Article:
Gitman, I.M. and Song, Y. (2018) Fibonacci sequence for modelling stop bands in random microstructure. ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik, 98 (2). pp. 270-276. ISSN: 0044-2267

https://doi.org/10.1002/zamm.201700010

Fibonacci sequence for modelling stop bands in random microstructure

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words Fibonacci sequence, composite material, randomness, wave propagation, stop band.
MSC (2010) 00-xx

In this paper the mathematical concept of the Fibonacci sequence has been introduced as an accurate and reliable tool to model randomness in a heterogeneous material. It is also argued, that this randomness plays an important role and can control the response of a heterogeneous material, subjected to dynamic loading, here an elastic wave propagating through the material. A particular dynamic phenomenon, the presence of band gaps, has been analysed. It has been shown that randomness, modelled using the Fibonacci sequence, introduced into the material's structure, increases the range of stop band frequencies.

1 Introduction

It is well known that by changing and controlling the microstructure of heterogeneous materials it is possible to control and influence the overall response of macroscopic media in static as well as dynamic loading conditions. This ability of controlling the overall response is particularly important in applications such as waveguide filters, devices that can allow waves at some frequencies to propagate, i.e. pass through the material, while at other frequencies waves are rejected, i.e. stopped from propagating. These ranges of frequencies are referred to as *pass band* and *stop band* correspondingly.

The band gap phenomenon [1] has mainly been studied in two-phase periodically structured materials [5],[7], [10]. However, realistic materials are rarely perfectly periodic, thus it is of importance to understand how waveguide technology can be also implemented in more realistic applications. Some attempts have been made in [6] and [9] to analyse the behaviour of materials with different types of disorders. Analyses of the influence of mechanical and geometrical disorder led to the conclusion [9] that geometrical disorder is significantly more dominant, in comparison to mechanical disorder, affecting the band gap: increasing geometrical randomness leads to the second pass band dropping to zero, i.e. even small perturbations in the geometry make it possible to turn a pass band into a stop band.

There are different ways of introducing geometrical disorder into the material microstructure: in [9] it was added in the form of small perturbations following a normal distribution and described by mean and standard deviation[1].

The above methodology, however, is based on the assumption of a continuous probability distribution (normal in this case), which, in general, should be argued for each particular application. Continuous probability distribution may or may not represent the actual material accurately, as continuous distribution implies continuous changes in properties of components in the analysed material, which is not always the case in reality. Thus, this methodology could lead to a reduction in usability for some practical industrial applications, for example, additively manufactured composites. The latter materials may be better described using discrete probability distribution. Appreciating the differences between the two discussed above methodologies and, perhaps, limitations of the Fibonacci based approach, the latter would be referred to as quasi- or pseudo-random [2].

Although not widely used, the idea of employing Fibonacci numbers in physics and mechanics is not novel in itself: among others should be, for example, mentioned the numerical statistical work of [4] in the field of quasicrystals. Theoretical research, reported in [8] is, perhaps, closer to the current study: the authors analysed, with the help of the Fibonacci sequence, band gaps in 1D periodic multilayered structures, which are influenced by the effect of linear graded index material. In [2] the Fibonacci sequence has also been used to analyse elastic wave propagation in one dimensional solid-fluid quasi-periodic phononic crystals. A more recent study of optical properties in a one dimensional quasi-periodic graphene

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[1] The term "small perturbation" refers to the case of small coefficient of variation.

[2] It is noted here that this Fibonacci sequence based approach can be considered as a particular case of more generic pseudo-random sequence approach, where the particular randomising algorithm can be controlled by a user.

