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Bayesian Probabilistic Power Flow Analysis Using Jacobian Approximate Bayesian Computation

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***Abstract*—A probabilistic power flow (PPF) study is an essential tool for the analysis and planning of a power system when specific variables are considered as random variables with particular probability distributions. The most widely used method for solving the PPF problem is Monte Carlo simulation (MCS). Although MCS is accurate for obtaining the uncertainty of the state variables, it is also computationally expensive, since it relies on repetitive deterministic power flow solutions. In this paper, we introduce a different perspective for the PPF problem. We frame the PPF as a probabilistic inference problem, and instead of repetitively solving optimization problems, we use Bayesian inference for computing posterior distributions over state variables. Additionally, we provide a likelihood-free method based on the Approximate Bayesian Computation philosophy, that incorporates the Jacobian computed from the power flow equations. Results in three different test systems show that the proposed methodologies are competitive alternatives for solving the PPF problem, and in some cases, they allow for reduction in computation time when compared to MCS.**

***Index Terms*—Approximate Bayesian Computation, Bayesian inference, Power system, Probabilistic power flow.**

I. Introduction

Probabilistic approaches have gained considerable attention within power flow analysis due to the uncertainty which is naturally present in power systems (PS). Such uncertainty appears due variability in power generation, variation in the demand, and changes in the network configuration. The first methods proposed to approach power flow analysis under a probabilistic formulation appeared in Borkowska [1] and Dopazo et. al [2] in the mid-1970s. From these studies, two philosophies have been used to analyze the uncertainty in a PS: the probabilistic power flow (PPF) [1] and the stochastic load flow (SLF) [2]. SLF uses a regression model together with a white Gaussian noise to model the uncertainty in the power flow equations [2]. PPF assumes that specific variables in the PS can be considered as random variables with particular probability distributions. The goal is then to obtain probability distributions for all the other variables in the system, particularly for voltages, angles and power flows between nodes [3]. In this paper, we consider the approach used in PPF towards uncertainty quantification in PS.

Different methods have been proposed to address the PPF problem. They can broadly be classified in three main categories: analytical, approximate and simulation-based methods [4]. Analytical methods are based on convolution processes which use a fast Fourier transform and linearized power flow equations [5]; or work with a multilinear model to deal with the power system nonlinearities [6]. Approximate methods replace the probability distributions of the random variables by their statistical moments [7]. They include the point estimate method [7] and the unscented transformation methods [6]. Finally, simulation-based methods are based on statistical sampling of input random variables (active and reactive powers injected) and the propagation of these samples through repetitive deterministic power flow (DPF) solutions [4]. Statistical sampling of the input random variables can be accomplished using different methods including Latin hypercube, uniform design sampling or Monte Carlo simulation (MCS) [6], which is the most popular. Other methods also use simulation for analytical or approximate representations of the power system, including the linear MCS [6] and the Taguchi method [8]. Although, analytical and approximate methods are computationally more effective than simulation-based methods, they require mathematical assumptions or approximations for obtaining feasible solutions [4]. Hence, the analytical methods may offer less accurate solutions than MCS [4].

MCS combines sampling approaches of input random variables with deterministic optimization [4]. Input random variables are assumed to follow particular probability distributions that model the uncertainty in the PS. Samples from these probability distributions are used to solve a DPF problem that computes the corresponding values of state variables, (angles at PV and PQ nodes, and voltages at PQ nodes). The process is repeated several times, until sufficient samples from the state variables are obtained in order to build an estimator for the marginal distributions of the state variables.

The process described above has its shortcomings. On one hand, it does not take into account the fact that previous knowledge of state variables might be available in terms of probability distributions, for example: from normal operating conditions of a PS, the magnitude of the voltage variables is close to one per unit, therefore, we can define a specific probability distribution for these variables. That is, MCS does not consider the state variables as random variables within the PPF problem before observing different configurations of the input random variables. On the other hand, this sampling approach is computationally expensive, since it relies on repetitive DPF solutions.

In this paper, we address these shortcomings by formulating the PPF problem as a Bayesian inference problem.

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Using Bayesian inference requires the specification of prior distributions for the state variables, and a likelihood function that relates the observations to the state variables. We apply Bayes theorem to obtain the posterior distribution over the state variables. By including prior distributions over the state variables into the PPF problem, it will be possible to exploit an additional source of information that has not been used in simulation methods before, like MCS. In principle, using this extra source of information allow us to reduce the computation time to solve the PPF problem. Furthermore, the way in which the Bayesian inference method is applied requires solving a forward computation problem, rather than an inverse optimization problem as needed in MCS. This means that we will not need to solve heavy computational optimization methods, every time we generate a new sample.

Different studies have used Bayesian methods to consider the uncertainty in a PS. For example: in [9], [10], the authors employed a Gaussian mixture model to approximate non-Gaussian distributions of loads in a PS. The authors put prior distributions over the parameters (mean, covariance and mixing coefficients) of each Gaussian distribution. In this mixture model, the likelihood function is defined using measurements of loads (observations) given the parameters of each Gaussian distribution. The goal of this Bayesian modeling is to obtain the posterior distribution of the parameters given the observations, where the parameters can be estimated by using the Expectation Maximization algorithm [9] or variational Bayesian inference [10]. These methods focus on modeling the uncertainty of the loads in a PPF problem. We are interested in using Bayesian inference as an holistic approach for uncertainty quantification in the random variables associated to a PS. Our aim goes beyond modeling uncertainty in the loads.

In a classical Bayesian inference problem, we would know beforehand the likelihood function. Since likelihood functions for power systems have not been discussed properly in the literature, we discuss alternatives for likelihood functions, but mainly appeal to likelihood-free methods for obtaining posterior distributions. In particular, we use Approximate Bayesian Computation (ABC) philosophy that replaces the calculation of the likelihood function by comparisons between simulated and observed data [11]. ABC methods have been applied in several fields of science and engineering for Bayesian inference. They have been employed for statistical inference in systems biology [12], ecological models [13] and were also applied by [14] in problems of parameter inference and model selection for dynamical models.

The ABC approach for analyzing PPF problems was originally proposed by the authors of [15]. They introduced an ABC approach based on Markov Chain Monte Carlo, capable of including prior distributions over the state variables into the PPF problem. However, their work suffers from three important limitations. First, their method has low acceptance rates when the prior distribution over the state variables is not close to the posterior distribution. Second, their approximate approach may get accepted samples with low probability. The third limitation is that their method obtains less accurate solutions than MCS.

Here, we propose a new ABC method tailored to power systems, in which the Jacobian of the power flow equations of the PS is used to guide the search for more probable samples from the posterior distribution over the state variables, overcoming the limitations of the approach introduced by [15]. We refer to this method as the Jacobian ABC or simply JABC. An additional advantage of JABC in the PPF context is that, in contrast to MCS, we do not need to solve costly optimization problems. We only need to compute several classical forward solutions of the PS, leading to an important decrease in computational complexity.

In this paper, we evaluate the performance of four likelihood-free methods, namely, ABC [11], JABC, ABC SMC [14] and JABC SMC for three test systems: IEEE $\{6, 39, 118\}$. ABC based on sequential Monte Carlo (SMC) is a modification of ABC that obtains an approximation of the true posterior using a series of sequential steps [14]. JABC SMC refers to an extension of ABC SMC that we also propose in this paper, and that uses the Jacobian for the power flow equations. The main contributions of this paper include the following:

1) A Bayesian inference perspective for addressing the Probabilistic power flow problem is introduced.
2) We discuss alternatives for likelihood functions. We also propose prior probability distributions over the state variables.
3) We also provide a Bayesian methodology that incorporates the Jacobian computed from the power flow equations for enhancing the posterior distribution estimation of the state variables.
4) We also propose ABC methods in combination with sequential Monte Carlo methods applied to the PPF problem.

