An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation*

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Abstract

We provide a lower bound for the ratio between the ordinary and uniform exponent of both simultaneous Diophantine approximation and Diophantine approximation by linear forms in any dimension. This lower bound was conjectured by Schmidt and Summerer and already shown in dimension 2 and 3. This lower bound is reached at regular graph presented in the context of parametric geometry of numbers, and thus optimal.

1 Introduction

Throughout this paper, the integer $n \geq 1$ denotes the dimension of the ambient space, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ denotes an n-tuple of real numbers such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent.

Given $n \geq 1$ and $\boldsymbol{\theta} \in \mathbb{R}^n$, we consider the irrationality measure function

$$\psi(t) = \min_{q \in \mathbb{Z}_+, q \leq t} \max_{1 \leq j \leq n} \|q\theta_j\|,$$

which gives rise to the ordinary exponent of simultaneous Diophantine approximation

$$\lambda(\boldsymbol{\theta}) = \sup\{\lambda : \liminf_{t \to +\infty} t^{\lambda}\psi(t) < +\infty\}$$

and the uniform exponent of simultaneous Diophantine approximation

$$\hat{\lambda}(\boldsymbol{\theta}) = \sup\{\lambda : \limsup_{t \to +\infty} t^{\lambda}\psi(t) < +\infty\}.$$

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The irrationality measure function

$$\varphi(t) = \min_{\boldsymbol{q}\in\mathbb{Z}^n, 0<\max_{1\le j\le n}|q_j|\le t} \|q_1\theta_1 + \cdots + q_n\theta_n\|$$

gives rise to the ordinary exponent of Diophantine approximation by linear form

$$\omega(\boldsymbol{\theta}) = \sup\{\omega : \liminf_{t\to+\infty} t^{\omega}\varphi(t) < +\infty\}$$

and the uniform exponent of Diophantine approximation by linear form

$$\hat{\omega}(\boldsymbol{\theta}) = \sup\{\omega : \limsup_{t\to+\infty} t^{\omega}\varphi(t) < +\infty\}.$$

These exponents were first introduced and studied by A. Khintchine [7, 8] and V. Jarník [4]. The idea is to study specific $\boldsymbol{\theta}$ for which it is possible to improve Dirichlet's *Schubfachprinzip*. The aim of this paper is to provide a lower bound for the ratios $\lambda/\hat{\lambda}$ and $\omega/\hat{\omega}$ as a function of $\hat{\lambda}$ and $\hat{\omega}$ respectively, in any dimension. In dimension $n = 1$, Khintchine [8] observed that the uniform exponent always takes the value 1 and it follows from Dirichlet's *Schubfachprinzip* that the ordinary exponent satisfy $\omega(\theta) = \lambda(\theta) \ge 1 = \hat{\omega}(\theta) = \hat{\lambda}(\theta)$. In dimension $n = 2$, Jarník proved in [5, 6] the optimal inequalities

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \ge \frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}, \tag{1}$$

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \ge \hat{\omega}(\boldsymbol{\theta}) - 1. \tag{2}$$

In [12], Moshchevitin proved the optimal bound for simultaneous approximation:

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \ge \frac{\hat{\lambda}(\boldsymbol{\theta}) + \sqrt{4\hat{\lambda}(\boldsymbol{\theta}) - 3\hat{\lambda}(\boldsymbol{\theta})^2}}{2(1-\hat{\lambda}(\boldsymbol{\theta}))} = \frac{1}{2}\left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})} + \sqrt{\left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}\right)^2 + \frac{4\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}}\right) \tag{3}$$

Schmidt and Summerer provided an alternative proof using parametric geometry of numbers in [20], and the following bound for approximation by linear forms:

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \ge \frac{\sqrt{4\hat{\omega}(\boldsymbol{\theta}) - 3} - 1}{2}. \tag{4}$$

A simple proof of this bound was given in [13]. In [6], Jarník also provided a lower bound in arbitrary dimension $n \ge 2$.

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \ \ge\ \hat{\omega}(\boldsymbol{\theta})^{1/(n-1)} - 3, \ \text{provided that } \hat{\omega}(\boldsymbol{\theta}) > (5n^2)^{n-1}, \tag{5}$$

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \ \ge\ \frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}. \tag{6}$$

In fact, these bounds also apply in a more general setting of simultaneous Diophantine approximation by a set of linear forms.

Using their new tools of parametric geometry of numbers, Schmidt and Summerer [18] provided the first general improvement working in the whole admissible interval of values of the uniform exponents $\hat{\omega}$ and $\hat{\lambda}$.

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq \frac{(n-2)(\hat{\omega}(\boldsymbol{\theta})-1)}{1+(n-3)\hat{\omega}(\boldsymbol{\theta})}, \tag{7}$$

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq \frac{\hat{\lambda}(\boldsymbol{\theta})+n-3}{(n-2)(1-\hat{\lambda}(\boldsymbol{\theta}))}. \tag{8}$$

Here relation (8) is sharper than relation (6). Relation (7) is valid for the whole interval of possible values of $\hat{\omega}(\boldsymbol{\theta})$, but Jarník's asymptotic relation (5) is better for large $\hat{\omega}(\boldsymbol{\theta})$. A simple proof of (8) was given in [3].

In [20] Schmidt and Summerer conjecture that, as in dimension $n = 3$, the general optimal lower bound is reached at *regular graphs*. In this paper we show that this conjecture holds. Let us first introduce some notation.

For given $n \geq 1$ and $1/n \leq \alpha < 1$, we consider the polynomial

$$R_{n,\alpha}(x) = x^{n-1} - \frac{\alpha}{1-\alpha}(x^{n-2} + \cdots + x + 1) \tag{9}$$

and denote by $G(n, \alpha)$ its unique real positive root. For $\alpha^* \geq n$ we denote by $1/G^*(n, \alpha^*)$ the unique positive root of $R_{n,1/\alpha^*}(x)$.

Theorem 1. *For $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent, one has*

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq G(n, \hat{\lambda}(\boldsymbol{\theta})) \quad \text{and} \quad \frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq G^*(n, \hat{\omega}(\boldsymbol{\theta})) \tag{10}$$

Furthermore, for any $\hat{\omega} \geq n$ and any $C \geq G^*(n, \hat{\omega})$, there exists infinitely many $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$$\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \omega(\boldsymbol{\theta}) = C\hat{\omega}$$

and for any $1/n \leq \hat{\lambda} \leq 1$ and any $C \geq G(n, \hat{\lambda})$, there exists infinitely many $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$$\hat{\lambda}(\boldsymbol{\theta}) = \hat{\lambda} \quad \text{and} \quad \lambda(\boldsymbol{\theta}) = C\hat{\lambda}.$$

It follows from Roy's theorem [16] applied to Schmidt-Summerer's *regular graphs* [20] [15] that the lower bound is reached and thus optimal. The second part of Theorem 1 refines this observation. Note that for any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent, we have $\hat{\omega}(\boldsymbol{\theta}) \geq n$ and $\hat{\lambda}(\boldsymbol{\theta}) \in [1/n, 1]$, (see for example [9]) hence the constraint on $\hat{\lambda}$ and $\hat{\omega}$ is not restrictive.

We can reformulate these lower bounds by the inequalities

$$1 + \omega(\boldsymbol{\theta}) - \hat{\omega}(\boldsymbol{\theta}) \geq \left(\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})}\right)^n \quad \text{and} \quad 1 - \frac{1}{\hat{\lambda}(\boldsymbol{\theta})} + \frac{1}{\lambda(\boldsymbol{\theta})} \leq \left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{\lambda(\boldsymbol{\theta})}\right)^n.$$

In these formulae appears clearly the natural symmetry property of spectra of Diophantine approximation pointed out by Schmidt and Summerer [18]. It is even more obvious in the proof, as the very same geometric analysis applies to both cases.

The main part of Theorem 1 is the lower bound. The proof uses determinants of best approximation vectors, following the idea of [12]. It deeply relies on an inequality of Schmidt [17] applied inductively to a well chosen subsequence of best approximation vectors. The second part of Theorem 1 is a consequence of the parametric geometry of numbers, and is proved independently in section 5.

2 Main tools

2.1 Sequences of best approximations

We denote by $(\boldsymbol{z}_l)_{l \in \mathbb{N}}$ a sequence of best approximations (or minimal points) to $\boldsymbol{\theta} \in \mathbb{R}^n$. This notion was introduced by Voronoi [21] as minimal points in lattices, it was first defined in our context by Rogers [14]. It has been used implicitly or explicitly in many proofs concerning exponents of Diophantine approximation.

In the context of best approximation vectors for simultaneous Diophantine approximation, we can write

$$\boldsymbol{z}_l = (q_l, a_{1,l}, a_{2,l}, \dots, a_{n,l}) \in \mathbb{Z}^{n+1}, l \in \mathbb{N}.$$

Set

$$\xi_l = \max_{1 \leq i \leq n} |q_l \theta_i - a_{i,l}|.$$

By definition of best approximations

$$1 < q_1 < q_2 < \cdots < q_l < q_{l+1} < \cdots \quad \text{and} \quad 1 > \xi_1 > \xi_2 > \cdots > \xi_l > \xi_{l+1} > \cdots$$

We may also assume that q_1 is large enough so that for every $l \geq 1$

$$\xi_l \leq q_{l+1}^{-\alpha}, \tag{11}$$

where $\alpha < \hat{\lambda}(\boldsymbol{\theta})$.

In the context of best approximation vector for approximation by linear forms, we can write

$$\boldsymbol{z}_l = (q_{1,l}, q_{2,l}, \ldots, q_{n,l}, a_l) \in \mathbb{Z}^{n+1}, l \in \mathbb{N}.$$

Set

$$L_l = |q_{1,l}\theta_1 + \cdots + q_{n,l}\theta_n - a_l| \text{ and } M_l = \max_{1\leq j\leq n} |q_j|.$$

By definition of best approximations

$$1 < M_1 < M_2 < \cdots < M_l < M_{l+1} < \cdots \text{ and } 1 > L_1 > L_2 > \cdots > L_l > L_{l+1} > \cdots$$

We may also assume that M_1 is large enough so that for every $l \geq 1$

$$L_l \leq M_{l+1}^{-\alpha^*}. \tag{12}$$

where $\alpha^* < \hat{\omega}(\boldsymbol{\theta})$.

