The University Of Sheffield.

This is a repository copy of *Information length in quantum system*.

White Rose Research Online URL for this paper:
http://eprints.whiterose.ac.uk/130166/

Version: Accepted Version

Article:

Kim, E. and Lewis, P. (2018) Information length in quantum system. Journal of Statistical Mechanics: Theory and Experiment. 043106. ISSN 1742-5468

https://doi.org/10.1088/1742-5468/aabbbe

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

White Rose university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

Information length in quantum systems

Eun-jin Kim[1] and Patrick Lewis[2]

[1] *School of Mathematics and Statistics, University of Sheffield, Sheffield, S3 7RH, UK*

[2] *Department of Physics and Astronomy, University of Sheffield, Sheffield, S3 7RH, UK*

Abstract

A probabilistic description is essential for understanding the dynamics in many systems due to uncertainty or fluctuations. We show how to utilise time-dependent probability density functions to compute the information length $\mathcal{L}$, as a Lagrangian measure that counts the number of different states that a quantum system evolves through in time. Using $\mathcal{L}$, we examine the information change associated with the evolution of initial Gaussian wave packets and elucidate consequences of quantum effects.

I. INTRODUCTION

A probabilistic description is essential for understanding the dynamics in many systems due to uncertainty or fluctuations. This is especially the case for out-of-equilibrium systems which exhibit significant fluctuations such as; turbulence in astrophysical and laboratory plasmas, forest fires, the stock market, and biological ecosystems [1–3]. A full knowledge of Probability Density Functions (PDFs), especially time-dependent PDFs, becomes essential to describe these systems. Once computed analytically, numerically, or constructed from data, time-dependent PDFs provide a system-independent way of quantifying the change in information during time-evolution by the number of statistically different states that a system passes through in time [4–10]. (Note that we use information for statistically different states, refraining ourselves from the debate on the exact definition of information (see, e.g. [11, 12])). Crudely, a unit of information change can be inferred when a PDF moves the distance equal to its width; two PDFs with the same width are indistinguishable when their mean positions differ less than the width. On the other hand, doubling/halving PDFs induces a logarithmic increase in information. In order to do this systematically, we first define the dynamical time $\tau(t)$ [4–10],

$$\mathcal{E} \equiv \frac{1}{[\tau(t)]^2} = \int \frac{1}{p(x,t)} \left[\frac{\partial p(x,t)}{\partial t}\right]^2 dx. \tag{1}$$

Where $\tau(t)$ is the characteristic time scale over which the information changes. Having units of time, $\tau(t)$ quantifies the correlation time of a PDF. Alternatively, $1/\tau$ quantifies the (average) rate of change of information in time. A particular path which gives a constant valued $\mathcal{E}$ is a geodesic along which the information propagates at the same speed [7] (c.f. see Appendix A). The total change in information between initial and final times, 0 and t respectively, is then defined by measuring the total elapsed time in units of τ as:

$$\mathcal{L}(t) = \int_0^t \frac{dt_1}{\tau(t_1)} = \int_0^t \sqrt{\int dx \frac{1}{p(x,t_1)} \left[\frac{\partial p(x,t_1)}{\partial t_1}\right]^2}\, dt_1. \tag{2}$$

$\mathcal{L}(t)$ is a Lagrangian quantity, uniquely defined as a function of time t for a given initial PDF, and represents the total number of statistically distinguishable states that a system evolves through. It thus provides a very convenient methodology for measuring the distance between $p(x,t)$ and $p(x,0)$ continuously in time for a given $p(x,0)$. More details on theoretical background (e.g. the relation to statistical distance in Hilbert space and the relative entropy) and applications of $\mathcal{E}$ and $\mathcal{L}$ can be found in Appendix A and [4–11, 13].

In Quantum Mechanics (QM), stochasticity arises due to the uncertainty relation $\Delta x \Delta p \geq \frac{\hbar}{2}$ even in the absence of external noise. In particular, in the semi-classical limit, h serves as a unit of information in the $p-x$ phase space since each quantum state corresponds to a classical volume h; the total number of states is the classical volume of phase space divided by h. Associated with the time evolution of a quantum system, exploring different parts of the phase space, is information change. A wider PDF corresponds to a wave function with large variance in QM and occupies a larger x region in the phase space; it is thus expected to cause more change in information, opposite to what is expected in classical systems. The main aim of this paper is to elucidate consequences of quantum effects (e.g. uncertainty relation, quantisation), addressing the issue noted above.

