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Reconstructing the topology on monoids and polymorphism clones of the rationals

Abstract. We show how to reconstruct the topology on the monoid of endomorphisms of the rational numbers under the strict or reflexive order relation, and the polymorphism clone of the rational numbers under the reflexive relation. In addition we show how automatic homeomorphicity results can be lifted to polymorphism clones generated by monoids.

Keywords: Rationals, Automatic homeomorphicity, Embedding, Endomorphism, Polymorphism clone

Mathematics Subject Classification: 08A35

1. Introduction

We write M and E for the monoids of endomorphisms of $(\mathbb{Q}, <)$ (coinciding with the self-embeddings of $(\mathbb{Q}, <)$ since $<$ is linear) and (\mathbb{Q}, \leq) respectively, and G for the automorphism group $\text{Aut}(\mathbb{Q}, <)$ (which equals $\text{Aut}(\mathbb{Q}, \leq)$), so that G is the family of invertible members of E . Our main results are that M and E have ‘automatic homeomorphicity’ in the sense of [3], with a corresponding result for the polymorphism clone of (\mathbb{Q}, \leq) . This intuitively means that the natural topology (see below for a precise definition) can be recognized inside the algebraic structure; the more formal definition says that any isomorphism from M to a closed submonoid of the full transformation monoid on a countable set is also necessarily a homeomorphism, with an analogous statement for E and $\text{Pol}(\mathbb{Q}, \leq)$. The case of $\text{Pol}(\mathbb{Q}, <)$ is not yet solved.

Examining automatic homeomorphicity and automatic continuity of the automorphism group of structures, that is of closed subgroups of the full symmetric group, is a topic in model theory, which has been studied for more than thirty years now. However, only recently these reconstruction notions have been generalized from permutations to unary transformations (i.e. to closed transformation monoids) and to operations of higher arity (closed clones, polymorphisms of relational structures), see [3], where a more thorough introduction to the topic and an illustrative collection of examples can be found.

A polymorphism clone can be thought of as a much more complex symmetry invariant of a structure than just its automorphism group; therefore a number of non-trivial properties are encoded in it. For instance, apart from being an interesting feature of a relational structure in itself, the presence of automatic homeomorphicity or even automatic continuity of the polymorphism clone greatly simplifies the process of providing concrete hardness certifi-

ates (via primitive positive interpretations) for the complexity of fixed template constraint satisfaction problems given by ω -categorical structures (cf. [2, Theorem 28, p. 2545], but compare with [1, Theorem 1.4] for an abstract algebraic criterion). Thus, reconstruction notions regarding the topology of closed clones are of more general interest. Studying such questions for endomorphism monoids, as well (see e.g. [6]), is a natural intermediate step between automorphism groups and polymorphism clones.

Concerning automatic homeomorphicity of $\text{End}(\mathbb{Q}, \leq)$ a precursor of our result should be mentioned. In [6, Corollary 4.13, p. 145] it is proved independently and by substantially different methods that the endomorphism monoid of the countable dense reflexive linear order without endpoints has automatic homeomorphicity with respect to the class of all countable posets (Theorem 4.12 of [6], from which Corollary 4.13 is derived, presents a less restrictive but more technical limitation of the same kind). This, in general, is a strict weakening of automatic homeomorphicity because the authors only require and demonstrate that isomorphisms between E and endomorphism monoids of countable posets are homeomorphisms. The latter constitutes just a small part of the full automatic homeomorphicity property. Notably, the additional assumption imposed in [6] on the monoid isomorphisms ensures that constant endomorphisms are mapped to constants, a fact which does not hold in the general case. Circumventing this issue is indeed one of the critical steps in section 4 where we establish automatic homeomorphicity of E , for which, as it turns out, a necessary intermediate step is to study the same property for the monoid M of self-embeddings.

According to the treatment given in [3], the main preliminary technical result needed to demonstrate automatic homeomorphicity for M is that any injective endomorphism of M which fixes G pointwise, also fixes every member of M (since M is the closure of G). We need a slightly more general variant of this fact for the proofs regarding the monoid E . We are also able to show the truth of the corresponding statement for automorphisms of E (though the deduction of automatic homeomorphicity requires more work, and a new and direct method, since G is not dense in E). We may identify M as the family of injective members of E ; another important monoid which plays a role in some of the proofs is that of the surjective maps in E , denoted by S , which we show to coincide with the epimorphisms of (\mathbb{Q}, \leq) . We shall see that, in fact, each of these is definable (in the monoid language) in E .

In order to establish the preliminary result concerning injective endomorphisms of M mentioned above, we somehow have to represent the members of M inside G . The most natural and obvious way to attempt to do this is via centralizers. Indeed in similar ‘interpretability’ results for G , this is often sufficient, and in our case, we can make quite good progress using this idea. To be more concrete, let us write ξ for the given injective endomorphism of M which fixes G pointwise. Given any $f \in M$, it is natural to consider $C_G(f) = \{g \in G : fg = gf\}$. For certain functions f , we can show that $C_G(f) = C_G(f_1) \Rightarrow f = f_1$. From this it easily follows that $\xi(f) = f$. This is because the centralizers of f and $\xi(f)$ are equal, as $g \in C_G(\xi(f)) \Leftrightarrow g\xi(f) = \xi(f)g \Leftrightarrow \xi(gf) = \xi(fg)$ (as ξ

fixes members of $G \Leftrightarrow gf = fg \Leftrightarrow g \in C_G(f)$, and so from $C_G(f) = C_G(\xi(f))$ we deduce that $\xi(f) = f$. Among elements f to which this applies are those of the form $f(x) = x$ if $x < \pi$, $x+1$ if $x > \pi$ (and similarly for any other irrational), as well as many others. An example of an f to which this does *not* apply is $f(x) = x$ if $x < 0$, $x+1$ if $x \geq 0$ (which shares a centralizer with f_1 given by $f_1(x) = x$ if $x \leq 0$, $x+1$ if $x > 0$). In the case of order-preserving permutation groups, arguments using centralizers are widespread, and solve many problems. See [4] for material on this.

To prove that $\xi(f) = f$ for general f is however more involved, and a technique described in [3] which uses sets of *pairs* of group elements rather than subsets of G is used instead. This is $S(f) = \{(\alpha, \beta) \in G^2 : \alpha f = f\beta\}$. Our method is then to find certain subfamilies of M , which we denote by Γ , Γ^+ , Γ^- , and Γ^\pm , and show that for $f \in \Gamma \cup \Gamma^+ \cup \Gamma^- \cup \Gamma^\pm$, $S(f) = S(f_1) \Leftrightarrow f = f_1$, from which by essentially the same proof as above, $\xi(f) = f$. Then we show that for any member f of M there are g_1, g_2 lying in one of Γ , Γ^+ , Γ^- , Γ^\pm such that $g_1 f = g_2$, and use a trick involving cancellation to conclude the proof. We need a few technicalities to achieve this. From this it immediately follows by results from [3] that M has automatic homeomorphicity (Theorem 2.6).

In the next section we move on to a discussion of the endomorphism monoid E of (\mathbb{Q}, \leq) . We can show that various natural subsets of E are definable in E , and we can lift the technical result concerning the map ξ to this context too (assuming that it is an automorphism).

In section 4, we discuss the analogous result for E . The key idea here is to analyze directly the possible actions of E on a countable set Ω , which is the set which features in the definition of ‘automatic homeomorphicity’. Any isomorphism of E to a closed submonoid of the transformation monoid on Ω gives rise to a monoid ‘action’ of E on Ω . We focus on the *group* orbits of $G \subseteq E$, and use them to guide our analysis. Provided we know that the restriction of the isomorphism to M maps it to a *closed* submonoid, we can deduce from Theorem 2.6 that it is a homeomorphism. Closedness of this image is easy to prove if the isomorphism sends constants to constants, but examples show that, in general, we cannot rely on this property. Thus, we have to invent another method to ensure closedness, which is done by generalizing Lemma 12 from [3]. After this is solved, we are able to demonstrate precisely how the members of M act on Ω , and using the technical lemmas from section 3, we can then *directly* describe how E acts, and show that the isomorphism assumed to exist must also be a homeomorphism.

In section 5, we use the earlier results to lift automatic homeomorphicity to the corresponding polymorphism clone. So far this argument only works in the reflexive case, since ‘idempotents’ with *finite* image are required in the proof, which exist in E but not in M . The problem highlighted in the previous paragraph for the monoid does not cause difficulties here however, since the fact that for clones the images of constants are necessarily constants avoids the difficulty.

In the final section, we give a method for lifting automatic homeomorphicity (and also automatic continuity—a variant, where every homomorphism into the

full transformation monoid / clone on a countable set is required to be continuous) from monoids to the clones they generate. In this context, we can immediately deduce that the polymorphism clones $\langle \text{End}(\mathbb{Q}, <) \rangle$ and $\langle \text{End}(\mathbb{Q}, \leq) \rangle$ generated by $\text{End}(\mathbb{Q}, <)$ and $\text{End}(\mathbb{Q}, \leq)$, respectively, have automatic homeomorphicity. These clones are (rather small) subclones of the corresponding full polymorphism clones.

To make sense of results about continuity, we need to recall what the topology is, on G , M , E , and indeed also on the clone. For E , we take as sub-basic open sets all sets of the form $\mathcal{B}_{qr} = \{f \in E: f(q) = r\}$, and the topologies on G and M are then the induced ones. Basic open sets are then finite intersections of these, so have the form $\{f \in E: f \upharpoonright_B = g\}$, where B is a finite subset of \mathbb{Q} and $g: B \rightarrow \mathbb{Q}$. In the polymorphism clone P , the same sets are used, but with higher ‘arities’. Thus for each n , and $q_1, q_2, \dots, q_n, r \in \mathbb{Q}$, $\mathcal{B}_{q_1 q_2 \dots q_n r} = \{f \in P^{(n)}: f(q_1, q_2, \dots, q_n) = r\}$ is taken as a sub-basic open set. Similarly, in the full transformation monoid $\text{Tr}(\Omega)$ on a set Ω (usually countable), sub-basic open sets have the form $\{f \in \text{Tr}(\Omega): f(x) = y\}$ for $x, y \in \Omega$, with basic open sets as finite intersections of these, and sub-basic open sets on the polymorphism clone similarly given by allowing increased arities. We remark that saying that G is dense in M thus says that any embedding of $(\mathbb{Q}, <)$ can be approximated by automorphisms on arbitrarily large finite sets. Therefore, saying that M is the closure of G says that any limit of members of G lies in M and any member of M may be expressed as such a limit. It is worth noting that, since G , M and E or the corresponding monoids on Ω live on a countable carrier set, their topology is actually metrizable by an ultrametric (see [6, 1.1, p. 132] for details). This enables us to use sequential convergence and continuity instead of the net analogues needed in general, and we shall exploit this, for instance, in section 6.

An alternative proof of the main technical result Corollary 2.5 was given independently by James Hyde [5] (not using methods from [3]).

