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On the Cauchy problem for a semilinear fractional elliptic equation

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Abstract

We study, for the first time in the literature on the subject, the Cauchy problem for a semilinear fractional elliptic equation. Under an *a priori* assumption on the solution, we propose the Fourier truncation method for stabilizing the ill-posed problem. A stability estimate of logarithmic type is established.
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1. Introduction

Fractional differential equations arise in many fields of science and engineering [11], and most of the previous studies have been devoted to fractional diffusion and wave equations [3, 10, 13]. More recently fractional elliptic equations have become the point of interest of some distinguished studies [1, 2, 6] and the present paper is aimed to contribute towards broadening the overall understanding of inverse problems associated to equations of this type. In this paper, we consider the boundary value problem for the semilinear fractional elliptic equation

$$\partial_t^\alpha u + \Delta u = F(x,t,u(x,t)), \qquad (x,t) \in \Omega \times (0,T) =: Q_T, \tag{1.1}$$

with the following boundary conditions:

$$\begin{cases} u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = f(x), & x \in \Omega, \\ u_t(x,0) = g(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain with a smooth boundary $\partial\Omega$, and $T > 0$ is a given number. In (1.1), $\alpha \in (1,2)$ is the fractional order and ∂_t^α denotes the Caputo fractional derivative with respect to t, (see [9, 12]),

$$\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u}{\partial s^2}(x,s)ds, \quad (x,t) \in Q_T,$$

where Γ is the Gamma function. We note that a modified equation to (1.1) as

$$\partial_t^\alpha u - \Delta u = F(x,t,u(x,t)), \qquad (x,t) \in Q_T, \tag{1.3}$$

called a semilinear fractional wave equation, subject to the conditions (1.2) has been studied in [9].

In the case $\alpha \searrow 1$, the problem (1.1) becomes an ill-posed backward problem for the parabolic heat equation [14], whilst in the case of $\alpha \nearrow 2$, the problem (1.1) becomes a classical elliptic inverse problem (called the Cauchy problem for the Laplace equation), [7]. It is well-known that this latter problem is ill-posed in the sense of Hadarmard and regularization results have been obtained in [15]. A natural question is whether the Cauchy problem for the

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fractional elliptic equation (1.1) is also as ill-posed. In contrast to the classical elliptic equation, even linear fractional elliptic equations are not very studied. In [8], the authors considered the ill-posedness (though no regularization was addressed) of problem (1.1)-(1.2) in the simpler linear case $F = 0$. To the best knowledge of the authors, there are no publications on the Cauchy problem for semilinear fractional elliptic equation (1.1) for general source function F.

The manuscript is organized as follows. In section 2, we introduce the nonlinear integral equation satisfied by the solution of the Cauchy problem (1.1)-(1.2). In section 3, we give the Fourier truncation method and obtain the stability estimate in the L^2 norm.

2. The integral equation

It is well-known [5] that the spectral problem

$$\begin{cases} -\Delta\phi_j(x) = \lambda_j\phi_j(x), & x \in \Omega \\ \phi_j(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_j \le ...$ and $\lambda_j \to \infty$ as $j \to \infty$. Let the corresponding eigenfunctions be $\phi_j \in H_0^1(\Omega)$.

Next, suppose that problem (1.1) has a solution u of the form $u(x,t) = \sum_{j=1}^{\infty} u_j(t)\phi_j(x)$. Then, $u_j(t)$ solves the following fractional ordinary differential equation with initial conditions:

$$\begin{cases} \partial_t^\alpha u_j - \lambda_j u_j(t) & = \left\langle F(x,t,u(x,t)), \phi_j \right\rangle, \quad t \in (0,T), \\ u_j(0) & = \left\langle f, \phi_j \right\rangle, \\ \frac{du_j}{dt}(0) & = \left\langle g, \phi_j \right\rangle, \end{cases} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\Omega)$. By applying the method of [9, 12], we obtain the solution of (2.5) as follows:

$$u_j(t) = E_{\alpha,1}(\lambda_j t^\alpha)\left\langle f, \phi_j \right\rangle + tE_{\alpha,2}(\lambda_j t^\alpha)\left\langle g, \phi_j \right\rangle + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (t-s)^\alpha)\left\langle F(\cdot, s, u(\cdot, s)), \phi_j \right\rangle ds, \tag{2.6}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha z + \beta)}, \quad z \in \mathbb{C} \tag{2.7}$$