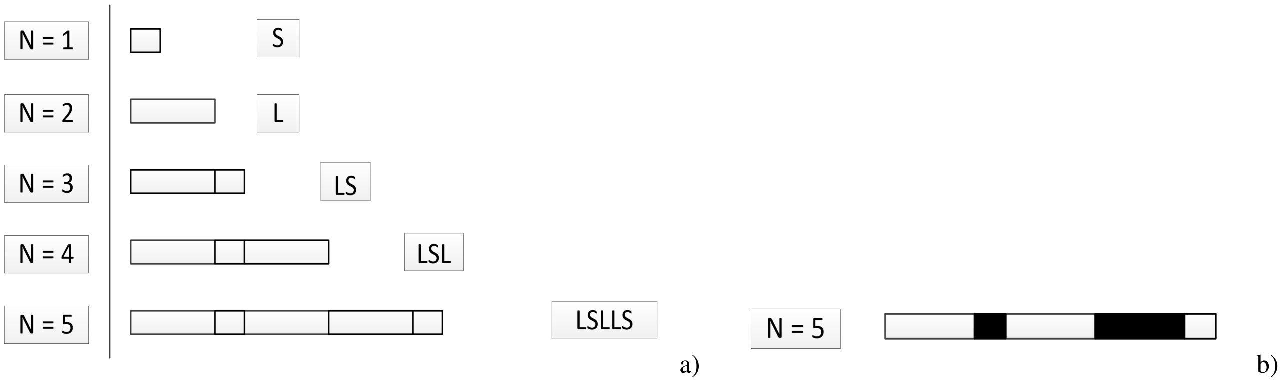

Fig. 1 Illustration of a Fibonacci bar construction a) – layers for $N = 1$ to $N = 5$; b) – material properties assigned.

photonic crystal using the Fibonacci sequence is presented in [12]. Researchers have looked into two dimensional phononic crystals as well: [11], for example, analysed the band structures of in-plane elastic waves propagating in such materials, experiencing one-dimensional random disorder and aperiodicity, while keeping the second dimension periodic.

In this paper we revisit and argue the validity and convenience of using the Fibonacci sequence in a practical mechanical context: the analysis of random heterogeneous material behaviour, subjected to propagating elastic waves. We will study the occurrence of stop bands (or wave filters) and identify the relevant parameters that influence these stop bands. We promote the use of the Fibonacci-based methodology, that not only allows to model stop bands, but also to control stop bands, and thus overall material response.

2 Fibonacci layout of material with quasi-random properties

We recall that the Fibonacci sequence can be defined as $F^N = F^{N-1} + F^{N-2}$, with $N \geq 2$, and seed values $F^1 = 1$ and $F^2 = 1$. Following a similar logic, a Fibonacci bar (F^N_{bar}) can be constructed. The Fibonacci bar is defined as a laminate material with N lamellae made of two geometrically different material components. The length of each lamella (component), for illustrative purpose, is referred to as either long (L) or short (S). These two parts become the seed parts of the Fibonacci bar:

– for $N = 1$ the Fibonacci bar (F^1_{bar}) is constructed from one short layer: $F^1_{bar} = S$;

– for $N = 2$ the Fibonacci bar is constructed by one long layer: $F^2_{bar} = L$;

– for $N = 3$ the Fibonacci bar is constructed by two parts, the first part being F^2_{bar} (L) and the second part F^1_{bar} (S) which is placed at the end of the first part: $F^3_{bar} = LS$;

– for $N >= 3$, F^N_{bar} is generated by F^{N-2}_{bar} placed at the end of F^{N-1}_{bar}, resulting in the right Fibonacci bar depicted in Figure 1–a.

Note here, that the starting point has been chosen arbitrary and similar logic can be used if starting from the long layer.

The length of each layer is determined by the Fibonacci sequence, hence, when $N = 5$, the bar is characterised as $LSLLS$ (see Figure 1–a). For large enough N the distribution of long – short layers, F^N_{bar}, can be considered random, or, referring to the note in Section 1, *quasi-random*. At the same time, the constructed Fibonacci bar is a laminate material with two different material phases, for illustrative purposes presented as black and white blocks, thus for $N = 5$, the material property of each layer is white-black-white-black-white (see Figure 1–b).

The procedure introduced above, resulting in controlled quasi-random laminate material, can be straightforwardly interpreted in terms of additive manufacturing processes.