2. Main technical lemmas for M

Throughout this section we suppose that ξ is an injective endomorphism of M which fixes G pointwise, and our goal is to show that it also fixes M pointwise. It is fairly easy to show by ‘bare hands’ that there are some members of M which must be fixed, for instance those f that are characterized by their centralizers (meaning that if f and f' have equal centralizers in G , then they are equal). However, this type of argument only applies to a limited range of members of M , and we need a more systematic approach. For this we isolate particular subfamilies of members of M , written Γ , Γ^+ , Γ^- , Γ^\pm , show that all their members are fixed, and then lift this to all members f of M by writing f in terms of members of $\Gamma \cup \Gamma^+ \cup \Gamma^- \cup \Gamma^\pm$. To describe what Γ is, we require the following definition.

The *2-coloured version of the rationals* denoted by \mathbb{Q}_2 , can be characterized as the set \mathbb{Q} of rational numbers, together with a colouring function $F: \mathbb{Q} \rightarrow$

$C = \{\text{red}, \text{blue}\}$ such that for every $x < y$ in \mathbb{Q} and $c \in C$ there is $z \in \mathbb{Q}$ with $x < z < y$ and $F(z) = c$. It is well known that this exists and is unique up to isomorphism.

A key observation in what we do is that the only relevant information about $f \in M$ for our present purposes is its *image*. This is because, if f_1 and f_2 in M have the same image, then $f_2^{-1}f_1$ is (defined and) an automorphism, so by hypothesis is fixed by ξ , and, since $f_2(f_2^{-1}f_1) = f_1$, it is immediate that f_1 is fixed if and only if f_2 is. So we really need to focus mainly on subsets of \mathbb{Q} , though we often construe them as images. In fact with regard to this, it is clear that a subset of \mathbb{Q} is the image of some self-embedding if and only if it is isomorphic to \mathbb{Q} .

With this in mind, for any $A \subseteq \mathbb{Q}$ isomorphic to \mathbb{Q} , let us define a relation \sim on \mathbb{Q} by $x \sim y$ if there is at most one point of A strictly between x and y . Then (rather surprisingly) this is an equivalence relation. For if $x \sim y \sim z$ and not $x \sim z$, then there must be distinct $a, b \in A$ between x and z . The interval between x and z cannot be contained in either of the intervals between x and y or between y and z (since then $x \sim z$ would be immediate). We assume without loss of generality that $x < y$, and so it follows that $y < z$. One of a, b lies in $[x, y]$ and the other in $[y, z]$, but now a and b are consecutive members of a copy of \mathbb{Q} , which is impossible.

The equivalence classes are clearly convex and can intersect A in at most one point. So this gives two main options, that is, equivalence classes intersecting A (in a singleton, which we call ‘red’, and those which are disjoint from A , which we call ‘blue’. Furthermore, being convex subsets of \mathbb{Q} , each \sim -class is a non-empty interval of \mathbb{Q} of the form (a, b) , $(a, b]$, $[a, b)$, or $[a, b]$ (where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and in the last case $a = b$ is allowed).

From the definition of \sim and in particular from convexity of the equivalence classes we observe the following properties. Putting $[x]_{\sim} < [y]_{\sim}$ for rationals x, y with $x \not\sim y$ if and only if $x < y$ gives a well-defined strict linear order on the equivalence classes. Moreover, by the choice of A , if $x, y \in A$ and $x < y$ then there are infinitely many points $z \in A$ satisfying $x < z < y$. That is, between two distinct red classes there are infinitely many red classes. Similarly, if x and y belong to distinct blue equivalence classes, there must be at least two and thus infinitely many points of A in between. Also if x is in a red class and y is in a blue one (and we may assume that actually $x \in A$), then there must be another (and hence infinitely many) point of A between them. Consequently, between any two distinct equivalence classes, we find infinitely many red classes. Thus, whenever we can show that at least one blue class lies between any two points of A , then between any two distinct \sim -classes there is a red as well as a blue class.

We denote by Γ the family of all $f \in M$ such that \mathbb{Q} may be written as the disjoint union $\bigcup\{A_q : q \in \mathbb{Q}_2\}$ of convex subsets A_q of \mathbb{Q} such that $q < r \Rightarrow A_q < A_r$, each A_q is isomorphic to \mathbb{Q} , and if q is a red point of \mathbb{Q}_2 then A_q is a red interval of \mathbb{Q} with respect to $\text{im}(f)$ (that is, $|A_q \cap \text{im}(f)| = 1$), and if q is a blue point of \mathbb{Q}_2 then A_q is blue (that is, $A_q \cap \text{im}(f) = \emptyset$). The intuition is that the points of the image of f are spread out as much as they possibly can be. To

handle members of M whose image may be bounded above or below, we also need to consider Γ^+ , which is defined similarly but using $\mathbb{Q}_2 \cup \{\infty\}$ (\mathbb{Q}_2 with a right endpoint added), and similarly Γ^- , Γ^\pm from $\mathbb{Q}_2 \cup \{-\infty\}$, $\mathbb{Q}_2 \cup \{\pm\infty\}$ (all infinite points coloured blue). The need to consider these variants was pointed out to us by Christian Pech.

A main technical lemma, adapted from [3], shows how certain pairs of finite partial automorphisms can be extended to pairs of automorphisms. For this purpose, for any $g \in M$ we let \sim be the equivalence relation defined above with respect to $\text{im}(g)$, and let P be the family of all pairs (a, b) of finite partial automorphisms of \mathbb{Q} satisfying the following properties:

- (1) a is colour-preserving (that is, an element $x \in \text{dom}(a)$ belongs to a red interval if and only if $a(x)$ belongs to a red interval), strongly \sim -preserving (meaning that for $x, y \in \text{dom}(a)$, $x \sim y \Leftrightarrow a(x) \sim a(y)$), and if there is a least or greatest blue interval, then a preserves it,
- (2) if $x \in \text{dom}(a)$ lies in a red interval containing a point y of $\text{im}(g)$, then $y \in \text{dom}(a)$,
- (3) if $x \in \text{im}(a)$ lies in a red interval containing a point y of $\text{im}(g)$, then $y \in \text{im}(a)$,
- (4) $g(\text{dom}(b)) \subseteq \text{dom}(a)$,
- (5) $g(\text{im}(b)) \subseteq \text{im}(a)$,
- (6) if $x \in \text{im}(g) \cap \text{dom}(a)$, then $g^{-1}(x) \in \text{dom}(b)$, and $gbg^{-1}(x) = a(x)$,
- (7) if $x \in \text{im}(g) \cap \text{im}(a)$, then $g^{-1}(x) \in \text{im}(b)$, and $gb^{-1}g^{-1}(x) = a^{-1}(x)$.

LEMMA 2.1. *Let $g \in \Gamma \cup \Gamma^+ \cup \Gamma^- \cup \Gamma^\pm$. Then any $(a, b) \in P$ can be extended to a pair of automorphisms (α, β) of $(\mathbb{Q}, <)$ such that $\alpha g = g\beta$.*

PROOF. We first treat the case $g \in \Gamma$. We define a finite partial automorphism \bar{a} of \mathbb{Q}_2 thus. Let $\mathbb{Q} = \bigcup\{A_q : q \in \mathbb{Q}_2\}$ as in the definition of $g \in \Gamma$, and let $\bar{a}(q) = r$ if there is $x \in A_q \cap \text{dom}(a)$ such that $a(x) \in A_r$. Clause (1) guarantees that \bar{a} is well-defined, and clauses (1), (2), (3) ensure that it is colour-preserving. Extend \bar{a} to an automorphism $\bar{\alpha}$ of \mathbb{Q}_2 , and let $\alpha \in \text{Aut}(\mathbb{Q}, <)$ be an extension of a satisfying $\alpha(\text{im}(g)) = \text{im}(g)$ (it is possible to achieve this because $a(\text{im}(g) \cap \text{dom}(a)) \subseteq \text{im}(g)$ by (6) and $a^{-1}(\text{im}(g) \cap \text{im}(a)) \subseteq \text{im}(g)$ by (7)) such that for each $q \in \mathbb{Q}_2$, $\alpha(A_q) = A_{\bar{\alpha}(q)}$. This is possible since for any q such that $A_q \cap \text{dom}(a) \neq \emptyset$, if $a(A_q \cap \text{dom}(a)) \subseteq A_r$, then $\bar{a}(q) = r$. Let $\beta = g^{-1}\alpha g$, which is also an automorphism, since α preserves $\text{im}(g)$.

A similar argument applies to Γ^+ , Γ^- , Γ^\pm , using $\mathbb{Q}_2 \cup \{\infty\}$, $\mathbb{Q}_2 \cup \{-\infty\}$, $\mathbb{Q}_2 \cup \{\pm\infty\}$ respectively in the argument in place of \mathbb{Q}_2 , noting that the final condition in clause (1) ensures that the greatest or least blue interval, if it exists, is preserved by $\bar{\alpha}$. \square

LEMMA 2.2. *Any injective monoid homomorphism $\xi: M \rightarrow E$ which fixes G pointwise also fixes every member of $\Gamma \cup \Gamma^+ \cup \Gamma^- \cup \Gamma^\pm$.*

PROOF. Recall the definition of $S(g) = \{(\alpha, \beta) \in G^2 : \alpha g = g\beta\}$ for $g \in E$. Now let $g \in \Gamma$, and consider elements u and s of \mathbb{Q} with $s \neq g(u)$. We construct (α, β) in $S(g)$ such that $\alpha(s) \neq s$ and $\beta(u) = u$. We consider two cases:

1. If $s \in \text{im}(g)$, then s and $g(u)$ lie in different red intervals. Without loss of generality we suppose that $g(u) < s$. Since $\text{im}(g) \cong \mathbb{Q}$, there is $t \in \text{im}(g)$ greater than s . Since g is order-reflecting (that is, its inverse preserves the order), $u < g^{-1}(s) < g^{-1}(t)$. Hence $a = \{(g(u), g(u)), (s, t)\}$ and $b = \{(u, u), ((g^{-1}(s), g^{-1}(t)))\}$ are finite partial automorphisms. We can verify that $(a, b) \in P$ (as defined before Lemma 2.1).
2. If $s \notin \text{im}(g)$, then since $g(u) \neq s$, without loss of generality we suppose that $g(u) < s$. We consider two cases:
 - (i) If s lies in a blue interval A_q , we choose $t \neq s$ in the same interval. Since A_q is convex, $a = \{(s, t), (g(u), g(u))\}$ and $b = \{(u, u)\}$ are finite partial automorphisms. Again $(a, b) \in P$.
 - (ii) If s lies in a red interval A_q containing $r \in \text{im}(g)$, we choose a point $t \in A_q \setminus \{g(u), r, s\}$ on the same side of r (which also allows for the possibility that $r = g(u)$). Then $a = \{(g(u), g(u)), (r, r), (s, t)\}$ and $b = \{(u, u), (g^{-1}(r), g^{-1}(r))\}$ are finite partial automorphisms, and once more we can verify that $(a, b) \in P$.