is the Mittag-Leffler function and u satisfies the integral equation

$$\begin{aligned} u(x,t) = & \sum_{j=1}^{\infty} \left[E_{\alpha,1}(\lambda_j t^\alpha)\left\langle f, \phi_j \right\rangle + tE_{\alpha,2}(\lambda_j t^\alpha)\left\langle g, \phi_j \right\rangle \right] \phi_j(x) \\ & + \sum_{j=1}^{\infty} \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (t-s)^\alpha)\left\langle F(\cdot, s, u(\cdot, s)), \phi_j \right\rangle ds \right] \phi_j(x), \quad (x,t) \in Q_T. \end{aligned} \tag{2.8}$$

2.1. *Properties of the Mittag-Leffler function*

The following lemmas state some properties of the Mittag-Leffler function (2.7), which will be useful for the main analysis of section 3.

Lemma 2.1. *(a) Let $\alpha \ge a_0, \beta \ge b_0$ and M be positive numbers. Then there exists a positive constant $C_E = C_E(a_0, b_0)$ such that*

$$0 \le E_{\alpha,\beta}(z) \le C_E E_{a_0,b_0}(M), \quad z \in [0, M]. \tag{2.9}$$

(b) Let α, β and z_1 be positive numbers. Then there exists a positive constant $C = C(z_1)$ such that

$$|E_{\alpha,\beta}(z) - \phi_0(\alpha, \beta, z)| \le \frac{C}{1+z}, \quad z \in [z_1, \infty), \tag{2.10}$$

where

$$\phi_0(\alpha, \beta, z) := \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp\left(z^{\frac{1}{\alpha}}\right).$$

Proof. See Lemma 2.3 of [4]. $\square$

Lemma 2.2. *Let* $\alpha_0, \alpha_1, \beta_0$ *and* $\beta_1 \in \mathbb{R}$ *satisfy* $1 < \alpha_0 < \alpha_1 < 2$, $1 \le \beta_0 < \beta_1 \le 2$. *Let* $\alpha \in [\alpha_0, \alpha_1]$ *and* $\beta \in [\beta_0, \beta_1]$. *Then there exists a constant* $\overline{C} > 0$ *such that*

$$E_{\alpha,\beta}(z) \le \overline{C}\phi_0(\alpha, \beta, z), \quad z \in (0, \infty). \tag{2.11}$$

Remark also that $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)} \le \frac{1}{\Gamma(1.4616)} \approx 1.13$.

Proof. From Part (a) of Lemma 2.1, $\alpha \ge 1$ and $\beta \ge 1$, we know that there exists a constant $C_E > 0$ such that

$$E_{\alpha,\beta}(z) \le C_E E_{1,1}(1) = eC_E, \quad \forall z \in [0, 1]. \tag{2.12}$$

Also, since $\alpha \ge 1$ and $\beta \ge 1$, we have that $z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) \ge 1$ for all $z \in (0, 1]$ and (2.12) gives

$$E_{\alpha,\beta}(z) \le eC_E\alpha\phi_0(\alpha, \beta, z), \quad z \in (0, 1]. \tag{2.13}$$

If $z \ge 1$, then we define $G(z, \alpha, \beta) := E_{\alpha,\beta}(z)/\phi_0(\alpha, \beta, z)$. From Part (b) of Lemma 2.1, we have that there exists $C > 0$ such that

$$|E_{\alpha,\beta}(z) - \phi_0(\alpha, \beta, z)| \le \frac{C}{1 + z}, \quad z \in [1, \infty).$$

This implies that

$$|G(z, \alpha, \beta) - 1| = \left| \frac{E_{\alpha,\beta}(z)}{\phi_0(\alpha, \beta, z)} - 1 \right| \le \frac{\alpha C z^{\frac{\beta-1}{\alpha}}}{(1 + z)e^{z^{1/\alpha}}} \le \frac{\alpha C z^{\frac{\beta_1-1}{\alpha_0}}}{(1 + z)e^{z^{1/\alpha_1}}} =: \psi(z), \quad z \in [1, \infty). \tag{2.14}$$