Specifically, to gain a key insight, we consider the evolution of Gaussian wave packets for a particle under no force and constant force, and for harmonic oscillators including damped harmonic oscillators [14]. We demonstrate an interesting dual role of PDF width and the effect of energy quantisation on $\mathcal{L}$. We utilise the known exact propagators to compute time-dependent PDFs for a given initial Gaussian wave packet, referring readers to [14–22]. Note that our approach is quite different from the traditional spectral analysis where the main focus is on the computation of energy eigenvalues and eigenstates or the transition probability. In fact, an interesting oscillation in PDF width is shown, which received much less attention.

II. FREE-PARTICLE GAUSSIAN WAVE PACKETS:

To make this paper self-contained, we start with a brief recap on free-particle wave packets. A propagator $K(x,x';t)$ for a particle with mass m in the 1-Dimension (1D) is well-known (e.g. see [22])

$$K(x,x';t) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[-\frac{m(x-x')^2}{2i\hbar t}\right]. \tag{3}$$

At $t=0$, we consider a Gaussian wave function localised around $x'=0$

$$\psi(x',0) = \left[\frac{2\beta_0}{\pi}\right]^{\frac{1}{4}} e^{-\beta_0 x'^2 + ik_0 x'}, \tag{4}$$

where $k_0 = p(t=0)/\hbar$ is the wave number at $t=0$, $d_x = (2\beta_0)^{-1/2}$ is the width of the wave packet, and p is the momentum. From $\psi(x,t) = \int_{-\infty}^{\infty} dx'\, K(x,x';t')\psi(x',0)$, we can then find the following PDF $P(x,t) = |\psi(x,t)|^2$:

$$P(x,t) = \sqrt{\frac{\beta}{\pi}} \exp\left[-\beta\left(x - \langle x\rangle\right)^2\right], \tag{5}$$

where

$$\beta = \frac{2\beta_0 m^2}{m^2 + (2\hbar\beta_0 t)^2}, \ \ \langle x\rangle = v_0 t = \frac{\hbar k_0 t}{m}. \tag{6}$$

Here, v_0 is the constant velocity and the angular brackets denote the average. Eq. (4) is obviously Gaussian with the mean $\langle x\rangle = v_0 t$ and the variance $\Delta(t) = \langle (x - \langle x\rangle)^2\rangle = 2d_x^2$,

$$\Delta(t) = \frac{1}{2\beta} = \frac{1}{4\beta_0} + \frac{\beta_0\hbar^2 t^2}{m^2} = \Delta(0) + \frac{\hbar^2 t^2}{4\Delta(0)m^2}. \tag{7}$$

The first term in Eq. (7) is due to the variance of the initial PDF, $\Delta(0) = \langle (x(0) - \langle x(0)\rangle)^2\rangle = \frac{1}{4\beta_0}$. The second term in Eq. (7) represents the spreading of the wave packet/PDF in time due to quantum effects, which disappears in the classical limit $\hbar \to 0$ as $\beta = \beta(0) = 2\beta_0$. These quantum effects give rises to a super-diffusion $\propto t^2$, occurring faster than the Brownian motion $\Delta \propto t$, in the limit of a very narrow initial wave packet (as $\beta(0) \to \infty$).

We can find $\mathcal{L}$ for the PDF in Eq. (5) (by using $\langle (x - \langle x\rangle)^4\rangle = 3\langle (x - \langle x\rangle)^2\rangle^2$) (e.g. see [7]) as:

$$\mathcal{E} = \frac{1}{2\beta^2}\left(\frac{d\beta}{dt}\right)^2 + 2\beta\left(\frac{d\langle x\rangle}{dt}\right)^2 \tag{8}$$