In the context of simultaneous Diophantine approximation, provided that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Q}$, it is known that a sequence of best approximation vectors ultimately spans the whole space $\mathbb{R}^{n+1}$. However in the context of approximation by linear forms, the situation is different: it may happen that vectors of best approximation span a strictly lower dimensional subspace of $\mathbb{R}^{n+1}$. See the surveys [11, 10] by Moshchevitin and the paper [1] by Chevallier for more detail. Fortunately, if best approximation do not span the whole space $\mathbb{R}^{n+1}$ we get a sharper result, since $G(n, \alpha)$ is a decreasing function of n. Thus, we may assume without loss of generality that in both contexts best approximation vectors ultimately span the whole space $\mathbb{R}^{n+1}$.

Whenever $1, \theta_1, \ldots, \theta_n$ are linearly dependent over $\mathbb{Q}$, consider $\tilde{\boldsymbol{\theta}} = (\theta_{i_1}, \ldots, \theta_{i_k})$ a largest subset of the components of $\boldsymbol{\theta}$ which satisfy the linear independence property over $\mathbb{Q}$ with 1. It is easy to check that $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$ have the same exponents, and thus results of lower dimension apply. Thus, we may assume without loss of generality that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Q}$.

Using sequences of best approximations, proving that

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq G$$

is equivalent to showing that for arbitrarily large indices k, one has $q_{k+1} \gg q_k^G$. Similarly, proving that

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq G$$

is equivalent to showing that for arbitrarily large indices k, one has $M_{k+1} \gg M_k^G$. This observation relies on the expression of exponents of Diophantine approximation in terms of best approximation vectors

$$\omega = \limsup_{k\to\infty}\left(-\frac{\log(L_k)}{\log(M_k)}\right) \quad , \quad \hat{\omega} = \liminf_{k\to\infty}\left(-\frac{\log(L_k)}{\log(M_{k+1})}\right),$$

$$\lambda = \limsup_{k\to\infty}\left(-\frac{\log(\xi_k)}{\log(q_k)}\right) \quad , \quad \hat{\lambda} = \liminf_{k\to\infty}\left(\frac{\log(\xi_k)}{\log(q_{k+1})}\right).$$

The proofs in the case of simultaneous approximation and approximation by linear forms rely on the same geometric analysis. The idea is to consider an arbitrarily large index k, and construct a pattern of best approximation vectors in which at least one pair of successive best approximation satisfies

$$q_{k+1} \gg q_k^G \quad \text{or} \quad M_{k+1} \gg M_k^G \tag{13}$$

for the required G.

We recall a well known fact about best approximations vectors and determinants (see for example [12] [13]).

Lemma 1. *For any $l \geq 1$, consider Λ_l the lattice with basis $\boldsymbol{z}_l, \boldsymbol{z}_{l+1}$ and the three dimensional fundamental volume Δ_l of the lattice Γ_l with basis $\boldsymbol{z}_{l-1}, \boldsymbol{z}_l, \boldsymbol{z}_{l+1}$. In the context of simultaneous approximation we have the estimates*

$$\det(\Lambda_l) \asymp \xi_l q_{l+1}, \tag{14}$$

$$\Delta_l = \det(\Gamma_l) \ll \xi_{l-1}\xi_l q_{l+1}, \tag{15}$$

while in the context of approximation by linear forms we have the estimates

$$\det(\Lambda_l) \asymp L_l M_{l+1}, \tag{16}$$

$$\Delta_l = \det(\Gamma_l) \ll L_{l-1} M_l M_{l+1}. \tag{17}$$

In particular, two consecutive best approximation vectors are linearly independent.

Notation We denote by calligraphic letter $\mathcal{S}$ sets of best approximation vectors $\{\boldsymbol{z}_k, \dots, \boldsymbol{z}_m\}$. Given such a set $\mathcal{S}$, we denote by greek letters $\Gamma = \langle \boldsymbol{z}_k, \dots, \boldsymbol{z}_m \rangle_{\mathbb{Z}}$ the lattice spanned by its elements, and by bold roman letters $\mathbf{S} = \langle \boldsymbol{z}_k, \dots, \boldsymbol{z}_m \rangle_{\mathbb{R}}$ the rational subspace spanned over $\mathbb{R}$. Finally, we denote with gothic letters $\mathfrak{S}$ the underlying lattice $\mathfrak{S} = \mathbf{S} \cap \mathbb{Z}^n$. Note that $\Gamma \subset \mathfrak{S}$. If our objects are 2-dimensional, we rather use the letters $\mathcal{L}, \Lambda, \mathbf{L}$ and $\mathfrak{L}$.

2.2 Key lemmas

Lemma 2 ($\Gamma_- \underset{\Lambda}{-} \Gamma_+$)**.** *Denote by $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to a point $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that $k > \nu$ and triples*

$$\mathcal{S}_- := \{\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\} \quad \textit{and} \quad \mathcal{S}_+ := \{\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\}$$

consist of linearly independent consecutive best approximation vectors. Consider the three-dimensional lattices

$$\mathfrak{S}_- = \langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n, \quad \textit{and} \quad \mathfrak{S}_+ = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n$$

and suppose that

$$\langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k \rangle_{\mathbb{Z}} =: \Lambda. \tag{18}$$

Suppose that for positive s and t the following estimate holds

$$(\det \mathfrak{S}_-)^s \, (\det \mathfrak{S}_+)^t \gg \det \Lambda. \tag{19}$$

In the context of simultaneous Diophantine approximation, suppose that our vectors are large enough so that for $\alpha < \hat{\lambda}(\boldsymbol{\theta})$.

$$\xi_j \leq q_{j+1}^{-\alpha} \quad \textit{for} \quad j = \nu - 1, \nu, k-1, k. \tag{20}$$

Define

$$g(s,t) = \frac{\alpha s}{(1-\alpha)(s - w(s,t))} = \frac{\alpha(t + w(s,t) - 1) - w(s,t) + 1}{(1-\alpha)t}. \tag{21}$$

where the second equality comes from $w(s,t) \in (0,1)$ being the root of the equation

$$w^2 - \left(s + 1 + \frac{\alpha}{1-\alpha} t\right) w + s = 0. \tag{22}$$

Then

$$\textit{either} \quad q_{\nu+1} \gg q_\nu^{g(s,t)} \quad \textit{or} \quad q_{k+1} \gg q_k^{g(s,t)}. \tag{23}$$

In the context of approximation by linear forms, suppose that our vectors are large enough so that for $\alpha^* < \hat{\omega}(\boldsymbol{\theta})$.

$$L_j \leq M_{j+1}^{-\alpha^*}, \quad \textit{for} \quad j = \nu - 1, \nu, k-1, k. \tag{24}$$

Define

$$g^*(s,t) = \frac{(1-\alpha^*)s}{(1-\alpha^*)s - w^*(s,t)} = \frac{(1-\alpha^*)(1-w^*(s,t)-t)}{t}. \quad (25)$$

where the second equality comes from $w^*(s,t) \in (0,1)$ *being the root of the equation*

$$w^{*2} - \left(1 - t - \frac{s}{\alpha^* - 1}\right) w^* + \frac{s}{\alpha^* - 1} = 0. \quad (26)$$

Then

$$\textit{either} \quad M_{\nu+1} \gg M_\nu^{g^*(s,t)} \quad \textit{or} \quad M_{k+1} \gg M_k^{g^*(s,t)}. \quad (27)$$

When the parameter are $s = t = 1$, this lemma provides directly the result for the approximation of 3 numbers (Proof from [12], see subsection 3.1 for details). Considering parameters s and t provides a tool to exhibit consecutive best approximations with the required properties in higher dimension. The value of the parameter $g(s,t)$ or $g^*(s,t)$ needs to be optimized over a range of values for the parameter s and t, with condition coming from the geometry and depending on the dimension. To prove Theorem 1, we show inductively that the optimized parameter $g(s,t)$ or $1/g^*(s,t)$ over the range of suitable parameters s, t is root of the polynomial R_n defined by (9).

Proof of Lemma 2. Substituting (14) in (19) in light of (18), since $\langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1} \rangle_{\mathbb{Z}} \subset \mathfrak{S}_-$ and $\langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1} \rangle_{\mathbb{Z}} \subset \mathfrak{S}_+$ it follows that

$$(\xi_{\nu-1}\xi_\nu q_{\nu+1})^s (\xi_{k-1}\xi_k q_{k+1})^t \gg (\xi_\nu q_{\nu+1})^{w(s,t)} (\xi_{k-1} q_k)^{1-w(s,t)}.$$

This means that either

$$(\xi_{\nu-1}\xi_\nu q_{\nu+1})^s \gg (\xi_\nu q_{\nu+1})^{w(s,t)}$$

or

$$(\xi_{k-1}\xi_k q_{k+1})^t \gg (\xi_{k-1} q_k)^{1-w(s,t)}.$$

Now we take into account (20). We have either

$$q_\nu^{s\alpha} \ll q_{\nu+1}^{(1-\alpha)(s-w(s,t))}$$

or

$$q_k^{1-w(s,t)+\alpha(t+w(s,t)-1)} \ll q_{k+1}^{t(1-\alpha)}.$$

Hence (23) by definition of g.
Similarly, substituting (16) in (19) in light of (18) and the sub-lattice remark, it follows that

$$(L_{\nu-1} M_\nu M_{\nu+1})^s (L_{k-1} M_k M_{k+1})^t \gg (L_\nu M_{\nu+1})^{w^*(s,t)} (L_{k-1} M_k)^{1-w^*(s,t)}.$$

This means that either

$$(L_{\nu-1} M_\nu M_{\nu+1})^s \gg (L_\nu M_{\nu+1})^{w^*(s,t)}$$

or

$$(L_{k-1}M_kM_{k+1})^t \gg (L_{k-1}M_k)^{1-w^*(s,t)}.$$

Now we take into account (24). We have either

$$M_{\nu+1}^{(1-\alpha^*)w^*(s,t)-s} \gg M_v^{(1-\alpha^*)s}$$

or

$$M_{k+1}^t \gg M_k^{(1-\alpha^*)(1-w^*(s,t)-t)}.$$

Hence (27) by definition of g^*. □

Our proof relies on Schmidt's inequality on height (see [17], in fact this inequality was already used in the last section in [12]). It provides the setting to apply Lemma 2 simultaneously for different parameters s, t.