In each case we can extend (a, b) to (α, β) such that $\alpha g = g\beta$ by appealing to Lemma 2.1, thus (α, β) lies in $S(g)$, and satisfies $\beta(u) = u$, $\alpha(s) = t \neq s$.

This means that for any u in \mathbb{Q} the element $g(u)$ can be recovered from $S(g)$, namely as the unique value s in \mathbb{Q} satisfying either side of the equivalence

$$g(u) = s \iff \forall (\alpha, \beta) \in S(g) (\beta(u) = u \rightarrow \alpha(s) = s) \quad (1)$$

For if $g(u) = s$ and the pair $(\alpha, \beta) \in S(g)$ verifies $\beta(u) = u$, then we have $\alpha(s) = \alpha(g(u)) = g(\beta(u)) = g(u) = s$. This implication is even true for any $g \in E$, not just for $g \in \Gamma$. Conversely, if $g \in \Gamma$ and $g(u) \neq s$, then by the above we can construct $(\alpha, \beta) \in S(g)$ such that $\beta(u) = u$ and $\alpha(s) \neq s$.

Note that since ξ is an injective homomorphism fixing G pointwise,

$$\begin{aligned} S(\xi(g)) &= \{(\alpha, \beta) \in G^2 : \alpha\xi(g) = \xi(g)\beta\} = \{(\alpha, \beta) \in G^2 : \xi(\alpha g) = \xi(g\beta)\} \\ &= \{(\alpha, \beta) \in G^2 : \alpha g = g\beta\} = S(g). \end{aligned}$$

From this and Condition (1) we obtain $\xi(g) = g$: namely, for $u \in \mathbb{Q}$ put $s := \xi(g)(u)$, then all of the following equivalent conditions hold:

$$\begin{aligned} \forall (\alpha, \beta) \in S(\xi(g)) (\beta(u) = u \rightarrow \alpha(s) = s) &\iff \\ \forall (\alpha, \beta) \in S(g) (\beta(u) = u \rightarrow \alpha(s) = s) &\stackrel{(1)}{\iff} g(u) = s. \end{aligned}$$

Similar proofs apply in the cases $g \in \Gamma^+$, Γ^- , Γ^\pm . We just note for instance in the case of Γ^+ that if s lies in the greatest blue interval, then so does t (Case (2i)). \square

Now we consider how the members of Γ and M interact. If $g \in \Gamma$ and $f \in M$ where $\text{im}(f)$ is ‘coterminal’ (that is, for every $x \in \mathbb{Q}$ there are $u, v \in \text{im}(f)$ with $u \leq x \leq v$), then any \sim_{gf} -class is a union of a convex family of \sim_g -classes. This is because $\text{im}(gf) \subseteq \text{im}(g)$ and so if $x \leq y$, then $x \sim_g y \Rightarrow |[x, y] \cap \text{im}(g)| \leq 1 \Rightarrow |[x, y] \cap \text{im}(gf)| \leq 1 \Rightarrow x \sim_{gf} y$. Since all \sim_g -classes are isomorphic to \mathbb{Q} , so are all the \sim_{gf} -classes. The family of red \sim_{gf} -classes is ordered like \mathbb{Q} , since it corresponds precisely to the image of gf , which is a copy of \mathbb{Q} . And the blue \sim_{gf} -classes occupy some cuts among the red ones. Two distinct blue \sim_{gf} -classes must occupy distinct cuts, as if they had no red \sim_{gf} -class between them, then by definition of \sim_{gf} , they would have to be in the same \sim_{gf} -class. This means that we may write \mathbb{Q} as a disjoint union of sets A_q for q lying in some subset Q of \mathbb{Q}_2 , where each A_q is isomorphic to \mathbb{Q} and all the red members of \mathbb{Q}_2 lie in Q . This describes the general set-up. Depending on the particular g and f , we may find that $gf \in \Gamma$ or not. We first see that if they both lie in Γ , then the product necessarily does too. Modified remarks apply in the cases where $\text{im}(f)$ is bounded above, or below, or both, in which case we use the appropriate class, Γ^+ or Γ^- or Γ^\pm .

LEMMA 2.3. *If g_1 and g_2 lie in Γ then so does g_2g_1 (and similarly for Γ^+ , Γ^- , Γ^\pm).*

PROOF. From the above remarks, we just need to see that between any two g_2g_1 -red intervals there is a g_2g_1 -blue one. Let $g_2g_1(x) < g_2g_1(y)$. As $g_1 \in \Gamma$, there is a g_1 -blue interval $(a, b) \subseteq (g_1(x), g_1(y))$, and its endpoints a and b are irrationals which are limits of points of $\text{im}(g_1)$. Let $a = \sup_{n \in \mathbb{N}} g_1(a_n)$, $b = \inf_{n \in \mathbb{N}} g_1(b_n)$ where (a_n) is an increasing sequence, and (b_n) is a decreasing sequence. From $(a, b) \cap \text{im}(g_1) = \emptyset$ it follows that $g_2(a, b) \cap \text{im}(g_2g_1) = \emptyset$. Let (c, d) be the g_2g_1 -interval containing $g_2(a, b)$. If $c \leq g_2g_1(a_n)$ for some $n \in \mathbb{N}$, then $c \leq g_2g_1(a_n) < g_2g_1(a_{n+1}) < g_2g_1(a_{n+2}) < d$ which would give more than one point of $\text{im}(g_2g_1)$ in (c, d) , contrary to its being a g_2g_1 -interval. Similarly we cannot have $g_2g_1(b_n) \leq d$ for any n . Hence, if $c < g_2g_1(z) < d$ for some $z \in \mathbb{Q}$, then $g_2g_1(a_n) < g_2g_1(z) < g_2g_1(b_n)$ for every n . This implies that $g_1(a_n) < g_1(z) < g_1(b_n)$ for all $n \in \mathbb{N}$, hence $a < g_1(z) < b$, contrary to $(a, b) \cap \text{im}(g_1) = \emptyset$. Consequently, $(c, d) \cap \text{im}(g_2g_1) = \emptyset$, and (c, d) is a g_2g_1 -blue interval. Furthermore, for t in (c, d) , we have $g_2g_1(a_n) < c < t$ for all n , thus $t \leq g_2g_1(x)$ would imply $g_1(x) > g_1(a_n)$ for every index n and so $g_1(x) > a$. This contradicts $(a, b) \subseteq (g_1(x), g_1(y))$, hence $g_2g_1(x) < t$. Analogously, we can prove $t < g_2g_1(y)$, and therefore, $(c, d) \subseteq (g_2g_1(x), g_2g_1(y))$.

From this and the basic properties of $\sim_{g_2g_1}$ observed earlier, it easily follows that the family of $\sim_{g_2g_1}$ -intervals is ordered like \mathbb{Q}_2 . \square

LEMMA 2.4. *For any $f \in M$ whose image is coterminal in \mathbb{Q} , there is $g \in \Gamma$ such that $gf \in \Gamma$ (with similar statements for the other classes Γ^+ , Γ^- , Γ^\pm).*

PROOF. It is no doubt possible to prove this directly, but it seems a little easier to go by way of the previous lemma. We start by taking any $g_1 \in \Gamma$, and then we see that we can describe g_1f fairly well. Then we take another $g_2 \in \Gamma$, which

will be chosen so that $g_2g_1f \in \Gamma$. Appealing to Lemma 2.3, we may let $g = g_2g_1$ to conclude the proof.

By the discussion above, there is a subset Q of \mathbb{Q}_2 containing all the red points, such that $\mathbb{Q} = \bigcup_{q \in Q} A_q$ where the A_q are copies of \mathbb{Q} such that $q < r$ in Q implies that $A_q < A_r$ and if $q \in Q$ is red, then A_q is a g_1f -red interval, and if it is blue, then A_q is a g_1f -blue interval. Let us also write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}_2} B_q$ where $B_q \cong \mathbb{Q}$ and $q < r \Rightarrow B_q < B_r$, and we choose $g_2 \in \Gamma$ mapping A_q to B_q for each $q \in Q$. More precisely, for this we let $B_q = \bigcup_{r \in \mathbb{Q}_2} B_{q,r}$ where $B_{q,r} \cong \mathbb{Q}$ and $r < s \Rightarrow B_{q,r} < B_{q,s}$ and ensure that if r is red, $|\text{im}(g_2) \cap B_{q,r}| = 1$, and if r is blue, $\text{im}(g_2) \cap B_{q,r} = \emptyset$. From this we can see that $g_2 \in \Gamma$. Furthermore, each B_q for red q is a g_2g_1f -red interval, and for blue q is a g_2g_1f -blue interval. Hence also $g_2g_1f \in \Gamma$. \square

COROLLARY 2.5. *Any injective monoid homomorphism $\xi: M \rightarrow E$ which fixes G pointwise also fixes every member of M .*

PROOF. Let $f \in M$. By Lemma 2.4, if $\text{im}(f)$ is coterminal, then there is $g \in \Gamma$ such that $gf \in \Gamma$. By Lemma 2.2, ξ fixes g and gf . Therefore $g\xi(f) = \xi(g)\xi(f) = \xi(gf) = gf$. Since g is in Γ and thus in M , it is left cancellable (see Lemma 3.2 below), and hence $\xi(f) = f$. If $\text{im}(f)$ is bounded above but not below, we argue similarly using Γ^+ in place of Γ , and Γ^-, Γ^\pm correspond in a similar way to the cases $\text{im}(f)$ bounded below and not above, and bounded above and below, respectively. \square

It clearly follows from this corollary that every injective endomorphism of M fixing G pointwise is the identity on M . This implies the following theorem.

THEOREM 2.6. *$M = \text{Emb}(\mathbb{Q}, \leq)$ has automatic homeomorphicity, meaning that any isomorphism between M and a closed submonoid of the full transformation monoid on a countable set is a homeomorphism.*

PROOF. This follows from Corollary 2.5 and [3, Lemma 12, p. 13], since G is dense in the closed monoid $M = \text{End}(\mathbb{Q}, <)$; for by [3, Proposition 7, p. 8] we know that G has automatic homeomorphicity, since it has the small index property [7] and hence automatic continuity [3, 3.6, p. 8]. \square

3. Preliminary results for the endomorphism monoid of (\mathbb{Q}, \leq)

Let us now consider (\mathbb{Q}, \leq) , and the associated four ‘natural’ monoids, namely its endomorphisms $\text{End}(\mathbb{Q}, \leq)$, embeddings $\text{Emb}(\mathbb{Q}, \leq)$ (being the same as the injective endomorphisms due to the order being linear), surjective endomorphisms $\text{Surj}(\mathbb{Q}, \leq)$, and automorphisms $\text{Aut}(\mathbb{Q}, \leq)$. The embeddings and automorphisms are the same as for $(\mathbb{Q}, <)$, so we continue to abbreviate these as M and G respectively. The others we write as E (for endomorphisms) and S (for ‘surjective’) respectively. Since we want to see what we can ‘recover’ from G

as before, we first look at which subsets of E are definable. We start by showing that the surjective endomorphisms coincide with the endomorphisms having a right inverse.