$$= 2t^2 \frac{1}{(T^2+t^2)^2} + 2\beta_0 \frac{T^2}{T^2+t^2} v_0^2. \tag{9}$$

Here, we defined a characteristic time $T = \frac{m}{2\hbar\beta_0}$. It is interesting to rewrite T using $\Delta x(0)\Delta p(0) \sim \frac{\hbar}{2}$ so that $\frac{1}{4\beta_0} = \Delta(0) = (\Delta x(0))^2 \sim \frac{\hbar^2}{4}\frac{1}{(\Delta p(0))^2} \sim \frac{\Delta x(0)}{2(\Delta k(0))}$:

$$T \sim \frac{m\Delta x(0)}{\hbar(\Delta k(0))} \sim \frac{\Delta x(0)}{\Delta v_0}, \tag{10}$$

where $\Delta v_0 = \Delta p(0)/m$. Thus, T represents the characteristic timescale for the spreading of the initial Gaussian wave packet. The first and second terms in Eqs. (8)-(9) are due to the change in the variance and mean value of the PDF, respectively. For $t \ll T$, the change in the mean value dominates over the first term, giving a constant value $\mathcal{E} \sim 2\beta_0 v_0^2$ (i.e. a geodesic). The second term in Eq. (9) in general gives $\mathcal{L} \sim \sqrt{2\beta_0}\, v_0 T \sinh^{-1}\left(\frac{t}{T}\right)$, leading to

$\mathcal{L} \propto t$ for small t; $\mathcal{L}$ increases at the rate $v_0 T/d_x$ (the distance the PDF moves during the spreading time T divided by the width of ψ). When $t \sim T$, the spreading of the free-particle wave packet no longer permits a geodesic. Interestingly, for $t \gg T$, both terms in Eq. (9) become $\propto t^{-2}$, and $\mathcal{L}$ increases logarithmically with t as $\propto \ln t$. The logarithmic dependence is essentially due to the increase in differential entropy $S(t) = -\int dx\, P(x,t) \ln P(x,t)$. The ratio η of the second to the first term in Eq. (9) is

$$\eta = T^2 \beta_0 v_0^2 \sim \left(\frac{p(0)\sqrt{\Delta(0)}}{\hbar} \right)^2,$$

where $p(0) = m v_0$ and $\Delta(0) = (4\beta_0)^{-1}$. Physically, $\eta^{1/2}$ represents the number of different states in the phase space covered by the movement of the PDF with the width $\sqrt{\Delta(0)}$ and momentum $p(0)$. When $\eta \gg 1$, the movement of the PDF is the main source of the information change while when $\eta \ll 1$, the change in the width of the PDF is mainly responsible.

III. G

aussian wave packets under a constant force: A robust geodesic solution is possible under the influence of a constant force F. To show this, we replace $\langle x \rangle$ in Eq. (5) by (e.g. see [23])

$$\langle x \rangle = \frac{\hbar k_0 t}{m} + \frac{F t^2}{2m}. \tag{11}$$

Eq. (9) is now to be replaced by

$$\mathcal{E} = 2t^2 \frac{1}{(T^2 + t^2)^2} + 2\beta_0 \frac{T^2}{T^2 + t^2} v_0^2 \left[1 + \frac{Ft}{\hbar k_0} \right]^2. \tag{12}$$

Eq. (12) shows the interesting possibility of having a constant value of $\mathcal{E}$ for a sufficiently large t as the increase in the momentum $\propto Ft$ compensates the increase in the width of PDF $\propto t$. Specifically, as $t \to \infty$, Eq. (12) is reduced to

$$\sqrt{\mathcal{E}} \to \frac{F}{\hbar \sqrt{2\beta_0}} \sim \frac{F d_x}{\hbar}, \quad \mathcal{L} \to \frac{(Ft) d_x}{\hbar}. \tag{13}$$

In Eq. (13), Ft represents the momentum due to the constant force F while $d_x = (2\beta_0)^{-1/2}$ is the width of the initial wave packet. Thus, $F t d_x$ in Eq. (13) represents the phase volume covered by the motion due to the constant force F in the $p - x$ phase space. Alternatively, one unit of the information change occurs whenever the constant force F causes the motion over the volume $\hbar$. We now show that similar results are also obtained when we compute $\mathcal{L}$ by using the PDF in the momentum space $P(p,t)$ (see [23]):

$$P(p,t) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha(p-(mv_0+Ft))^2},$$
$$\mathcal{E} = 2\alpha F^2, \quad \mathcal{L} = \sqrt{2\alpha} F t, \tag{14}$$

where $\alpha = \frac{1}{2\hbar^2\beta_0}$. Thus, a (non-zero) constant $\mathcal{E}$ (geodesic) is induced by the force F. Furthermore, $\mathcal{L} = \sqrt{2\alpha}Ft \sim (Ft)\Delta(0)/\hbar$, similarly to $\mathcal{L}$ in Eq. (13). In view of the complementary relations between position and momentum in quantum systems, this is an elegant result since it shows the robustness of $\mathcal{L}$ in either position or momentum representation for a geodesic solution.