II. Bayesian modeling for probabilistic power flow problems

According to Su in [7] and given a network configuration, the power flow equations can be written as follows,

$$\mathbf{b} = \mathbf{g}\left(\mathbf{x}\right), \tag{1}$$

$$\mathbf{z} = \mathbf{h}\left(\mathbf{x}\right), \tag{2}$$

where $\mathbf{g}$ and $\mathbf{h}$ are nonlinear power flow equations. The vector $\mathbf{x} \in \mathbb{R}^{N_{\mathbf{x}}}$ includes the state variables, angles and voltages. If we assume N nodes and N_g nodes in PV nodes, hence, $N_{\mathbf{x}} = 2N - N_g - 2$, being the number of the unknown angles and voltages equal to $N_\theta = N - 1$ and $N_V = N - N_g - 1$, respectively [16]. The vector $\mathbf{z}$ has elements given by the power flows through lines; and $\mathbf{b}$ is a vector with entries given by the net active and reactive powers injected, which are known. Besides, $\mathbf{b}$ depends on the powers generated (P_g) and loads (P_d). In PPF studies, P_g and P_d are modeled through probability distributions. It is common to use Gaussian, discrete and Weibull distributions to model the uncertainty over the loads and powers generated [4].

In contrast to Eq. (1), authors in [17] proposed to model the PPF problem assuming that $\mathbf{y} = \mathbf{f}\left(\mathbf{x}\right)$, where $\mathbf{x}$ is the input variable vector that contains loads, network configuration and powers generated by distributed generation; $\mathbf{y}$ is the output vector that includes voltage magnitudes, angles and some powers generated; and $\mathbf{f}(\mathbf{x})$ determines the state of the system as a function of the input variables.

In [18], the authors used the model of Eq. (1), and assumed that $\mathbf{b}$ is the vector of input random variables and $\mathbf{x}$ is the vector of state variables; $\mathbf{b}$ is modeled by a multivariate Gaussian distribution. The mean vector and the covariance matrix for $\mathbf{b}$ are assumed known. Based on these assumptions, the mean vector for $\mathbf{x}$ can be computed through DPF methods and its covariance matrix, $\mathbf{\Sigma_x}$, can be obtained as $\mathbf{\Sigma_x} = \mathrm{diag}\left(\mathbf{J}^\top \mathbf{\Sigma_b}^{-1}\mathbf{J}\right)$, where $\mathbf{J}$ is the Jacobian of the power flow equations; and $\Sigma_\mathbf{b}$ is the covariance matrix of the power injected, $\mathbf{b}$. According to the authors, $\mathbf{x}$ can be assumed to follow a multivariate Gaussian distribution. Such assumption is valid if all random variables are Gaussian distributed and the power flow equations are linearized around an operation point $\mathbf{x}_0$ [2]. However, if the PS includes renewable energy, for example wind or solar energy, the random variables associated to these energies are non-Gaussian anymore [17], and the assumption about $\mathbf{x}$ is not necessarily Gaussian.

In this paper, we also use Eq. (1) for obtaining an approximation of the probability distribution over the voltages and angles, given that the input random variables are modeled by particular probability distributions. We use Bayesian inference for estimating an approximate probability distribution over $\mathbf{x}$ given powers injected. Using Bayes theorem,

$$p(\mathbf{x}|\,\mathcal{D}) = \frac{p(\mathcal{D}|\,\mathbf{x})\,p(\mathbf{x})}{p(\mathcal{D})}, \tag{3}$$

where $p(\mathbf{x})$ is the prior distribution (prior) for $\mathbf{x}$, that encodes prior assumptions over $\mathbf{x}$; the term $p(\mathcal{D}|\,\mathbf{x})$ is the likelihood function (likelihood) that expresses how probable the observed data set is for different settings of $\mathbf{x}$; and $p(\mathbf{x}|\,\mathcal{D})$ is the posterior probability distribution (posterior) of the state variable $\mathbf{x}$ given observed data $\mathcal{D}$. For the PPF problem, the modeling of the particular input random variables can be included into the likelihood. In this paper, $\mathcal{D}$ refers to the reactive and active powers injected ($\mathbf{b}$ in Eq. (1)). The posterior quantifies the knowledge about the unknown variables and evaluate the uncertainty in $\mathbf{x}$ after observing $\mathcal{D}$ [19]. The term $p(\mathcal{D})$ is a normalization constant for ensuring that the posterior is a valid probability distribution. It is often called the evidence and it is given by $p(\mathcal{D}) = \int p(\mathcal{D}|\,\mathbf{x})\,p(\mathbf{x})\,d\mathbf{x}$ [19].

It can be noticed from Eq. (3) that a likelihood is necessary to compute the posterior and the evidence. In a typical regression problem, a common likelihood assumes that the relationship between $\mathbf{b}$ and $\mathbf{x}$ is linear, and that the observation noise follows a Gaussian distribution [2]. However, for real PS, the relationship between $\mathbf{b}$ and $\mathbf{x}$ is not linear. Also, one may argue that the observation noise is far from being Gaussian, particularly when the PS includes renewable energy.

Rather than attempting to build a likelihood for the PPF problem, we use likelihood-free methods for computing the posterior. We employ ABC methods, that define the likelihood using simulator outputs. For the PPF problem, the simulator corresponds to expression $\mathbf{g}(\mathbf{x})$ in Eq. (1). ABC methods have gained attention in the last years due to their flexibility, easiness of implementation, and the fact that they can be applied to any model for which forward simulation is possible. In the next section, we explain the ABC methods and how to apply them to solve the Bayesian PPF problem.

III. Approximate Bayesian Computation Methods

Given a prior $p(\mathbf{x})$, the goal in ABC is to approximate the posterior $p(\mathbf{x}|\,\mathcal{D})$ by using simulator outputs $\mathcal{D}'$ localized in the same space as $\mathcal{D}$. Hence, assuming that we have an auxiliary variable $\mathcal{D}'$, Eq. (3) can be rewritten as,

$$p(\mathbf{x}|\,\mathcal{D}) \approx \frac{p(\mathbf{x}) \int p_\epsilon(\mathcal{D}|\,\mathbf{g}(\mathbf{x}) = \mathcal{D}')\,p(\mathcal{D}'|\,\mathbf{x})\,d\mathcal{D}'}{\int p(\mathbf{x}) \int p_\epsilon(\mathcal{D}|\,\mathbf{g}(\mathbf{x}) = \mathcal{D}')\,p(\mathcal{D}'|\,\mathbf{x})\,d\mathcal{D}' d\mathbf{x}}, \tag{4}$$

where $p(\mathcal{D}'|\,\mathbf{x})$ is the probabilistic simulator, which for the PPF problems is $\mathbf{g}(\mathbf{x})$; and $p_\epsilon(\mathcal{D}|\,\mathbf{g}(\mathbf{x}) = \mathcal{D}')$ is a distribution that measures how similar $\mathcal{D}'$ is to $\mathcal{D}$. This distribution depends on ϵ and controls the acceptable discrepancy between $\mathcal{D}$ and $\mathcal{D}'$. To measure the similarity between $\mathcal{D}$ and $\mathcal{D}'$, it is common to use a distance measure $d(\mathcal{D}, \mathcal{D}')$. From Eq. (4), the principle behind ABC is to approximate the likelihood $\int p_\epsilon(\mathcal{D}|\,\mathbf{g}(\mathbf{x}) = \mathcal{D}')\,p(\mathcal{D}'|\,\mathbf{x})\,d\mathcal{D}'$ by $p(d(\mathcal{D}, \mathcal{D}') \le \epsilon|\,\mathbf{x})$ using a comparison between the observed data $\mathcal{D}$, and the simulated data $\mathcal{D}'$, which implies that the posterior is given by

$$p(\mathbf{x}|\,\mathcal{D}) \propto p(d(\mathcal{D}, \mathcal{D}') \le \epsilon|\,\mathbf{x})\,p(\mathbf{x}). \tag{5}$$

To obtain samples from the posterior shown in the Eq. (5), the ABC rejection sampler [20] can be used. This sampler is the most common ABC method and is shown in algorithm 1.