Since $\lim_{z\to+\infty} \psi(z) = 0$, we can find an $M > 1$ such that $0 \le \psi(z) \le 1/2$ for $z \ge M$. It follows that $G(z, \alpha, \beta) \le 3/2$ for $z \in [M, \infty)$. Now, denote $D := [1, M] \times [\alpha_0, \alpha_1] \times [\beta_0, \beta_1]$ and $c^+ := \sup_D G(z, \alpha, \beta)$. Using a compactness argument, we obtain $c^+ = \max_D G(z, \alpha, \beta) > 0$. Denoting $\tilde{C} := \max\{c^+, \frac{3}{2}\}$, we have that $G(z, \alpha, \beta) \le \tilde{C}$ for $z \in [1, \infty)$. Finally, defining $\overline{C} := \max\{\tilde{C}, \alpha e C_E\}$ we obtain that (2.11) holds for all $z \in (0, \infty)$. $\square$

Lemma 2.3. *Let* $\alpha \in [\alpha_0, \alpha_1]$ *with* $1 < \alpha_0 < \alpha_1 < 2$. *Then, for any* $t \ge 0$, *the following inequalities hold:*

$$E_{\alpha,1}(\lambda_j t^\alpha) \le \frac{\overline{C}}{\alpha} \exp(\lambda_j^{\frac{1}{\alpha}} t), \tag{2.15}$$

$$tE_{\alpha,2}(\lambda_j t^\alpha) \le \frac{\overline{C}}{\alpha} \lambda_j^{\frac{-1}{\alpha}} \exp(\lambda_j^{\frac{1}{\alpha}} t), \tag{2.16}$$

$$t^{\alpha-1}E_{\alpha,\alpha}(\lambda_j t^\alpha) \le \frac{\overline{C}}{\alpha} \lambda_j^{\frac{1-\alpha}{\alpha}} \exp(\lambda_j^{\frac{1}{\alpha}} t). \tag{2.17}$$

Proof. Let $z = \lambda_j t^\alpha$ and putting $\beta = 1, 2$ and α in (2.11) of Lemma 2.2, we obtain (2.15)-(2.17), respectively. $\square$

3. Fourier truncation method

Let $N \in \mathbb{N}^*$ be a positive integer, which later on will play the role of the regularization parameter, and denote

$$\mathcal{S}_N(t)f := \sum_{j=1}^{N} E_{\alpha,1}(\lambda_j t^\alpha) \left\langle f, \phi_j \right\rangle \phi_j, \quad \mathcal{P}_N(t)g := \sum_{j=1}^{N} tE_{\alpha,2}(\lambda_j t^\alpha) \left\langle g, \phi_j \right\rangle \phi_j, \tag{3.18}$$

for f and $g \in L^2(\Omega)$. Then we define the solution by truncated Fourier series as satisfying the nonlinear integral equation

$$u_N(x, t) = \mathcal{S}_N(t)f(x) + \mathcal{P}_N(t)g(x) + \int_0^t \mathcal{Q}_N(t - s)F(u_N)(x, s)ds, \tag{3.19}$$

where

$$F(u_N)(x,s) := F(x,s,u_N(x,s)), \quad Q_N(t-s)F(u_N)(x,s) := \sum_{j=1}^{N}(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda_j(t-s)^{\alpha})\left\langle F(u_N)(\cdot,s),\phi_j\right\rangle \phi_j(x). \quad (3.20)$$

Using that $(\lambda_i)_{i\geq 1}$ is an increasing sequence of positive numbers and Lemma 2.3 it is easy to derive that

$$\|\mathcal{S}_N(t)\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{\overline{C}}{\alpha}\exp\left(\lambda_N^{\frac{1}{\alpha}}t\right), \quad \|\mathcal{P}_N(t)\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{\overline{C}}{\alpha}\lambda_1^{\frac{-1}{\alpha}}\exp\left(\lambda_N^{\frac{1}{\alpha}}t\right), \quad t\in[0,T], \tag{3.21}$$

where $\mathcal{L}(L^2(\Omega))$ denotes the space of all linear and continuous maps from $L^2(\Omega)$ onto itself, and

$$|Q_N(t-s)F(u_N)(x,s)| \leq \frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}\exp\left(\lambda_N^{\frac{1}{\alpha}}(t-s)\right)\sqrt{\sum_{j=1}^{N}| < F(u_N)(\cdot,s),\phi_j > |^2}, \quad x\in\Omega,\ s,t\in[0,T]. \tag{3.22}$$

Let now the Cauchy data in (1.2) be in error $\delta \geq 0$ satisfying

$$\|f - f^{\delta}\| \leq \delta, \quad \|g - g^{\delta}\| \leq \delta, \tag{3.23}$$

where, unless otherwise specified, the norm $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. Then, we can define the regularized solution by truncated Fourier series as satisfying the nonlinear integral equation

$$u_N^{\delta}(x,t) = \mathcal{S}_N(t)f^{\delta}(x) + \mathcal{P}_N(t)g^{\delta}(x) + \int_0^t Q_N(t-s)F(u_N^{\delta})(x,s)ds, \tag{3.24}$$

where the regularization parameter N will be chosen depending on the amount of noise δ.

