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BOUNDED ORBITS OF DIAGONALIZABLE FLOWS ON FINITE VOLUME QUOTIENTS OF PRODUCTS OF $\mathrm{SL}_2(\mathbb{R})$

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Abstract. We prove a number field analogue of W. M. Schmidt's conjecture on the intersection of weighted badly approximable vectors and use this to prove an instance of a conjecture of An, Guan and Kleinbock [4]. Namely, let $G := \mathrm{SL}_2(\mathbb{R}) \times \cdots \times \mathrm{SL}_2(\mathbb{R})$ and Γ be a lattice in G. We show that the set of points on G/Γ whose forward orbits under a one parameter Ad-semisimple subsemigroup of G are bounded, form a *hyperplane absolute winning* set.

1. Introduction

Let G be a Lie group. We will say that $g \in G$ is Ad-semisimple if Ad_g is diagonalizable over $\mathbb{C}$ and Ad-diagonalizable if Ad_g is diagonalizable over $\mathbb{R}$. In this paper, we prove the following theorem that verifies some cases of [4, Conjecture 7.1]:

Theorem 1.1. *Let $G = \mathrm{SL}_2(\mathbb{R}) \times \cdots \times \mathrm{SL}_2(\mathbb{R})$ be a finite product of copies of $\mathrm{SL}_2(\mathbb{R})$ and let Γ be a lattice subgroup of G. Then for any one parameter Ad-semisimple subsemigroup $F^+ = \{g_t : t \geq 0\}$ of G, the set*

$$E(F^+) := \{x \in G/\Gamma : F^+x \text{ is bounded}\}$$

is Hyperplane Absolute Winning (HAW).

The action of a subsemigroup as in Theorem 1.1 on the finite volume homogeneous space G/Γ is ergodic and as a consequence, the set $E(F^+)$ has zero (Haar) measure. One consequence of the HAW property proved in Theorem 1.1 is that it nevertheless has full Hausdorff dimension. In fact, the HAW property is much richer and HAW sets exhibit many more interesting properties in addition to being of full Hausdorff dimension. The conjecture of An, Guan and Kleinbock predicts that $E(F^+)$ is HAW for G any Lie group, Γ any lattice in G and F^+ any Ad-diagonalizable subsemigroup of G. In the same paper, this conjecture is verified for $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. This type of result goes back to the work of S. G. Dani [10], from whose work the AGK conjecture can be verified for real rank 1 Lie groups. As observed by Dani, the study of bounded orbits of diagonalizable flows on homogeneous spaces is intimately related to the study of *badly approximable* numbers or matrices. This connection will also be important in the present work. In particular, along the way to proving the main theorem, we will prove (cf. Proposition 3.5 below) a number

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field analogue of W. M. Schmidt's [23] celebrated conjecture on intersections of weighted badly approximable vectors. We believe this result to be of independent interest.

Following Dani's influential paper, there have been significant advances both in the understanding of bounded orbits of diagonalizable flows on homogeneous spaces, as well as in the study of badly approximable numbers and vectors. On the homogeneous side, we mention the conjectures of Margulis, resolved by Kleinbock and Margulis [15], the work of Kleinbock [13], Kleinbock-Weiss [16, 17] and An-Guan-Kleinbock [4]. On the number theoretic side, we mention conjectures of W. M. Schmidt, resolved by Badziahin, Pollington and Velani [5] and their subsequent strengthening in different contexts, by An [1, 2], Beresnevich [6] and An, Beresnevich and Velani [3]. We refer the reader to these works for the history of the problems as well as a more comprehensive list of results and references. Pertinent to the present work is the paper [11] of Einsiedler, Ghosh and Lytle where some special cases of Theorem 1.1 were established, namely the cases

(1) $G = \mathrm{SL}_2(\mathbb{R}) \times \cdots \times \mathrm{SL}_2(\mathbb{R}), \Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ and $\{F^+ = g_t \ : \ t \geq 0\}$ where
$$g_t := \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \dots, \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right).$$
In [11], $E(F^+)$ was shown to be winning for Schmidt's game. In fact, a more general result, involving points in C^1 curves whose forward orbits are bounded, was proved. In [14], it was subsequently shown that the set $E(F^+)$ is winning for a stronger version of Schmidt's game.

(2) In [11], the case of K a real quadratic field, $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ and $\{F^+ = g_t \ : \ t \geq 0\}$ where
$$g_t := \left(\begin{pmatrix} e^{r_{\sigma_1} t} & 0 \\ 0 & e^{-r_{\sigma_1} t} \end{pmatrix}, \begin{pmatrix} e^{r_{\sigma_2} t} & 0 \\ 0 & e^{-r_{\sigma_2} t} \end{pmatrix} \right)$$
was also considered. Here $r_{\sigma_i} \geq 0$ and $r_{\sigma_1} + r_{\sigma_2} = 1$.

In §2 we record preliminaries on the hyperplane absolute game and the hyperplane potential game. These are variants of the classical game introduced by W. M. Schmidt [21]. The subsequent two sections are devoted to the proof of a special case of Theorem 1.1, namely when $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ where K is a totally real field of degree d over $\mathbb{Q}$, $\mathcal{O}_K$ is its ring of integers and Γ is a lattice in $G = \mathrm{SL}_2(\mathbb{R}) \times \cdots \times \mathrm{SL}_2(\mathbb{R})$ via the Galois embedding. This particular case of our theorem is connected to Diophantine approximation of vectors in $\mathbb{R}^d$ by rationals in the number field K. Indeed, this case is the generalisation of the result of [11] in (2) above. This case forms the bulk of our paper and is intimately connected to the number field analogue of Schmidt's conjecture mentioned above. We use a transference ("the Dani correspondence") to relate this case to the HAW property of certain vectors badly approximable by rationals in K and prove this latter property. Finally we use the structure theory of Lie groups and Margulis arithmeticity theorem to conclude the proof of Theorem 1.1. We conclude the introduction with some remarks:

(1) Let G be a Lie group, Γ be a lattice in G and $F^+ = \{g_t \,:\, t \geq 0\}$ a one-parameter diagonalizable subsemigroup of G. Consider the expanding horospherical subgroup of G relative to F, namely

$$H(F^+) := \{h \in G \,:\, \lim_{t \to +\infty} g_t^{-1} h g_t = e\}.$$

In [4, Theorem 1.3], it is shown that for $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and any $\Lambda \in G/\Gamma$, the set

$$\{h \in H \,:\, h\Lambda \in E(F^+)\}$$

is HAW on H. They further conjecture the same statement for arbitrary Lie groups and lattices. This conjecture can also be proved in the setting of the present paper by adapting the technique of [4].