Algorithm 1: ABC rejection

```
Draw x from p(x)
Simulate D' using p(D'|x)
Accept x if d(D, D') ≤ ϵ
```

In the algorithm above, ϵ determines the accuracy of the algorithm. In this paper, algorithm 1 will be referred to as ABC. It is important to mention that the empirical distribution over accepted samples for $\mathbf{x}$ is an approximation that can be expressed by Eq. (5). For the ideal case of $\epsilon = 0$, the samples that we draw will come from the true posterior. However, the algorithm would need to perform a large amount of simulations for accepting any sample [11]. As for a distance measure d, several alternatives can be used in practice, including the Euclidean or Mahalanobis distance, or the root mean squared error (RMSE). In this paper, we use RMSE. In the next sections, we explain how we extend the ABC method to incorporate information from the PS. We will also describe how to embed these ideas in other ABC algorithms.

IV. JACOBIAN ABC FOR PPF PROBLEMS

When using algorithm 1, it is possible to have low acceptance rates when the prior is not close to the posterior. This problem is addressed by an ABC extension presented in [15], which is based on Markov Chain Monte Carlo (MCMC) [21]. ABC MCMC randomly explores the state space by modifying the current accepted samples. It uses a proposal distribution $q\left(\mathbf{x}^{*} \middle| \mathbf{x}_{i}\right)$ and the Hasting correction given by $\alpha = \min\left(1, \frac{p(\mathbf{x}^{*})q(\mathbf{x}_{i}|\mathbf{x}^{*})}{p(\mathbf{x}_{i})q(\mathbf{x}^{*}|\mathbf{x}_{i})}\right)$ to move from the i-th state accepted $\mathbf{x}_i$ to a new state $\mathbf{x}^*$. Where $p(\mathbf{x}^*)$ and $p(\mathbf{x}_i)$ is the prior distribution evaluated at $\mathbf{x}^*$ and $\mathbf{x}_i$, respectively. ABC MCMC also obtains samples from an approximated posterior over $\mathbf{x}$. However, according to [14], ABC MCMC may get accepted samples with low probability. To deal with the problems mentioned before, we propose to use the Jacobian of the power flow equations as part of the ABC MCMC method. If we assume a symmetric proposal distribution, that is, $q\left(\mathbf{x}^{*} \middle| \mathbf{x}_{i}\right) = q\left(\mathbf{x}_{i} \middle| \mathbf{x}^{*}\right)$, the Hasting correction is given by $\alpha = \min\left(1, \frac{p(\mathbf{x}^{*})}{p(\mathbf{x}_{i})}\right)$. We note that if $\mathbf{x}^*$ is more probable than $\mathbf{x}_i$, we definitely accept $\mathbf{x}^*$. For this, we employ the Jacobian matrix information to do an improved state space exploration. Specifically, we use a multivariate Gaussian distribution as symmetric proposal distribution q, where its mean value is updated as follows [22],

$$\mathbf{x}_{i+1} = \mathbf{x}^{*} + \mathbf{J}_{i}^{-1}\left(\mathcal{D} - \mathcal{D}'\right), \tag{6}$$

where the matrix $\mathbf{J}_i : \mathbb{R}^{N_{\mathbf{x}}} \rightarrow \mathbb{R}^{N_{\mathbf{x}}}$ is the Jacobian of $\mathbf{g}(\mathbf{x})$, that is, $\mathbf{J}_i = \left.\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_i}$ [16]; and $\mathcal{D} - \mathcal{D}'$ is a vector of relative errors between observed and simulated data. Eq. (6) can be seen as a correction step, which searches that $\mathcal{D}' = \mathcal{D}$ through $\mathbf{x}^*$. We also need to define the covariance matrix for the distribution q, $\mathbf{\Sigma_x}$. In this paper, we use a diagonal matrix with N_θ elements $\sigma^2_{\theta_q}$ for angles and N_V diagonal entries $\sigma^2_{v_q}$ for voltages. The basic idea of using Eq. (6) is to move from a current state $\mathbf{x}_i$ to a more probable new state $\mathbf{x}^*$. We refer to this new ABC algorithm as JABC. The JABC algorithm can be summarized as follows,

Algorithm 2: JABC

1 Draw $\mathbf{x}^*$ from $q\left(\mathbf{x} \middle| \mathbf{x}_i, \Sigma_{\mathbf{x}}\right)$
2 Simulate $\mathcal{D}'$ using $p\left(\mathcal{D}' \middle| \mathbf{x}\right)$
3 if $d\left(\mathcal{D}, \mathcal{D}'\right) \leq \epsilon$
4 $\mathbf{x}_{i+1} = \mathbf{x}^{*} + \mathbf{J}_{i}^{-1}\left(\mathcal{D} - \mathcal{D}'\right)$
5 Otherwise reject $\mathbf{x}^*$

For using algorithm 2, it is necessary to adequately choose the initial condition $\mathbf{x}_0$. In this paper, we take an all-ones vector for voltages, and the solution of the DC power flow algorithm for angles.

Another point to mention is that when using ABC, one needs to specify a suitable value for ϵ. The optimal value for ϵ depends on the similarity between the prior and the posterior, which is difficult to know beforehand. One way to avoid the manual selection of ϵ is to use an ABC algorithm coupled with sequential Monte Carlo (ABC SMC) [23], where the value for ϵ is specified adaptively with each iteration of the algorithm. The goal of ABC SMC is to obtain an approximation of the true posterior using a series of sequential steps, expressed by $p\left(\theta \middle| d\left(\mathcal{D}, \mathcal{D}'\right) \leq \epsilon_t\right)$, for $t = 1, \cdots, T$, where ϵ_t is a threshold that decreases at each step $(\epsilon_1, >, \ldots, > \epsilon_T)$, refining the approximation towards the target distribution. ABC SMC is computationally much more efficient than ABC, however the computation time depends on the number of the sequential steps [14]. ABC SMC has a first stage based on ABC. We can replace this stage with JABC, leading to what we call in the paper as the JABC SMC. Details of the ABC SMC method can be found in [14].

V. EXPERIMENTAL EVALUATION

In this section, we present several experiments that illustrate different aspects of this new approach for tackling the PPF problem. In the first experiment, we show an example where the likelihood functions are actually known and compare the performance between MCS, Hamiltonian Monte Carlo, and ABC in an IEEE 6-bus test system. For the second experiment, we evaluate how the ABC algorithms work, including the sequential Monte Carlo variants, compare with MCS, the point-estimation method and the Taguchi method in terms of the number of nodes of a PS. We use the IEEE $\{6, 118\}$-bus test systems. For the third experiment, we show the performance of the ABC methods when including renewable sources of energy in the IEEE 39-bus test system.