Proposition 1 (Schmidt's inequality)**.** *Let A, B be two rational subspaces in $\mathbb{R}^n$, we have*

$$H(A+B)\cdot H(A\cap B) \ll H(A)\cdot H(B). \tag{28}$$

where the height $H(A)$ is the determinant of the underlying lattice $\det(\mathfrak{A}) = \det(A\cap\mathbb{Z}^n)$.

2.3 Properties of the polynomial R_n and the optimized g

In this subsection, we state various properties needed for the proof.

The polynomial R_n defined in (9) can be defined inductively the following way.

$$\begin{cases} R_2(X) = X-\beta \\ R_{n+1}(X) = XR_n(X)-\beta \end{cases}$$

where β is respectively $\frac{\alpha}{1-\alpha}$ and $\frac{1}{\alpha^*-1}$. From now on, we describe the geometry of best approximations that does not depend on whether we consider exponents λ or ω.

Multiplying the two values defining $g = g(s,t)$ in (21), we see that g satisfies the equation

$$g^2 - \left(\beta + \frac{1-s}{t}\right)g - \frac{s\beta}{t} = 0. \tag{29}$$

Multiplying the two values defining $g^* = g^*(s,t)$ in (25), we see that $1/g^*$ satisfies the equation

$$1/g^{*2} - \left(\beta + \frac{1-t}{s}\right)1/g^* - \frac{t\beta}{s} = 0. \tag{30}$$

In particular, we can use this equation to compute the optimal value of either s or t when the other parameter is 1. Namely,

$$
\begin{aligned}
s &= \frac{g^2-\beta g-g}{\beta-g}, \quad \text{for} \quad g=g(s,1) && (31)\\
t &= \frac{\beta}{g(g-\beta)}, \quad \text{for} \quad g=g(1,t) && (32)\\
s &= \frac{g^2-\beta g-\beta}{g-\beta}=\frac{R_3(g)}{g-\beta}, \quad \text{for} \quad g=g(1-s,1), && (33)\\
t &= \frac{g^2-\beta g-\beta}{g(g-\beta)}=\frac{R_3(g)}{g(g-\beta)}, \quad \text{for} \quad g=g(1,1-t). && (34)
\end{aligned}
$$

Mutatis mutandis, the same holds with $1/g^*$ with symmetry in the parameter s and t.

3 Examples: approximation to three and four numbers.

In this section, we describe in details the computations in the cases of approximation to three and four numbers. The aim is to provide a concrete exemple of the construction of patterns of best approximation vectors on simple examples before moving to arbitrary dimension in section 4.

3.1 Approximation to three numbers

Consider $\boldsymbol{\theta} \in \mathbb{R}^3$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta}$, without loss of generality we can assume that the sequence $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ spans the whole space $\mathbb{R}^4$.

Lemma 3. *For arbitrarily large indexes k_0, there exists indexes $k > \nu > k_0$ and triples of consecutive best approximation vectors*

$$\mathcal{S}_- := \{\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\} \ \textit{and}\ \mathcal{S}_+ := \{\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\}$$

consisting of linearly independent vectors. Setting

$$\mathfrak{S}_- := \langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_{\mathbb{R}} \cap \mathbb{Z}^n \quad \textit{and} \quad \mathfrak{S}_+ := \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_{\mathbb{R}} \cap \mathbb{Z}^n$$

we have

$$\mathfrak{S}_- \cap \mathfrak{S}_+ := \Lambda = \langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k\rangle_{\mathbb{Z}} \ \textit{and}\ \langle \mathfrak{S}_- \cup \mathfrak{S}_+\rangle_{\mathbb{R}} = \mathbb{R}^4. \tag{35}$$

In other words, for arbitrarily large indexes, the pattern of Lemma 2 appears in the sequence of best approximation vectors. Denote by $\mathcal{S}_4$ the pattern of best approximation vectors described in Lemma 3. Here we chose k_0 sufficiently large for (20) resp. (24) to hold. Schmidt's inequality (28) provides (19) with parameters $s = t = 1$.

In the context of simultaneous Diophantine approximation, Lemma 2 provides that for any $\alpha < \hat{\lambda}(\boldsymbol{\theta})$,

$$q_{l+1} \gg q_l^{g_\alpha}$$

for $l = \nu$ or k, where g_α is solution of the equation (29) with $s = t = 1$. Namely

$$g_\alpha^2 - \beta g_\alpha - \beta = R_3(g_\alpha) = 0$$

which provides

$$g_\alpha = \frac{\beta + \sqrt{\beta^2 + 4\beta}}{2} = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1-\alpha)}.$$

Hence for every $\alpha < \lambda(\boldsymbol{\theta})$, we have

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq g_\alpha = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1-\alpha)}.$$

We deduce the lower bound (3).

In the context of approximation by linear forms, Lemma 2 provides that for any $\alpha^* < \hat{\omega}(\boldsymbol{\theta})$,

$$M_{l+1} \gg M_l^{g^*_{\alpha^*}}$$

for $l = \nu$ or k, where $1/g^*_{\alpha^*}$ is solution of the equation (30) with $s = t = 1$. Namely

$$1/g^{*2}_{\alpha^*} - \beta 1/g^*_{\alpha^*} - \beta = R_3(1/g^*_{\alpha^*}) = 0 = \beta g^{*2}_{\alpha^*} + \beta g^*_{\alpha^*} - 1$$

which provides

$$g^*_{\alpha^*} = \frac{\sqrt{\beta^2 - 4\beta} - \beta}{2\beta} = \frac{\sqrt{4\alpha^* - 3} - 1}{2}.$$

Hence for every $\alpha^* < \omega(\boldsymbol{\theta})$, we have

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq g^*_{\alpha^*} = \frac{\sqrt{4\alpha^* - 3} - 1}{2}.$$

We deduce the lower bound (4).

We now explain how to obtain the pattern of best approximation vectors in Lemma 3. It is the basic step for a more general construction in higher dimension.

Proof of Lemma 3. Figure 1 may be usefull to understand the construction.
Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta} \in \mathbb{R}^3$, and an arbitrarily large index k_0. Since $(\boldsymbol{z}_l)_{l\geq k_0}$ spans a 4-dimensional subspace, we can define $k+1$ to be the smallest index such that the sequence of best approximations $(\boldsymbol{z}_l)_{k_0\leq l\leq k+1}$ spans this 4-dimensional subspace. Note that $\boldsymbol{z}_{k+1}$ is not in the 3-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0\leq l\leq k}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}$ are linearly independent. Set $\nu - 1 \geq k_0$ to be the largest index such that $(\boldsymbol{z}_l)_{\nu-1\leq l\leq k+1}$ spans a 4-dimensional subspace. Note that $\boldsymbol{z}_{\nu-1}$ is not in the 3-dimensional subspace spanned by $(\boldsymbol{z}_l)_{\nu\leq l\leq k+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}$ are linearly independent. Moreover, combining both observations we get that

$$\Lambda := \langle (\boldsymbol{z}_l)_{\nu\leq l\leq k} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k \rangle_{\mathbb{Z}}$$

is 2-dimensional. Hence, the considered indexes ν and k provide best approximation vectors satisfying Lemma 3.

□

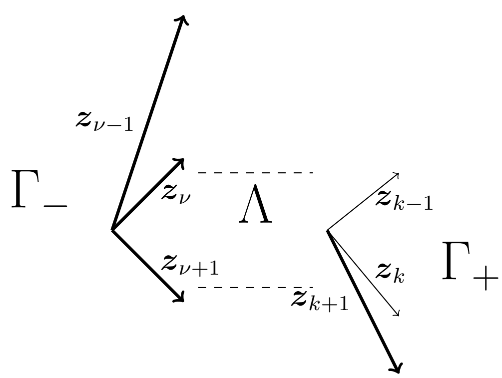

Figure 1: All best approximation vectors with index between ν and k lie in the 2-dimensional subspace spanned by Λ.

3.2 Approximation to four numbers

In the case of approximation to four numbers, we select a pattern $\mathcal{S}_5$ of best approximation vectors that combines two patterns $\mathcal{S}_4$ coming from Lemma 2. This is the first step of the induction for arbitrary dimension, where we combine two patterns of lower dimension. Thus, it is an enlightening example. Note that in this simple case, a proper choice of parameters was made in [2, equalities after formula (13) from the case $\mathfrak{i}(\Theta) = 1$].

Consider $\boldsymbol{\theta} \in \mathbb{R}^4$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta}$, without loss of generality we can assume that the sequence $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ spans the whole space $\mathbb{R}^5$.

Lemma 4. *Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta} \in \mathbb{R}^4$. Let k_0 be an arbitrarily large index. There exists indexes $k_0 < r_0 < r_1 \leq s_1 < r_2$ such that the following holds.*

1. *The triples of consecutive best approximation vectors*

$$\mathcal{S}_{r_i} := \{\boldsymbol{z}_{r_i-1}, \boldsymbol{z}_{r_i}, \boldsymbol{z}_{r_i+1}\}, \quad 0 \leq i \leq 2$$

consist of linearly independent vectors spanning a 3-dimensional subspace $\mathbf{S}_i := \langle \mathcal{S}_{r_i} \rangle_{\mathbb{R}}$.

2. *The triple of consecutive best approximation vectors*

$$\mathcal{S}_{s_1} := \{\boldsymbol{z}_{s_1-1}, \boldsymbol{z}_{s_1}, \boldsymbol{z}_{s_1+1}\}$$

consists of linearly independent vectors spanning $\mathbf{S}_1$.