(2) It is plausible that the method of proof developed in the present paper can be used to deal with the case where G consists of products of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. Indeed the main argument would then be carried out with an arbitrary number field rather than a totally real number field.

(3) Proposition 3.5 below, i.e. the number field analogue of Schmidt's conjecture, can be formulated for arbitrary number fields rather than just totally real ones. The proof is identical to the one presented here; we have restricted ourselves to totally real fields for notational ease.

(4) In Theorem 4.2 in his thesis [18], the last named author proved a more general version of Proposition 3.1 below. Specifically, the notion of winning used is slightly more general and a higher dimensional analogue of $\mathbf{Bad}(K, \mathbf{r})$ (defined below) is considered. This result can be used to verify [4, Conjecture 7.1] in some more cases, namely for certain Ad-semisimple one parameter flows on some special quotients of products of $\mathrm{SL}_n(\mathbb{R})$.

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2. Preliminaries on Schmidt games

In this section, we will recall definitions of certain recent variants of Schmidt games, namely, the hyperplane absolute game and the hyperplane potential game. We follow the exposition in [4]. They are both variants of the (α, β)-game introduced by Schmidt in [21]. Since we do not make a direct use of the (α, β)-game in this paper, we omit its definition here and refer the interested reader to [21, 22]. Instead, we list here some nice properties of the α-winning sets:

(1) If the game is played on a Riemannian manifold, then any α-winning set is thick.
(2) The intersection of countably many α-winning sets is α-winning.

2.1. **Hyperplane absolute game.** The hyperplane absolute game was introduced in [8]. It is played on a Euclidean space $\mathbb{R}^d$. Given a hyperplane L and a $\delta > 0$, we denote by $L^{(\delta)}$ the δ-neighborhood of L, i.e.,

$$L^{(\delta)} := \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}(\mathbf{x}, L) < \delta\}.$$

For $\beta \in (0, \frac{1}{3})$, the *$\beta$-hyperplane absolute game* is defined as follows. Bob starts by choosing a closed ball $B_0 \subset \mathbb{R}^d$ of radius ρ_0. In the i-th turn, Bob chooses a closed ball B_i with radius ρ_i, and then Alice chooses a hyperplane neighborhood $L_i^{(\delta_i)}$ with $\delta_i \leq \beta\rho_i$. Then in the $(i+1)$-th turn, Bob chooses a closed ball $B_{i+1} \subset B_i \setminus L_i^{(\delta_i)}$ of radius $\rho_{i+1} \geq \beta\rho_i$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots .$$

We say that a subset $S \subset \mathbb{R}^d$ is *β-hyperplane absolute winning* (*β-HAW* for short) if no matter how Bob plays, Alice can ensure that

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \emptyset.$$

We say S is *hyperplane absolute winning* (*HAW* for short) if it is β-HAW for any $\beta \in (0, \frac{1}{3})$.

We have the following lemma collecting the basic properties of β-HAW subsets and HAW subsets of $\mathbb{R}^d$ ([8], [17]):

Lemma 2.1. (1) *A HAW subset is always $\frac{1}{2}$-winning.*
(2) *Given $\beta, \beta' \in (0, \frac{1}{3})$, if $\beta \geq \beta'$, then any β'-HAW set is β-HAW.*
(3) *A countable intersection of HAW sets is again HAW.*
(4) *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 diffeomorphism. If S is a HAW set, then so is $\varphi(S)$.*

The notion of HAW was extended to subsets of C^1 manifolds in [17]. This is done in two steps. First, one defines the hyperplane absolute game on an open subset $W \subset \mathbb{R}^d$. It is defined just as the hyperplane absolute game on $\mathbb{R}^d$, except for requiring that Bob's first move B_0 be contained in W. Now, let M be a d-dimensional C^1 manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be a C^1 atlas on M. A subset $S \subset M$ is said to be HAW on M if for each α, $\phi_\alpha(S \cap U_\alpha)$ is HAW on $\phi_\alpha(U_\alpha)$. The definition is independent of the choice of atlas by the property (4) listed above. We have the following lemma that collects the basic properties of HAW subsets of a C^1 manifold (cf. [17]).

Lemma 2.2. (1) *HAW subsets of a C^1 manifold are thick.*
(2) *A countable intersection of HAW subsets of a C^1 manifold is again HAW.*
(3) *Let $\phi : M \to N$ be a diffeomorphism between C^1 manifolds, and let $S \subset M$ be a HAW subset of M. Then $\phi(S)$ is a HAW subset of N.*
(4) *Let M be a C^1 manifold with an open cover $\{U_\alpha\}$. Then, a subset $S \subset M$ is HAW on M if and only if $S \cap U_\alpha$ is HAW on U_α for each α.*
(5) *Let M_1, M_2 be C^1 manifolds, and let $S_i \subset M_i$ $(i = 1, 2)$ be HAW subsets of M_i. Then $S_1 \times S_2$ is a HAW subset of $M_1 \times M_2$.*

Indeed, everything except (5) is proved in [17]. So we provide a proof of (5) here.

Proof of Lemma 2.2 (5). In view of (3) and (4) of Lemma 2.2, (5) is a direct consequence of the following claim:

$$\text{Let } U_i\ (i=1,2) \text{ be Euclidean opens and } S_i \subset U_i \text{ be } \beta^2\text{-HAW. Then } S_1 \times S_2 \text{ is } \beta\text{-HAW.} \tag{2.1}$$

Let B_{2i} be the closed ball of radius ρ_{2i} chosen by Bob at the $2i$-th turn of a β-hyperplane absolute game. Take closed balls $V_i \subset U_1$ (resp. $W_i \subset U_2$) of radius ρ_{2i} such that $B_{2i} \subset V_i \times W_i$. Since $S_i \subset U_i$ are β^2-HAW, we are given hyperplane neighborhoods $L_{i,1}^{(\delta_{i,1})} \subset U_1$ and $L_{i,2}^{(\delta_{i,2})} \subset U_2$ with $\delta_{i,1}, \delta_{i,2} \leq \beta^2 \rho_i$ according to the winning strategy. By choosing hyperplane neighborhoods $L_{i,1}^{(\delta_{i,1})} \times U_2$ (resp. $U_1 \times L_{i,2}^{(\delta_{i,2})}$) at the $2i$-th turn (resp. $(2i+1)$-th turn), Alice can make sure that

$$B_{2i+2} \subset V_i \times W_i \setminus \left(L_{i,1}^{(\delta_{i,1})} \times U_2 \cup U_1 \times L_{i,2}^{(\delta_{i,2})} \right).$$

By this process, we can make sure that the outcome point is contained in $S_1 \times S_2$. Hence claim (2.1) is proved. □

2.2. **Hyperplane potential game.** The hyperplane potential game was introduced in [12] and also defines a class of subsets of $\mathbb{R}^d$ called *hyperplane potential winning* (*HPW* for short) sets. The following lemma allows one to prove the HAW property of a set $S \subset \mathbb{R}^d$ by showing that it is winning for the hyperplane potential game. And this is exactly the game we will use in this paper.