IV. QUANTUM HARMONIC OSCILLATOR:

A Quantum Damped Harmonic Oscillator (QDHO) is described by the following Hamiltonian H

$$H = \frac{p^2}{2m} e^{-\gamma t} + \frac{1}{2} m\omega^2 x^2 e^{\gamma t}. \tag{15}$$

Here, $\omega = \sqrt{\frac{\kappa}{m}}$ is the natural frequency of the Quantum Simple Harmonic Oscillator (QSHO) when $\gamma = 0$; κ is a spring constant; γ is the damping parameter. For the significance and details on the CK oscillator and/or controversial issues, please see [14–21] and references therein. The propagator for the Hamilton in Eq. (15) (e.g. see [14–22]) is

$$K(x,t;x',0) = \frac{e^{\gamma t/4}}{\left[\left(\frac{2\pi i\hbar}{m\Omega}\right)\sin(\Omega t)\right]^{\frac{1}{2}}} \exp(i\phi), \tag{16}$$

where

$$\Omega^2 = \sqrt{\omega^2 - \frac{1}{4}\gamma^2},$$
$$\phi = \frac{m\Omega}{2\hbar}\left[\cot(\Omega t)\left(\overline{x}^2 + x'^2\right) - \frac{2\overline{x}x'}{\sin(\Omega t)}\right] - \frac{\gamma m}{4\hbar}\left(\overline{x}^2 - x'^2\right), \tag{17}$$

where $\overline{x} = e^{\frac{\gamma t}{2}} x$.

A. QSHO

When $\gamma = 0$, Eqs. (15)-(16) recover the QSHO, and the ground state has the initial wave function $\psi(x', 0)$ given by Eq. (4) with $\beta_0 = \frac{m\omega}{2\hbar}$ and $k_0 = 0$. We are interested in the time evolution of the Gaussian wave packet with an arbitrary β_0 and k_0. Thus, from Eq. (4) and Eq. (16) with $\Omega = \omega$ and $\gamma = 0$, we obtain (also see [23])

$$\psi(x,t) = \left[\frac{2\beta_0}{\pi}\right]^{\frac{1}{4}} \frac{e^{\gamma t/4}}{\left[\left(\frac{2\pi i\hbar}{m\Omega}\right)\sin(\Omega t)\right]^{\frac{1}{2}}} \sqrt{\frac{\pi}{Q}} e^{-G(x-\langle x\rangle)^2}, \tag{18}$$

where

$$\begin{aligned} \langle x\rangle &= \frac{\hbar k_0}{m\omega}\sin(\omega t), \\ Q &= \beta_0 - \frac{im}{2\hbar}\omega\cot(\omega t), \\ G &= \frac{(m\omega)^2}{4\hbar^2 Q\sin^2(\omega t)^2} - \frac{im\omega}{2\hbar}\cot(\omega t). \end{aligned} \tag{19}$$

Eqs. (18)-(19) then give us

$$P(x,t) = \sqrt{\frac{\beta}{\pi}} e^{-\beta(x-\langle x\rangle)^2}, \tag{20}$$

where

$$\beta = \frac{2\beta_0}{\chi\sin^2(\omega t) + \cos^2(\omega t)}, \tag{21}$$

where $\chi = \left(\frac{2\hbar\beta_0}{m\omega}\right)^2$. It is worth noting that the apparent periodic variation of β in Eq. (21) disappears when the initial wave function in Eq. (4) is the energy eigenfunction for the ground state (i.e. $\chi = 1$); Eq. (21) reduces to $\beta = 2\beta_0 = \beta(0)$. That is, the ground state remains in the ground state, ψ keeping the original width, as expected. When the width of $\psi(x', 0)$ in Eq. (4) deviates from that of the ground state wave function, β in Eq. (21) oscillates in time with the frequency 2ω, the amplitude of oscillation increasing with the deviation. Examples are shown in Figure 1 for $\chi = \left(\frac{2\hbar\beta_0}{m\omega}\right)^2 = 0.5$ and 2 for $\omega = 1$.