LEMMA 3.1. *Each map which is right inverse to some $f \in S$ belongs to M . In particular a member of E belongs to S if and only if it has a right inverse endomorphism. Furthermore, the sets of right inverse endomorphisms of distinct members of S are unequal.*

PROOF. Let $f \in S$ and suppose $g: \mathbb{Q} \rightarrow \mathbb{Q}$ satisfies $fg = \text{id}_{\mathbb{Q}}$. We show that $g \in \text{End}(\mathbb{Q}, <) = M$. For this consider $x, y \in \mathbb{Q}$ such that $x < y$. Then $f(g(x)) = x \not\leq y = f(g(y))$, which implies that $g(x) \not\leq g(y)$ since f is order-preserving. As the order is linear, it follows that $g(x) < g(y)$.

As every $f \in S$ is surjective, it has a right inverse map, which belongs to $M \subseteq E$ by the above. Moreover, if $f \in E$ has a right inverse g , then $fg = \text{id}_{\mathbb{Q}}$ implies that f is surjective.

To prove the final remark, we observe how to ‘recover’ (i.e. define) $f \in S$ from its family of right inverses. In fact the equation $f(x) = y$ is equivalent to $(\exists g \in E)(fg = \text{id}_{\mathbb{Q}} \wedge g(y) = x)$. For from the above, it is clear that if $f(x) = y$, there is a right inverse map (and hence endomorphism) taking y to x , which gives ‘ \Rightarrow ’. Conversely, if $g(y) = x$ for some right inverse g of f , then $f(x) = fg(y) = y$. \square

Let us write C for the family of constant maps, namely $\{c_a: a \in \mathbb{Q}\}$, where $c_a(x) = a$ for all x . Thus $C \subseteq E$ (but of course $C \cap M = \emptyset = C \cap S$). All the mentioned sets are indeed definable in E .

LEMMA 3.2. *Each of C, M, S, G is a definable subset of E : C contains precisely all left absorbing (left zero) elements in E , M are the monomorphisms, S coincides with the epimorphisms, and G consists of the isomorphisms inside E .*

PROOF. We have to show that $C = \{g \in E: (\forall f \in E)gf = g\}$. To see that this correctly defines C , first let $a \in \mathbb{Q}$, and note that for any $f \in E$, $c_a f = c_a$ since $c_a f(x) = a = c_a(x)$ for all $x \in \mathbb{Q}$. Conversely, suppose that $gf = g$ for all $f \in E$, and pick any $a \in \mathbb{Q}$. Then $gc_a = g$ and hence for any $x \in \mathbb{Q}$, $g(x) = gc_a(x) = g(a)$, so g is constant.

We would like to characterize M as the members of E with left inverses, but this is incorrect, as one sees for instance by considering the function $f(x) = x$ if $x < \pi$, $x + 1$ if $x > \pi$. If this had a left inverse g say, then for all a and b such that $a < \pi < b$, $f(a) < 4 < f(b)$, and so $a < g(4) < b$, which forces $g(4)$ to be π which is not rational. Instead we use a related condition, of left cancellability (i.e., of being a ‘monomorphism’). So we shall show that a member f of E lies in M if and only if for any g and h in E , $fg = fh \Rightarrow g = h$. If $f \in M$ then this property holds, since for any g and h such that $fg(x) = fh(x)$ holds for any x in \mathbb{Q} , we have $g(x) = h(x)$ due to f being injective. Conversely, suppose that f is a monomorphism in E . Whenever $x, y \in \mathbb{Q}$ are such that $f(x) = f(y)$, then $fc_x = fc_y$ holds for the constant endomorphisms $c_x, c_y \in E$. As f is

left cancellable, this implies $c_x = c_y$, and so $x = y$. Hence, f is an injective endomorphism, thus it belongs to M .

Clearly, by Lemma 3.1 the set S is definable as the collection of endomorphisms of (\mathbb{Q}, \leq) having a right inverse endomorphism. However, we can also show that $f \in S$ if and only if it is right cancellable (so is an ‘epimorphism’). Certainly, if $f \in S$, then it is right cancellable because there is a right inverse for f . Conversely, suppose that f is not surjective, and let y not lie in its image. Let $g(x) = h(x) = x$ if $x < y$, $g(x) = h(x) = x + 1$ if $x > y$, and $g(y) = y$, $h(y) = y + 1$. Then $g, h \in E$, and they agree on $\mathbb{Q} \setminus \{y\} \supseteq \text{im}(f)$, and hence $gf = hf$. However, $g \neq h$, so f is not right cancellable.

Finally, $G = M \cap S$, so it too is definable as the set of isomorphisms (i.e. the morphisms having two-sided inverses). \square

LEMMA 3.3. *For any $h \in E$ there are $f \in M$ and $g \in S$ such that $h = gf$.*

PROOF. If $q \in \mathbb{Q}$ then $h^{-1}(\{q\})$ is a convex subset of \mathbb{Q} , as $h(x_1) = h(x_2) = q$ and $x_1 \leq y \leq x_2$ imply that $h(y) = q$. Let $X = \bigcup_{q \in \mathbb{Q}} X_q$ where X_q equals $\{q\} \times h^{-1}(\{q\})$ if q lies in the image of h , and is $\{q\}$ otherwise. We order X lexicographically, i.e., we define the order on X by keeping the linear order induced by \mathbb{Q} within each X_q and by stipulating $x_1 < x_2$ for $x_1 \in X_{q_1}$ and $x_2 \in X_{q_2}$ if and only if $q_1 < q_2$. Then X is countable densely linearly ordered without endpoints, so there is an isomorphism $\theta: \mathbb{Q} \rightarrow X$. Let $\varphi: X \rightarrow \mathbb{Q}$ be given by $\varphi((q, y)) = q$ if $h(y) = q$, and $\varphi(q) = q$ if q does not lie in the image of h . Finally, let $f(x) = \theta^{-1}((h(x), x))$ and $g = \varphi\theta$.

We verify the desired properties. To see that $f \in M$, let $x < y$. Then $h(x) \leq h(y)$, implying $(h(x), x) < (h(y), y)$, so $\theta^{-1}((h(x), x)) < \theta^{-1}((h(y), y))$. Also, since θ and φ are order-preserving and surjective, so is g . Finally, to see that $h = gf$, take any $x \in \mathbb{Q}$. Then $q = h(x) \in \text{im}(h)$, so $(h(x), x) \in X_q$ and $f(x) = \theta^{-1}((h(x), x))$, so $gf(x) = \varphi\theta^{-1}((h(x), x)) = \varphi((h(x), x))$, which, by definition, is equal to $h(x)$. \square

COROLLARY 3.4. *Any monoid automorphism ξ of E which fixes G pointwise is the identity.*

PROOF. The key point here is that since by Lemma 3.2 M is definable in E as the family of left cancellable elements, ξ must map M to itself, so we can appeal to Corollary 2.5 to deduce that it also fixes M pointwise. The first part of Lemma 3.1 implies that ξ fixes S setwise, so it follows from the contrapositive of the second part that it fixes S pointwise. Now it is immediate from Lemma 3.3 that ξ fixes every member of E . \square

Note that we would really like this to hold for injective endomorphisms, and not just for automorphisms. This may be true, but our proof does not show it at present; that is because for a possibly not surjective ξ , it is not clear that the defining property of M inside E (namely left cancellability) carries over to its image under ξ . A more detailed analysis of the proof of Lemma 3.2 however shows that the property does hold for injective endomorphisms ξ whose image contains at least one constant operation.

We conclude this section by showing the definability of some other concepts, related to what we have already done.

LEMMA 3.5. *The relation $f, g \in M \wedge \text{im}(f) \subseteq \text{im}(g)$ is definable in the monoid E .*

PROOF. We already know that membership in M is definable. We can then define the given relation by $(\exists h \in M)f = gh$. Clearly if this formula is true, then the image of f is contained in the image of g . Conversely, if $\text{im}(f) \subseteq \text{im}(g)$, we can define h by $h(q) = r \Leftrightarrow f(q) = g(r)$. This defines h since $\text{im}(f) \subseteq \text{im}(g)$, and it is well-defined because g is 1-1. Finally, h preserves the (strict) order since f does and g reflects it. \square

This result may be used to give a ‘representation’ of \mathbb{Q} inside M , namely we can characterize those f in M whose image omits precisely one point of \mathbb{Q} by the formula $f \in M \setminus G \wedge (\forall g \in M \setminus G(\text{im}(f) \subseteq \text{im}(g) \rightarrow \text{im}(g) \subseteq \text{im}(f)))$, representing that f has a maximal image among non-automorphisms. And of course we can also characterize when two such maps ‘encode’ the same point by saying that they have the same image.

We remark that in E , by contrast, we already have the constant maps c_q available, so we have an immediate and direct way of representing the points of \mathbb{Q} inside the monoid.

Finally in this section, we show how finite subsets of \mathbb{Q} can be represented in E .

LEMMA 3.6. *For any $f \in E$, $\text{im}(f) = \{q \in \mathbb{Q} : (\exists h \in E)fh = c_q\}$. Hence $|\text{im}(f)| = n \Leftrightarrow$ there are exactly n constants k such that $(\exists h \in E)fh = k$.*

PROOF. If $q = f(r)$ for some $r \in \mathbb{Q}$, then $fc_r = c_q$, so we may choose $h \in E$ as c_r . Conversely, if $fh = c_q$ for some endomorphism $h \in E$, then $\{q\} = \text{im}(c_q) = \text{im}(fh) \subseteq \text{im}(f)$. \square

We remark that the situation for these maps is radically different in the cases $n = 1$ and $n > 1$. For $n = 1$ there are exactly \aleph_0 maps having image of that size, namely the constant maps c_q . But if $n > 1$, for each B of size n there are 2^{\aleph_0} maps having image B . For if $B = \{b_0, b_1, \dots, b_{n-1}\}$ then $f^{-1}(\{b_i\})$ are pairwise disjoint intervals with endpoints a_i, a_{i+1} say, $-\infty = a_0 < a_1 < \dots < a_n = \infty$ (open or closed or semi-open) and a_i may take any real value. All the same, these maps are quite easy to visualize, and will play an important part in what follows.

