nu)} d\Theta = 0 \quad \text{A.16}$$

533 **2. Stress boundary conditions**

534 The normal and shear stresses at the boundary must lie on the Mohr circle that touches the
 535 failure envelope. For a strip footing problem with anisotropic soil mass, the shear stress acting
 536 on the boundary is zero. Following a geometrical calculation, the mean stress p can be solved
 537 as follows:

$$538 \quad p = \frac{\sigma_n \mp c \cot \phi_{max} \sin \phi(\Theta)}{1 \pm \sin \phi(\Theta)} \quad A.17$$

539 where the first sign $n=1.0$ applies to the case in which σ_n is the major principal stress, and the
 540 second sign $n=2.0$ applies to the case in which it is the minor principal stress.

541

542 As shown in Fig. 8, the family of straight β -lines indicate the characteristics within the region
 543 OCD , which demonstrate an angle of θ . The extent of the region OCD is governed by the
 544 condition that OA is smooth. In other words, $\theta = 0^\circ$ on \overline{OA} . This statement implies that the
 545 angle $\angle COD$ is a right angle. Hence, following Equation A.17, two stress variables (p_1, θ_1) and
 546 (p_2, θ_2) can be obtained. When the two stress variables are given, and assuming the two stress
 547 variables are located at two points along the same family of β -lines, we can write the expression
 548 of vertical pressure at plastic collapse:

$$549 \quad q_t = \left(1 + \sqrt{\frac{2}{M}} n \sin \phi_{max}\right) \left(e^{\int_0^{\pi/2} G(\Theta) d\Theta} \cdot \frac{q\sqrt{M} + \sqrt{M} c \cot \phi_{max}}{\sqrt{M} - \sqrt{2n \sin \phi_{max}}}\right) - c \cot \phi_{max} \quad A.18$$

550 and:

$$551 \quad G(\Theta) = \frac{2\sqrt{2n \sin \phi_{max}} (C^2 + D^2)}{\sqrt{2n \sin \phi_{max}} \cdot D \cdot C + \sqrt{C^5 + D^2} \cdot C^3 - 2C^4 (n \sin \phi_{max})^2} \quad A.19$$

$$552 \quad C = 2[(1 - n^2) \sin^2(2\Theta - 2\beta) + n^2] \quad A.20$$

$$553 \quad D = (n^2 - 1) \sin(4\Theta - 4\beta) \quad A.21$$

$$554 \quad M = 2[(1 - n^2) \sin^2(2\beta) + n^2] \quad A.22$$

555 where n and β are the shape parameters of the anisotropic yield criterion.

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