The oscillation in β has an interesting consequence on $\mathcal{L}$. To see this, by using Eqs. (18)-(20) in Eq. (8), we find

$$\mathcal{E} = \frac{1}{2}\left|\frac{\omega(\chi-1)\sin(2\omega t)}{(\chi-1)\sin^2(\omega t)+1}\right|^2 + \frac{\chi\omega^2 k_0^2\cos^2(\omega t)}{(\chi-1)\sin^2(\omega t)+1}, \tag{22}$$

where again $\chi = \frac{4\hbar^2\beta_0^2}{m^2\omega^2}$. The first and second terms in Eq. (22) represent the change in the information due to the spreading of the wave packet/PDF and the movement of the

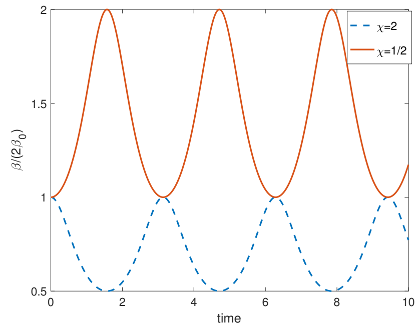

FIG. 1: The evolution of $\frac{\beta}{2\beta_0}$ for QSHO and $\chi = \left(\frac{2\hbar\beta_0}{m\omega}\right)^2 = 0.5$ and 2: $\omega = 1$.

wavepacket/PDF, respectively. In general, both terms in $\mathcal{E}$ oscillate with the frequency 2ω. For instance, the first term in Eq. (22) becomes zero whenever $t = n\frac{\pi}{2\omega}$, while the second term becomes zero at $t = (2n+1)\frac{\pi}{2\omega}$ (where n is any positive integer including zero). The periodicity in $\mathcal{E}$ leads to equally spaced increments in $\mathcal{L}$ every time interval $\frac{\pi}{2\omega}$.

To appreciate this, we first look at the case $\chi = 1$ when the initial $\psi(x', 0)$ is in the ground state. In this case, there is no change in the width of the PDF; the first term in Eq. (22) vanishes. $\mathcal{E}$ is solely from the second term due to the movement of the PDF peak, which leads to

$$\mathcal{L} = \frac{2\sqrt{\beta_0}\hbar k_0}{m} \int_0^t dt_1 \big|\cos(\omega t_1)\big| = l_* k_0 \Big[N + |\sin(\omega t)|_{t_0}^t\Big]. \tag{23}$$

Here, $N = \mathrm{mod}\big(\frac{t\omega}{\pi}\big)$ ($\beta_0 = \frac{m\omega}{2\hbar}$) and $t_0 = \frac{N\pi}{2\omega}$. $l_* = \sqrt{\frac{2\hbar}{m\omega}}$ is the characteristic length scale in QSHO, representing the amount by which the oscillator must be displaced from its centre in order for the potential energy to be equal to the quantized ground state energy $\frac{1}{2}\hbar\omega$ (i.e., $\frac{1}{2}\hbar\omega \sim \frac{1}{2}m\omega^2 l_*^2$). Alternatively, l_* provides the resolution in space in view of the change in potential energy. Two spatial points with a distance less than l_* apart are indistinguishable as the difference in the potential energy between these two points is less than the ground state energy. The periodic part of $\mathcal{L}$ in Eq. (23) permits us to define $\mathcal{L}_0 = l_* k_0$ as a quantum $\mathcal{L}$ so that $\mathcal{L} = N\mathcal{L}_0+$ (continuous part). Eq. (23) then means that whenever the QSHO changes its mean position by l_*, it increases the information length by $\mathcal{L}_0$. Using $k_0 = p_0/\hbar$, we can see that $\mathcal{L}_0 = l_* k_0 = \frac{l_* p_0}{\hbar}$, representing one unit of the information change due to the change in phase volume by $\hbar$ in the $p - x$ phase space, similarly to the case of free particle wave packets discussed above.