4. Automatic homeomorphicity of $\text{End}(\mathbb{Q}, \leq)$

In this section we give a discussion of the automatic homeomorphicity question for E . Here, since G is not dense in E , we are obliged to use a more direct method, which may be of some independent interest (and will also be used in section 5). In the hypothesis of automatic homeomorphicity we are asked to consider an isomorphism θ of E with a closed submonoid E' of the full transformation monoid $\text{Tr}(\Omega)$ on some countable set Ω , and show that it is a

homeomorphism. This θ may be viewed as a (faithful) monoid action of E on Ω (which we write as a left action). Our strategy is to try to demonstrate directly that θ is a homeomorphism, by describing explicitly what it can be. To that end, let us study the G -orbits of θ . If $X \subseteq \Omega$ is one such orbit, then for some $x \in X$, $X = \{\theta(g)(x) : g \in G\}$. By the orbit-stabilizer theorem, the orbit is in natural 1–1 correspondence with the left cosets of the stabilizer $G_x = \{g \in G : \theta(g)(x) = x\}$. Since $X \subseteq \Omega$, it is countable, and so $|G : G_x|$ is countable. By the small index property for G [7, Theorem 3.5], $G_x = G_B$ for some finite $B \subseteq \mathbb{Q}$. (Strictly speaking the ‘small index property’ just says that a subgroup of index $< 2^{\aleph_0}$ contains the pointwise stabilizer of some finite set B , but by taking such B to be minimal, in this case one easily verifies that it is actually equal to the pointwise stabilizer.) Furthermore, this gives rise to an identification of X with the set $[\mathbb{Q}]^n$ of the n -element subsets of \mathbb{Q} respecting the action as follows: Let $a_{g(B)} = \theta(g)(x)$. Then $a_{g_1(B)} = a_{g_2(B)} \Leftrightarrow \theta(g_1)(x) = \theta(g_2)(x) \Leftrightarrow g_2^{-1}g_1 \in G_x \Leftrightarrow g_2^{-1}g_1 \in G_B \Leftrightarrow g_1(B) = g_2(B)$. Since $[\mathbb{Q}]^n$, the set of n -element subsets of \mathbb{Q} , forms an orbit under the action of G , this means that we may write X as $\{a_{g(B)} : g \in G\} = \{a_C : C \in [\mathbb{Q}]^n\}$, and the action of θ is given by $\theta(g)(a_C) = a_{g(C)}$. Under these circumstances we say that this G -orbit has rank n .

The conclusion of the discussion in the previous paragraph is that Ω may be written as the union of G -orbits, each having finite rank, and θ provides a natural action of G on each G -orbit. Let us write $\Omega = \bigcup_{i \in I} \Omega_i$, where Ω_i are the G -orbits, and let Ω_i have rank n_i , so that we may write $\Omega_i = \{a_B^i : B \in [\mathbb{Q}]^{n_i}\}$. The action is therefore given by $\theta(g)(a_B^i) = a_{g(B)}^i$ for each $i \in I$ and $B \in [\mathbb{Q}]^{n_i}$. What we now want to do is to show how this action extends to an action of E , first treating members of M . To do this, we need to know that the restriction $\theta \upharpoonright_M : M \rightarrow M'$, where $M' = \theta(M)$, is continuous. We could infer this from Theorem 2.6 once we knew that M' is a closed submonoid of E' . However, it turns out we first need to prove continuity of the restriction before we can verify this assumption, so using Theorem 2.6 does not seem to be the right way to do it.

LEMMA 4.1. *For an isomorphism $\theta : E \rightarrow E'$ to a closed submonoid E' of $\text{Tr}(\Omega)$ on a countable set Ω , the monoid $M' = \theta(M)$ is closed in $\text{Tr}(\Omega)$ and the restriction $\theta \upharpoonright_M : M \rightarrow M'$ is a homeomorphism.*

PROOF. This is an almost verbatim copy of the proof of Lemma 12 in [3], but with the ending modified as we are in a slightly different situation.

Let us denote by G' the monoid reduct of the group of invertible elements of E' (that is, G' is the group of invertible elements of E' , but viewed as a monoid, so we ‘forget’ that inverses exist). Furthermore, let $\overline{G'}$ be the closure of G' in E' (or in $\text{Tr}(\Omega)$); this is again a transformation monoid, and G' is dense in it. We also know that G comprises the set of invertible elements of E , and it is dense in the closed monoid M . It is easy to see that $\theta(G) \subseteq G'$ as θ is a monoid homomorphism. Moreover, since θ is an isomorphism, $\theta^{-1}(G') \subseteq G$ follows by a symmetric argument, and hence $\theta(G) = G'$ so that the restriction $\theta \upharpoonright_G : G \rightarrow G'$ is a well-defined bijective monoid homomorphism. As the monoids G and G'

are group reducts, $\theta \upharpoonright_G$ actually is a group isomorphism, too. Moreover, density of G' in the closed monoid $\overline{G'}$ implies that $G' = \overline{G'} \cap \text{Sym}(\Omega)$, and similarly G is a closed subgroup of the full symmetric group on \mathbb{Q} . As the automorphism group G has automatic homeomorphicity, $\theta \upharpoonright_G$ is a homeomorphism. Now applying Proposition 11 of [3], there is an extension $\overline{\theta \upharpoonright_G}: M \rightarrow \overline{G'}$ of $\theta \upharpoonright_G$, which is a monoid isomorphism and a homeomorphism. As E' is closed, $\overline{G'} \subseteq E'$, and we let $\iota: \overline{G'} \rightarrow E'$ be the inclusion map, which is a monoid embedding. Then $\xi := \theta^{-1} \iota \overline{\theta \upharpoonright_G}$ is an injective monoid homomorphism from M into E , which fixes every member of G . By Corollary 2.5, $\xi(f) = f$ for every $f \in M$, that is, $\theta(f) = \theta(\xi(f)) = \iota(\overline{\theta \upharpoonright_G}(f)) = \overline{\theta \upharpoonright_G}(f)$. This finally proves that $M' = \theta(M) = \overline{\theta \upharpoonright_G}(M) = \overline{G'}$, and hence M' is closed in $\text{Tr}(\Omega)$. Moreover, $\theta \upharpoonright_{M'}: M' \rightarrow M'$ coincides with $\overline{\theta \upharpoonright_G}$ and consequently is a homeomorphism. \square

LEMMA 4.2. *For each $f \in M$, $i \in I$, and $a_B^i \in \Omega$, i.e. every $B \in [\mathbb{Q}]^{n_i}$, $\theta(f)(a_B^i) = a_{f(B)}^i$.*

PROOF. As G is dense in M , we may find a sequence (g_n) in G such that $g_n \rightarrow f$. Now the topologies on M and M' are generated by sub-basic open sets of the form $\mathcal{B}_{qr} = \{h \in M: h(q) = r\}$ for $q, r \in \mathbb{Q}$ and $\mathcal{C}_{ijBC} = \{h \in \theta(M): h(a_B^i) = a_C^j\}$ for $i, j \in I$, $B \in [\mathbb{Q}]^{n_i}$ and $C \in [\mathbb{Q}]^{n_j}$. Let $B = \{q_1, \dots, q_m\}$ and $r_k = f(q_k)$. Since $g_n \rightarrow f$ and $f \in \mathcal{B}_{q_k r_k}$, there is N_k such that $(\forall n \geq N_k) g_n \in \mathcal{B}_{q_k r_k}$, so for all $n \geq \max_{1 \leq k \leq m} N_k$, $g_n(B) = f(B)$. By Lemma 4.1, the restriction of θ to M is continuous, so $\theta(g_n) \rightarrow \theta(f)$. Let $j \in I$ and $C \in [\mathbb{Q}]^{n_j}$ be determined by $\theta(f)(a_B^i) = a_C^j$, thus $\theta(f) \in \mathcal{C}_{ijBC}$. From $\theta(g_n) \rightarrow \theta(f)$ it follows that $(\exists N)(\forall n \geq N) \theta(g_n) \in \mathcal{C}_{ijBC}$. So for this N , $(\forall n \geq N) \theta(g_n)(a_B^i) = a_C^j$. But we know that $\theta(g_n)(a_B^i) = a_{g_n(B)}^i$ since $g_n \in G$. Hence for such n , $j = i$ and $g_n(B) = C$. Taking now $n \geq N, \max_{1 \leq k \leq m} N_k$, it follows that $j = i$ and $C = g_n(B) = f(B)$. Thus $\theta(f)(a_B^i) = a_{f(B)}^i$ as required. \square

We can extend the statement of Lemma 4.2 to certain members of E , provided that they act ‘like’ members of M on the relevant set.

LEMMA 4.3. *If $f \in E$, $i \in I$, and $a_B^i \in \Omega_i$, where $|f(B)| = n_i = |B|$, then $\theta(f)(a_B^i) = a_{f(B)}^i$.*

PROOF. First consider the case where $f \in S$. As in the proof of Lemma 3.1 there is a right inverse $h \in M$ for f , and in addition, h may be chosen so that for each $x \in B$, $hf(x) = x$. Then, applying Lemma 4.2 to $h \in M$, $\theta(f)(a_B^i) = \theta(f)(a_{hf(B)}^i) = \theta(f)\theta(h)(a_{f(B)}^i) = \theta(\text{id}_{\mathbb{Q}})(a_{f(B)}^i) = a_{f(B)}^i$. Now consider any $h \in E$ such that $|h(B)| = n_i$. By Lemma 3.3, we may write $h = gf$ where $f \in M$ and $g \in S$, and $|g(f(B))| = |h(B)| = n_i$. Hence by what we have just shown, $\theta(g)(a_{f(B)}^i) = a_{g(f(B))}^i$, so, by Lemma 4.2 applied to $f \in M$, $\theta(h)(a_B^i) = \theta(g)\theta(f)(a_B^i) = \theta(g)(a_{f(B)}^i) = a_{g(f(B))}^i = a_{h(B)}^i$. \square

If $f \in E$ ‘collapses’ a set B , then we can certainly not deduce that $\theta(f)(a_B^i) = a_C^j$ for $j = i$, since Ω_i and Ω_j will have different ranks. For the proof of openness

in the main theorem, we would still need some information about C , namely that it is contained in $f(B)$.

LEMMA 4.4. *Let $i \in I$ and $B \in [\mathbb{Q}]^{n_i}$. Then the following statements hold:*

- (i) *If $B \neq \emptyset$, then there is an idempotent endomorphism $h \in E$ having B as image such that $\theta(h)(a_B^i) = a_B^i$. If $B = \emptyset$, then $\theta(h)(a_B^i) = a_B^i$ holds for every $h \in E$.*
- (ii) *$\theta(f_1)(a_B^i) = \theta(f_2)(a_B^i)$ whenever $f_1, f_2 \in E$ satisfy $f_1 \upharpoonright_B = f_2 \upharpoonright_B$.*
- (iii) *If, for $f \in E$, $j \in I$ and $C \subseteq \mathbb{Q}$ are given by $\theta(f)(a_B^i) = a_C^j$, then $C \subseteq f(B)$.*

PROOF. (i) Let B be non-empty. By subdividing \mathbb{Q} into $|B|$ pairwise disjoint intervals each containing a unique member of B , and mapping the whole of each such interval to the member of B it contains, we obtain an endomorphism $h \in E$ fixing all elements of B and satisfying $\text{im}(h) = B$, which is clearly idempotent. Since $h(B) = B \in [\mathbb{Q}]^{n_i}$, we can apply Lemma 4.3 to get $\theta(h)(a_B^i) = a_{h(B)}^i = a_B^i$.

If $B = \emptyset$, then $n_i = 0$ and $\Omega_i = \{a_\emptyset^i\}$ is a singleton. Now Lemma 4.3 gives $\theta(h)(a_\emptyset^i) = a_{h(\emptyset)}^i = a_\emptyset^i$ for every $h \in E$.