3 Stop band prediction based on one dimensional Fibonacci bar

In order to illustrate the use of the Fibonacci sequence in practice, first a one dimensional numerical test has been carried out. For this purpose, a 220 mm bar with fixed ends has been subjected to a longitudinal elastic wave with frequency ranging from $f = 0.1 \times 10^5$Hz to 6.4×10^5Hz with intervals of 0.1×10^5Hz. A bar has been constructed from the actual heterogeneous test material (100 mm), surrounding by 10 mm homogeneous parts, representing the source point (input

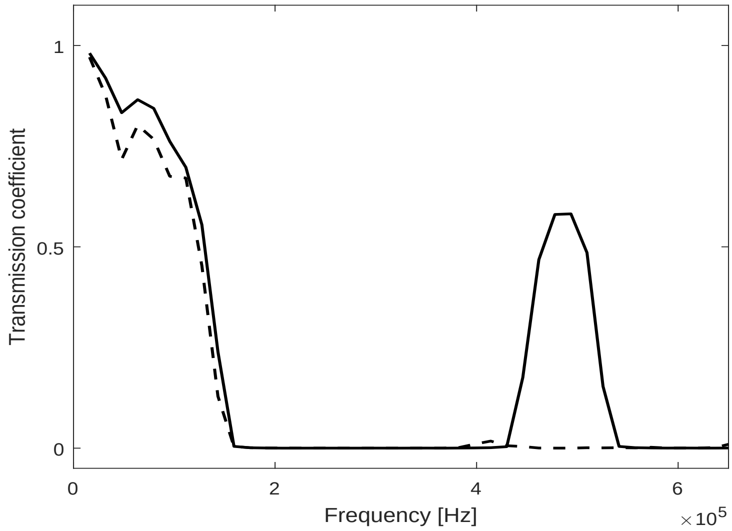

Fig. 2 Quasi-random (dashed line) compared with periodic case (solid line).

of propagating waves) and the recording point (output of the waves propagating through the heterogeneous part). The material properties in these homogeneous parts correspond to the harmonic mean of the Young's modulus and arithmetic mean of the density of heterogeneous material's properties. In order to ensure that no reflections can be generated (see [9] for details), the 50 mm perfect match layers (PML) are added on both sides of the test material with input and output parts, as alternatives to absorbing boundary conditions.

For a representative statistical analyses, 10 realisations of 100 mm long 2-phase heterogeneous material have been generated (see [3] for arguments toward the number of realisations). First a periodic material with $L = 5$ mm and $S = 5$ mm was considered. As the length of each layer (lamellae) was 5 mm, and the total length of the heterogeneous material bar was 100 mm, 20 layers in the Fibonacci bar needed to be generated. Thus, in order to accommodate 10 independent statistical realisations, F_{bar}^N with at least 200 layers needed to be constructed[3]. Choosing $N = 13$ produces a sufficient number of layers in the Fibonacci bar: F_{bar}^{13} amounts to 233 layers. Mechanical properties, Young's moduli and densities, of the material phases M_1 and M_2 are chosen as follows: $E_{M_1} = 2 \times 10^{11}$Pa, $\rho_{M_1} = 8 \times 10^3$kg/m^3, $E_{M_2} = 8 \times 10^{10}$Pa and $\rho_{M_2} = 4.4 \times 10^3$kg/m^3.

The results are analysed from the position of a transmission coefficients – the ratio of amplitudes of waves traveling through heterogeneous material $A_{het}(f)$ over amplitudes of waves traveling through homogeneous material $A_{hom}(f)$:

$$T(f) = \frac{A_{het}(f)}{A_{hom}(f)} \tag{1}$$

Amplitudes $A_{het}(f)$ and $A_{hom}(f)$ are obtained after Fourier transform of the monitored displacement, following a continuous sin wave traveling through homogeneous and heterogeneous materials correspondingly; here f is the varying frequency of the sine wave signal.

Signals have been averaged over 10 realisations of Fibonacci bar material. The transmission coefficient of quasi-random Fibonacci based bar are then compared with the transmission coefficient results of a wave propagating through a non-periodic material with the length of long and short layers taken as $L = 6$ mm and $S = 4$ mm correspondingly. This comparison is shown in Figure 2. As it can be seen, the quasi-random Fibonacci bar material can significantly reduce the value of the transmission coefficient in the second pass band compared to periodic laminate material. It should also be noted that the outcomes of different realisations have low variability (see Figure 3, where, for illustration, outcomes of 5 realisations are presented; all realisations produce qualitatively similar results, meaning that any of the realisations could serve as a potential template for an additive manufacturing process.