3. *The pairs of consecutive best approximation vectors $\boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ and $\boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}$ span the same 2-dimensional subspace*

$$\Lambda_0 := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1} \rangle_{\mathbb{Z}} = \mathbf{S}_0 \cap \mathbf{S}_1 \cap \mathbb{Z}^5.$$

4. *The pairs of consecutive best approximation vectors $\boldsymbol{z}_{s_1}, \boldsymbol{z}_{s_1+1}$ and $\boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}$ span the same 2-dimensional subspace*

$$\Lambda_1 := \langle \boldsymbol{z}_{s_1}, \boldsymbol{z}_{s_1+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2} \rangle_{\mathbb{Z}} = \mathbf{S}_1 \cap \mathbf{S}_2 \cap \mathbb{Z}^5.$$

5. *The whole space $\mathbb{R}^5$ is spanned by*

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \boldsymbol{z}_{r_1+1}, \boldsymbol{z}_{r_2+1} \rangle_{\mathbb{R}} = \langle \mathbf{S}_0 \cup \mathbf{S}_1 \cup \mathbf{S}_2 \rangle_{\mathbb{R}} = \mathbb{R}^5.$$

The 5-dimensional pattern described in Lemma 4 is denoted by $\mathcal{S}_5 = \mathcal{S}_0 \underset{\Lambda_0}{-} \mathcal{S}_1 \underset{\Lambda_1}{-} \mathcal{S}_2$. Note that it consists of two 4-dimensional patterns

$$\mathcal{S}_{4,0} = \mathcal{S}_0 \underset{\Lambda_0}{-} \mathcal{S}_1$$

given by indexes $\nu = r_0$ and $k = r_1$ in Lemma 3.1 and

$$\mathcal{S}_{4,1} = \mathcal{S}_1 \underset{\Lambda_1}{-} \mathcal{S}_2$$

given by indexes $\nu = s_1$ and $k = r_2$ in Lemma 3.1. These two 4-dimensional patterns $\mathcal{S}_{4,0}, \mathcal{S}_{4,1}$ intersect on the 3-dimensional subspace $\mathbf{S}_1$. We denote it by

$$\mathcal{S}_5 = \mathcal{S}_{4,0} \underset{\mathbf{S}_1}{-} \mathcal{S}_{4,1}$$

For the pattern $\mathcal{S}_5$, Schmidt's inequality (28) provides

$$\det \mathfrak{S}_0 \det \mathfrak{S}_1 \det \mathfrak{S}_2 \gg \det \Lambda_0 \det \Lambda_1$$

where $\mathfrak{S}_i = \mathbf{S}_i \cap \mathbb{Z}^5$. It can be rewritten as

$$\frac{\det \mathfrak{S}_0 \, (\det \mathfrak{S}_1)^x}{\det \Lambda_0} \cdot \frac{(\det \mathfrak{S}_1)^{1-x} \det \mathfrak{S}_2}{\det \Lambda_2} \gg 1$$

with arbitrary $x \in (0, 1)$. This means that

$$\text{either} \quad \frac{\det \mathfrak{S}_0 \, (\det \mathfrak{S}_1)^x}{\det \Lambda_0} \gg 1 \quad \text{or} \quad \frac{(\det \mathfrak{S}_1)^{1-x} \det \mathfrak{S}_2}{\det \Lambda_2} \gg 1.$$

Applying Lemma 2 two times with parameters $(s, t) = (1, x)$ and $(s, t) = (1 - x, 1)$ we get the lower bound

$$\frac{\lambda}{\hat{\lambda}} \geq g \quad \text{or} \quad \frac{\omega}{\hat{\omega}} \geq g^*$$

where g and g^* are given by the optimization equations

$$g = g(1, x) = g(1 - x, 1) \quad \text{or } g^* = g^*(1, x^*) = g^*(1 - x^*, 1). \tag{36}$$

From (32,33) we have

$$x = \frac{\beta}{g(g - \beta)} = \frac{R_3(g)}{g - \beta}$$

and so g satisfies the equation

$$R_4(g) = gR_3(g) - \beta = 0.$$

Similarly, $R_4(1/g^*) = 0$.

This proves first part of Theorem 1 for approximation to four numbers. □

Here, there is one parameter x to optimize. In higher dimension, we have many more, and need to compute the optimization of these parameters inductively.

Proof of Lemma 4. Figure 2 may be usefull to understand the construction.
Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta} \in \mathbb{R}^4$, and an arbitrarily large index k_0. Set r_2+1 to be the smallest index such that the sequence of best approximations $(\boldsymbol{z}_l)_{k_0\leq l\leq r_2+1}$ spans the whole 5-dimensional space. Note that $\boldsymbol{z}_{r_2+1}$ is not in the 4-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0\leq l\leq r_2}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1}$ are linearly independent and span a 3-dimensional lattice Γ_2. Set $r_0-1 \geq k$ to be the largest index such that $(\boldsymbol{z}_l)_{r_0-1\leq l\leq r_2+1}$ spans the whole 5-dimensional space. Note that $\boldsymbol{z}_{r_0-1}$ is not in the 4-dimensional subspace spanned by $(\boldsymbol{z}_l)_{r_0\leq l\leq r_2+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ are linearly independent and span the 3-dimensional lattice Γ_0. Moreover, combining both observations we get that

$$\Gamma_1 = \langle (\boldsymbol{z}_l)_{r_0\leq l\leq r_2} \rangle_{\mathbb{Z}}$$

is a 3-dimensional lattice.
Now appears the induction step: we apply the construction of Lemma 3 to the two 4-dimensional subspaces

$$\mathbf{S}_{4,0} := \langle (\boldsymbol{z}_l)_{r_0-1\leq l\leq r_2} \rangle_{\mathbb{R}} \quad \text{and} \quad \mathbf{S}_{4,1} := \langle (\boldsymbol{z}_l)_{r_0\leq l\leq r_2+1} \rangle_{\mathbb{R}}.$$

Set r_1+1 to be the smallest index such that $(\boldsymbol{z}_l)_{r_0-1\leq l\leq r_1+1}$ spans $\mathbf{S}_{4,0}$. Note that $\boldsymbol{z}_{r_1+1}$ is not in the 3-dimensional subspace $\mathbf{S}_0$ spanned by $(\boldsymbol{z}_l)_{r_0-1\leq l\leq r_1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}, \boldsymbol{z}_{r_1+1}$ are linearly independent and span a 3-dimensional lattice Γ_2. By construction, r_0-1 is already the largest index such that $\langle (\boldsymbol{z}_l)_{r_0-1\leq l\leq r_1} \rangle_{\mathbb{R}} = \mathbf{S}_{4,0}$. Hence, $\langle (\boldsymbol{z}_l)_{r_0\leq l\leq r_1} \rangle_{\mathbb{Z}} =: \Lambda_0$ is a 2-dimensional lattice spanned by either $\langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1} \rangle_{\mathbb{Z}}$ or $\langle \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1} \rangle_{\mathbb{Z}}$, and is the intersection $\mathbf{S}_0 \cap \mathbf{S}_1 \cap \mathbb{Z}^5$.
Set s_1-1 to be the largest index such that $(\boldsymbol{z}_l)_{s_1-1\leq l\leq r_2+1}$ spans $\mathbf{S}_{4,1}$. Note that $\boldsymbol{z}_{s_1-1}$ is not in the 3-dimensional subspace $\mathbf{S}_2$ spanned by $(\boldsymbol{z}_l)_{s_1\leq l\leq r_2+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{s_1-1}, \boldsymbol{z}_{s_1}, \boldsymbol{z}_{s_1+1}$ are linearly independent and span a 3-dimensional lattice Γ_1. By construction, r_2+1 is already the smallest index such that $\langle (\boldsymbol{z}_l)_{s_1-1\leq l\leq r_2+1} \rangle_{\mathbb{R}} = \mathbf{S}_{4,1}$. Hence, $\langle (\boldsymbol{z}_l)_{s_1\leq l\leq r_2} \rangle = \Lambda_1$ is a 2-dimensional lattice spanned by $\langle \boldsymbol{z}_{s_1}, \boldsymbol{z}_{s_1+1} \rangle_{\mathbb{Z}}$ or $\langle \boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2} \rangle_{\mathbb{Z}}$, and is the intersection $\mathbf{S}_1 \cap \mathbf{S}_2 \cap \mathbb{Z}^5$.

□

Note that we may have $r_1 = s_1$, which is indeed the case for *regular graphs*.

In Figure 2, the dashed lines should be interpreted as follows. The best approximation vectors $(\boldsymbol{z}_l)_{r_0\leq l\leq r_1}$ generate the 2-dimensional lattice Λ_0. The best approximation vectors $(\boldsymbol{z}_l)_{s_1\leq l\leq r_2}$ generate the 2-dimensional lattice Λ_1. The best approximation vectors $(\boldsymbol{z}_l)_{r_1-1\leq l\leq s_1+1}$ generate the 3-dimensional lattice Γ_1. The five bold vectors span the whole

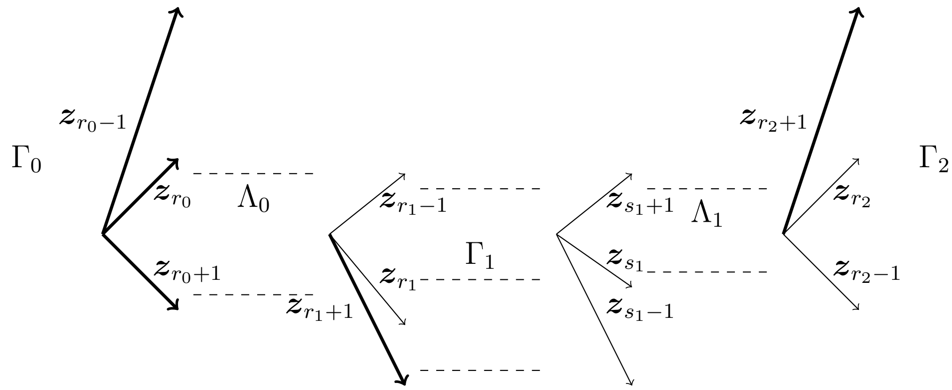

Figure 2: Selected sequence of best approximation vectors.

space $\mathbb{R}^5$.

4 Arbitrary dimension

Consider $\boldsymbol{\theta} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta}$, without loss of generality we can assume that the sequence $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ spans the whole space $\mathbb{R}^{n+1}$.

Lemma 5. *Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta} \in \mathbb{R}^n$. Let k_0 be an arbitrarily large index. There exists 2^{n-3} indexes $k_0 < r_0 < r_1, \dots, r_{2^{n-2}-2} < r_{2^{n-2}-1}$ such that the following holds.*

1. *The triples of consecutive best approximation vectors*

$$\mathcal{S}_{3,l} = \{\boldsymbol{z}_{r_l-1}, \boldsymbol{z}_{r_l}, \boldsymbol{z}_{r_l+1}\}, \quad 0 \le l \le 2^{n-2} - 1$$

   consist of linearly independent vectors spanning a 3-dimensional rational subspace $S_{3,l}$.