We assume that F satisfies the global Lipschitz property, i.e. there exists $K \geq 0$ such that

$$|F(x,t,u) - F(x,t,v)| \leq K\|u-v\|_{L^2(\Omega)}, \quad u,v\in L^2(\Omega),\ (x,t)\in Q_T. \tag{3.25}$$

Now we state our main results in the following theorem.

Theorem 3.1. *The nonlinear integral equation* (3.24) *has a unique solution* $u_N^{\delta} \in C([0,T];L^2(\Omega))$. *Assume further that there exists a postive* $\gamma > 0$ *such that*

$$\left(\sum_{j=1}^{\infty}\lambda_j^{2\gamma}\exp\left(2\lambda_j^{\frac{1}{\alpha}}(T-t)\right)|u_j(t)|^2\right)^{1/2} \leq A, \quad t\in[0,T], \tag{3.26}$$

for some constant $A > 0$. *Then, we have the following stability estimate:*

$$\|u(\cdot,t) - u_N^{\delta}(\cdot,t)\|$$
$$\leq \exp\left(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}t\right)\exp\left(-\lambda_N^{\frac{1}{\alpha}}(T-t)\right)\left[\frac{\overline{C}}{\alpha}\left(1+\lambda_1^{\frac{-1}{\alpha}}\right)\exp\left(\lambda_N^{\frac{1}{\alpha}}T\right)\delta + A\lambda_N^{-\gamma}\right], \quad t\in[0,T]. \tag{3.27}$$

Remark 3.1. *Choosing* $N = N(\delta)$ *such that* $\lambda_N \leq \left(\frac{\sigma}{T}\ln\left(\frac{1}{\delta}\right)\right)^{\alpha}$ *for some* $\sigma \in (0,1)$, *then the error* $\|u(\cdot,t) - u_N^{\delta}(\cdot,t)\|$ *is of logarithmic order* $\left|\ln(\frac{1}{\delta})\right|^{-\gamma\alpha}$.

Proof. **Part 1.** The existence and uniqueness of a solution to the nonlinear integral equation (3.24).

For $w \in C([0,T];L^2(\Omega))$, we put

$$\mathcal{J}(w)(x,t) := \mathcal{S}_N(t)f^{\delta}(x) + \mathcal{P}_N(t)g^{\delta}(x) + \int_0^t Q_N(t-s)F(w)(x,s)ds. \tag{3.28}$$

We shall prove by induction if $w_1, w_2 \in C([0,T];L^2(\Omega))$ then, for any $m \in \mathbb{N}^*$,

$$\left\|\mathcal{J}^m(w_1)(\cdot,t) - \mathcal{J}^m(w_2)(\cdot,t)\right\| \leq A_m t^m \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}, \quad t\in[0,T], \tag{3.29}$$

where

$$A_m := \frac{\left(\frac{K\overline{C}\lambda_1^{\frac{1-\alpha}{\alpha}}}{\alpha} e^{T\lambda_N^{\frac{1}{\alpha}}}\right)^m}{m!}.$$

For $m = 1$, using (3.22) and (3.25), we have

$$\|\mathcal{J}(w_1)(\cdot,t) - \mathcal{J}(w_2)(\cdot,t)\| = \left\| \int_0^t Q_N(t-s)(F(w_1) - F(w_2))(\cdot,s)ds \right\| \le A_1 t \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}. \tag{3.30}$$

Assume that (3.29) holds for $m = p$ and we show that it also holds for $m = p + 1$. Using again (3.22) and (3.25), we have

$$\begin{aligned}\|\mathcal{J}^{p+1}(w_1)(\cdot,t) - \mathcal{J}^{p+1}(w_2)(\cdot,t)\| &= \left\| \int_0^t Q_N(t-s)(F(\mathcal{J}^p(w_1)) - F(\mathcal{J}^p(w_2)))(\cdot,s)ds \right\| \\ &\le \frac{K\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} A_p \|w_1 - w_2\|_{C([0,T];L^2(\Omega))} e^{T\lambda_N^{\frac{1}{\alpha}}} \int_0^t s^p ds \le A_{p+1} t^{p+1} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}.\end{aligned} \tag{3.31}$$