Lemma 2.3. (cf. [12, Theorem C.8]) *A subset S of $\mathbb{R}^d$ is HPW if and only if it is HAW.*

The hyperplane potential game involves two parameters $\beta \in (0,1)$ and $\gamma > 0$. Bob starts the game by choosing a closed ball $B_0 \subset \mathbb{R}^d$ of radius ρ_0. In the i-th turn, Bob chooses a closed ball B_i of radius ρ_i, and then Alice chooses a countable family of hyperplane neighborhoods $\{L_{i,k}^{(\delta_{i,k})} : k \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{\infty} \delta_{i,k}^{\gamma} \leq (\beta \rho_i)^{\gamma}.$$

Then in the $(i+1)$-th turn, Bob chooses a closed ball $B_{i+1} \subset B_i$ of radius $\rho_{i+1} \geq \beta \rho_i$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots.$$

We say a subset $S \subset \mathbb{R}^d$ is (β,γ)-*hyperplane potential winning* ((β,γ)-*HPW* for short) if no matter how Bob plays, Alice can ensure that

$$\bigcap_{i=0}^{\infty} B_i \cap \left(S \cup \bigcup_{i=0}^{\infty} \bigcup_{k=1}^{\infty} L_{i,k}^{(\delta_{i,k})} \right) \neq \emptyset.$$

We say S is *hyperplane potential winning* (*HPW* for short) if it is (β,γ)-HPW for any $\beta \in (0,1)$ and $\gamma > 0$.

3. A SPECIAL CASE

This and the next section are devoted to prove a special case of Theorem 1.1. We begin by introducing some notation. Let K be a totally real field of degree d over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers, and S be the set of field embeddings $K \hookrightarrow \mathbb{R}$. Then we have $|S| = d$. Set

$$\theta : K \to \prod_{\sigma\in S} \mathbb{R}, \quad \theta(p) = (\sigma(p))_{\sigma\in S}.$$

Let $\mathrm{Res}_{K/\mathbb{Q}}$ denote Weil's restriction of scalar's functor. It is well known ([7, Theorem 7.8]) that the group $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{R})$. The latter coincides with the product of d copies of $\mathrm{SL}_2(\mathbb{R})$. For simplicity, in this section and the next, we set

$$G = \mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{R}) = \prod_{\sigma\in S} \mathrm{SL}_2(\mathbb{R}), \quad \Gamma = \mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{Z}).$$

It follows from the definition that the subgroup $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{Z})$ coincides with the subgroup $\theta(\mathrm{SL}_2(\mathcal{O}_K))$, where θ is the map defined by $\theta(g) = (\sigma(g))_{\sigma\in S}$. Now we are ready to state the following special case of Theorem 1.1.

Proposition 3.1. *Let $\mathbf{r} \in \mathbb{R}^d$ be a real vector with $r_\sigma \geq 0$ for $\sigma \in S$ and $\sum_{\sigma\in S} r_\sigma = 1$, set*

$$g_{\mathbf{r}}(t) := \left(\begin{pmatrix} e^{r_\sigma t} & 0 \\ 0 & e^{-r_\sigma t}\end{pmatrix}\right)_{\sigma\in S} \tag{3.1}$$

and $F_{\mathbf{r}}^+ = \{g_{\mathbf{r}}(t) : t \geq 0\}$, then the set

$$E(F_{\mathbf{r}}^+) := \{x \in G/\Gamma : F_{\mathbf{r}}^+ x \text{ is bounded }\}$$

is HAW.

We will fix $\mathbf{r}$ in this and the next section. Set

$$S_1 = \{\sigma \in S : r_\sigma > 0\}, \text{ and } S_2 = S \setminus S_1.$$

Assume $|S_1| = d_1, |S_2| = d_2$. Choose and fix $\omega \in S$ with $r_\omega = r$, where

$$r = \max_{\sigma\in S} r_\sigma.$$

Define a weighted norm, called the $\mathbf{r}$-*norm*, on $\prod_{\sigma\in S}\mathbb{R}$ by

$$\|\mathbf{x}\|_{\mathbf{r}} = \max_{\sigma\in S_1} |x_\sigma|^{\frac{1}{r_\sigma}}.$$

Definition 3.2. Say a vector $\mathbf{x} = (x_\sigma)_{\sigma\in S} \in \prod_{\sigma\in S}\mathbb{R}$ is $(K,\mathbf{r})$-*badly approximable* if

$$\inf_{\substack{q\in\mathcal{O}_K\setminus\{0\}\\ p\in\mathcal{O}_K}} \max\left\{\max_{\sigma\in S_1} \|q\|_{\mathbf{r}}^{r_\sigma}|\sigma(q)x_\sigma + \sigma(p)|, \max_{\sigma\in S_2}\max\{|\sigma(q)x_\sigma+\sigma(p)|, |\sigma(q)|\}\right\} > 0.$$

The set of $(K,\mathbf{r})$-badly approximable vectors is denoted as $\mathbf{Bad}(K,\mathbf{r})$.

Remark 3.3. The notation of $(K,\mathbf{r})$-badly approximable vector is the weighted case of K-badly approximable vector introduced in [11].

Denote the expanding horospherical subgroup of the semigroup $F_{\mathbf{r}}^+$ as $H = H(F_{\mathbf{r}}^+)$. Then it can be easily seen that H can be identified with $\prod_{\sigma\in S}\mathbb{R}$ through the map:

$$u : \prod_{\sigma\in S}\mathbb{R} \to \prod_{\sigma\in S}\mathrm{SL}_2(\mathbb{R}), \quad u((x_\sigma)_{\sigma\in S}) = \left(\begin{pmatrix} 1 & x_\sigma \\ 0 & 1\end{pmatrix}\right)_{\sigma\in S}$$

Then we have the following correspondence between $(K,\mathbf{r})$-badly approximable vector and bounded $F_{\mathbf{r}}^+$ trajectories known in the literature as the "Dani" correspondence.