A. PPF analysis using MCS and Bayesian inference

In this experiment, we compared Bayesian inference methods and MCS for solving PPF problems, assuming a particular form for the likelihood. For this we used the IEEE 6-bus system and assumed to know the true posterior of $\mathbf{x}$. To define such posterior, we use the input random variables as in [7] (no uncertainty in line parameters) and employ MCS for obtaining the probability distribution of $\mathbf{x}$, that we use as the true posterior of $\mathbf{x}$. We initially drew 1000 samples from this posterior and computed $\{\mathbf{b}_i\}_{i=1}^{1000}$ through Eq. (1). [1] To calculate $\mathbf{b}_i$, which corresponds to $\mathcal{D}$ for the ABC methods, we used two different likelihood functions: a multivariate Gaussian likelihood function (MGLF) with mean $\mathbf{g}(\mathbf{x})$ and covariance matrix $\mathbf{\Sigma} = 3 \times 10^{-4}\mathbf{I}$, that is, $p\left(\mathbf{b} \middle| \mathbf{x}\right) = \prod_{i=1}^{1000} \mathcal{N}\left(\mathbf{b}_i \middle| \mathbf{g}(\mathbf{x}), \mathbf{\Sigma}\right)$; and a multivariate t likelihood function (MTLF) with mean $\mathbf{g}(\mathbf{x})$, precision matrix $\mathbf{\Lambda} = \mathbf{\Sigma}^{-1}$ and 5 degrees of freedom ($\nu = 5$), that is, $p\left(\mathbf{b} \middle| \mathbf{x}\right) = \prod_{i=1}^{1000} \mathcal{S}\left(\mathbf{b}_i \middle| \mathbf{g}(\mathbf{x}), \mathbf{\Lambda}, \nu\right)$. [2]

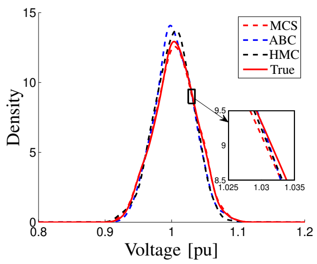
From the PPF perspective, the MGLF means that each $\mathbf{b}_i$ is modeled by Gaussian distributions (input Gaussian variables).

[1] We have used a number of samples of 1000 for our experiments with the goal of ensuring a coefficient of variation of 2% (for more details see [7]). The coefficient of variation is defined as the ratio between the standard deviation and mean value.

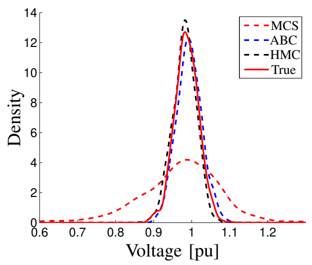
[2] The MGLF is given by $\mathcal{N}\left(\mathbf{b} \middle| \mathbf{g}(\mathbf{x}), \mathbf{\Sigma}\right) = \frac{1}{(2\pi)^{N_{\mathbf{x}}/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{\nabla}^2\right)$, where $\mathbf{\nabla}^2 = \left(\mathbf{b} - \mathbf{g}(\mathbf{x})\right)^{\top} \mathbf{\Sigma}^{-1} \left(\mathbf{b} - \mathbf{g}(\mathbf{x})\right)$. The MTLF is given by $\mathcal{S}\left(\mathbf{b} \middle| \mathbf{g}(\mathbf{x}), \mathbf{\Lambda}, \nu\right) = \frac{\Gamma(\nu/2 + N_{\mathbf{x}}/2)}{\Gamma(\nu/2)} \frac{|\mathbf{\Lambda}|^{1/2}}{(\nu\pi)^{N_{\mathbf{x}}/2}} \left[1 + \frac{\mathbf{\Delta}^2}{\nu}\right]^{-\nu/2-1/2}$, where $\mathbf{\Delta}^2 = \left(\mathbf{b} - \mathbf{g}(\mathbf{x})\right)^{\top} \mathbf{\Lambda} \left(\mathbf{b} - \mathbf{g}(\mathbf{x})\right)$ and $\Gamma(x)$ is the gamma function defined by $\Gamma(x) \equiv \int_0^\infty u^{x-1} e^{-u} du$.

If we assume that the relationship between $\mathbf{b}_i$ and $\mathbf{x}$ is linear, we can have a similar likelihood used by [2]. On the other hand, a MTLF can be considered as a probabilistic model from the observations that it is tolerant to the negative effect of outliers, for example: due to a measurement equipment malfunction of the input variables.

Once we generated the different datasets using these two likelihood functions, we used the Bayesian inference framework described above to make inference for the posterior of $\mathbf{x} = \begin{bmatrix} \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & V_4 & V_5 & V_6 \end{bmatrix}^\top$ given $\mathbf{b}$, $p\left(\mathbf{x}|\,\mathbf{b}\right)$. For each voltage, we use Gaussian distributions as prior distributions with mean equal to one, and variance $\sigma_v^2 = 0.0015$. We use uniform distributions as prior distributions for each angle, i. e., $\theta_i \sim \mathcal{U}\left(a_i, b_i\right)$. Parameters a_i and b_i are computed as $a_i = \theta_i^{DC} - \Delta\theta$ and $b_i = \theta_i^{DC} + \Delta\theta$, where θ_i^{DC} is the ith DC power flow solution [18]; and $\Delta\theta$ quantifies the error present in the solution obtained by the DC power flow algorithm with respect to the AC power flow solution (Eq. (1)). We chose $\Delta\theta$ equal to 0.07. We employed ABC and MCS to infer the posterior of $\mathbf{x}$. For ABC, we put $\epsilon = 0.7$, the simulation function was set as $\mathbf{g}\left(\mathbf{x}\right)$ and we ran one simulation by each $\mathbf{b}_i$. We used MATPOWER to implement MCS.[3] We also compared these two methods against the most popular likelihood-based Bayesian method, Hamiltonian Monte Carlo (HMC) [19]. Fig 1 shows the probability distribution of V_6 generated by the MCS, ABC and HMC, when we assumed a MGLF (see Fig. 1(a)) and a MTLF (see Fig. 1(b)).



(a) Posterior for V_6 using a MGLF



(b) Posterior for V_6 using a MTLF

Fig. 1. Posterior for V_6 when two likelihood functions are considered. The dashed red, blue and black lines are the responses obtained by MCS, ABC and HMC. The solid red lines are the true posteriors. Figs 1(a) and 1(b) show the posterior for V_6 by using a MGLF and a MTLF, respectively.

Fig. 1(a) shows the posteriors inferred when a MGLF was assumed. We observe that MCS, ABC and HMC obtain satisfactory posteriors. When we used a MTLF (see Fig. 1(b)), which is an intractable likelihood, we note that MCS did not infer appropriately the posterior for V_6, however, ABC and HMC properly estimated the posterior of V_6. From this results, we see that the MCS is affected by outliers, obtaining bad convergence properties and high variability. In contrast to HMC, ABC did not need a particular form for the likelihood to obtain the posterior of V_6. On the other hand, these results confirm the importance of including additional source of information (a prior over $\mathbf{x}$) to analyze the uncertainty of $\mathbf{x}$. We observed the results for other variables of $\mathbf{x}$, and they are very similar to the results of V_6.