Therefore, by the induction principle, we have that (3.29) holds. From it, we also obtain that

$$\left\|\mathcal{J}^m(w_1) - \mathcal{J}^m(w_2)\right\|_{C([0,T];L^2(\Omega))} \le B_m \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}, \tag{3.32}$$

where $B_m := A_m T^m$. Since $\lim_{m\to+\infty} B_m = 0$ there exists a positive integer number m_0 such that $\mathcal{J}^{m_0}$ is a contraction. It follows that the equation $\mathcal{J}^{m_0} w = w$ has a unique solution $u_N^\delta \in C([0,T];L^2(\Omega))$. We claim that $\mathcal{J}(u_N^\delta) = u_N^\delta$. In fact, since $\mathcal{J}^{m_0}(u_N^\delta) = u_N^\delta$, we know that $\mathcal{J}\left(\mathcal{J}^{m_0}(u_N^\delta)\right) = \mathcal{J}(u_N^\delta)$. This is equivalent to $\mathcal{J}^{m_0}\left(\mathcal{J}(u_N^\delta)\right) = \mathcal{J}(u_N^\delta)$. Hence, $\mathcal{J}(u_N^\delta)$ is a fixed point of $\mathcal{J}^{m_0}$. Moreover, as noted above, u_N^δ is a fixed point of $\mathcal{J}^{m_0}$.

Part 2. Let v_N be the solution of the nonlinear integral equation

$$v_N(x,t) = \mathcal{S}_N(t) f(x) + \mathcal{P}_N(t) g(x) + \int_0^t Q_N(t-s) F(v_N)(x,s)ds. \tag{3.33}$$

Step 1. Estimate $\|u_N^\delta(\cdot,t) - v_N(\cdot,t)\|_{L^2(\Omega)}$. Using (3.21)-(3.25) and (3.33), we have

$$\begin{aligned}\|u_N^\delta(\cdot,t) - v_N(\cdot,t)\| &\le \left\|\mathcal{S}_N(t)\big(f - f^\delta\big)\right\| + \left\|\mathcal{P}_N(t)\big(g - g^\delta\big)\right\| + \left\| \int_0^t Q_N(t-s)\Big(F(u_N^\delta)(\cdot,s) - F(v_N)(\cdot,s)\Big)ds \right\| \\ &\le \frac{\overline{C}}{\alpha}\exp(\lambda_N^{\frac{1}{\alpha}} t)\|f - f^\delta\| + \frac{\overline{C}}{\alpha}\lambda_1^{\frac{-1}{\alpha}} \exp(\lambda_N^{\frac{1}{\alpha}} t)\|g - g^\delta\| + \frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} \int_0^t \exp(\lambda_N^{\frac{1}{\alpha}}(t-s)) \left\|F(u_N^\delta)(\cdot,s) - F(v_N)(\cdot,s)\right\| ds \\ &\le \frac{\overline{C}}{\alpha}\Big(1 + \lambda_1^{\frac{-1}{\alpha}}\Big)\exp(\lambda_N^{\frac{1}{\alpha}} t)\delta + K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} \int_0^t \exp(\lambda_N^{\frac{1}{\alpha}}(t-s)) \left\|u_N^\delta(\cdot,s) - v_N(\cdot,s)\right\| ds.\end{aligned} \tag{3.34}$$

Multiplying both sides to $\exp(-\lambda_N^{\frac{1}{\alpha}} t)$, we derive that

$$\exp(-\lambda_N^{\frac{1}{\alpha}} t)\|u_N^\delta(\cdot,t) - v_N(\cdot,t)\| \le \frac{\overline{C}}{\alpha}\Big(1 + \lambda_1^{\frac{-1}{\alpha}}\Big)\delta + K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} \int_0^t \exp(-\lambda_N^{\frac{1}{\alpha}} s)\left\|u_N^\delta(\cdot,s) - v_N(\cdot,s)\right\| ds. \tag{3.35}$$