Proposition 3.4. *A vector $\mathbf{x} = (x_\sigma)_{\sigma\in S}$ is $(K,\mathbf{r})$-badly approximable if and only if the trajectory $F_{\mathbf{r}}^+u(\mathbf{x})\Gamma$ is bounded in G/Γ. In other words,*

$$\mathbf{Bad}(K,\mathbf{r}) = u^{-1}\big(\pi^{-1}(E(F_{\mathbf{r}}^+))\cap H\big), \tag{3.2}$$

where π denotes the projection $G\to G/\Gamma$.

Proof. For simplicity, denote the elements in S as $\{\sigma_1,\dots,\sigma_d\}$ and the weights r_{σ_i} as r_i. Without loss of generality, we may assume that $r_i > 0$ for $1\le i\le d_1$ and $r_i = 0$ for $d_1 < i \le d$. It is easily seen that $D_K^{-\frac{1}{2d}}\theta(\mathcal{O}_K)$ forms a unimodular lattice of $\mathbb{R}^d$, where D_K is the discriminant of K. Write the lattice $D_K^{-\frac{1}{2d}}\theta(\mathcal{O}_K)\times D_K^{-\frac{1}{2d}}\theta(\mathcal{O}_K)\subset\mathbb{R}^{2d}$ simply as L_K. Then define a homomorphism $\psi : G\to \mathrm{SL}_{2d}(\mathbb{R})$ by

$$\psi(g)_{ij} = \begin{cases} a_i, & \text{if } 1\le i = j\le d,\\ b_i, & \text{if } 1\le i = j-d\le d,\\ c_{i-d}, & \text{if } 1\le i-d = j\le d,\\ d_{i-d}, & \text{if } d+1\le i = j\le 2d,\\ 0, & \text{otherwise,}\end{cases} \qquad \text{where } g = \left(\begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}\right)_{1\le i\le d}. \tag{3.3}$$

Now we claim that

$$\{g\in G : \psi(g)L_K = L_K\} = \Gamma \tag{3.4}$$

Let g be as in (3.3). At first, we focus on the study of $(\psi(g))_1 = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$. If $\psi(g)L_K = L_K$, it follows that $a_1\sigma_1(k)+b_1\sigma_1(k'), c_1\sigma_1(k)+d_1\sigma_1(k')\in\sigma_1(\mathcal{O}_K)$ for all $k,k'\in\mathcal{O}_K$. By choosing k or k' to be 0, we can show that $f\sigma_1(\mathcal{O}_K) = \sigma_1(\mathcal{O}_K)$ for $f = a_1,b_1,c_1,d_1$. Hence it follows from the definition of $\mathcal{O}_K$ that the matrix $(\psi(g))_1$ has all its entries in $\sigma_1(\mathcal{O}_K)$. Consequently,

$$(\psi(g))_1\in M_{2\times 2}(\sigma_1(\mathcal{O}_K))\cap\mathrm{SL}_2(\mathbb{R}) = \sigma_1(\mathrm{SL}_2(\mathcal{O}_K)).$$

Then since an element in $\mathrm{SL}_2(\mathbb{R})$ is uniquely determined by its action on $\mathbb{R}^2$, it follows that, if $\psi(g)L_K = L_K$, then $(\psi(g))_i = \sigma_i\sigma_1^{-1}\big((\psi(g))_1\big)$. This shows that $\psi(g)\in\theta(\mathrm{SL}_2(\mathcal{O}_K)) = \Gamma$, hence proves claim (3.4).

As Γ is a lattice in G, in view of claim (3.4) and [20, Theorem 1.13], we find that the embedding

$$\phi : G/\Gamma\to\mathrm{SL}_{2d}(\mathbb{R})/\mathrm{SL}_{2d}(\mathbb{Z}), \quad \phi(g\Gamma) = \psi(g)L_K$$

is a proper map. Note that here we use the fact that the space $\mathrm{SL}_{2d}(\mathbb{R})/\mathrm{SL}_{2d}(\mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^{2d}$ implicity. Hence it follows that:

$$F_{\mathbf{r}}^{+}u(\mathbf{x})\Gamma \text{ is bounded in } G/\Gamma \iff \psi(F_{\mathbf{r}}^{+}u(\mathbf{x}))L_K \text{ is bounded in } \mathrm{SL}_{2d}(\mathbb{R})/\mathrm{SL}_{2d}(\mathbb{Z}). \tag{3.5}$$

Note that we have

$$\psi(g_{\mathbf{r}}(t)) = \mathrm{diag}(e^{r_1 t}, \dots, e^{r_d t}, e^{-r_1 t}, \dots, e^{-r_d t}),$$

and

$$\psi(u(\mathbf{x})) = \begin{pmatrix} I_d & \mathrm{diag}(\mathbf{x}) \\ & I_d \end{pmatrix}, \text{ where } \mathrm{diag}(\mathbf{x}) = \mathrm{diag}(x_1, \dots, x_d).$$