It is also of interest to discuss the behaviour of the transmission coefficient and its dependence on the ratio of long to short lengths of layers. This dependence can be observed in Figure 4, where two cases of different ratios $L = 6$ mm and

[3] We took here an approach of having 10 independent realisations, arranged in series in the Fibonacci bar. An alternative approach could also be taken, as to identify realisations from a shorter the Fibonacci bar by randomly picking a 100 mm samples from it. In this case further research is needed to confirm representativity of the sets of samples. This alternative approach, could potentially further optimize the Fibonacci approach.

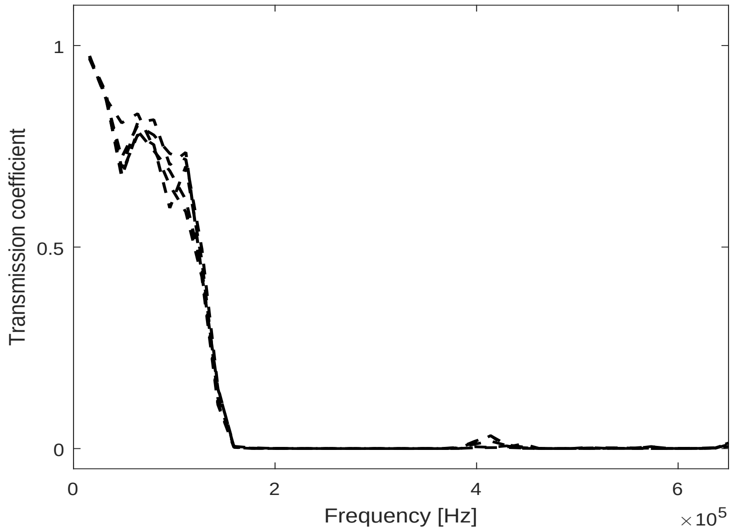

Fig. 3 Realisations of Fibonacci bar simulations

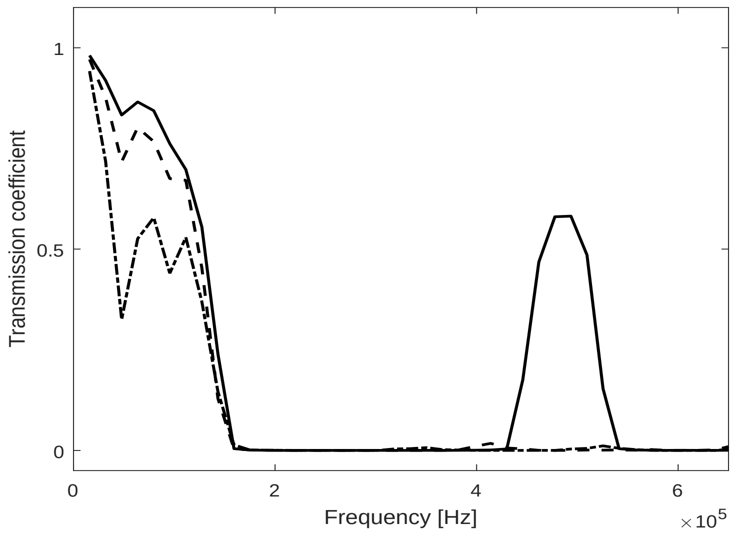

Fig. 4 Low (dashed line) and high (dashed-dotted line) contrasts Fibonacci bar, periodic case (solid line).

$S = 4$ mm (dashed line) and $L = 7.5$ mm and $S = 2.5$ mm (dashed-dotted line) were compared with the periodic material (solid line). Note here that first stop band frequencies for all periodic, large and small contrast samples are similar, and that both low and high contrast in layers length resulted in the removal of the second pass band. On the other hand it should be noted that higher contrast in Fibonacci seed values leads to much lower values of transmission coefficient in the first pass band.