2. *For $4 \le k \le n+1$ and $0 \le l \le 2^{n-k+1} - 1$, denote by $\mathcal{S}_{k,l}$ the set of best approximation vectors*

$$\mathcal{S}_{k,l} = \cup_{\nu=0}^{2^{k-3}-1} \mathcal{S}_{3,2^{k-3}l+\nu}.$$

   $\mathcal{S}_{k,l}$ spans the k-dimensional rational subspace $\mathbf{S}_{k,l}$.

3. *The rational subspaces $\mathbf{S}_{k,l}$ satisfy the relations*

$$\mathbf{S}_{k,2l} \cup \mathbf{S}_{k,2l+1} = \mathbf{S}_{k+1,l} \quad (37)$$

$$\mathbf{S}_{k,2l} \cap \mathbf{S}_{k,2l+1} = \mathbf{S}_{k-1,4l+1} = \mathbf{S}_{k-1,4l+2} =: \mathbf{Q}_{k-1,l}. \quad (38)$$

   In particular, $\mathbf{Q}_{2,l}$ is spanned by both $\boldsymbol{z}_{r_{4l+1}}, \boldsymbol{z}_{r_{4l+1}+1}$ and $\boldsymbol{z}_{r_{4l+2}-1}, \boldsymbol{z}_{r_{4l+2}}$.

4. *The full space* $\mathbb{R}^{n+1}$ *is spanned by*

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \boldsymbol{z}_{r_1+1}, \boldsymbol{z}_{r_2+1}, \dots, \boldsymbol{z}_{r_{2^{n-3}-1}+1} \rangle_{\mathbb{R}} = \langle \cup_{l=0}^{2^{n-k+1}-1} \mathbf{S}_{k,l} \rangle_{\mathbb{R}}, \quad 3 \leq k \leq n+1.$$

In particular, $\mathbf{S}_{n+1,0} = \mathbb{R}^{n+1}$*.*

Here, the first index always denote the dimension of the considered object. For a given dimension k, two subspaces $\mathbf{S}_{k,l_1}$ and $\mathbf{S}_{k,l_2}$ may coincide, but at least $n-k-1$ of them are distinct since they all together span the whole space $\mathbb{R}^{n+1}$. In particular for dimension $k=3$, this means that two indexes $r_0 < r_{l_1} \leq r_{l_2} < r_{2^{n-2}-1}$ may coincide. Indeed for *regular graphs* we have that $\mathbf{S}_{k-1,4l+1} = \mathbf{S}_{k-1,4l+2}$ are the same.

Lemma 5 coincide with Lemma 3 for the approximation to three numbers and with Lemma 4 for the approximation to four numbers. In the later case, we have:

$$\begin{aligned} \mathbf{S}_i \sim \mathbf{S}_{3,i} &\quad \text{for} \quad 0 \leq i \leq 2, \\ \Lambda_j \sim \mathbf{Q}_{2,j} &\quad \text{for} \quad 0 \leq j \leq 1, \\ \mathbf{S}_{4,k} \sim \mathbf{S}_{4,k} &\quad \text{for} \quad 0 \leq k \leq 1. \end{aligned}$$

We can partially describe the situation with the binary tree from Figure 3, where each child is included in its parent. In particular, the parent of a given rational subspace $\mathbf{S}_{k,l}$ is $\mathbf{S}_{k+1,\sigma(l)}$ where σ is the usual shift on the binary expansion.

We may write the recursive step of the construction of patterns as follows:

$$\mathcal{S}_{n+1} = \mathcal{S}_{n+1,0} = \mathcal{S}_{n,0} \underset{\mathbf{Q}_{n-1,0}}{-} \mathcal{S}_{n,1}$$

where $\mathbf{Q}_{n-1,0}$ is a $n-1$ dimensional subspace. For $\mathbf{S}_{n,0}$, $\mathbf{S}_{n,1}$ and $\mathbf{Q}_{n-1,0}$ the rational subspaces and their underlying lattices $\mathfrak{S}_{n,0}$, $\mathfrak{S}_{n,1}$ and $\mathfrak{Q}_{n-1,0}$, Schmidt's inequality (28) provides

$$\frac{\det \mathfrak{S}_{n,0} \cdot \det \mathfrak{S}_{n,1}}{\det \mathfrak{Q}_{n-1,0}} \gg 1. \tag{39}$$

This relation enables us to shift the optimization equation in the next dimension obtained in the next lemma.

Lemma 6. *Consider* $\boldsymbol{\theta} \in \mathbb{R}^n$ *with* $\mathbb{Q}$*-linearly independent coordinates with* 1*. Consider* $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ *a sequence of best approximations to* $\boldsymbol{\theta}$ *spanning the whole space* $\mathbb{R}^{n+1}$*. Consider the pattern of best approximation vectors* $\mathcal{S}_{n+1,0}$ *and its sub-patterns given by Lemma 5. The following formula holds.*

$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\det(\mathfrak{S}_{3,4l}) \det(\mathfrak{Q}_{3,l})^{1-y_{n-4}}}{\det(\mathfrak{Q}_{2,2l})} \right)^{w_{n-4,l}} \cdot \prod_{l=0}^{2^{n-4}-1} \left(\frac{\det(\mathfrak{Q}_{3,l})^{1-z_{n-4}} \det(\mathfrak{S}_{3,4l+3})}{\det(\mathfrak{Q}_{2,2l+1})} \right)^{w'_{n-4,l}} \gg 1 \tag{40}$$

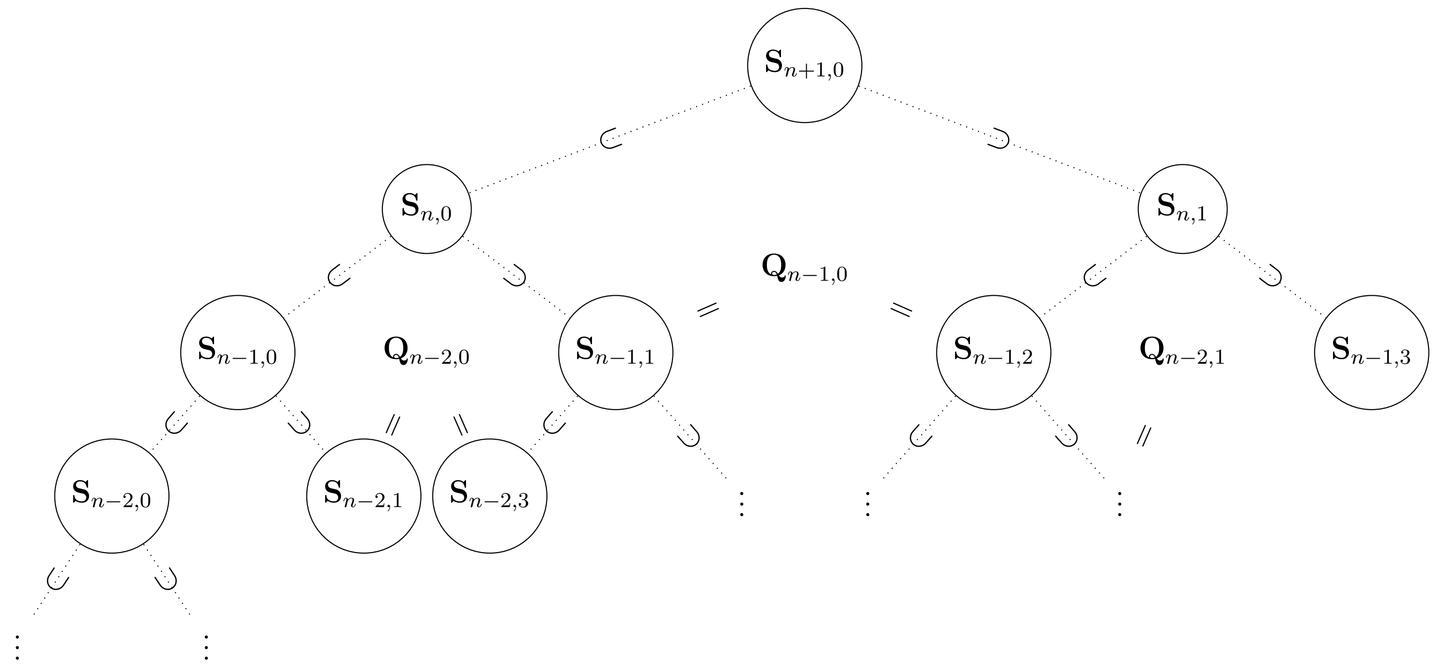

Figure 3: Binary tree sketching the situation described in Lemma 5.

where the parameters $w_{k,l}, w'_{k,l}, y_k$ and z_k are defined inductively by

$$
\begin{aligned}
0 &= y_0 + z_0 - 1 && (41)\\
(y_{k+1}, z_{k+1}) &= F(y_k, z_k) = \left(\frac{y_k}{y_k + z_k - y_k z_k}, \frac{z_k}{y_k + z_k - y_k z_k}\right) && (42)\\
1 &= w_{0,0} = w'_{0,0}\\
w_{k+1,2l} &= w_{k,l}\\
w_{k+1,2l+1} &= (1 - z_k) w'_{k,l}\\
w'_{k+1,2l} &= (1 - y_k) w_{k,l}\\
w'_{k+1,2l+1} &= w'_{k,l}
\end{aligned}
$$

Here as before, $\mathfrak{S}_{k,l} = \mathbf{S}_{k,l} \cap \mathbb{Z}^{n+1}$ is the underlying lattice of the rational subspace $\mathbf{S}_{k,l}$.

We do not need to compute the values of $w_{k,l}$ or $w'_{k,l}$. From formula (40), we deduce that there exists an index l such that either

$$\frac{\det(\mathfrak{S}_{3,4l}) \det(\mathfrak{Q}_{3,l})^{1-y_{n-4}}}{\det(\mathfrak{Q}_{2,2l})} \gg 1 \quad \text{or} \quad \frac{\det(\mathfrak{Q}_{3,l})^{1-z_{n-4}} \det(\mathfrak{S}_{3,4l+3})}{\det(\mathfrak{Q}_{2,2l+1})} \gg 1$$

Applying Lemma 2 twice, the optimization constant g is given by

$$g = g(1, 1 - z_{n-4}) = g(1 - y_{n-4}, 1)$$

where $(y_0, z_0) = F^{-n+4}(y_{n-4}, z_{n-4})$ satisfies $y_0 + z_0 - 1 = 0$.