B. PPF analysis with different sample sizes and number of nodes

We evaluated the performance of the different methods when the sample size for the input random variables and the number of the nodes of the PS were increased. We considered two case studies: in the first case, we analyzed the PPF problem for the IEEE 6-bus system assuming that the true posterior of $\mathbf{x}$ was also known and then observed how the simulation methods worked with different sample sizes. For the second case, we examined classic PPF problems in terms of the sample and system size. For classic PPF problems, we generated N_s samples from the input random variables to obtain $\{\mathbf{b}_i\}_{i=1}^{N_s}$ from Eq. (1). We then inferred the posterior $p\left(\mathbf{x}|\,\mathbf{b}\right)$ using all methods over the IEEE $\{6, 118\}$ bus systems.

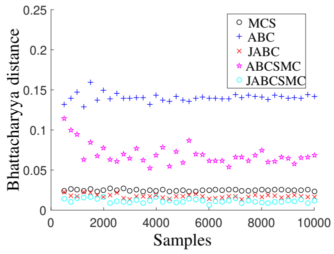
For the first case, we used the input random variables shown in [7] and we increased the number of samples of these variables from 500 to 10000 samples, in steps of 250 samples. For each step, we ran ABC, JABC, ABC SMC JABC SMC, MCS and computed the Bhattacharyya distance (BD) [24] to measure the similarity between the true distribution over $\mathbf{x}$ and the distribution computed by each method. For ABC and JABC, we used $\epsilon = 0.7$. For ABC SMC and JABC SMC, we used $\{\epsilon_t\}_{t=1}^{T} = \{3.0, 2.0, 1.0, 0.9, 0.7\}$. From now on, we will continue to employ Gaussian and uniform distributions as prior distributions for each voltage and angle, respectively, as it was described in the previous subsection. For ABC SMC, we employed a multivariate Gaussian distribution as proposal distribution for $\mathbf{x}$, with a covariance matrix that depends on $\sigma_{v_q}^2 = 10^{-6}$ and $\sigma_{\theta_q}^2 = 10^{-7}$ as diagonal entries for voltages and angles, respectively.

For JABC and JABC SMC, we employed a multivariate Gaussian distribution as symmetric proposal distribution, $q \sim \mathcal{N}\left(\mathbf{x}_i, \mathbf{\Sigma}_\mathbf{x}\right)$, where $\mathbf{x}_i$ and $\mathbf{\Sigma}_\mathbf{x}$ are the mean and covariance matrix for the proposal distribution. The mean $\mathbf{x}_i$ can be computed using Eq. (6). The covariance $\mathbf{\Sigma}_\mathbf{x}$ is assumed to be a diagonal matrix with parameters, $\sigma_{v_q}^2 = 10^{-5}$ and $\sigma_{\theta_q}^2 = 10^{-6}$. For $\mathbf{x}_0$, we used a vector of ones for voltages, and the DC power flow solution for angles.

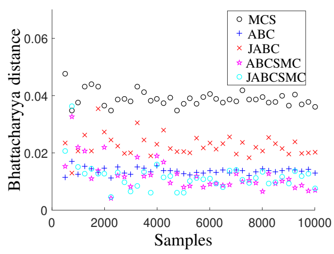
In Fig. 2, we present the BD in terms of the number of samples. We show the BD for voltages (see Fig. 2(a)) and angles (see Fig. 2(b)). From Fig. 2(a), we observe that the BD obtained by MCS is the most steady. We also see that JABC and JABC SMC compute better BD than MCS, ABC and ABC SMC; being the BD computed by JABC SMC, the lowest. For angles, we note that the BD calculated by ABC is the most steady, but the BD achieved by JABC SMC is the lowest. BD values are summarized in the Table I. These results validate our hypothesis of using the Jacobian of the power flow equations into the ABC algorithm to search more probable samples for $\mathbf{x}$. As can be seen in Fig. 2, the JABC SMC is a proper estimator of the posterior over $\mathbf{x}$ in terms of the number of samples. From Fig. 2, ABC and ABC SMC computed better BDs for angles than the BDs for voltages, this

[3]It is available at `http://www.pserc.cornell.edu/matpower/`

is due to the information included by DC power flow solution in the prior over angles. These results show that it is necessary (for the ABC and ABC SMC) to use an improved prior for the voltages, that is, the ABC and ABC SMC performance could be enhanced by using an informative prior over $\mathbf{x}$, for example: to use a multivariate Gaussian distribution over $\mathbf{x}$, where the mean vector depends on a previous DPF solution and the covariance matrix could be assumed known.



(a) BD for voltages



(b) BD for angles

Fig. 2. BD for voltages and angles for different number of samples. The circles, plus signs, cross, stars and the squares represent the BD obtained by MCS, ABC, JABC, ABCMCS and JABCSMC, respectively. Figs. 2(a) and 2(b) show the BD for voltages and angles.

Method	BD_V	BD_θ
MCS	0.0248 ± 0.0009	0.0389 ± 0.0027
ABC	0.1404 ± 0.0049	0.0136 ± 0.0011
ABCSMC	0.0690 ± 0.0126	0.0121 ± 0.0054
JABC	0.0174 ± 0.0022	0.0225 ± 0.0037
JABCSMC	$\mathbf{0.0117 \pm 0.0024}$	$\mathbf{0.0117 \pm 0.0051}$

TABLE I
BD OBTAINED BY ALL ABC METHODS AND MCS.

For the second case study, we solved two classic PPF problems with different number of nodes. In particular, we analyzed the IEEE 6 and 118 bus systems using input Gaussian variables [7]. Due to the ABC and ABC SMC obtain poor results, we applied JABC, JABC SMC and MCS to these two systems to infer the posterior $p(\mathbf{x}|\,\mathbf{b})$ over $N_s = 1000$ samples from the input random variables. For comparison purposes, we also applied the point-estimation method based on two points (PEM) [7] and the Taguchi method (TM) [8] to analyze these two PPF problems. For the PEM, due to this method must be combined with some series expansion to acquire the probability distribution of the PPF results, we used a Gram-Charlier series expansion [25]. For the TM, we employed a nonparametric density estimator [26] to obtain the probability distributions of the PPF results. We used the parameters mentioned in the previous experiment for the IEEE 6 bus system. For the IEEE 118 bus system, we used $\epsilon = 4.0$ in the JABC algorithm. For JABC SMC, we have reduced the sequential steps with regarding to the previous experiment, we specifically used $\{\epsilon_t\}_{t=1}^{T} = \{6.0, 5.0, 4.0\}$. For JABC and JABC SMC, we employed $\sigma^2_{v_q} = 10^{-5}$ and $\sigma^2_{\theta_q} = 10^{-6}$ as parameters of the proposal distributions. Figs. 3 and 4 compare each method against the MCS when some variables of these systems are inferred. Specifically, we show the probability distribution over V_6 (see Fig. 3) for the 6 bus system and the distribution for θ_{108} (see Fig. 4) for the 118 bus system, since the results for other variables in $\mathbf{x}$ are similar to the results shown in Figs. 3 and 4. From Figs. 3(a) and 4(a), we observe that the PEM does not infer appropriately the probability distributions of both variables. We also note that the samples obtained by the TM in both variables (see Figs 3(b) and 4(b)) are around the DPF solution (see the blue vertical solid line), however the shapes of the probability distributions obtained by this method are not similar to the probability distributions obtained by MCS. However, JABC and JABC SMC provide results close to the distributions obtained by MCS (see Figs. 3(c), 4(c), 3(d) and 4(d)), confirming the importance of the improved state space exploration of $\mathbf{x}$ in the Jacobian ABC-type methods.