4 Stop band prediction based on two dimensional Fibonacci square

Next, Fibonacci geometries are explored in 2D. Similarly to Section 3, the test geometry has again been divided into three parts: perfectly matched layers, signal input / output layers and the actual heterogeneous test material. A $20\mathrm{mm} \times 20\mathrm{mm}$ heterogeneous test material has been constructed from 400 squares ($1\mathrm{mm} \times 1\mathrm{mm}$), with two different material components, laid-out as a chess board. Similarly, again, to Section 3 a continuous longitudinal harmonic wave has been initiated as input signal. However, unlike the point source in 1D, the input (and thus output) in 2D is a line source/receiver located at the

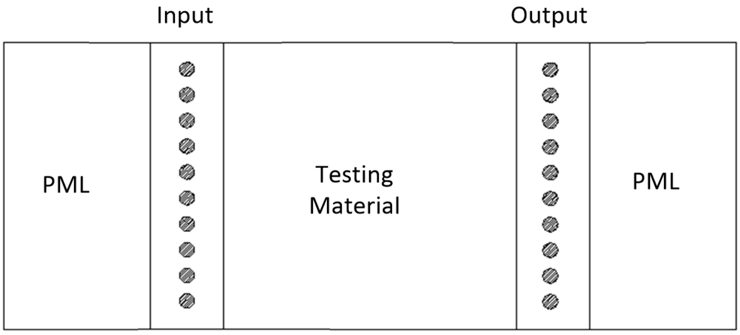

Fig. 5 Simulation model of wave propagation in 2D.

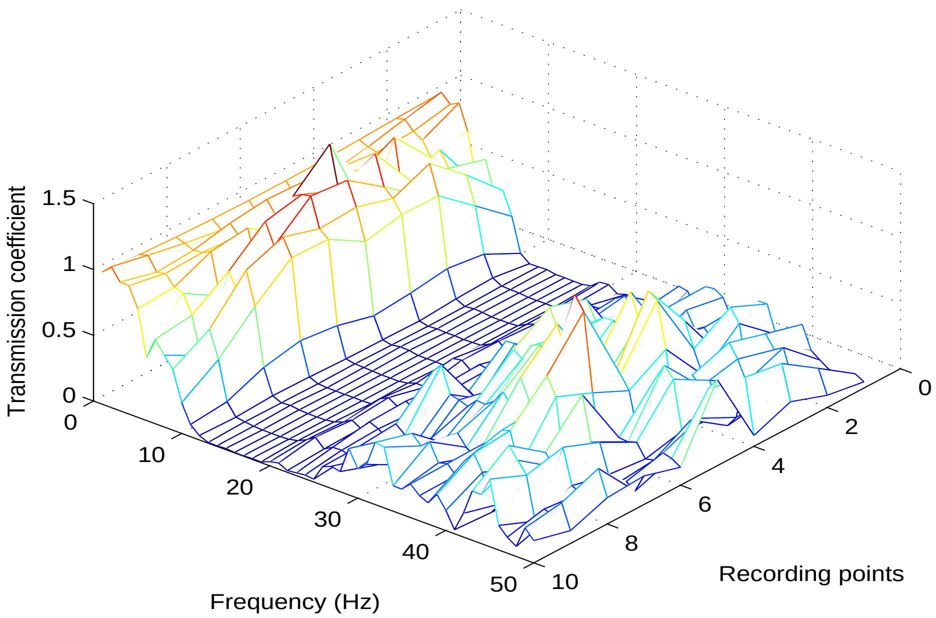

Fig. 6 Wave propagating through periodic material.

horizontal center of the input/output parts. In this line source, 10 points have been chosen to represent the displacement distribution (see Figure 5).

The following material properties for the two components M_1 and M_2 were used in the numerical simulations: $E_{M_1} = 1.6 \times 10^6$Pa, $\rho_{M_1} = 1 \times 10^9$kg/m^3, $E_{M_2} = 0.4 \times 10^6$Pa and $\rho_{M_2} = 0.1 \times 10^9$kg/m^3. Numerical tests were performed for frequencies ranging from 1Hz to 50Hz. First periodic material with $L = 1$ mm and $S = 1$ mm was studied, the transmission coefficients results of which are shown in Figure 6.

As a next step, the Fibonacci sequence has been used in order to mimic randomness in the 2D test material. Following the technique introduced in Section 2 to generate the Fibonacci bar, here the Fibonacci square was constructed. Three different orientations (directions) of randomness have been considered: material with (i) horizontally random but vertically periodic structure, (ii) vertically random but horizontally periodic structure, and (iii) horizontally and vertically random structure.