By formulae (33) (34), we get

$$y_{n-4} = \frac{R_3(g)}{R_2(g)} \quad \text{and} \quad z_{n-4} = \frac{R_3(g)}{gR_2(g)}.$$

Then, the recurrence formula (42) provides that for $4 \le k \le n-3$

$$y_{n-k} = \frac{R_{k-1}(g)}{R_{k-2}(g)} \quad \text{and} \quad z_{n-k} = \frac{R_{k-1}(g)}{gR_{k-2}(g)}$$

if k is odd and

$$y_{n-k} = \frac{R_{k-1}(g)}{gR_{k-2}(g)} \quad \text{and} \quad z_{n-k} = \frac{R_{k-1}(g)}{R_{k-2}(g)}$$

if k is even.

Proof. Suppose $y_{n-k} = \frac{R_{k-1}}{R_{k-2}}$ and $z_{n-k} = \frac{R_{k-1}}{gR_{k-2}}$. Since

$$F^{-1}(y_{n-k}, z_{n-k}) = \left(\frac{y_{n-k} + z_{n-k} - 1}{z_{n-k}}, \frac{y_{n-k} + z_{n-k} - 1}{y_{n-k}}\right)$$

we get

$$\begin{aligned} y_{n-k-1} &= \frac{y_{n-k} + z_{n-k} - 1}{z_{n-k}} = \frac{R_{k-1}/R_{k-2} + R_{k-1}/gR_{k-2} - 1}{R_{k-1}/gR_{k-2}} = \frac{R_k}{R_{k-1}}, \\ z_{n-k-1} &= \frac{y_{n-k} + z_{n-k} - 1}{y_{n-k}} = \frac{R_k}{gR_{k-1}}. \end{aligned}$$

Hence the formulae by symmetry and initialization for $k = 4$. □

In particular, $(y_0, z_0) = \left(\frac{R_{n-1}(g)}{gR_{n-2}(g)}, \frac{R_{n-1}(g)}{R_{n-2}(g)}\right)$ or $\left(\frac{R_{n-1}(g)}{R_{n-2}(g)}, \frac{R_{n-1}(g)}{gR_{n-2}(g)}\right)$ depending on the parity of n. This leads to

$$0 = y_0 + z_0 - 1 = \frac{gR_{n-1}(g) + R_{n-1}(g) - gR_{n-2}(g)}{gR_{n-2}(g)} = \frac{R_n(g)}{gR_{n-2}(g)}.$$

That is, $R_n(g) = 0$. So we proved the bound (13) holds for arbitrary large indices, and in arbitrary dimension n, for the required g.
Mutatis mutandis by replacing g by $1/g^*$, we get $R_n(1/g^*) = 0$.

This proves first part of Theorem 1. □

Proof of Lemma 5. Figure 3 may be usefull to understand the construction.
Let $k_0 \gg 1$. We prove the lemma by induction. Suppose that we can construct a pattern $\mathfrak{S}_k$ of 2^{k-3} triples of consecutive best approximation vectors given by indexes $k_0 < r_0 < r_1, \dots, r_{2^{k-2}-2} < r_{2^{k-2}-1}$ spanning a k-dimensional rational space. Such a construction for $k = 4, 5$ holds via Lemmas 3 and 4. This provides the initialization.

Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations spanning a $(k+1)$-dimensional rational space $\mathbf{S}_{k+1}$. Set $r_{2^{k-1}-1} + 1$ to be the smallest index such that the sequence of best approximations $(\boldsymbol{z}_l)_{k_0 \le l \le r_{2^{k-1}-1}+1}$ spans $\mathbf{S}_{k+1}$. Note that $\boldsymbol{z}_{r_{2^{k-1}-1}+1}$ is not in the k-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0 \le l \le r_{2^{k-1}-1}}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_{2^{k-1}-1}-1}, \boldsymbol{z}_{r_{2^{k-1}-1}}, \boldsymbol{z}_{r_{2^{k-1}-1}+1}$ are linearly independent and span a 3-dimensional subspace $\mathbf{Q}_{2,2^{k-1}-1}$. Set $r_0 - 1 \ge k_0$ to be the largest index such that $(\boldsymbol{z}_l)_{r_0-1 \le l \le r_{2^{k-1}-1}+1}$ spans $\mathbf{S}_{k+1}$. Note that $\boldsymbol{z}_{r_0-1}$ is not in the k-dimensional subspace spanned by $(\boldsymbol{z}_l)_{r_0 \le l \le r_{2^{k-1}-1}+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ are linearly independent and span the 3-dimensional subspace $\mathbf{S}_{3,0}$. Moreover, combining both observations we get that

$$\mathbf{Q}_{k-1,0} := \langle (\boldsymbol{z}_l)_{r_0 \le l \le r_{2^{k-1}-1}} \rangle_{\mathbb{R}}$$

is a $k-1$-dimensional subspace.
We use the induction hypothesis for the two k-dimensional subspaces

$$\mathbf{S}'_k := \langle (\boldsymbol{z}_l)_{r_0-1 \le l \le r_{2^{k-1}-1}} \rangle_{\mathbb{R}} \quad \text{and} \quad \mathbf{S}''_k := \langle (\boldsymbol{z}_l)_{r_0 \le l \le r_{2^{k-1}-1}+1} \rangle_{\mathbb{R}}$$

for $k_0 = r_0 - 1$ resp. $k_0 = r_0$. This provides two patterns $\mathcal{S}'_k$ and $\mathcal{S}''_k$ of triples of best approximation vectors defined by indexes $r_0 \le r'_0 < r'_1, \dots, r'_{2^{k-2}-2} < r'_{2^{k-2}-1}$ and $r_0 + 1 \le r''_0 < r''_1, \dots, r''_{2^{k-2}-2} < r''_{2^{k-2}-1}$ satisfying the conditions of Lemma 5. A key observation is that by definition of r_0, we necessarily have $r'_0 = r_0$. Similarly, by definition of $r_{2^{k-1}-1}$, we necessarily have $r_{2^{k-1}-1} = r''_{2^{k-2}-1}$. It follows that both sub-patterns $\mathcal{S}'_{k-1,1}$ and $\mathcal{S}''_{k-1,0}$ span the rational subspace $\mathbf{Q}_{k-1,0}$. Hence, the pattern $\mathcal{S}$ defined by the triples given by indexes

$$r_i = r'_i \quad \text{and} \quad r_{i+2^{k-2}} = r''_i \quad \text{for } 0 \le i \le 2^{k-2} - 1$$

combining the two sub-patterns $\mathcal{S}'_k$ and $\mathcal{S}''_k$ satisfies the required properties at the rank $k+1$.

$$\mathcal{S} = \mathcal{S}'_k \underset{\mathbf{Q}_{k-1,0}}{-} \mathcal{S}''_k.$$

Lemma 5 follows applied to $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to $\boldsymbol{\theta} \in \mathbb{R}^n$ spanning the whole space $\mathbb{R}^{n+1}$. $\square$

Proof of Lemma 6. We prove by induction the more general formula

$$\prod_{l=0}^{2^{k-1}-1}\left(\frac{\det\left(\mathfrak{S}_{n-k,4l}\right)\det\left(\mathfrak{Q}_{n-k,l}\right)^{1-y_{k-1}}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)}\right)^{w_{k-1,l}}\times \prod_{l=0}^{2^{k-1}-1}\left(\frac{\det\left(\mathfrak{Q}_{n-k,l}\right)^{1-z_{k-1}}\det\left(\mathfrak{S}_{n-k,4l+3}\right)}{\det\left(\mathfrak{Q}_{n-k-1,2l+1}\right)}\right)^{w'_{k-1,l}}\gg 1 \tag{43}$$

Initialization follows the steps of approximation to four numbers, and Lemma 6 is the formula for $k=n-3$. Namely, Schmidt's inequality (28) provides

$$\det(\mathfrak{S}_{n,0})\det(\mathfrak{S}_{n,1}) \gg \det(\mathfrak{Q}_{n-1,0})\det(\mathfrak{S}_{n+1,0}), \tag{44}$$

$$\det(\mathfrak{S}_{n-1,0})\det(\mathfrak{S}_{n-1,1}) \gg \det(\mathfrak{Q}_{n-2,0})\det(\mathfrak{S}_{n,0}), \tag{45}$$

$$\det(\mathfrak{S}_{n-1,2})\det(\mathfrak{S}_{n-1,3}) \gg \det(\mathfrak{Q}_{n-2,1})\det(\mathfrak{S}_{n,1}). \tag{46}$$

Since $\mathcal{S}_{n+1,0}$ spans the whole space $\mathbb{R}^{n+1}$, we have $\det\mathfrak{S}_{n+1,0}=1$ and using the fact that $\det\mathfrak{Q}_{n-1,0}=\det\mathfrak{S}_{n-1,1}=\det\mathfrak{S}_{n-1,2}$ (by (38)), we get the formula

$$\frac{\det(\mathfrak{S}_{n-1,0})\det(\mathfrak{Q}_{n-1,0})\det(\mathfrak{S}_{n-1,3})}{\det(\mathfrak{Q}_{n-2,0})\det(\mathfrak{Q}_{n-2,1})}\gg 1.$$

Setting $w_{0,0}=w'_{0,0}=1$ and y_0 and z_0 such that $y_0+z_0-1=0$, we can rewrite

$$\left(\frac{\det(\mathfrak{S}_{n-1,0})\det(\mathfrak{Q}_{n-1,0})^{1-y_0}}{\det(\mathfrak{Q}_{n-2,0})}\right)^{w_{0,0}}\left(\frac{\det(\mathfrak{Q}_{n-1,0})^{1-z_0}\det(\mathfrak{S}_{n-1,3})}{\det(\mathfrak{Q}_{n-2,1})}\right)^{w'_{0,0}}\gg 1.$$

This establishes the expected formula for $k=1$. The inductive step sees Schmidt's inequality (28) split each term of the product in two terms involving rational subspaces of lower dimension, and shift the values of the parameters y_k and z_k.
Indeed, for $3\le j\le n+1$ and $0\le j\le 2^{n+1-j}-1$ we have

$$\frac{\det(\mathfrak{Q}_{j-1,2i})\det(\mathfrak{S}_{j-1,8i+3})}{\det(\mathfrak{Q}_{j-2,4i+1})} \gg \det(\mathfrak{Q}_{j,i}), \tag{47}$$

$$\frac{\det(\mathfrak{S}_{j-1,8i+4})\det(\mathfrak{Q}_{j-1,2i+1})}{\det(\mathfrak{Q}_{j-2,4i+2})} \gg \det(\mathfrak{Q}_{j,i}), \tag{48}$$

$$\frac{\det(\mathfrak{S}_{j-1,4i})\det(\mathfrak{Q}_{j-1,i})}{\det(\mathfrak{Q}_{j-2,2i})} \gg \det(\mathfrak{S}_{j,2i}), \tag{49}$$

$$\frac{\det(\mathfrak{Q}_{j-1,2i+1})\det(\mathfrak{S}_{j-1,8i+7})}{\det(\mathfrak{Q}_{j-2,4i+3})} \gg \det(\mathfrak{S}_{j,4i+3}). \tag{50}$$

Here, (47) is a straight application of (28) applied to the rational subspaces $\mathbf{S}_{j-1,8i+3}$ and $\mathbf{S}_{j-1,8i+2}=\mathbf{Q}_{j-1,2i}$ whose union spans $\mathbf{S}_{j,4i+1}=\mathbf{Q}_{j,i}$ and whose intersection is $\mathbf{Q}_{j-2,4i+1}$.