We also recorded the computation time (CT) required for each method to solve the PPF problems shown in Figs. 3 and 4. All simulations were conducted on an Intel Core i7 PC with a 2.1GHz processor. Table II lists the CTs took for the different methods. Notice that the CTs, for JABC and JABC SMC, are lower than the CT took by MCS. From Table II, we also note that the proposed methods require more CT compared to the PEM and TM. However, the proposed methods do not require DPF solutions to obtain the probability densities shown in Figs. 3 and 4. [4]

		MCS	PEM	TM	JABC	JABC SMC
CT [s]	IEEE-6	13.540	0.1248	0.0936	0.6365	1.7156
	IEEE-118	17.097	6.0528	5.2884	5.8464	16.757

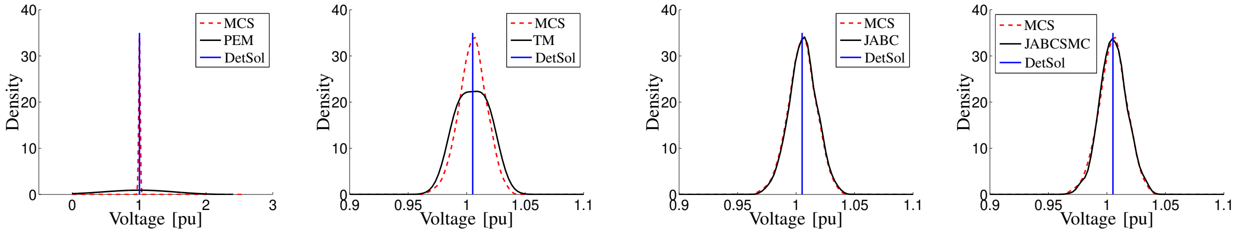
TABLE II
COMPUTATION TIME (CT), IN SECONDS [S], REQUIRED BY MCS AND ALL ABC METHODS TO SOLVE TWO PPF PROBLEMS WHEN ASSUMING INPUT GAUSSIAN VARIABLES.

Considering input Gaussian variables, we compared the DPF solution and the estimated posterior mean using the relative error (RE), [5] for each variable using the different solution methods. In what follows, the numerical analysis is only done for the state variables (state variables can be used to compute other variables, i.e. active and reactive power flows between buses). For the comparison, we drew 25 subsets of input random variables, we then applied PEM, TM, JABC, JABC SMC and MCS to obtain 25 REs for angles and voltages. With these errors, we computed the mean value and an one standard deviation for each method. We used the $\{6, 118\}$-bus systems and 1000 samples for all input random variables. We slightly changed the application of ABC methods. Here we ran 1000 simulations for the ABC methods by each input random variable configuration, since in the previous experiments, we used one simulation by each $\mathbf{b}_i$. Due to the ABC methods are approximate inference approaches, we wanted to show if it was possible to obtain REs close to the results using MCS. Table III presents the RE obtained by MCS and ABC methods.

For the 6-bus system, notice that the REs obtained by JABC and JABC SMC are lower than the REs computed by MCS, PEM and TM for voltages and angles. For the 118-bus system,

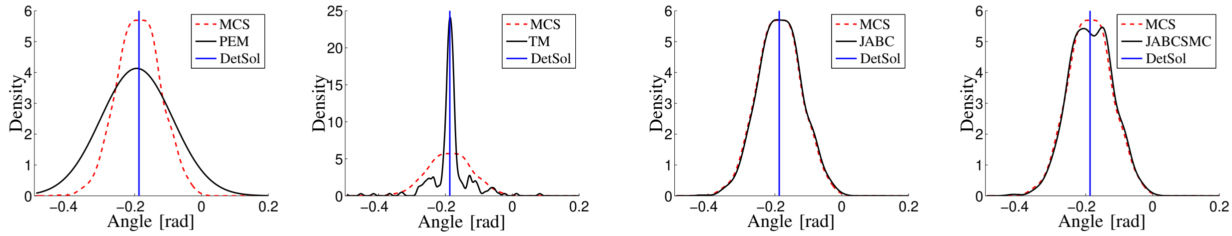
[4]It is necessary to comment that the goal of the PEM and TM is to reduce of the number of simulations in MCS, therefore the number of samples is also reduced. From this point of view, a fair comparison between these two methods and the remaining methods applied would not be possible.

[5]The RE is computed as $RE = \left|\frac{x_{ds} - x_{ps}}{x_{ds}}\right|$, where x_{ds} is the DPF solution, and x_{ps} is the estimated posterior mean obtained by each method.



(a) PEM (b) TM (c) JABC (d) JABCSMC

Fig. 3. Posterior for V_6. The dashed red line is the responses obtained by MCS. The solid black lines are the probability densities calculated by PEM, TM, JABC and JABC SMC. The blue vertical solid line is the DPF solution. Figs. 3(a) to 3(d) show the posteriors for V_6, obtained by PEM, TM, JABC and JABC SMC, respectively.



(a) PEM (b) TM (c) JABC (d) JABCSMC

Fig. 4. Posterior for θ_{108}. The dashed red line is the responses obtained by MCS. The solid black lines are the probability densities calculated by PEM, TM, JABC and JABC SMC. The blue vertical solid line is the DPF solution. Figs. 3(a) to 3(d) show the posteriors for V_6, obtained by PEM, TM, JABC and JABC SMC, respectively.

Index	Method	IEEE 6	IEEE 118
RE_θ [%]	MCS	1.5190 ± 0.8843	3.7290 ± 1.9822
	PEM	1.5268 ± 0.8923	3.7841 ± 2.0324
	TM	1.5494 ± 0.9092	3.7674 ± 2.0281
	JABC	$\mathbf{1.5073 \pm 0.8793}$	2.3344 ± 1.0952
	JABC SMC	1.5149 ± 0.8813	$\mathbf{2.3341 \pm 1.0951}$
RE_V [%]	MCS	0.0598 ± 0.0302	$\mathbf{0.0212 \pm 0.0054}$
	PEM	0.0603 ± 0.0300	0.0213 ± 0.0055
	TM	0.0605 ± 0.0300	0.0212 ± 0.0054
	JABC	$\mathbf{0.0585 \pm 0.0302}$	0.0234 ± 0.0019
	JABC SMC	0.0596 ± 0.0302	0.0235 ± 0.0019

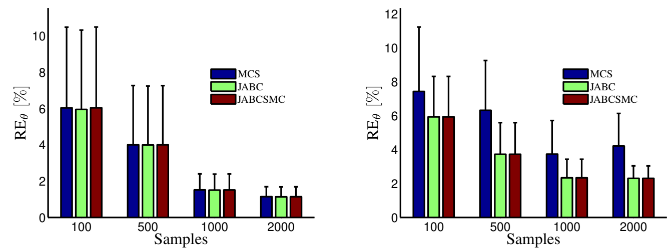
TABLE III
RELATIVE ERROR (RE) FOR ANGLES AND VOLTAGES USING ALL METHODS WHEN USING INPUT GAUSSIAN VARIABLES.

the REs computed by JABC and JABC SMC are lower than the results calculated by MCS, PEM and TM for angles. We also note that MCS and TM computed the lowest RE for the voltages, however the REs from JABC and JABC SMC are close to the results achieved by MCS. [6]

After showing the results when we had 1000 samples from the input random variables, we repeated the experiment of Table III with 100, 500, 1000 and 2000 samples. For this experiment, we only used the MCS, JABC and JABC SMC. Figs 5 and 6 report the RE, for angles and voltages, versus the number of samples.