In view of Mahler's criterion and (3.5), we have

$$\begin{aligned}
& F_{\mathbf{r}}^{+}u(\mathbf{x})\Gamma \text{ is bounded in } G/\Gamma \\
\iff & \psi(F_{\mathbf{r}}^{+}u(\mathbf{x}))L_K \text{ is bounded in } \mathrm{SL}_{2d}(\mathbb{R})/\mathrm{SL}_{2d}(\mathbb{Z}) \\
\iff & \inf_{p,q\in\mathcal{O}_K} \inf_{t>0} \max \left\{ \begin{array}{l} \max\limits_{1\le i\le d_1} \max\{e^{r_i t}|\sigma_i(q)x_i + \sigma_i(p)|, e^{-r_i t}|\sigma_i(q)|\}, \\ \max\limits_{d_1<i\le d} \max\{|\sigma_i(q)x_i + \sigma_i(p)|, |\sigma_i(q)|\} \end{array} \right\} > 0 \\
\iff & \inf_{\substack{q\in\mathcal{O}_K\setminus\{0\} \\ p\in\mathcal{O}_K}} \inf_{t>0} \max \left\{ \begin{array}{l} \max\limits_{1\le i\le d_1} \max\{e^{t}|\sigma_i(q)x_i + \sigma_i(p)|^{\frac{1}{r_i}}, e^{-t}|\sigma_i(q)|^{\frac{1}{r_i}}\}, \\ \max\limits_{d_1<i\le d} \max\{|\sigma_i(q)x_i + \sigma_i(p)|, |\sigma_i(q)|\} \end{array} \right\} > 0 \\
\iff & \inf_{\substack{q\in\mathcal{O}_K\setminus\{0\} \\ p\in\mathcal{O}_K}} \inf_{t>0} \max \left\{ \begin{array}{l} \max\{\max\limits_{1\le i\le d} e^{t}|\sigma_i(q)x_i + \sigma_i(p)|^{\frac{1}{r_i}}, e^{-t}\|q\|_{\mathbf{r}}\}, \\ \max\limits_{d_1<i\le d} \max\{|\sigma_i(q)x_i + \sigma_i(p)|, |\sigma_i(q)|\} \end{array} \right\} > 0 \\
\iff & \inf_{\substack{q\in\mathcal{O}_K\setminus\{0\} \\ p\in\mathcal{O}_K}} \max \left\{ \begin{array}{l} \|q\|_{\mathbf{r}} \left(\max\limits_{1\le i\le d} |\sigma_i(q)x_i + \sigma_i(p)|^{\frac{1}{r_i}} \right), \\ \max\limits_{d_1<i\le d} \max\{|\sigma_i(q)x_i + \sigma_i(p)|, |\sigma_i(q)|\} \end{array} \right\} > 0 \\
\iff & \inf_{\substack{q\in\mathcal{O}_K\setminus\{0\} \\ p\in\mathcal{O}_K}} \max \left\{ \begin{array}{l} \max\limits_{1\le i\le d} \|q\|_{\mathbf{r}}^{r_i}|\sigma_i(q)x_i + \sigma_i(p)|, \\ \max\limits_{d_1<i\le d} \max\{|\sigma_i(q)x_i + \sigma_i(p)|, |\sigma_i(q)|\} \end{array} \right\} > 0.
\end{aligned}$$

This completes the proof. □

Now we are ready to state the following Proposition which establishes a number field analogue of W. M. Schmidt's conjecture. The proof of the Proposition is postponed to the next section.

Proposition 3.5. $\mathbf{Bad}(K, \mathbf{r})$ *is HAW.*

Proof of Proposition 3.1 modulo Proposition 3.5. Write

$$P := \prod_{\sigma\in S} \left(\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right)_{\sigma\in S}.$$

As for any $p \in P$ the set $\{\mathrm{Ad}(g)p : g \in F_{\mathbf{r}}^+\}$ is bounded, we have

$$\Lambda \in E(F_{\mathbf{r}}^+) \Longleftrightarrow p\Lambda \in E(F_{\mathbf{r}}^+) \quad \forall p \in P. \tag{3.6}$$

We claim that

$$\pi(PH) = G/\Gamma. \tag{3.7}$$

Indeed, according to the Bruhat decomposition, the set PH is Zariski open in G. Suppose to the contrary that $g\Gamma \notin \pi(PH)$ for some $g \in G$, then we will have $\Gamma \cap g^{-1}PH = \emptyset$. This contradicts the Borel density theorem, hence proves our claim.

To prove $E(F_{\mathbf{r}}^+)$ is HAW, it suffices to prove that for any $\Lambda \in G/\Gamma$, there exists a neighborhood Ω of Λ in G/Γ such that the set $E(F_{\mathbf{r}}^+) \cap \Omega$ is HAW. In view of (3.7), we can find $b_0 \in P$ and $u_0 \in H$ such that $b_0 u_0 \Gamma = \Lambda$. Then choose a neighborhood Ω_P (resp. Ω_H) of p_0 (resp. u_0) in P (resp. H) small enough that the map $\phi : \Omega_P \times \Omega_H \to G/\Gamma$ is an homeomorphism onto its image Ω. Hence we are reduced to proving that the set

$$\phi^{-1}(E(F_{\mathbf{r}}^+) \cap \Omega) = \{(p, u) \in \Omega_P \times \Omega_H : pu\Gamma \in E(F_{\mathbf{r}}^+)\}$$

is HAW. In view of (3.6), the set defined above coincides with

$$\Omega_P \times \left(\pi^{-1}(E(F_{\mathbf{r}}^+)) \cap \Omega_H\right) \tag{3.8}$$

And the HAW property of the set (3.8) follows from (3.2) and Lemma 3.5. $\square$

4. Proof of Proposition 3.5

First we introduce another formulation of the set $\mathbf{Bad}(K, \mathbf{r})$. For $\varepsilon > 0$, set

$$\mathcal{O}_K(\mathbf{r}, \varepsilon) = \{q \in \mathcal{O}_K \setminus \{0\} : \max_{\sigma \in S_2} |\sigma(q)| \leq \varepsilon\}.$$

For $(p, q) \in \mathcal{O}_K \times \mathcal{O}_K(\mathbf{r}, \varepsilon)$, define

$$\Delta_\varepsilon(p, q) = \prod_{\sigma \in S_1} \left[\frac{\sigma(p)}{\sigma(q)} \pm \frac{\varepsilon}{|\sigma(q)| \|q\|_{\mathbf{r}}^{r_\sigma}}\right] \times \prod_{\sigma \in S_2} \left[\frac{\sigma(p)}{\sigma(q)} \pm \frac{\varepsilon}{|\sigma(q)|}\right] \subset \prod_{\sigma \in S} \mathbb{R},$$

where $[A \pm B]$ denotes the interval $[A - B, A + B] \subset \mathbb{R}$. Then set

$$\mathbf{Bad}_\varepsilon(K, \mathbf{r}) := \prod_{\sigma \in S} \mathbb{R} \setminus \bigcup_{(p,q) \in \mathcal{O}_K \times \mathcal{O}_K(\mathbf{r}, \varepsilon)} \Delta_\varepsilon(p, q). \tag{4.1}$$

It is not hard to check:

Lemma 4.1.

$$\mathbf{Bad}(K, \mathbf{r}) = \bigcup_{\varepsilon > 0} \mathbf{Bad}_\varepsilon(K, \mathbf{r}).$$

Proof. It suffices to show that the set of vectors $\mathbf{x} = (x_\sigma)_{\sigma \in S} \in \prod_{\sigma \in S} \mathbb{R}$ satisfying

$$\inf_{\substack{q \in \mathcal{O}_K \setminus \{0\} \\ p \in \mathcal{O}_K}} \max \left\{ \max_{\sigma \in S_1} \|q\|_{\mathbf{r}}^{r_\sigma} |\sigma(q) x_\sigma + \sigma(p)|, \max_{\sigma \in S_2} \max\{|\sigma(q) x_\sigma + \sigma(p)|, |\sigma(q)|\} \right\} > \varepsilon \tag{4.2}$$

coincides with $\mathbf{Bad}_\varepsilon(K,\mathbf{r})$. By definition of $\mathcal{O}_K(\mathbf{r},\epsilon)$, the equation (4.2) is equivalent to the following

$$\inf_{\substack{q\in\mathcal{O}_K(\mathbf{r},\varepsilon)\\ p\in\mathcal{O}_K}} \max\left\{\max_{\sigma\in S_1}\|q\|_{\mathbf{r}}^{r_\sigma}|\sigma(q)x_\sigma+\sigma(p)|, \max_{\sigma\in S_2}|\sigma(q)x_\sigma+\sigma(p)|\right\} > \varepsilon. \tag{4.3}$$