In general when $\chi \neq 1$, it is instructive to look at the effects of the first and second terms in Eq. (22) separately. First, the spreading of wave packets/PDF, the first term in Eq. (22), gives

$$\mathcal{L}\left(\frac{\pi}{2\omega},0\right)=\mathcal{L}\left(\frac{\pi}{\omega},\frac{\pi}{2\omega}\right)=\mathcal{L}\left(\frac{3\pi}{2\omega},\frac{\pi}{\omega}\right)=\mathcal{L}\left(\frac{2\pi}{\omega},\frac{3\pi}{2\omega}\right)=\cdots=\frac{1}{\sqrt{2}}|\ln\chi|, \tag{24}$$

where $\chi=\frac{4\hbar^2\beta_0^2}{m^2\omega^2}$. Thus, at time t, we have

$$\sqrt{2}\mathcal{L}(t,0)=M|\ln\chi|+\left|\ln\left[(\chi-1)\sin^2(\omega t)+1\right]\right|_{t_0}^{t}, \tag{25}$$

where $M=\mathrm{mod}\left(\frac{2t\omega}{\pi}\right)$ and $t_0=\frac{M\pi}{2\omega}$. Therefore, the first term in Eq. (25) gives the periodic increase in $\sqrt{2}\mathcal{L}$ by $\sqrt{2}\mathcal{L}_0=|\ln\chi|=2\left|\ln\frac{2\hbar\beta_0}{m\omega}\right|=2|\ln(l_*^2\beta_0)|$. The factor of $l_*\beta_0^{1/2}\sim\frac{l_*}{d_x}$ is quite interesting; it demonstrates the increase in $\mathcal{L}$ by a factor $\propto\ln\left(\frac{d_x}{l_*}\right)$ for a wider PDF with $d_x>l_*$ while by a factor $\propto\ln\left(\frac{l_*}{d_x}\right)$ for a narrow PDF with $d_x<l_*$, the smaller between d_x and l_* serving a length unit. This is a purely quantum effect resulting from energy quantisation - energy setting the characteristic length scale l_x. Here, the logarithmic dependence of $\mathcal{L}_0$ on $l_*^2\beta_0$ reflects that the change in the information is associated with the change in the PDF width (or entropy increase).

Secondly, the movement of the wave packets/PDF, the second term in Eq. (22), gives

$$\mathcal{L}=\frac{2\sqrt{\beta_0}\hbar k_0}{m\omega\sqrt{|\chi-1|}}\sinh^{-1}\left(\sqrt{|\chi-1|}\,\sin(\omega t)\right), \tag{26}$$

recovering Eq. (23) in the limit of $\chi\to1$. Eq. (26) also shows a periodic increase in $\mathcal{L}$, similarly to Eq. (23), the amount of the increase $\propto\frac{2\sqrt{\beta_0}\hbar k_0}{m\omega}=l_*^2k_0\sqrt{\beta_0}$. The difference now is that $\mathcal{L}$ is determined by a geometric mean of l_*k_0 and $l_*\sqrt{\beta_0}$ (the initial width of the PDF) due to the PDF broadening.

B. QDHO

When $\gamma\neq0$, the exact ground state wave function at $t=0$ is

$$\psi(x',0)=\left[\frac{m\Omega}{\pi\hbar}\right]^{\frac{1}{4}}e^{-\frac{m\Omega}{2\hbar}(1+i\gamma/2\Omega)x'^2}. \tag{27}$$

For simplicity, we now consider an arbitrary width $1/2\beta_0$ but $k_0=0$:

$$\psi(x',0)=\left[\frac{2\beta_0}{\pi}\right]^{\frac{1}{4}}e^{-\beta_0(1+i\gamma/2\Omega)x'^2}. \tag{28}$$