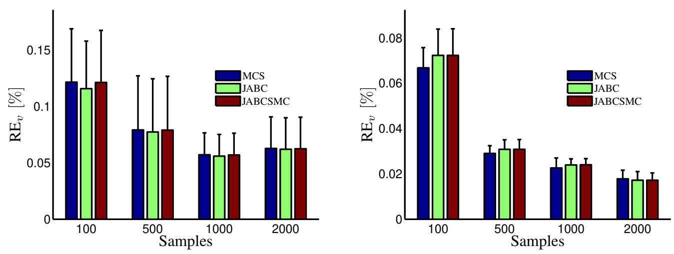
Figs 5 and 6 show that the RE decreases when the number of samples increases in both systems. From Figs 5(a), 6(a) and 6(b) the REs obtained by JABC and JABC SMC tend to REs computed by MCS. However from Fig 5(b), the REs

[6]Although, we have included averaged results, a close inspection of the individual performance on every variable confirms our analysis, in the sense that the errors obtained by JABC and JABC SMC are lower than or equal the errors computed by MCS.



(a) 6 Bus system (b) 118 Bus system

Fig. 5. REs for angles in terms of the number of samples. Blue, green and brown bars are the RE obtained by MCS, JABC and JABC SMC, respectively. The vertical line is one standard deviation for each variable.



(a) 6 Bus system (b) 118 Bus system

Fig. 6. REs for voltages in terms of the number of samples. Blue, green and brown bars are the RE obtained by MCS, JABC and JABC SMC, respectively. Vertical lines are one standard deviation for each variable.

obtained by JABC and JABC SMC are lower than the REs using MCS. These high REs of MCS are due to poor estimates for the angle at node 26, causing high variability (see the standard deviation in each scenario). These results confirm that the ABC methods, which are approximate Bayesian inference approaches, provide satisfactory results with respect to the optimization-based simulation methods.

C. PPF analysis with renewable energy

We now perform an experiment where we consider renewable energy in a IEEE 39-bus test system. In this case, we model all input random variables as in [4], but we do not consider correlated loads. We add a wind farm at bus 32, where the output power from one wind turbine, P_w, in terms of the wind speed v_w, is given by

$$P_w(v_w) = \begin{cases} 0 & v_w \le v_{cin}, \\ 0.5\rho A_w C_p v_w^3 & v_{cin} < v_w \le v_r, \\ P_r & v_{cout} < v_w, \end{cases} \tag{7}$$

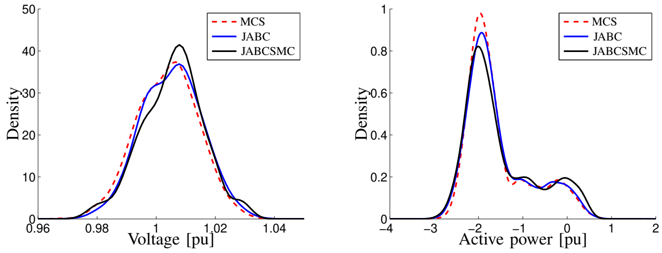
where v_{cin} is the cut-in wind speed; ρ is the air density; $A_w = \pi R^2$ is the area of the wind turbine rotor; R is the radius of the rotor; C_p is a coefficient of power, at which the wind turbine generator starts generating power [4]; v_r is the nominal rotational speed; P_r is the nominal wind power; and v_{cout} is the cut-out wind speed, at which the wind turbine generator is shut down for safety reasons [4]. Furthermore, we assumed that v_w followed a Weibull distribution and we adopted the parameters used by [4]. [7] We also modeled the wind farm output power as a Gaussian random variable. The output power can be modeled as $P_{wt}(v_w) = P_w(v_w) + \omega$, where $P_{wt}(v_w)$ is the wind turbine output power; $P_w(v_w)$ is the deterministic output power expressed by Eq. (7); and ω is a white Gaussian noise with variance $\sigma_\omega^2 = 10^{-6}$.

In this experiment, we drew 1000 samples from the input random variables and we used the same prior distributions over $\mathbf{x}$ as it was mentioned in the first experiment. However, we increased the variance in the Gaussian prior for each voltage to $\sigma_v^2 = 0.005$. We only compared the MCS, JABC and JABC SMC. We used $\epsilon = 2.0$ in the JABC algorithm. For JABC SMC, we used $\{\epsilon_t\}_{t=1}^{T} = \{3.0, 2.75, 2.5, 2.25, 2.0\}$. For JABC and JABC SMC, we employed $\sigma_{v_q}^2 = 10^{-5}$ and $\sigma_{\theta_q}^2 = 10^{-6}$ as parameters of the proposal distributions. Fig. 7 compares the posterior distributions for V_{15} and P_{39-9} obtained by MCS, JABC and JABC SMC. [8] From this figure, we notice that the posterior for V_{15} (see Fig. 7(a)) obtained by JABC is closer to the distributions obtained by MCS, than the posterior computed by JABC SMC.

From Fig. 7(b), notice that the posterior for P_{39-9} has two modes, despite this, the distributions obtained by JABC and JABC SMC are consistent with the distribution for P_{39-9} using MCS. MCS, JABC and JABC SMC took 15.0853s, 1.5288s and 5.3040s, respectively, for solving this PPF problem. These results show how the Bayesian inference is an alternative for solving the PPF problems, allowing a non-linear relationship between $\mathbf{b}$ and $\mathbf{x}$, and dealing with input non-Gaussian variables. We also analyzed the results for the remaining variables, and they are similar to the results shown in Fig. 7.

[7]For our experiments, we used $a = 15$ and $b = 2.5$. For the Eq. (7), we used $v_{cin} = 3$ m/s, $v_{cout} = 25$m/s, $v_r = 10.28$m/s, $C_p = 0.473$ and $R = 45$m.

[8]After obtaining samples from the posterior of $\mathbf{x}$, we employ the Eq. (2) for computing the posterior over P_{39-9}.



(a) Posterior for V_{15} (b) Posterior for P_{39-9}

Fig. 7. Posterior for V_{15} and P_{39-9}. The dashed red line, solid blue line and solid black line are the posteriors calculated by MCS, JABC and JABC SMC, respectively. Figs. 7(a) and 7(b) show the posteriors for V_{15} and P_{39-9}, respectively, obtained by each method.

Since we do not have a ground-truth solution, we observed how close the mean and standard deviation of angles and voltages obtained by the ABC methods are to the values computed by MCS, that is, we used the mean and standard deviation obtained by MCS as reference values. Using these values, we computed the relative error for ABC methods. [9] We generated 25 different sets of input random variables, with 1000 samples for each variable. We then applied the JABC and JABC SMC to infer the posterior mean and standard deviation of each input random variable configuration. Finally, we computed 25 relative errors using the previous information. Table IV lists the mean and one standard deviation for the relative errors that compare the reference values obtained by MCS, and the estimated values using the ABC methods. Notice that JABC gives better estimated results than JABC SMC, since in some cases the estimates obtained from JABC SMC have a large variance. However, the errors obtained by JABC SMC do not exceed 3%.