Now we are reduced to show the set of vectors $\mathbf{x}=(x_\sigma)_{\sigma\in S}\in\prod_{\sigma\in S}\mathbb{R}$ satisfying (4.3) coincides with $\mathbf{Bad}_\varepsilon(K,\mathbf{r})$, which is straightforward to verify, and hence omitted. □

To prove the set $\mathbf{Bad}(K,\mathbf{r})$ is HAW, it suffices to prove that it is (β,γ)-hyperplane potential winning for any $\beta\in(0,1),\gamma>0$. We choose and fix a pair of such (β,γ) in this section. Furthermore, we denote the ball chosen by Bob in the first round of the game by B_0. By letting Alice making empty moves at the first rounds and relabeling the index, we may assume $\rho_0=\rho(B_0)<1$ without loss of generality. Choose and fix $R>0$ satisfying

$$\frac{d}{R^\gamma-1}\le\left(\frac{\beta^2}{2}\right)^\gamma. \tag{4.4}$$

Then set

$$\varepsilon=\frac{1}{4}\rho_0R^{-4d}\quad\text{ and }\quad H_n=\varepsilon\rho_0^{-1}R^n\quad(n\ge1). \tag{4.5}$$

Now for $n\ge0$, we define a class of closed balls $\mathscr{B}_n$ as

$$\mathscr{B}_n:=\{B\subset B_0:\beta R^{-n}\rho_0<\rho(B)\le R^{-n}\rho_0\}.$$

We are going to define a subdivision of $\mathcal{O}_K(\mathbf{r},\varepsilon)$. To begin, we shall need the following height function:

$$H:\mathcal{O}_K(\mathbf{r},\varepsilon)\to\mathbb{R},\quad H(q)=\max_{\sigma\in S_1}|\sigma(q)|\|q\|_{\mathbf{r}}^{r_\sigma}.$$

We have the following lemma controlling the size of $H(q)$ and $\|q\|_{\mathbf{r}}$.

Lemma 4.2. *For all $q\in\mathcal{O}_K(\mathbf{r},\varepsilon)$, there holds*

$$1\le\|q\|_{\mathbf{r}}^{\frac{1}{d}}\le H(q)\le\|q\|_{\mathbf{r}}^{2r}. \tag{4.6}$$

Proof. For the second inequality in (4.6), we have

$$H(q)^{d_1}\ge\prod_{\sigma\in S_1}|\sigma(q)|\|q\|_{\mathbf{r}}^{r_\sigma}\ge\left(\prod_{\sigma\in S_2}\sigma(q)\right)^{-1}|N(q)|\|q\|_{\mathbf{r}}\ge\|q\|_{\mathbf{r}}.$$

The third inequality in (4.6) is a direct consequence of the following estimate

$$|\sigma(q)|\le\|q\|_{\mathbf{r}}^{r_\sigma},\qquad\text{for all }\sigma\in S_1, \tag{4.7}$$

which is easy to check by the definition of $\|q\|_{\mathbf{r}}$. Finally, according to (4.7), we have

$$\|q\|_{\mathbf{r}}\ge\prod_{\sigma\in S_1}|\sigma(q)|\ge\left(\prod_{\sigma\in S_2}\sigma(q)\right)^{-1}|N(q)|\ge1.$$

This gives the first inequality. □

Now we can define the subdivision of $\mathcal{O}_K(\mathbf{r},\varepsilon)$. Set

$$\mathscr{P}_n = \{q \in \mathcal{O}_K(\mathbf{r},\varepsilon) : H_n \le H(q) < H_{n+1}\},$$

and

$$\mathscr{P}_{n,k} = \{q \in \mathscr{P}_n : H_n R^{(4k-4)d} \le \|q\|_{\mathbf{r}}^{2r} < H_n R^{4kd}\}.$$

In view of (4.6) and the trivial estimate $H_1 < 1$, we have

$$\mathcal{O}_K(\mathbf{r},\varepsilon) = \bigcup_{n\ge 0} \mathscr{P}_n.$$

The following lemma is important.

Lemma 4.3.

$$\mathcal{O}_K(\mathbf{r},\varepsilon) = \bigcup_{n\ge 0}\bigcup_{k\ge 1} \mathscr{P}_{n+k,k}.$$

Proof. To prove this lemma, it is equivalent to prove that

$$\mathscr{P}_{n,k} = \emptyset \qquad \text{for all } k \ge n. \tag{4.8}$$

Assuming the contrary that there is $q \in \mathscr{P}_{n,k}$ for some $k \ge n$, then we have

$$\|q\|_{\mathbf{r}}^{2} \ge \|q\|_{\mathbf{r}}^{2r} \ge H_n R^{(4n-4)d} > H_{n+1}^{2d}$$

by (4.5). This contradicts (4.6), hence proves (4.8). □

We shall need the following lemma:

Lemma 4.4. *Let $B \in \mathscr{B}_n$. Then for any $k \ge 1$, the map $F : \mathcal{O}_K \times \mathcal{O}_K(\mathbf{r},\varepsilon) \to K^*$ defined by*

$$F(p,q) = \frac{p}{q}$$

is constant on the set

$$\mathscr{P}_{n+k,k}(B) := \{(p,q) : q \in \mathscr{P}_{n+k,k} \text{ and } \Delta_\varepsilon(p,q) \cap B \ne \emptyset\}.$$

Proof. For any $B \in \mathscr{B}_n$ and $q \in \mathscr{P}_{n+k,k}$, we have

$$\rho(B) \le \frac{R^{k+1}\varepsilon}{H(q)} \tag{4.9}$$

by (4.5). Suppose the contrary that we have two pairs (p_1, q_1) and (p_2, q_2) with

$$\frac{p_1}{q_1} \ne \frac{p_2}{q_2} \tag{4.10}$$

satisfying

$$\Delta_\varepsilon(p_1,q_1) \cap B \ne \emptyset \text{ and } \Delta_\varepsilon(p_2,q_2) \cap B \ne \emptyset. \tag{4.11}$$