Note that Eq. (28) with $\beta_0 = \frac{m\Omega}{2\hbar}$ in Eq. (28) recovers Eq. (27). Using Eq. (28) and Eqs. (17)-(18), we find

$$\psi(x,t) = \left[\frac{2\beta_0}{\pi}\right]^{\frac{1}{4}} \frac{e^{\gamma t/4}}{\left[\left(\frac{2\pi i\hbar}{m\Omega}\right)\sin(\Omega t)\right]^{\frac{1}{2}}} \sqrt{\frac{\pi}{\overline{Q}}} e^{-\overline{G}\overline{x}^2}, \tag{29}$$

where

$$\begin{aligned} \overline{Q} &= \beta_0\left(1 + i\frac{\gamma}{2\Omega}\right) - \frac{im}{2\hbar}\left[\Omega\cot(\Omega t) + \frac{\gamma}{2}\right], \\ \overline{G} &= \frac{(m\Omega)^2}{4\hbar^2\overline{Q}\sin^2(\Omega t)^2} - \frac{im}{2\hbar}\left[\Omega\cot(\Omega t) - \frac{\gamma}{2}\right]. \end{aligned} \tag{30}$$

From Eqs. (29)-(30), we then find $P(x,t) = |\psi(x,t)|^2$ as follows:

$$P(x,t) = \sqrt{\frac{\overline{\beta}}{\pi}} e^{-\overline{\beta}x^2}, \tag{31}$$

where

$$\overline{\beta} = \left(\frac{m\Omega}{2\hbar}\right)^2 \frac{2\beta_0 e^{\gamma t}}{\beta_0^2\sin^2(\Omega t) + \left(\frac{m}{2\hbar}\right)^2\left[\Omega\cos(\Omega t) + \frac{\overline{\gamma}}{2}\sin(\Omega t)\right]^2}, \tag{32}$$

where $\overline{\gamma} = \gamma(1 - \frac{2\beta_0\hbar}{\Omega m})$. It is of interest to look at the case where $\psi(x',0)$ is the ground state satisfying $\beta_0 = \frac{\Omega m}{2\hbar}$, $\overline{\gamma} = 0$. Thus, Eq. (32) is reduced to

$$\overline{\beta} = \frac{m\Omega}{\hbar} e^{\gamma t}, \tag{33}$$

leading to

$$\sqrt{2\mathcal{E}} = \gamma, \quad \sqrt{2}\mathcal{L} = \gamma t. \tag{34}$$

Eq. (34) means that the damping γ is the very source of the information change in QDHO, the information length linearly increasing with time.

When $\beta_0 \neq \frac{\Omega m}{2\hbar}$, we can show that

$$\sqrt{2}\mathcal{L} = \left|\gamma t - \left[S|\ln\lambda| + \left|\ln\left[(\lambda-1)\sin^2(\Omega t) + 1\right]\right|_{t_0}^{t}\right]\right|, \tag{35}$$

where $\lambda = \frac{\overline{\gamma}}{2\Omega} + \left(\frac{2\hbar\beta_0}{m\Omega}\right)^2$, $S = \mathrm{mod}\left(\frac{2t\Omega}{\pi}\right)$ and $t_0 = \frac{S\pi}{2\Omega}$. While the second term in Eq. (35) gives the periodic increase in $\mathcal{L}$ as in Eq. (24), the first term dominates for large t, leading to $\mathcal{L} \propto \gamma t/\sqrt{2}$. That is, damping is the main source for $\mathcal{L}$.

V. CONCLUSION

In summary, we demonstrated that $\mathcal{L}$ proves to be valuable in measuring the information change as a quantum system continuously evolves in time. In particular, we elucidated consequences of quantum effects (uncertainty relation, energy quantisation) and the dual role of the width of PDF in quantum systems (PDF width can either increase or decrease $\mathcal{L}$). An interesting geodesic was obtained for Gaussian wave packets under a constant force as the constant force compensates the effect of PDF spreading via quantum effects. We also demonstrated the utility of $\mathcal{L}$ in quantifying the periodic information change due to the oscillation in phase space in QSHO and exponential damping in QDHO. The exploration of the relation between $\tau(t) = \mathcal{E}^{-1/2}$ and life time of quantum states (e.g., see [24] and references therein) and extension of this work are left for future papers.