IEEE 39		ε_θ [%]	ε_V [%]
JABC	μ	0.6866 ± 0.0768	0.1251 ± 0.0026
	σ	0.2017 ± 0.1254	0.3751 ± 0.1300
JABC SMC	μ	2.6610 ± 1.8072	0.1173 ± 0.0375
	σ	2.0547 ± 1.6072	2.3299 ± 1.0785

TABLE IV
RELATIVE ERROR FOR VOLTAGES AND ANGLES USING IEEE 39-BUS SYSTEM.

Table IV clearly shows that the JABC and JABC SMC methods proposed here, consistently outperforms the approach presented in [15].

[9]We compute relative errors with respect to the values calculated by MCS, that is, we employ $\varepsilon_x^\mu = \left|\frac{\mu_x - \mu_x^*}{\mu_x}\right|$ and $\varepsilon_x^\sigma = \left|\frac{\sigma_x - \sigma_x^*}{\sigma_x}\right|$ [7], where ε_x^μ and ε_x^σ are relative errors for the mean and standard deviation values; μ_x and σ_x are the mean and standard deviation obtained using MCS; μ_x^* and σ_x^* are the mean and standard deviation computed with the ABC methods proposed.

VI. Conclusions

In this paper, we introduced an alternative for solving PPF problems using the approximate Bayesian computation method and the Jacobian of the power flow equations. We also proposed priors for voltages and angles for the PPF problem under Bayesian inference perspective. We demonstrated that ABC and ABC SMC can work for an small power system using input Gaussian variables. However, it is necessary to define an informative prior over the state variables, for example, to use a multivariate Gaussian distribution where the mean vector depends on a previous AC power flow solution. We also showed that the posteriors of the state variables obtained by JABC and JABC SMC are close to the results using MCS, similarly JABC took less computation time for obtaining the PPF solution with respect to MCS. As future works, it would be possible to consider: uncertainty in line parameters, correlated random variables, likelihood functions that combine continuous and discrete random variables, and the application of the proposed methods to analyze distribution systems.

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References

[1] B. Borkowska, "Probabilistic load flow," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-93, no. 3, pp. 752–759, May 1974.

[2] J. Dopazo, O. Klitin, and A. Sasson, "Stochastic load flows," *IEEE Transactions on Power Apparatus and Systems*, vol. 94, no. 2, pp. 299–309, Mar 1975.

[3] J. Morales, L. Baringo, A. Conejo, and R. Minguez, "Probabilistic power flow with correlated wind sources," *IET Generation, Transmission Distribution*, vol. 4, no. 5, pp. 641–651, May 2010.

[4] N. Soleimanpour and M. Mohammadi, "Probabilistic load flow by using nonparametric density estimators," *Power Systems, IEEE Transactions on*, vol. 28, no. 4, pp. 3747–3755, Nov 2013.

[5] R. Allan, A. Leite da Silva, and R. Burchett, "Evaluation methods and accuracy in probabilistic load flow solutions," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-100, no. 5, pp. 2539–2546, May 1981.

[6] B. Prusty and D. Jena, "A critical review on probabilistic load flow studies in uncertainty constrained power systems with photovoltaic generation and a new approach," *Renewable and Sustainable Energy Reviews*, vol. 69, pp. 1286 – 1302, 2017.

[7] C.-L. Su, "Probabilistic load-flow computation using point estimate method," *Power Systems, IEEE Transactions on*, vol. 20, no. 4, pp. 1843–1851, Nov 2005.

[8] Y. Y. Hong, F. J. Lin, and T. H. Yu, "Taguchi method-based probabilistic load flow studies considering uncertain renewables and loads," *IET Renewable Power Generation*, vol. 10, no. 2, pp. 221–227, 2016.

[9] C. Carmona-Delgado, E. Romero-Ramos, and J. Riquelme-Santos, "Probabilistic load flow with versatile non-Gaussian power injections," *Electric Power Systems Research*, vol. 119, pp. 266 – 277, 2015.

[10] Z. Dong, Y. Wang, and J. Zhao, "Variational Bayesian inference for the probabilistic model of power load," *IET Generation, Transmission Distribution*, vol. 8, no. 11, pp. 1860–1868, 2014.

[11] R. Wilkinson, "Approximate Bayesian Computation (ABC) gives exact results under the assumption of model error," Tech. Rep., 2008.

[12] F. Bonassi and M. West, "Sequential Monte Carlo with adaptive weights for approximate Bayesian computation," *Bayesian Analysis*, 2014.

[13] D. J. Nott, Y. Fan, L. Marshall, and S. A. Sisson, "Approximate Bayesian Computation and Bayes' Linear Analysis: Toward High-Dimensional ABC," *Journal of Computational and Graphical Statistics*, vol. 23, no. 1, pp. 65–86, Dec. 2012.

[14] T. Toni, D. Welch, N. Strelkowa, A. Ipsen, and M. Stumpf, "Approximate Bayesian computation scheme for parameter inference and model selection in dynamical systems," *Journal of The Royal Society Interface*, vol. 6, no. 31, pp. 187–202, 2009.

[15] C. D. Zuluaga and M. A. Álvarez, *Approximate Probabilistic Power Flow*. Cham: Springer International Publishing, 2017, pp. 43–53.

[16] J. Grainger and W. Stevenson, *Power system analysis*, ser. McGraw-Hill series in electrical and computer engineering: Power and energy. McGraw-Hill, 1994.

[17] M. Aien, M. Fotuhi-Firuzabad, and M. Rashidinejad, "Probabilistic optimal power flow in correlated hybrid wind –photovoltaic power systems," *Smart Grid, IEEE Transactions on*, vol. 5, no. 1, pp. 130–138, Jan 2014.

[18] J. Zhu, *Optimization of power system operation (IEEE press series on power engineering); 2nd ed.* Wiley-Blackwell, Feb 2015.

[19] C. M. Bishop, *Pattern Recognition and Machine Learning*. Springer, 2006.

[20] J. Pritchard, M. Seielstad, A. Perez-Lezaun, and M. Feldman, "Population growth of human Y chromosomes: a study of Y chromosome microsatellites." *Molecular Biology and Evolution*, vol. 16, no. 12, p. 1791, 1999.

[21] P. Marjoram, J. Molitor, V. Plagnol, and S. Tavaré, "Markov Chain Monte Carlo without Likelihoods," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 100, no. 26, pp. 15 324–15 328, 2003.

[22] A. Bracale, G. Carpinelli, A. R. Di Fazio, and A. Russo, *Probabilistic Approaches for the Steady-State Analysis of Distribution Systems with Wind Farms*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 245–282.

[23] S. Sisson, Y. Fan, and M. Tanaka, "Sequential monte carlo without likelihoods," *Proceedings of the National Academy of Sciences*, vol. 104, no. 6, pp. 1760–1765, 2007.

[24] T. Kailath, "The divergence and bhattacharyya distance measures in signal selection," *IEEE Transactions on Communication Technology*, vol. 15, no. 1, pp. 52–60, February 1967.

[25] J. Morales and J. Perez-Ruiz, "Point estimate schemes to solve the probabilistic power flow," *Power Systems, IEEE Transactions on*, vol. 22, no. 4, pp. 1594–1601, Nov 2007.

[26] A. W. Bowman and A. Azzalini, *Applied smoothing techniques for data analysis : the kernel approach with S-Plus illustrations*. Clarendon Press ; Oxford University Press, 1997.

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