Then it follows (4.10) that

$$\left|\prod_{\sigma\in S}\left(\frac{\sigma(p_1)}{\sigma(q_1)} - \frac{\sigma(p_2)}{\sigma(q_2)}\right)\right| \ge \left|\frac{N(p_1q_2 - p_2q_1)}{N(q_1q_2)}\right| \ge \frac{1}{|N(q_1q_2)|}. \tag{4.12}$$

Now we claim that we can also prove the following inequality

$$\left|\prod_{\sigma\in S}\left(\frac{\sigma(p_1)}{\sigma(q_1)}-\frac{\sigma(p_2)}{\sigma(q_2)}\right)\right|<\frac{1}{|N(q_1q_2)|}, \tag{4.13}$$

which contradicts (4.12), hence completes the proof of the lemma. Indeed it follows from (4.11) and the definition of $\Delta_\varepsilon(p,q)$ that, for all $\sigma\in S_1$, we have

$$\left|\frac{\sigma(p_1)}{\sigma(q_1)}-\frac{\sigma(p_2)}{\sigma(q_2)}\right|\le\frac{\varepsilon}{|\sigma(q_1)|\|q_1\|_{\mathbf{r}}^{r_\sigma}}+\frac{\varepsilon}{|\sigma(q_2)|\|q_2\|_{\mathbf{r}}^{r_\sigma}}+2\rho(B). \tag{4.14}$$

In view of (4.9) and (4.14), we have

$$\begin{aligned}
\left|\prod_{\sigma\in S_1}\left(\frac{\sigma(p_1)}{\sigma(q_1)}-\frac{\sigma(p_2)}{\sigma(q_2)}\right)\right| &\le\prod_{\sigma\in S_1}\left(\frac{\varepsilon}{|\sigma(q_1)|\|q_1\|_{\mathbf{r}}^{r_\sigma}}+\frac{\varepsilon}{|\sigma(q_2)|\|q_2\|_{\mathbf{r}}^{r_\sigma}}+2\rho(B)\right)\\
&\le\prod_{\sigma\in S_1}\left(\frac{\varepsilon}{|\sigma(q_1)|\|q_1\|_{\mathbf{r}}^{r_\sigma}}+\frac{\varepsilon}{|\sigma(q_2)|\|q_2\|_{\mathbf{r}}^{r_\sigma}}+\frac{2R^{k+1}\varepsilon}{\max\{H(q_1),H(q_2)\}}\right)\\
&\le(R^{k+1}+1)^d\prod_{\sigma\in S_1}\frac{\varepsilon}{|\sigma(q_1q_2)|}\left(\frac{|\sigma(q_1)|}{\|q_2\|_{\mathbf{r}}^{r_\sigma}}+\frac{|\sigma(q_2)|}{\|q_1\|_{\mathbf{r}}^{r_\sigma}}\right)\\
&\le 2^dR^{dk+d}\frac{\varepsilon^d}{|\prod_{\sigma\in S_1}\sigma(q_1q_2)|}\prod_{\sigma\in S_1}R^{4r_\sigma d}\left(\frac{|\sigma(q_1)|}{\|q_1\|_{\mathbf{r}}^{r_\sigma}}+\frac{|\sigma(q_2)|}{\|q_2\|_{\mathbf{r}}^{r_\sigma}}\right)\\
&\le 2^{2d-1}R^{dk+5d}\varepsilon^d\frac{1}{|\prod_{\sigma\in S_1}\sigma(q_1q_2)|}\left(\frac{|\omega(q_1)|}{\|q_1\|_{\mathbf{r}}^{r}}+\frac{|\omega(q_2)|}{\|q_2\|_{\mathbf{r}}^{r}}\right)\\
&\le 2^{2d-1}R^{dk+5d}\varepsilon^d\frac{1}{|\prod_{\sigma\in S_1}\sigma(q_1q_2)|}\left(\frac{|H(q_1)|}{\|q_1\|_{\mathbf{r}}^{2r}}+\frac{|H(q_2)|}{\|q_2\|_{\mathbf{r}}^{2r}}\right)\\
&\le 2^{2d}R^{dk+5d}R^{-(4k-4)d}\varepsilon^d\frac{1}{|\prod_{\sigma\in S_1}\sigma(q_1q_2)|}\\
&<\frac{1}{|\prod_{\sigma\in S_1}\sigma(q_1q_2)|}.
\end{aligned} \tag{4.15}$$

On the other hand, it follows from (4.11) and the definition of $\Delta_\varepsilon(p,q)$ that, for all $\sigma\in S_2$, we have

$$\left|\frac{\sigma(p_1)}{\sigma(q_1)}-\frac{\sigma(p_2)}{\sigma(q_2)}\right|\le\frac{\varepsilon}{|\sigma(q_1)|}+\frac{\varepsilon}{|\sigma(q_2)|}+2\rho(B). \tag{4.16}$$

In view of (4.16) and the assumption $\rho_0<1$, we have

$$\begin{aligned}
\left|\prod_{\sigma\in S_2}\left(\frac{\sigma(p_1)}{\sigma(q_1)}-\frac{\sigma(p_2)}{\sigma(q_2)}\right)\right| &\le\prod_{\sigma\in S_2}\left(\frac{\varepsilon}{|\sigma(q_1)|}+\frac{\varepsilon}{|\sigma(q_2)|}+2\rho(B)\right)\\
&\le\prod_{\sigma\in S_2}\frac{4\varepsilon^2}{|\sigma(q_1q_2)|}<\frac{1}{\prod_{\sigma\in S_2}|\sigma(q_1q_2)|}.
\end{aligned} \tag{4.17}$$

Note that we have used the fact that $|\sigma(q)|\le\varepsilon$ for $\sigma\in S_2$ and $q\in\mathcal{O}_K(\mathbf{r},\varepsilon)$ and an elementary inequality saying that $4ab\ge a+b+2$ for $a,b\ge 1$. Now (4.13) follows from (4.15) and (4.17). Hence our proof is completed. □

Now we are in a position to prove Proposition 3.5.