Appendix A: Statistical distance

In Hilbert space, Wootters [13] defined the statistical distance between $\psi_1 = \psi(x, 0)$ and $\psi_2 = \psi(x, t)$ by using the Fisher-Rao metric (Fisher information metric) as

$$l(\psi_1, \psi_2) = \frac{1}{2} \int_0^t dz_1 \sqrt{\int dx \frac{1}{p(x, z_1)} \left[\frac{\partial p(x, z_1)}{\partial z_1} \right]^2}, \tag{A1}$$

where z_1 is a parameterisation of a curve between ψ_1 and ψ_2. The shortest path connecting ψ_1 and ψ_2 was shown to be the geodesic with the distance $l = \cos^{-1}\left(| \langle \psi_1 | \psi_2 \rangle | \right)$, depending only on the angle between ψ_1 and ψ_2; the number of distinguishability of probability distributions between ψ_1 and ψ_2 is proportional to the angle between ψ_1 and ψ_2. Our $\mathcal{L}$ in Eq. (1) is different from Eq. (A1) since t_1 in Eq. (1) is a real clock time and a path between between $\psi(x, 0)$ and $\psi(x, t)$ is determined by the solution $\psi(t)$. That is, for a given solution $\psi(t)$, there is a unique path that is connecting ψ_1 and ψ_2, which is not in general a shortest path; $\psi(t)$ follows a geodesic only in the case when $\mathcal{E}$ in Eq. (1) is constant.

---

[1] E.-W. Saw, D. Kuzzay, D. Faranda, A. Guittonneau, F. Daviaud, C. Wiertel-Gasquet, V. Padilla and B. Dubrulle, Nat. Commun. **7**:12466, doi: 10.1038/ncomms12466 (2016).

[2] A.P.L. Newton, E. Kim and H.-L. Liu, Phys. Plasmas **20**, 092306 (2013).

[3] H.J. Jensen, Self-organized Criticality: Emergent Complex Behavior in Physical and Biological Systems (Cambridge Univ. Press, 1998).

[4] S.B. Nicholson and E. Kim, Phys. Lett. A. **379**, 8388 (2015).

[5] S.B. Nicholson and E. Kim, Entropy **18**, 258, e18070258 (2016).

[6] J. Heseltine and E. Kim, J. Phys. A **49**, 175002 (2016).

[7] E. Kim, U. Lee, J. Heseltine and R. Hollerbach, Phys. Rev. E **93**, 062127 (2016).

[8] E. Kim and R. Hollerbach, Phys. Rev. E **95**, 022137 (2017).

[9] R. Hollerbach and E. Kim, Entropy **19**, 268 (2017).

[10] E. Kim, L.-M. Tenkès, R. Hollerbach and O. Radulescu, Entropy **19**, 511 (2017).

[11] B.R. Frieden, Physics from Fisher Information (Cambridge Univ. Press, 2000).

[12] M. M Wilde, arXiv:1106.1445 (2011).

[13] W.K. Wootters, Phys. Rev. D, **23**, 357 (1981).

[14] P. Caldirola, Nuovo Cimento **18**, 393 (1941); E. Kanai, Prog. Theor. Phys. **3**, 440 (1948).

[15] J. R. Brinati and S. S. Mizrahi, J. Math. Phys. **21**, 2154 (1980); E. S. Hernandez and S. S. Mizrahi, Physica A **121**, 587 (1983).

[16] M. A. Marchiolli and S. S. Mizrahi, J. Phys. A **30**, 2619 (1997).

[17] Z.-Q. Shao, Y.-Q. Li and X.-Y. Pan, J. Chem. Phys. **141**, 224110 (2014).

[18] E. J. Jang and W. S. Chung, Adv. Stud. Theo. Phys. **11**, 263 (2017).

[19] L.F. Landovitz, A.M. Levine and W.M. Schreiber, Phys. Rev. A **29**, 1162 (1979).

[20] L.H. Yu and C.-P. Sun, Phys. Rev. A **49**, 592 (1994).

[21] T.G. Philbin, New J. Phys. **14**, 083043 (2012).

[22] A.D. Jannussis, G.N. Brodimas and A. Streclas, Phys. Lett. A **74**, 6 (1979).

[23] R.W. Robinett and L.C. Bassett, Foundation of Phys. Lett. **17**, 607 (2004).

[24] S. Deffner and S. Campbell, J. Phys. A: Math. & Theo. **50**, 453001 (2017).