- (ii) By the second statement of (i), this claim is trivial in the case that B is the empty set. Otherwise, consider the idempotent $h \in E$ constructed in (i). We see by inspection that $f_1 h = f_2 h$, thus

$$\begin{aligned} \theta(f_1)(a_B^i) &= \theta(f_1)\theta(h)(a_B^i) = \theta(f_1 h)(a_B^i) \\ &= \theta(f_2 h)(a_B^i) = \theta(f_2)\theta(h)(a_B^i) = \theta(f_2)(a_B^i). \end{aligned}$$

- (iii) Now suppose for a contradiction that there is $c \in C \setminus f(B)$. Then there is $h \in G$ moving c to $h(c) \notin C$ but fixing all members of $f(B)$. Hence $f \upharpoonright_B = f' \upharpoonright_B$, where $f' = hf$, since h fixes $f(B)$ pointwise. As shown in (ii), $\theta(f')(a_B^i) = \theta(f)(a_B^i) = a_C^j$. However, on the other hand, $\theta(f')(a_B^i) = \theta(hf)(a_B^i) = \theta(h)\theta(f)(a_B^i) = \theta(h)(a_C^j) = a_{h(C)}^j$, contrary to $h(C) \neq C$. We conclude that $C \subseteq f(B)$ as required. \square

Using the ideas from above, we can demonstrate automatic homeomorphicity of $E = \text{End}(\mathbb{Q}, \leq)$.

THEOREM 4.5. *$E = \text{End}(\mathbb{Q}, \leq)$ has automatic homeomorphicity, meaning that any isomorphism θ between E and a closed submonoid $E' \subseteq \text{Tr}(\Omega)$ on a countable set Ω is a homeomorphism.*

PROOF. The sub-basic open sets in the monoids E and E' are of the form $\mathcal{B}_{qr} = \{f \in E: f(q) = r\}$ and $\mathcal{C}_{ijBC} = \{f \in E': f(a_B^i) = a_C^j\}$ for $B \in \Omega_i$, $C \in \Omega_j$, so to establish continuity we have to show that each $\theta^{-1}(\mathcal{C}_{ijBC})$ is open in E . Now B is a finite set, so we may let $B = \{q_1, q_2, \dots, q_m\}$, and,

for an arbitrary $f \in \theta^{-1}(\mathcal{C}_{ijBC})$, we let $r_k = f(q_k)$. Thus $f \in \bigcap_{k=1}^m \mathcal{B}_{q_k r_k}$. We show that $\bigcap_{k=1}^m \mathcal{B}_{q_k r_k} \subseteq \theta^{-1}(\mathcal{C}_{ijBC})$, and this is what is required, since it shows that $\theta^{-1}(\mathcal{C}_{ijBC})$ is a union of open sets, hence open in E . For let $f' \in \bigcap_{k=1}^m \mathcal{B}_{q_k r_k}$. Then $f'(q_k) = r_k$ for each k , which means that f and f' agree on B . By Lemma 4.4(ii), it follows that $\theta(f')(a_B^i) = \theta(f)(a_B^i) = a_C^j$. Hence $f' \in \theta^{-1}(\mathcal{C}_{ijBC})$.

To show that θ is also open, it suffices to show that the image of any sub-basic open set is open. So consider $\theta(\mathcal{B}_{qr})$ for any rationals q and r . Look at any member of this set, which may be written as $\theta(f)$ where $f \in \mathcal{B}_{qr}$; we shall find $i, j \in I$ and $B, C \subseteq \mathbb{Q}$ so that $\theta(f) \in \mathcal{C}_{ijBC} \subseteq \theta(\mathcal{B}_{qr})$. Since $f \in \mathcal{B}_{qr}$, $f(q) = r$. We shall show that there is some $i \in I$ such that $|\text{im}(f)| \geq n_i > 0$. Then we can find B and C of size n_i such that $f(B) = C$ with $q \in B$. Now we take $j = i$, and observe using Lemma 4.3 that, $\theta(f)(a_B^i) = a_{f(B)}^i = a_C^i$, which tells us that $\theta(f) \in \mathcal{C}_{iiBC}$. Furthermore, for any $g \in \mathcal{C}_{iiBC}$, since we are in E' , $g = \theta(h)$ for some $h \in E$, and $\theta(h)(a_B^i) = a_C^i$. Therefore, Lemma 4.4(iii) yields $C \subseteq h(B)$, and as $|B| = |C|$, finiteness of B implies $C = h(B)$. As f maps q to r , and so q and r are the corresponding entries of B and C when enumerated in increasing order, it follows that h also maps q to r . Hence $h \in \mathcal{B}_{qr}$, which shows that $g = \theta(h) \in \theta(\mathcal{B}_{qr})$, as required.

To see that such $i \in I$ exists, suppose otherwise. This means that for every $i \in I$, if $n_i > 0$ then $|\text{im}(f)| < n_i$. Consider any $i \in I$ and $a_B^i \in \Omega_i$ and let $\theta(f)(a_B^i) = a_C^j$. Then $C \subseteq f(B)$ by Lemma 4.4 and so $n_j \leq |\text{im}(f)|$. It follows that $n_j = 0$, and $C = \emptyset$. Choose $g \in G$ such that $g(f(x)) \neq f(x)$ holds for some $x \in \mathbb{Q}$, e.g. $g(y) = y + 1$ for $y \in \mathbb{Q}$. For every $i \in I$ and $B \in [\mathbb{Q}]^{n_i}$, $\theta(gf)(a_B^i) = \theta(g)\theta(f)(a_B^i) = \theta(g)(a_\emptyset^j) = a_{g(\emptyset)}^j = a_\emptyset^j = \theta(f)(a_B^i)$, showing that $\theta(gf) = \theta(f)$. However, $gf \neq f$ by the choice of g , contrary to the injectivity of θ . \square

We would like to have more precise information about the action of the image of θ on Ω . We have partial information about this from Lemmas 4.2, 4.3 and 4.4 but this still seems to leave many options open. We now set about describing the most general situation we are aware of under which there is such an action. The conjecture will then be that this describes everything that actually can occur.

The most natural way for E to act is just directly on \mathbb{Q} , and we can see that any action on any orbit of rank 1 must be like this, since the condition used in Lemma 4.3 (namely that $|f(B)| = |B|$) is immediately verified. The next most natural action is on $\bigcup_{1 \leq i \leq n} [\mathbb{Q}]^i$ for some fixed $n \geq 1$. Each $[\mathbb{Q}]^i$ is an orbit of the action of G , and the natural way to extend this so that E acts is to let $\theta(f)(B) = f(B)$ even when $f \notin G$. We can say that the action ‘cascades’ (like a waterfall) through the different $[\mathbb{Q}]^i$ depending on the action of f , as i decreases from n to 1. As a generalization of this, we can also let f act more drastically, since as far as we know all that is required by Lemma 4.4(iii) is that the point mapped to should be *contained in* the image under f . Thus if $n = n_k > n_{k-1} > \dots > n_0 = 0$ is a sequence of not necessarily consecutive

integers, we let $X = \bigcup_{0 \leq i \leq k} [\mathbb{Q}]^{n_i}$ and define $\theta(f)(B)$ to be the first n_i elements of $f(B)$ where i is greatest such that $n_i \leq |f(B)|$ (which is therefore the empty set if $i = 0$). It is straightforward to verify that this is an action.

The general action that we have in mind is built up from ones of this kind using a ‘tree’. There seems no reason why in the above action, n or k should be restricted to be finite, or why the G -orbits should be arranged linearly. So the following can also be seen to be a possible action using the same ideas. Let (T, \leq) be a countable partially ordered set in which for each $t \in T$, $\{s \in T : s \leq t\}$ is a finite linearly ordered set, with a labelling $l: T \rightarrow \mathbb{N}$ such that $t_1 < t_2 \Rightarrow l(t_1) < l(t_2)$ (strictly speaking, this is a ‘forest’). Given such T , which has at least one point labelled by a non-zero number, (or else, infinitely many labelled 0), we can form $\Omega = \bigcup\{[\mathbb{Q}]^{l(t)} \times \{t\} : t \in T\}$, and the action is given as above ‘down each branch’. That is, $\theta(f)(B_1, t_1) = (B_2, t_2)$ if B_2 is the first $l(t_2)$ elements of $f(B_1)$ if t_2 is the greatest point below t_1 in T such that $l(t_2) \leq |f(B_1)|$. This is similarly easily verified to be an action. So the main question remaining here is whether all such actions are of this form.

5. Automatic homeomorphicity of $\text{Pol}(\mathbb{Q}, \leq)$

In this section we use ideas from earlier in the paper to prove automatic homeomorphicity for the polymorphism clone $\text{Pol}(\mathbb{Q}, \leq)$. For definitions of the relevant notions here we refer the reader to [3], but mention a few notations and ideas that are needed. Denoting by \mathcal{O}_A the collection of all finitary operations $f: A^n \rightarrow A$ ($n \geq 0$) on a set A , a subset $C \subseteq \mathcal{O}_A$ is called a (‘concrete’) clone on A if it is closed under the operations of composition when defined (that is, the ‘arities’ are correct) and it contains all ‘projections’. These are the maps $\pi_i^{(n)}: A^n \rightarrow A$ given by $\pi_i^{(n)}(a_1, a_2, \dots, a_n) = a_i$, where $1 \leq i \leq n$. The collection of all polymorphisms of a relational structure always forms a clone, and clones arising in this way are precisely the ones that are topologically closed. Of central interest here is the clone $\text{Pol}(\mathbb{Q}, \leq)$ of polymorphisms of (\mathbb{Q}, \leq) , which is the family of all n -ary functions on \mathbb{Q} for $n \geq 0$ that preserve \leq , i.e. that are monotone maps from $(\mathbb{Q}, \leq)^n$ to (\mathbb{Q}, \leq) . Spelling out precisely what this means, $f: \mathbb{Q}^n \rightarrow \mathbb{Q}$ lies in the clone provided that if $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \mathbb{Q}^n$ and $a_i \leq b_i$ for all i , then $f(a_1, a_2, \dots, a_n) \leq f(b_1, b_2, \dots, b_n)$. There is a corresponding notion of ‘abstract clone’, which we do not require here. Let us note also that the set \mathcal{O}_A of all finitary operations on A forms a clone, even a polymorphism clone (e.g., $\mathcal{O}_A = \text{Pol}(A, =)$). This is the analogue of $\text{Sym}(A)$ for the automorphism group and $\text{Tr}(A)$ for the endomorphism monoid.

Relying on results of [3], when proving automatic homeomorphicity of the clone $\text{Pol}(\mathbb{Q}, \leq)$, it will suffice to verify that any clone isomorphism between $\text{Pol}(\mathbb{Q}, \leq)$ and a closed clone on some countable set is continuous. To exhibit the general method we are using here, we first prove the following result, which is based on adapting the strategy used to demonstrate the first part of Theorem 4.5.