Proof of Lemma 3.5. For any $B \in \mathscr{B}_n$ and $k \geq 1$, denote the unique point given by Lemma 4.4 as

$$\mathbf{s}(k, B) = (s_\sigma(k, B))_{\sigma \in S}.$$

Then it follows from Lemma 4.4 and the definition of $\mathscr{P}_{n+k,k}$ that

$$\bigcup_{(p,q) \in \mathscr{P}_{n+k,k}} \Delta_\varepsilon(p, q) \cap B \subset \bigcup_{\tau \in S} E_\tau(k, B)^{(R^{-n-k}\rho_0)},$$

where the hyperplane $E_\tau(k, B)$ is defined as

$$E_\tau(k, B) := \{\mathbf{x} \in \prod_{\sigma \in S} \mathbb{R} : x_\tau = s_\tau(k, B)\}. \tag{4.18}$$

As those $\mathscr{B}_n$ are mutually disjoint, hence for each $i \geq 0$ there exists at most one $n \geq 0$ with $B_i \in \mathscr{B}_n$. According to the definition of (β, γ)-hyperplane potential game, we have $\rho_{i+1} \geq \beta\rho_i$. In view of [4, Remark 2.4], we may assume that $\rho_0 \to 0$. Hence for each $n \geq 0$, there exists an $i \geq 0$ with $B_i \in \mathscr{B}_n$. Let $i(n)$ denote the smallest i with $B_i \in \mathscr{B}_n$. Then, the map $n \mapsto i(n)$ is an injective one from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$. Let Alice play according to the following strategy: each time after Bob chooses a closed ball B_i, if $i = i(n)$ for some $n \geq 0$, then Alice chooses the family of hyperplane neighborhoods

$$\{E_\tau(k, B_{i(n)})^{(R^{-n-k}\rho_0)} : \tau \in S, k \in \mathbb{N}\}.$$

where $E_\tau(k, B_{i(n)})$ is the hyperplane given by (4.18). Otherwise Alice makes an empty move. Since $B_{i(n)} \in \mathscr{B}_n$, $\rho_{i(n)} > \beta R^{-n}\rho_0$. Then, (4.4) implies that

$$\sum_{\tau \in S, k=1}^{\infty} (R^{-n-k}\rho_0)^\gamma = d(R^{-n}\rho_0)^\gamma (R^\gamma - 1)^{-1} \leq \left(\frac{\rho_i}{\beta}\right)^\gamma \left(\frac{\beta^2}{2}\right)^\gamma < (\beta\rho_i)^\gamma.$$

Hence Alice's move is legal. Then we have

$$\begin{aligned}
\bigcap_{i=0}^{\infty} B_i &= \bigcap_{i=0}^{\infty} B_i \cap \Big(\mathbf{Bad}(K, \mathbf{r}) \cup \bigcup_{(p,q) \in \mathcal{O}_K \times \mathcal{O}_K(\mathbf{r}, \epsilon)} \Delta_\varepsilon(p, q)\Big) \\
&= \bigcap_{i=0}^{\infty} B_i \cap \Big(\mathbf{Bad}(K, \mathbf{r}) \cup \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{q \in \mathscr{P}_{n+k,k}} \bigcup_{p \in \mathcal{O}_K} \Delta_\varepsilon(p, q)\Big) \\
&\subset \mathbf{Bad}(K, \mathbf{r}) \cup \Big(\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{q \in \mathscr{P}_{n+k,k}} \bigcup_{p \in \mathcal{O}_K} \Delta_\varepsilon(p, q) \cap B_{i(n)}\Big) \\
&= \mathbf{Bad}(K, \mathbf{r}) \cup \Big(\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{(p.q) \in \mathscr{P}_{n+k,k}(B_{i(n)})} \Delta_\varepsilon(p, q) \cap B_{i(n)}\Big) \\
&\subset \mathbf{Bad}(K, \mathbf{r}) \cup \Big(\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\tau \in S} E_\tau(k, B_{i(n)})^{(R^{-n-k}\rho_0)}\Big).
\end{aligned}$$

Thus the unique point $\mathbf{x}_\infty \in \bigcap_{i=0}^{\infty} B_i$ lies in

$$\mathbf{Bad}(K,\mathbf{r}) \cup \Big(\bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\tau\in S} E_\tau(k, B_{i(n)})^{(R^{-n-k}\rho_0)} \Big).$$

Hence, Alice wins. □

5. Proof of the main theorem

We shall need the following simple observation.

Lemma 5.1. *Let Γ and Γ' be lattices in G such that Γ is commensurable with Γ'. Then for any subsemigroup F^+ of G, there holds*

$$E(F^+) \text{ is HAW on } G/\Gamma \Longleftrightarrow E(F^+) \text{ is HAW on } G/\Gamma'.$$

Proof. As Γ, Γ' are commensurable with each other, the group $\Gamma'' = \Gamma \cap \Gamma'$ is of finite index in both Γ and Γ', and hence is a lattice subgroup of G. By replacing Γ' with Γ'', the proof of the lemma can be reduced to the case when $\Gamma' \subset \Gamma$. In this case, the natural projection map $\pi : G/\Gamma \mapsto G/\Gamma'$ is a finite covering map. Now the lemma follows from Lemma 2.2. □

Proof of Theorem 1.1. Let G be a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and Γ a lattice. Then according to [20], the lattice Γ is commensurable with $\Gamma_1 \times \cdots \times \Gamma_k$, where Γ_i $(1 \le i \le k)$ is an irreducible lattice in G_i $(1 \le i \le k)$. In view of Lemma 5.1, we are reduced to consider the case when $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$. Moreover, since an orbit is bounded on G/Γ if and only if its projection is bounded on each G_i/Γ_i, we are reduced to consider the case when Γ itself is irreducible and not cocompact by applying Lemma 2.2 (5). Now there are two cases:

Case 1. Suppose $G = \mathrm{SL}_2(\mathbb{R})$. Then it follows essentially from [10], although not stated explicitly there.

Case 2. Suppose G is a product of more than two copies of $\mathrm{SL}_2(\mathbb{R})$. Then it follows from Margulis arithmeticity theorem [19, Chapter IX, Theorem 1.9A] that this Γ is arithmetic, i.e., Γ is commensurable with $\mathbf{G}(\mathbb{Z})$ with $\mathbf{G}$ a $\mathbb{Q}$-simple semisimple group. Then $\mathbf{G} = \mathrm{Res}_{K/\mathbb{Q}}\mathbf{G}'$ with $\mathbf{G}'$ a K-form of SL_2 for some totally real field K. Since Γ is not cocompact, we have $\mathbf{G}'$ is K-isotropic. Hence $\mathbf{G}' = \mathrm{SL}_2$ and Γ is commensurable with $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_2(\mathbb{Z})$. And in view of the Lemma 2.2 and Proposition 3.1, what remains is to check that any one-parameter Ad-semisimple subsemigroup F^+ is conjugate to some $F_{\mathbf{r}}^+$, which is straightforward. □

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