For (48), we apply (28) to the rational subspaces $\mathbf{S}_{j-1,8i+4}$ and $\mathbf{S}_{j-1,8l+5} = \mathbf{Q}_{j-1,2i+1}$ whose union spans $\mathbf{S}_{j,4i+2} = Q_{j,i}$ and whose intersection is $\mathbf{Q}_{j-2,4i+1}$. For (49), we apply (28) to the rational subspaces $\mathbf{S}_{j-1,4i}$ and $\mathbf{S}_{j-1,4i+1} = \mathbf{Q}_{j-1,i}$ whose union spans $\mathbf{S}_{j,2i}$ and whose intersection is $\mathbf{Q}_{j-2,2i}$. For (50), we apply (28) to the rational subspaces $\mathbf{S}_{j-1,8i+6} = \mathbf{Q}_{j-1,2i+1}$ and $\mathbf{S}_{j-1,8i+7} = \mathbf{Q}_{j-1,2i+1}$ whose union spans $\mathbf{S}_{j,4i+3}$ and whose intersection is $\mathbf{Q}_{j-2,4i+3}$.

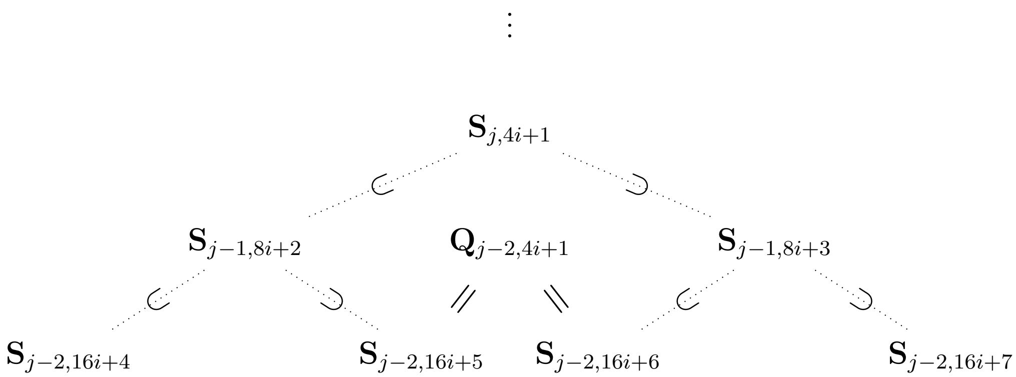

Figure 4: Situation to apply Schmidt's inequality for (47).

Inductive step. Assume that (43) holds for some $1 \leq k < n-3$. For every $0 \leq l \leq 2^{k-1} - 1$, we can split

$$1 \ll \left(\frac{\det\left(\mathfrak{S}_{n-k,4l}\right) \det\left(\mathfrak{Q}_{n-k,l}\right)^{1-y_{k-1}}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)} \right)^{w_{k-1,l}} \tag{51}$$

$$\ll \left(\frac{\left(\frac{\det(\mathfrak{S}_{n-k-1,8k}) \det(\mathfrak{Q}_{n-k-1,2l})}{\det(\mathfrak{Q}_{n-k-2,4l})} \right) \left(\frac{\det(\mathfrak{Q}_{n-k-1,2l}) \det(\mathfrak{S}_{n-k-1,8l+3})}{\det(\mathfrak{Q}_{n-k-2,4l+1})} \right)^{1-y_{k-1}}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)} \right)^{w_{k-1,l}} \tag{52}$$

where we used (49) with parameters $j = n-k$ and $i = 2l$ and (47) with parameters $j = n-k$ and $i = l$. For any $y_{k+1} \in (0,1)$ we can write

$$\left(\frac{\det(\mathfrak{S}_{n-k-1,8l}) \det(\mathfrak{Q}_{n-k-1,2l})^{y_{k+1}(1-y_k)}}{\det(\mathfrak{Q}_{n-k-2,4l})} \right)^{w_{k,l}} \times \left(\frac{\det(\mathfrak{Q}_{n-k-1,2l})^{1-y_{k+1}} \det(\mathfrak{S}_{n-k-1,8l+3})}{\det(\mathfrak{Q}_{n-k-2,4l+1})} \right)^{(1-y_k)w_{k,l}} \gg 1 \tag{53}$$

Similarly, for any $z_{k+1} \in (0,1)$, using (48) with $j = n-k$ and $i = l$ and (50) with $j = n-k$ and $i = l$ we get

$$1 \ll \left(\frac{\det(\mathfrak{S}_{n-k-1,8l+4}) \det(\mathfrak{Q}_{n-k-1,2l+1})^{1-z_{k+1}}}{\det(\mathfrak{Q}_{n-k-2,4l+2})} \right)^{(1-z_k)w'_{k,l}} \times \left(\frac{\det(\mathfrak{Q}_{n-k-1,2l+1})^{z_{k+1}(1-z_k)} \det(\mathfrak{S}_{n-k-1,8l+7})}{\det(\mathfrak{Q}_{n-k-2,4l+3})} \right)^{w'_{k,l}} \tag{54}$$

For the sake of optimization, we want

$$y_{k+1}(1-y_k) = 1 - z_{k+1} \quad \text{and} \quad 1 - y_{k+1} = z_{k+1}(1-z_k).$$

That is

$$y_k = \frac{y_{k+1} + z_{k+1} - 1}{z_{k+1}} \quad \text{and } z_k = \frac{y_{k+1} + z_{k+1} - 1}{y_{k+1}}$$

or equivalently

$$y_{k+1} = \frac{y_k}{y_k + z_k - y_k z_k} \quad \text{and } z_{k+1} = \frac{z_k}{y_k + z_k - y_k z_k}.$$

Setting for every $0 \leq l \leq 2^{k-1} - 1$

$$w_{k+1,2l} = w_{k,l}, \quad w_{k+1,2l+1} = (1-z_k)w'_{k,l}, \quad w'_{k+1,2l} = (1-y_k)w_{k,l} \quad \text{and} \quad w'_{k+1,2l+1} = w'_{k,l},$$

we established formula (43) for $k+1$ with the required induction fomulae for the parameters. □

5 Construction of points with given ratio

In this last section, we prove the second part of Theorem 1. To construct points with given ratio, we place ourselves in the context of parametric geometry of numbers introduced by Schmidt and Summerer in [19]. We refer the reader to [9, §2] for the notation used in this paper and the presentation of the parametric geometry of numbers. We use the notation introduced by D. Roy in [16] which is essentially dual to the one of W. M. Schmidt and L. Summerer [18]. We fully use Roy's theorem [16] as stated in [9, Theorem 5] to deduce the existence of a point with expected properties from an explicit family of generalized $(n+1)$-systems with three parameters. The construction shows how the values $G(n,\alpha)$ and $G^*(n,\alpha^*)$ appear naturally in the context of parametric geometry of numbers, and why they are reached at *regular graphs*.

Fix the dimension $n \geq 2$, and consider the case of approximation by a linear form. Fix the three parameters $\hat{\omega} \geq n$, $\rho = G^*(n,\hat{\omega})$ and $c \geq 1$. Consider the generalized $(n+1)$-system

$\boldsymbol{P}$ on the interval $[1, c\rho]$ depending on these parameters whose combined graph is given below by Figure 5, where

$$P_1(1) = \frac{1}{1+\hat{\omega}}, \ \ P_k(1) = \rho^{k-2}P_1(1) \ \text{ for } 2 \le k \le n+1 \text{ and } P_k(c\rho) = c\rho P_k(1) \text{ for } 1 \le k \le n+1.$$