LEMMA 5.1. *Let A and B be sets, P and P' be clones on A and B , respectively, and $\theta: P \rightarrow P'$ be a clone homomorphism. If for every $b \in B$ there is some unary function $h \in P^{(1)}$ with finite image such that $\theta(h)(b) = b$, then θ is continuous.*

PROOF. Under the given assumptions we have to verify that $\theta^{-1}(C)$ is open in P for any sub-basic open set C of P' . By definition of the topology of the clone P' there are $n \in \mathbb{N}$, $(b_1, \dots, b_n) = b \in B^n$ and $b' \in B$ such that C equals $\{g \in P'^{(n)}: g(b) = b'\}$. We want to show that every $f \in \theta^{-1}(C)$ is surrounded by an open neighbourhood in $\theta^{-1}(C) = \{f \in P^{(n)}: \theta(f)(b) = b'\}$, showing that f is an interior point of $\theta^{-1}(C)$.

By the assumption of the lemma, we can find maps $h_1, \dots, h_n \in P^{(1)}$ satisfying $\theta(h_i)(b_i) = b_i$ and having finite image $\text{im}(h_i) \subseteq A$ for every index $i \in \{1, \dots, n\}$. Therefore, the Cartesian product $A' = \prod_{i=1}^n \text{im}(h_i) \subseteq A^n$ is finite, too and, thus the set $P_f = \bigcap_{a \in A'} \{f' \in P^{(n)}: f'(a) = f(a)\}$ is a basic open neighbourhood of f in the topology of P . Hence, the result is proved once we establish that $P_f \subseteq \theta^{-1}(C)$.

For this let f' be any function in P_f , that is, we assume $f'(a) = f(a)$ for every $a \in A'$. Thus the n -ary functions f and f' coincide on the finite set $A' = \prod_{i=1}^n \text{im}(h_i)$, which then implies the equation

$$\begin{aligned} f \circ (h_1 \circ \pi_1^{(n)}, h_2 \circ \pi_2^{(n)}, \dots, h_n \circ \pi_n^{(n)}) \\ = f' \circ (h_1 \circ \pi_1^{(n)}, h_2 \circ \pi_2^{(n)}, \dots, h_n \circ \pi_n^{(n)}). \end{aligned}$$

From here we can conclude that $\theta(f')(b) = \theta(f)(b) = b'$, i.e., $f' \in \theta^{-1}(C)$, as follows:

$$\begin{aligned} \theta(f)(b) &= \theta(f)(b_1, \dots, b_n) = \theta(f)(\theta(h_1)(b_1), \dots, \theta(h_n)(b_n)) \\ &= \theta(f)\left(\theta(h_1)(\theta(\pi_1^{(n)})(b)), \dots, \theta(h_n)(\theta(\pi_n^{(n)})(b))\right) \\ &= \theta(f) \circ \left(\theta(h_1) \circ \theta(\pi_1^{(n)}), \dots, \theta(h_n) \circ \theta(\pi_n^{(n)})\right)(b) \\ &= \theta\left(f \circ (h_1 \circ \pi_1^{(n)}, \dots, h_n \circ \pi_n^{(n)})\right)(b). \end{aligned}$$

Similarly, $\theta(f')(b) = \theta\left(f' \circ (h_1 \circ \pi_1^{(n)}, \dots, h_n \circ \pi_n^{(n)})\right)(b)$. From the above equation it follows that $\theta(f')(b) = \theta(f)(b) = b'$, as required. \square

Proving automatic homeomorphicity of $P = \text{Pol}(\mathbb{Q}, \leq)$ now basically boils down to verifying the assumptions of the preceding result.

THEOREM 5.2. *$\text{Pol}(\mathbb{Q}, \leq)$ has automatic homeomorphicity, meaning that any isomorphism θ from $P = \text{Pol}(\mathbb{Q}, \leq)$ to a closed subclone P' of \mathcal{O}_Ω , for a countable set Ω , is a homeomorphism.*

PROOF. Note that, unlike in the case of the monoid E , where we would have had to prove both continuity and openness of the given isomorphism θ , here we

only need to check continuity, since openness follows from Proposition 27 of [3], being an easy consequence of P containing all (unary) constant operations. This avoids the need for proving the analogue of Lemma 4.4 (though this analogue still holds).

To demonstrate that θ is continuous, we use the machinery from section 4 to provide the assumptions of Lemma 5.1. Note that these properties are determined entirely by the restriction $\theta \upharpoonright_E: P^{(1)} \rightarrow P'^{(1)}$, which is a monoid isomorphism between the unary parts $P^{(1)} = E$ and $E' := P'^{(1)}$ (these are closed monoids because P and $\text{Tr}(\mathbb{Q})$, and P' and $\text{Tr}(\Omega)$ are closed sets). Namely, we have to verify that for every $b \in \Omega$ we can find an endomorphism $h \in E$ with finite image such that $\theta(h)(b) = \theta \upharpoonright_E(h)(b) = b$. However, this is precisely the content of part (i) of Lemma 4.4 applied to $\theta \upharpoonright_E$. \square

6. Automatic homeomorphicity of clones generated by monoids

In this final section we show that automatic homeomorphicity results can be lifted from monoids to the polymorphism clones they generate, under appropriate conditions. Given a submonoid E of the full transformation monoid $\text{Tr}(\Omega)$ on a set Ω , there is a least clone $\langle E \rangle$ on Ω containing E ; it may be formed by including all projections, and then closing up under compositions of functions where these are defined; it may be explicitly written as $\bigcup_{k \in \mathbb{N} \setminus \{0\}} \{f \circ \pi_j^{(k)} : j \in \{1, \dots, k\} \wedge f \in E\}$. This is of course a rather small subclone of \mathcal{O}_Ω , so any results obtained about it do not really give us information about the general situation. Our main result here is that if $E \subseteq \text{Tr}(\Omega)$ is a closed transformation monoid which has automatic homeomorphicity and its group of invertible members acts transitively on Ω , then $\langle E \rangle$ also has automatic homeomorphicity.

Because the definition of automatic homeomorphicity, as described in section 5, is given for closed clones we start by recalling that $\langle E \rangle$ is closed. This result belongs to the folklore of clone theory.

LEMMA 6.1. *If A is a set, and $E \subseteq \text{Tr}(A)$ is a closed transformation monoid, then the clone $\langle E \rangle$ is also closed.*

PROOF. Consider the relation $\rho = \{(x, y, z, u) \in A^4 : x = y \vee z = u\}$. We see that $\text{Pol}(A, \{\rho, \emptyset\}) = \langle \text{Tr}(A) \rangle$. Since every function in $\text{Pol}(A, \{\rho, \emptyset\})$ must preserve the empty relation, its arity must be larger than zero. If an n -ary (for $n > 0$) function $f \in \text{Pol}(A, \{\rho, \emptyset\})$ has at least two essential arguments, for indices i and j with $1 \leq i < j \leq n$ say, then there are $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A^n$ and $a', b' \in A$ such that the values $a, b, c, d \in A$ given by

$$\begin{aligned} f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n) &= a \\ f(a_1, \dots, a_{i-1}, a', a_{i+1}, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n) &= b \\ f(b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_{j-1}, b_j, b_{j+1}, \dots, b_n) &= c \\ f(b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_{j-1}, b', b_{j+1}, \dots, b_n) &= d \end{aligned}$$

satisfy $a \neq b$ and $c \neq d$. This means that $(a, b, c, d) \notin \rho$, which violates the condition that f preserves ρ , since $(a_i, a', b_i, b_i), (a_j, a_j, b_j, b') \in \rho$. Hence f has at most one essential position, the i th say ($1 \leq i \leq n$). As f depends at most on its i th position, for every tuple $(x_1, \dots, x_n) \in A^n$ we have $f(x_1, \dots, x_n) = f(x_i, x_2, \dots, x_n) = f(x_i, x_i, \dots, x_n) = \dots = f(x_i, \dots, x_i)$, which means that $f = g \circ \pi_i^{(n)}$ for the unary function $g = f \circ (\text{id}_A, \dots, \text{id}_A)$, and so $f \in \langle \text{Tr}(A) \rangle$. The reverse inclusion is trivial.

Now as E is a closed transformation monoid there is a set of finitary relations Q on A (e.g. the set of all invariant relations of E) such that $E = \text{End}(A, Q) = \text{Tr}(A) \cap \text{Pol}(A, Q)$. This implies that $\langle E \rangle$ is equal to $\text{Pol}(A, Q) \cap \langle \text{Tr}(A) \rangle = \text{Pol}(A, Q) \cap \text{Pol}(A, \{\rho, \emptyset\}) = \text{Pol}(A, Q \cup \{\rho, \emptyset\})$, from which it follows that $\langle E \rangle$ is a closed clone. Indeed, $\langle E \rangle \subseteq \text{Pol}(A, Q) \cap \langle \text{Tr}(A) \rangle$ is immediate. Conversely, if $f \in \text{Pol}(A, Q) \cap \langle \text{Tr}(A) \rangle$, then there is an arity $n > 0$, an index i such that $1 \leq i \leq n$, and a unary operation $g \in \text{Tr}(A)$ such that $f = g \circ \pi_i^{(n)}$. Consequently, $f \circ (\text{id}_A, \dots, \text{id}_A) = g$, and so $g \in \text{Pol}(A, Q)$ since $f \in \text{Pol}(A, Q)$. As g is unary, $g \in \text{End}(A, Q) = E$. It follows that $f = g \circ \pi_i^{(n)} \in \langle E \rangle$. \square

LEMMA 6.2. *Let $E \subseteq \text{Tr}(\Omega)$ be a transformation monoid on a countable set A and $\theta: \langle E \rangle \rightarrow \mathcal{O}_\Omega$ be a clone homomorphism from $\langle E \rangle$ into the clone of all operations on a countable set Ω . If the restriction of θ to its unary part $\theta \upharpoonright_E: E \rightarrow \text{Tr}(\Omega)$ is continuous, then θ is continuous.*

PROOF. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of k -ary operations of $\langle E \rangle$ that converges to $g \in \langle E \rangle^{(k)}$ say. We want to prove that $\lim_{n \rightarrow \infty} \theta(g_n) = \theta(g)$. Since

$$\langle E \rangle = \bigcup_{k \in \mathbb{N} \setminus \{0\}} \{f \circ \pi_j^{(k)} : j \in \{1, \dots, k\} \wedge f \in E\},$$

we have $g_n = f_n \circ \pi_{j_n}^{(k)}$ for all $n \in \mathbb{N}$ with $f_n \in E$ and $1 \leq j_n \leq k$ (the index j_n may not be uniquely determined in the case that f_n is constant, but then we make an arbitrary choice, for instance $j_n = 1$), and $g = f \circ \pi_j^{(k)}$ for some $f \in E$ and j such that $1 \leq j \leq k$. Let us first note that

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n &= \lim_{n \rightarrow \infty} (g_n \circ (\text{id}_A, \dots, \text{id}_A)) = \left(\lim_{n \rightarrow \infty} g_n \right) \circ (\text{id}_A, \dots, \text{id}_A) \\ &= g \circ (\text{id}_A, \dots, \text{id}_A) = \left(f \circ \pi_j^{(k)} \right) \circ (\text{id}_A, \dots, \text{id}_A) = f \circ \text{id}_A = f \end{aligned}$$

since composition of functions is continuous with regard to the product topology.