Applying Gronwall's inequality, we obtain that

$$\exp(-\lambda_N^{\frac{1}{\alpha}} t)\|u_N^\delta(\cdot,t) - v_N(\cdot,t)\| \le \frac{\overline{C}}{\alpha}\Big(1 + \lambda_1^{\frac{-1}{\alpha}}\Big)\delta \exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} t\Big).$$

Hence

$$\|u_N^\delta(\cdot,t) - v_N(\cdot,t)\| \le \frac{\overline{C}}{\alpha}\Big(1 + \lambda_1^{\frac{-1}{\alpha}}\Big)\exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}} t\Big)\exp(\lambda_N^{\frac{1}{\alpha}} t)\delta. \tag{3.36}$$

Step 2. Estimate $\|u(\cdot,t) - v_N(\cdot,t)\|$. First, it is easy to see that

$$\sum_{j=1}^N u_j(t)\phi_j(x) = \mathcal{S}_N(t) f(x) + \mathcal{P}_N(t) g(x) + \int_0^t Q_N(t-s) F(u)(x,s)ds. \tag{3.37}$$

Then, from (3.22), (3.25), (3.26), (3.33) and (3.37), we obtain

$$\begin{aligned}
\|u(\cdot,t)-v_N(\cdot,t)\| &\le \Big\|u(\cdot,t)-\sum_{j=1}^{N}u_j(t)\phi_j\Big\| + \Big\|\sum_{j=1}^{N}u_j(t)\phi_j - v_N(\cdot,t)\Big\| \\
&\le \sqrt{\sum_{j=N+1}^{\infty}\lambda_j^{-2\gamma}\exp\big(-2\lambda_j^{\frac{1}{\alpha}}(T-t)\big)\lambda_j^{2\gamma}\exp\big(2\lambda_j^{\frac{1}{\alpha}}(T-t)\big)|u_j(t)|^2} \\
&\quad + \Big\|\int_0^t Q_N(t-s)\Big(F(u)(\cdot,s)-F(v_N)(\cdot,s)\Big)ds\Big\| \\
&\le \lambda_N^{-\gamma}\exp\big(-\lambda_N^{\frac{1}{\alpha}}(T-t)\big)A + K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}\int_0^t \exp\big(\lambda_N^{\frac{1}{\alpha}}(t-s)\big)\|u(\cdot,s)-v_N(\cdot,s)\|ds.
\end{aligned}$$

Multiplying both sides by $\exp\big(\lambda_N^{\frac{1}{\alpha}}(T-t)\big)$, we have

$$\exp\big(\lambda_N^{\frac{1}{\alpha}}(T-t)\big)\|u(\cdot,t)-v_N(\cdot,t)\| \le \lambda_N^{-\gamma}A + K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}\int_0^t \exp\big(\lambda_N^{\frac{1}{\alpha}}(T-s)\big)\|u(\cdot,s)-v_N(\cdot,s)\|ds.$$

By using Gronwall's inequality, we thus obtain

$$\|u(\cdot,t)-v_N(\cdot,t)\| \le \exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}t\Big)A\lambda_N^{-\gamma}\exp\big(-\lambda_N^{\frac{1}{\alpha}}(T-t)\big). \tag{3.38}$$

Finally, from (3.36) and (3.38), we deduce that

$$\begin{aligned}
&\|u(\cdot,t)-u_N^{\delta}(\cdot,t)\| \le \|u(\cdot,t)-v_N(\cdot,t)\| + \|v_N(\cdot,t)-u_N^{\delta}(\cdot,t)\| \\
&\le \frac{\overline{C}}{\alpha}\Big(1+\lambda_1^{\frac{-1}{\alpha}}\Big)\exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}t\Big)\exp\big(\lambda_N^{\frac{1}{\alpha}}t\big)\delta + \exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}t\Big)A\lambda_N^{-\gamma}\exp\big(-\lambda_N^{\frac{1}{\alpha}}(T-t)\big) \\
&= \exp\Big(K\frac{\overline{C}}{\alpha}\lambda_1^{\frac{1-\alpha}{\alpha}}t\Big)\exp\big(-\lambda_N^{\frac{1}{\alpha}}(T-t)\big)\Big[\frac{\overline{C}}{\alpha}\Big(1+\lambda_1^{\frac{-1}{\alpha}}\Big)\exp\big(\lambda_N^{\frac{1}{\alpha}}T\big)\delta + A\lambda_N^{-\gamma}\Big].
\end{aligned}$$

This completes the proof of the theorem. □

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