Furthermore, as it has been mentioned in Section 3, attention has also been focused on contrast between material phases lengths, resulting in two types of Fibonacci sequences: a low contrast Fibonacci sequence with $L = 1.2$mm and $S = 0.8$mm which represents the same degree of randomness as small perturbation case, discussed in [9], and a high contrast Fibonacci sequence with $L = 1.5$mm and $S = 0.5$mm which contains a higher degree of randomness. The only difference between

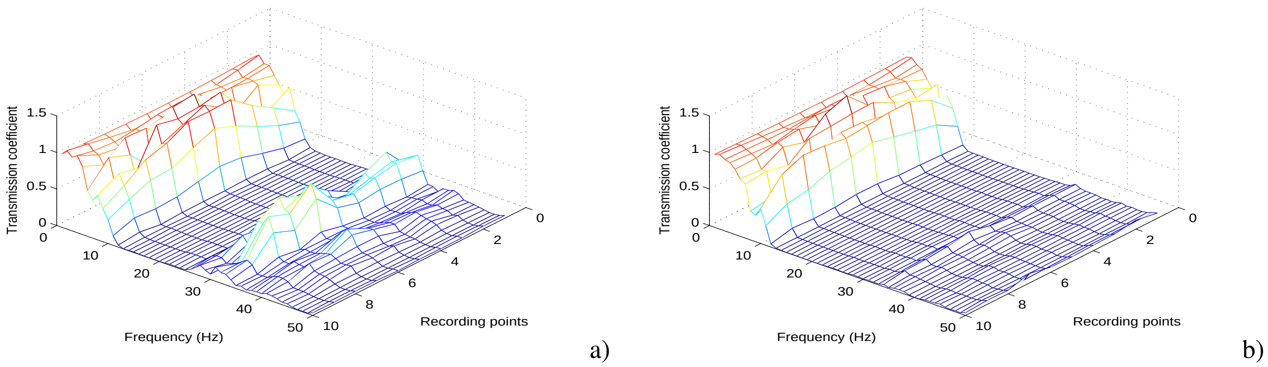

Fig. 7 Randomness added to horizontal direction a) – low contrast and b) – high contrast Fibonacci sequence.

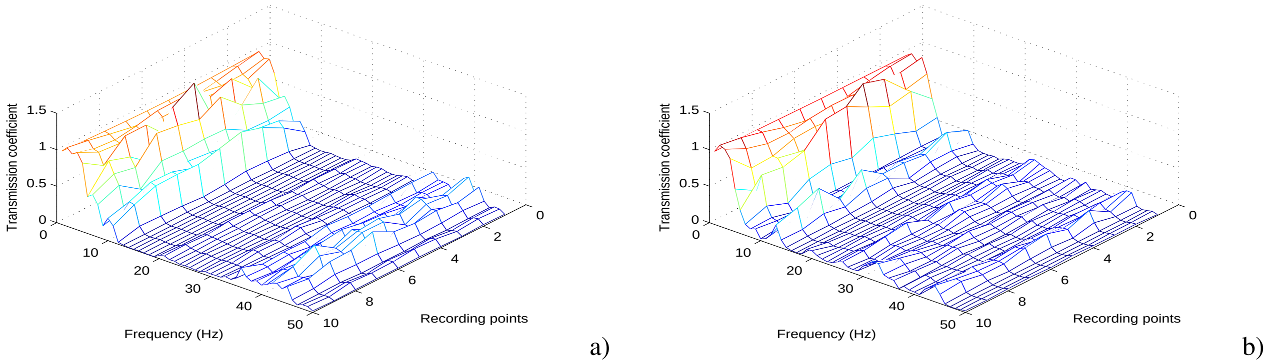

Fig. 8 Randomness added to vertical direction a) – low contrast and b) – high contrast Fibonacci sequence.