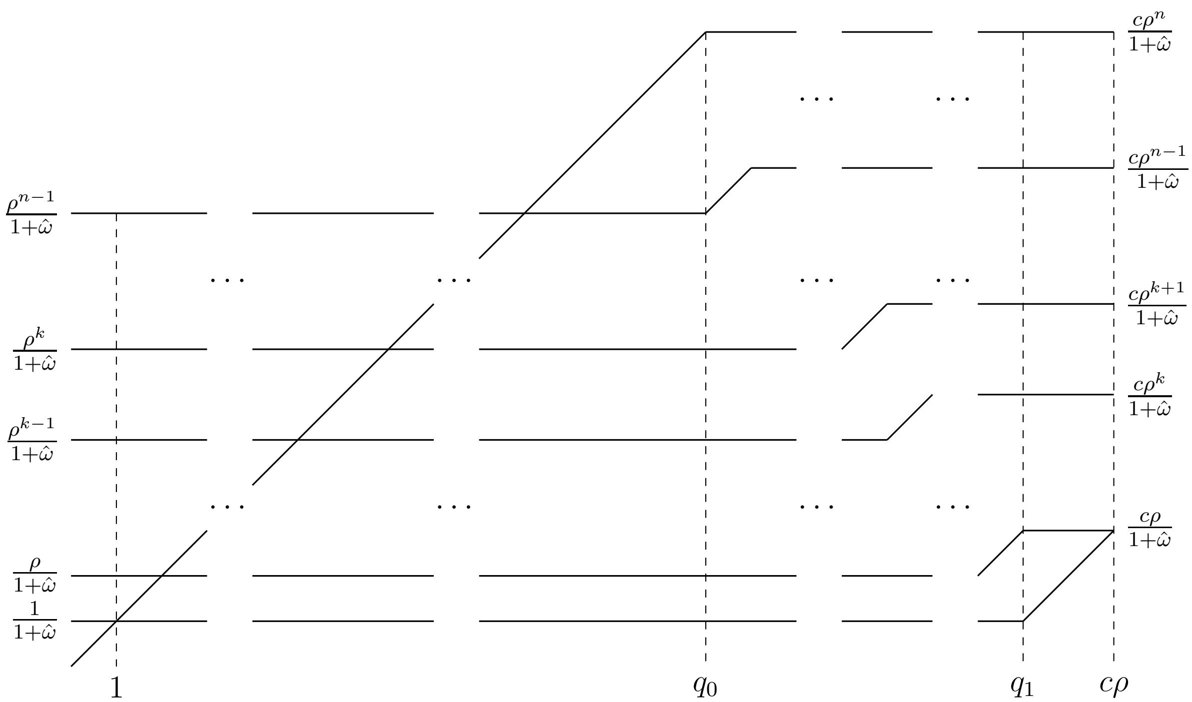

Figure 5: Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, c\rho]$

The fact that all coordinates sum up to 1 for $q = 1$ follows from ρ being the root of the polynomial R_n (9). On each interval between two consecutive division points, there is only one line segment with slope 1. On $[1, q_0]$, there is one line segment of slope 1 starting from the value $\frac{1}{1+\hat{\omega}}$ and reaching the value $\frac{c\rho^n}{1+\hat{\omega}}$. Then, each component P_k increases from $\frac{\rho^{k-1}}{1+\hat{\omega}}$ to $\frac{c\rho^{k-1}}{1+\hat{\omega}}$ with slope 1 where k decreases from $k = n$ down to $k = 2$.

We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity. This means, $\boldsymbol{P}(q) = (c\rho)^m \boldsymbol{P}((c\rho)^{-m} q)$ for all integers m. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and $c\rho$, one sees that the extension provides a generalized $(n+1)$-system on $[1, \infty)$.

Note that for $c = 1$, the parameter q_0 and q_1 coincide and we constructed a *regular graph*.

Roy's Theorem provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that

$$\frac{1}{1+\hat{\omega}(\boldsymbol{\theta})} = \limsup_{q\to+\infty} \frac{P_1(q)}{q},$$
$$\frac{1}{1+\omega(\boldsymbol{\theta})} = \liminf_{q\to+\infty} \frac{P_1(q)}{q}.$$

Here, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval $[1, c\rho[$. Thus,

$$\frac{1}{1+\hat{\omega}(\boldsymbol{\theta})} = \max_{[1,c\rho[} \frac{P_1(q)}{q} = \frac{P_1(1)}{1} = \frac{1}{1+\hat{\omega}},$$
$$\frac{1}{1+\omega(\boldsymbol{\theta})} = \min_{[1,c\rho[} \frac{P_1(q)}{q} = \frac{P_1(q_0)}{q_0} = \frac{1}{c\rho\hat{\omega}+1}$$

where $q_0 = \dfrac{c(\rho^n + \cdots + \rho^2 + \rho) + 1}{1+\hat{\omega}} = \dfrac{c(\rho\hat{\omega}) + 1}{1+\hat{\omega}}$. Hence,

$$\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \omega(\boldsymbol{\theta}) = c\rho\hat{\omega}.$$

and we constructed the required points since $c \geq 1$ and $\rho = G^*(n, \hat{\omega})$.

Consider the case of simultaneous approximation. Fix the three parameters $1 \geq \hat{\lambda} \geq 1/n$, $\rho = G(n, \hat{\lambda})$ and $c \geq 1$. Consider the generalized $(n+1)$-system $\boldsymbol{P}$ on the interval $[1, c\rho]$ depending on these parameters whose combined graph is given below by Figure 6, where

$$P_{n+1}(1) = \frac{\hat{\lambda}}{1+\hat{\lambda}}, \;\; P_k(1) = \rho^{n-k} P_1(1) \text{ for } 1 \leq k \leq n \text{ and } P_k(c\rho) = c\rho P_k(1) \text{ for } 2 \leq k \leq n+1.$$

The fact that all coordinates sum up to 1 for $q = 1$ follows from $1/\rho$ being the root of the polynomial R_n (9). Up to change of origin and rescaling, this is the same pattern as shown by Figure 5. We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity. This means, $\boldsymbol{P}(q) = (c\rho)^m \boldsymbol{P}((c\rho)^{-m} q)$ for all integers m. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and $c\rho$, one sees that the extension provides a generalized $(n+1)$-system on $[1, \infty)$.

For $c = 1$, the parameter q_0 and q_1 coincide and we constructed a *regular graph*.

Roy's Theorem provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that

$$\frac{\hat{\lambda}(\boldsymbol{\theta})}{1+\hat{\lambda}(\boldsymbol{\theta})} = \liminf_{q\to+\infty} \frac{P_{n+1}(q)}{q},$$
$$\frac{\lambda(\boldsymbol{\theta})}{1+\lambda(\boldsymbol{\theta})} = \limsup_{q\to+\infty} \frac{P_{n+1}(q)}{q}.$$

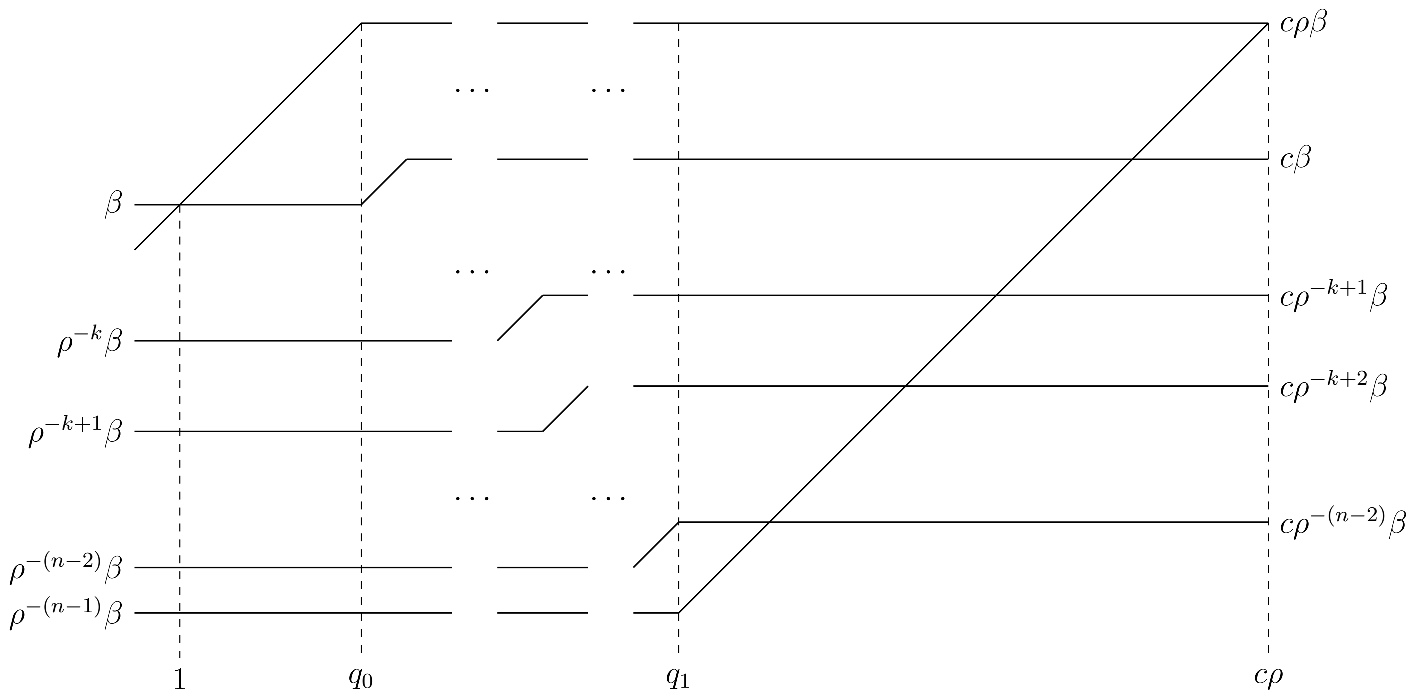

Figure 6: Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, c\rho]$, where $\beta = \frac{\hat{\lambda}}{1+\hat{\lambda}}$.

Here, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval $[1, c\rho[$. Thus,

$$\frac{\hat{\lambda}(\boldsymbol{\theta})}{1+\hat{\lambda}(\boldsymbol{\theta})} = \min_{[1,c\rho[} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(1)}{1} = \frac{\hat{\lambda}}{1+\hat{\lambda}},$$
$$\frac{\lambda(\boldsymbol{\theta})}{1+\lambda(\boldsymbol{\theta})} = \max_{[1,c\rho[} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(q_1)}{q_1} = \frac{c\rho\hat{\lambda}}{1+c\rho\hat{\lambda}}$$

where $q_1 = \dfrac{\hat{\lambda}(c(\rho^n + \cdots + \rho^2 + \rho) + 1)}{1+\hat{\lambda}} = \dfrac{\hat{\lambda}(c\rho + 1/\hat{\lambda})}{1+\hat{\lambda}}$. Hence,

$$\hat{\lambda}(\boldsymbol{\theta}) = \hat{\lambda} \quad \text{and} \quad \lambda(\boldsymbol{\theta}) = c\rho\hat{\lambda}.$$

and we constructed the required points since $c \geq 1$ and $\rho = G(n, \hat{\lambda})$.

Such self-similar generalized $(n+1)$-systems provide infinitely many distinct points $\boldsymbol{\theta} \in \mathbb{R}^n$ via Roy's theorem with $\mathbb{Q}$-linearly independent coordinates with 1, as explained in [9] at the end of §3. The $\mathbb{Q}$-linear independence comes from $P_1(q) \to \infty$ when $q \to \infty$. The construction of infinitely many points follows from a change of origin with the same pattern and self-similarity. □

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