The collection $\{\{n \in \mathbb{N} : j_n = t\} : 1 \leq t \leq k\}$ consists of disjoint subsets of \mathbb{N} whose union covers \mathbb{N} . Since this collection is finite, for at least one $t \in \{1, \dots, k\}$ the set $\{n \in \mathbb{N} : j_n = t\}$ must be infinite. By shifting the index of the sequence $(g_n)_{n \in \mathbb{N}}$ past the largest element of the finite members of the collection, we may assume that each of these sets is either infinite or empty. Let t_1, \dots, t_ℓ denote the distinct indices in $\{1, \dots, k\}$ for which $I_\nu = \{n \in \mathbb{N} : j_n = t_\nu\}$ ($1 \leq \nu \leq \ell$) is infinite. By enumerating I_ν in strictly increasing order we get subsequences $(n_{\nu,i})_{i \in \mathbb{N}}$ such that $I_\nu = \{n_{\nu,i} : i \in \mathbb{N}\}$ and thus $j_{n_{\nu,i}} = t_\nu$ is constant for all $i \in \mathbb{N}$.

Now we distinguish two cases. Suppose first that g , equivalently f , is a constant map. Then $f \circ \pi_j^{(k)} = f \circ \pi_{t_\nu}^{(k)}$ holds for every $\nu \in \{1, \dots, \ell\}$, and so we can infer that

$$\begin{aligned} \lim_{i \rightarrow \infty} \theta(g_{n_{\nu,i}}) &= \lim_{i \rightarrow \infty} \theta \left(f_{n_{\nu,i}} \circ \pi_{j_{n_{\nu,i}}}^{(k)} \right) = \lim_{i \rightarrow \infty} \theta \left(f_{n_{\nu,i}} \circ \pi_{t_\nu}^{(k)} \right) \\ &= \lim_{i \rightarrow \infty} \left(\theta(f_{n_{\nu,i}}) \circ \theta \left(\pi_{t_\nu}^{(k)} \right) \right) = \left(\lim_{i \rightarrow \infty} \theta(f_{n_{\nu,i}}) \right) \circ \left(\lim_{i \rightarrow \infty} \theta \left(\pi_{t_\nu}^{(k)} \right) \right) \\ &\stackrel{\dagger}{=} \theta \left(\lim_{i \rightarrow \infty} f_{n_{\nu,i}} \right) \circ \theta \left(\pi_{t_\nu}^{(k)} \right) = \theta \left(\lim_{n \rightarrow \infty} f_n \right) \circ \theta \left(\pi_{t_\nu}^{(k)} \right) \\ &= \theta(f) \circ \theta \left(\pi_{t_\nu}^{(k)} \right) = \theta \left(f \circ \pi_{t_\nu}^{(k)} \right) = \theta \left(f \circ \pi_j^{(k)} \right) = \theta(g) \end{aligned}$$

for each ν such that $1 \leq \nu \leq \ell$ (the equation marked by \dagger follows from the continuity of θ for unary operations). Now we have a partition of a sequence into a finite number of subsequences each of which converges to the same limit $\theta(g)$. It is straightforward to verify that $\lim_{n \rightarrow \infty} \theta(g_n) = \theta(g)$.

The second case of the proof is when f is not constant. We show that $\ell = 1$ and $t_1 = j$. In order to obtain a contradiction, let us assume that there is $\nu \in \{1, \dots, \ell\}$ where $t = t_\nu \neq j$. No generality is lost in assuming that $t < j$. Since f is not constant, there are arguments x, y such that $f \circ \pi_t^{(k)}(x, \dots, x, y, \dots, y) = f(x) \neq f(y) = f \circ \pi_j^{(k)}(x, \dots, x, y, \dots, y)$, and the last x occurs in the t th position. Hence $f \circ \pi_t^{(k)} \neq f \circ \pi_j^{(k)}$, and consequently $\varepsilon = d \left(f \circ \pi_t^{(k)}, f \circ \pi_j^{(k)} \right) > 0$. The subsequence $(f_{n_{\nu,i}})_{i \in \mathbb{N}}$ converges to f , and as composition of functions is continuous, the same holds for the sequence $(f_{n_{\nu,i}} \circ \pi_{j_{n_{\nu,i}}}^{(k)})_{i \in \mathbb{N}} = (f_{n_{\nu,i}} \circ \pi_{t_\nu}^{(k)})_{i \in \mathbb{N}} = (f_{n_{\nu,i}} \circ \pi_t^{(k)})_{i \in \mathbb{N}}$ and $f \circ \pi_t^{(k)}$. Let us choose $i \in \mathbb{N}$ large enough that $d \left(f_{n_{\nu,i}} \circ \pi_t^{(k)}, f \circ \pi_t^{(k)} \right) < \frac{\varepsilon}{2}$. By the triangle inequality,

$$\begin{aligned} \varepsilon &= d \left(f \circ \pi_j^{(k)}, f \circ \pi_t^{(k)} \right) \leq d \left(f \circ \pi_j^{(k)}, f_{n_{\nu,i}} \circ \pi_t^{(k)} \right) + d \left(f_{n_{\nu,i}} \circ \pi_t^{(k)}, f \circ \pi_t^{(k)} \right) \\ &< d \left(f \circ \pi_j^{(k)}, f_{n_{\nu,i}} \circ \pi_t^{(k)} \right) + \frac{\varepsilon}{2}, \end{aligned}$$

i.e.

$$\begin{aligned} d(g, g_{n_{\nu,i}}) &= d \left(f \circ \pi_j^{(k)}, f_{n_{\nu,i}} \circ \pi_{j_{n_{\nu,i}}}^{(k)} \right) = d \left(f \circ \pi_j^{(k)}, f_{n_{\nu,i}} \circ \pi_{t_\nu}^{(k)} \right) \\ &= d \left(f \circ \pi_j^{(k)}, f_{n_{\nu,i}} \circ \pi_t^{(k)} \right) > \frac{\varepsilon}{2} \end{aligned}$$

for all sufficiently large $i \in \mathbb{N}$. This means that $(g_n)_{n \in \mathbb{N}}$ cannot converge to g , contrary to our overall assumption. Therefore, all (distinct) t_ν have to be equal to j , and thus there can only be one such $t_\nu = t_1 = j$. This means that for all but finitely many $n \in \mathbb{N}$ (which we safely ignored above) we have $j_n = j$ and thus $g_n = f_n \circ \pi_{j_n}^{(k)} = f_n \circ \pi_j^{(k)}$ for almost all $n \in \mathbb{N}$. This enables us to conclude

that

$$\begin{aligned} \lim_{n \rightarrow \infty} \theta(g_n) &= \lim_{n \rightarrow \infty} \theta \left(f_n \circ \pi_j^{(k)} \right) = \lim_{n \rightarrow \infty} \left(\theta(f_n) \circ \theta \left(\pi_j^{(k)} \right) \right) \\ &\stackrel{\ddagger}{=} \theta \left(\lim_{n \rightarrow \infty} f_n \right) \circ \theta \left(\pi_j^{(k)} \right) = \theta(f) \circ \theta \left(\pi_j^{(k)} \right) = \theta \left(f \circ \pi_j^{(k)} \right) = \theta(g), \end{aligned}$$

where the equation marked by \ddagger holds because of the assumed continuity of θ restricted to unary operations and the continuity of the composition of operations. \square

COROLLARY 6.3. *Let \mathbb{A} be a relational structure on a countable set and $\theta: \langle \text{End}(\mathbb{A}) \rangle \rightarrow C$ be a clone isomorphism between $\langle \text{End}(\mathbb{A}) \rangle$ and a closed clone C over a countable set Ω . If the restriction of θ to its unary part $\theta \upharpoonright_{\text{End}(\mathbb{A})}: \text{End}(\mathbb{A}) \rightarrow C^{(1)}$ is continuous, then θ is continuous.*

PROOF. Let us denote by $\iota: C \rightarrow \mathcal{O}_\Omega$ and $\iota': C^{(1)} \rightarrow \text{Tr}(\Omega)$ the inclusion homomorphisms of C and $C^{(1)}$ into the full clone and the full transformation monoid on Ω , respectively. By definition of the subspace topology on $C^{(1)}$, ι' is continuous, so $\iota\theta$ is a clone homomorphism from an essentially at most unary clone on a countable set into the clone of all operations on the countable carrier set Ω , whose restriction to the unary part is $\iota'\theta \upharpoonright_{\text{End}(\mathbb{A})}$ and hence continuous. Letting $E = \text{End}(\mathbb{A})$ in Lemma 6.2 we deduce that $\iota\theta$ is continuous; since $\text{im}(\theta) \subseteq C$, it follows that θ is continuous, too. \square

As another consequence of Lemma 6.2, automatic continuity can be lifted from closed transformation monoids to their generated clones.

COROLLARY 6.4. *If A is a countable set, and $E \subseteq \text{Tr}(A)$ is a closed transformation monoid with automatic continuity, then the essentially at most unary clone $\langle E \rangle$ generated by it inherits this property.*

PROOF. By Lemma 6.1, the clone $\langle E \rangle$ is closed. If $\theta: \langle E \rangle \rightarrow \mathcal{O}_\Omega$ is a clone homomorphism into the full clone on a countable set Ω , then its restriction to the unary part is the monoid homomorphism $\theta \upharpoonright_E: E \rightarrow \text{Tr}(\Omega)$, which is continuous by the assumption on E . By Lemma 6.2, θ is continuous. \square

LEMMA 6.5. *If A is a countable set, and $E \subseteq \text{Tr}(A)$ is a closed transformation monoid which has automatic homeomorphicity and its group of invertible members G acts transitively on A , then $\langle E \rangle$ also has automatic homeomorphicity.*

PROOF. Let $\theta: \langle E \rangle \rightarrow C$ be a clone isomorphism between $\langle E \rangle$ and another closed clone C on a countable set Ω . Since E has automatic homeomorphicity and the unary part of C is closed—because $C^{(1)} = C \cap \text{Tr}(\Omega)$ and both sets are closed—the restriction $\theta \upharpoonright_E: E \rightarrow C^{(1)}$ is a homeomorphism. By Corollary 6.3 we conclude that θ is continuous. To see that it must be open too, we use Proposition 32 from [3], which holds for clone isomorphisms and is applicable here since G acts transitively on A and $\theta \upharpoonright_E$ is open. \square

From the previous lemma and Theorems 2.6 and 4.5 we obtain the result mentioned in the introduction.

COROLLARY 6.6. *The clones $\langle \text{End}(\mathbb{Q}, <) \rangle$ and $\langle \text{End}(\mathbb{Q}, \leq) \rangle$ have automatic homeomorphicity.*

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