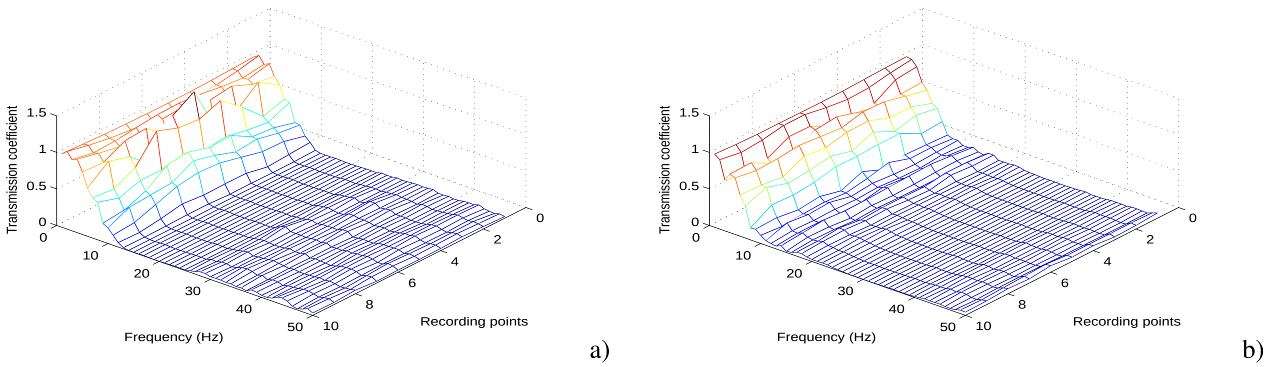

Fig. 9 Randomness added to horizontal and vertical direction a) – low contrast and b) – high contrast Fibonacci sequence.

the two types of randomness is the length of material components populating the two Fibonacci sequences. The results of the numerical tests are presented in Figures 7, 8 and 9.

As can be seen, in line with the conclusions made in Section 3, introducing Fibonacci-type quasi-randomness to the material considerably increases the frequency range of the first stop band. Moreover, sending a longitudinal harmonic wave to propagate through quasi-random heterogeneous material leads to the following observations:

: in case of low contrast Fibonacci sequence, introducing randomness in any of the orientations increases the stop band frequency range; moreover, randomness in the direction perpendicular to the wave front increases the stop band range even further, and randomness in both vertical and horizontal (along and perpendicular to the wave front) direction, results in virtual disappearance of the second pass band;

: in case of high contrast Fibonacci sequence, introducing randomness in any direction would lead to the virtual disappearance of the second pass band with the smoothest results, and thus the lowest transition coefficient, being produced in the case of material random in both directions. Transmission coefficients in the first pass band have lower values (a similar trend was observed in the 1D case as well). Generally high contrasts in material components results in larger stop band frequency range.

Thus it can be seen, that qualitatively similar results in terms of stop band could be obtained through randomness in either vertical or horizontal directions for both low and high contrast materials; on the other hand randomness introduced in both vertical and horizontal directions have more pronounced effects, as expected. Comparing Figures 7–b and 9–a, either a high contrast Fibonacci sequence with randomness added in one direction, or a low contrast Fibonacci sequence with randomness in both directions give very similar effects, thus providing two alternative but equivalent approaches to control stop bands and, thus, overall material response to the elastic wave propagation.

5 Conclusions

The main aim of this article was the exploration of the usability of the Fibonacci sequence in practical engineering applications. A two phase pseudo-random material, created using the Fibonacci sequence, was tested in an elastic wave propagation context. The response of the composite material was analysed from the position of randomness in the material's geometry and its influence on wave stop bands. It has been concluded that, first of all, introducing randomness to the material geometry considerably increases the stop band frequency range; this conclusion has confirmed the research published in [9]. Fibonacci based methodology is successful in recreating the aforementioned effect. Both low and high contrasts in the Fibonacci sequence obey the above conclusions, in particular when introduced in both directions. Different approaches to control stop bands, and thus overall material response, have been offered in the paper. Furthermore, using the Fibonacci sequence for particular engineering applications (mentioned above) will, considering the simplicity of the algorithm, increase the usability of the numerical implementation, as this methodology allows to generate a random sequence from a very compact set of recursive rules. It is straightforward to program the Fibonacci sequences, which in turn means that the findings of this paper can easily be extended to experimentation of realistic microstructures, for example using additive manufacturing.

Acknowledgements We gratefully acknowledge the Leverhulme Trust for financial support under grant number F/00 120/CC.

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