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# On the eigenvector algebra of the product of elements with commutator one in the first Weyl algebra

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**Abstract**

Let $A_1 = K\langle X, Y \,|\, [Y, X] = 1\rangle$ be the (first) Weyl algebra over a field $K$ of characteristic zero. It is known that the set of eigenvalues of the inner derivation $\mathrm{ad}(YX)$ of $A_1$ is $\mathbb{Z}$. Let $A_1 \to A_1$, $X \mapsto x$, $Y \mapsto y$, be a $K$-algebra homomorphism, i.e. $[y, x] = 1$. It is proved that the set of eigenvalues of the inner derivation $\mathrm{ad}(yx)$ of the Weyl algebra $A_1$ is $\mathbb{Z}$ and the eigenvector algebra of $\mathrm{ad}(yx)$ is $K\langle x, y\rangle$ (this would be an easy corollary of the Problem/Conjecture of Dixmier of 1968 [still open]: *is an algebra endomorphism of $A_1$ an automorphism?*).

*Key Words: the centralizer, the Weyl algebra, a locally nilpotent map (derivation), the drop of a map.*

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## 1 Introduction

The following notation is fixed throughout the paper: $A_1 = K\langle X, Y \,|\, [Y, X] = 1\rangle$ is the (first) *Weyl algebra* over a field $K$ of characteristic zero where $[a, b] := ab - ba$ is the *commutator* of elements $a$ and $b$; $H := YX$; $K^* := K\backslash\{0\}$; $C(a) := \{b \in A_1 \,|\, ab = ba\}$ is the *centralizer* of $a$ in $A_1$; $\mathrm{ad}(a) := [a, \cdot]$ is the *inner derivation* associated with an element $a$; the subalgebra $D(a)$ of $A_1$ generated by all the eigenvectors of $\mathrm{ad}(a)$ in $A_1$ is called the *eigenvector algebra* of $\mathrm{ad}(a)$ or $a$; a pair of elements $x$ and $y$ of the Weyl algebra $A_1$ such that $[y, x] = 1$ or, equivalently, a $K$-algebra endomorphism $A_1 \to A_1$, $X \mapsto x$, $Y \mapsto y$; $A_1' := K\langle x, y\rangle$ is the Weyl algebra and $\mathrm{Frac}(A_1')$ is its total skew field of fractions ($A_1' \subseteq A_1$ and $\mathrm{Frac}(A_1') \subseteq \mathrm{Frac}(A_1)$).

**The First Problem/Conjecture of Dixmier, [8]**: *is an algebra endomorphism of the Weyl algebra $A_1$ an automorphism?*

For a connection of the Conjecture of Dixmier with the Jacobian Conjecture the reader is referred to the papers of Tsuchimoto [11], Belov-Kanel and Kontsevich [7], and the author [6].

The aim of the paper is to prove the following four theorems.

**Theorem 1.1** *If $[y, x] = 1$ for some elements $x$ and $y$ of the Weyl algebra $A_1$ then $C(yx) = K[yx]$.*

**Theorem 1.2** *If $[y, x] = 1$ for some elements $x$ and $y$ of the Weyl algebra $A_1$ then eigenvector algebra $D(yx)$ is equal to the subalgebra $K\langle x, y\rangle$ of $A_1$. In particular, the set of eigenvalues of the inner derivation $\mathrm{ad}(yx)$ of the Weyl algebra $A_1$ is $\mathbb{Z}$, and, for each $i \in \mathbb{Z}$, the vector space of eigenvectors for $\mathrm{ad}(yx)$ with eigenvalue $i$ is $K[yx]v_i'$ where $v_i' := x^i$ if $i \geq 0$, and $v_i := y^{-i}$ if $< 0$.*

**Theorem 1.3** $\mathrm{Frac}(A_1') \cap A_1 = A_1'$ *where the intersection is taken in* $\mathrm{Frac}(A_1)$.

Dixmier [8] proved that each maximal commutative subalgebra $C$ of the Weyl algebra $A_1$ coincides with the centralizer $C(a)$ of every non-scalar element $a$ of $C$. Let $V$ be a vector space. A linear map $\varphi : V \to V$ is called a *locally nilpotent map* if $V = \bigcup_{n\geq 1} \ker(\varphi^n)$, i.e. for each element $v \in V$ there exists a natural number $n$ such $\varphi^n v = 0$. The linear map $\varphi$ is called a *semi-simple* linear map if $V = \bigoplus_{\lambda\in \mathrm{Ev}(\varphi)} \ker(\varphi - \lambda)$ where $\mathrm{Ev}(\varphi)$ is the set of eigenvalues of the map $\varphi$ in the field $K$. The next theorem classifies (up to isomorphism) the maximal commutative subalgebras $C$ of the Weyl algebra $A_1$ that admit either a locally nilpotent derivation or a semi-simple derivation $\delta$ (or both). In each case, the derivation $\delta$ and its eigenvalues are found.

**Theorem 1.4** *Let $C$ be a maximal commutative subalgebra of the Weyl algebra $A_1$ and $0 \neq \delta \in \mathrm{Der}_K(C)$. Then*

1. *The derivation $\delta$ is a locally nilpotent derivation of the algebra $C$ iff there exists an element $c \in C \backslash \ker(\delta)$ such that $\delta^n(c) = 0$ for some $n \geq 2$ iff $C = K[t]$ for some element $t \in C$ and $\delta = \frac{d}{dt}$.*
2. *The derivation $\delta$ is a semi-simple derivation of the algebra $C$ iff $\delta(c) = \lambda c$ for some $0 \neq c \in C$ and $\lambda \in K^*$ iff $C = \bigoplus_{\lambda \in \mathrm{Ev}(\delta)} \ker(\delta - \lambda)$, $\dim_K(\ker(\delta - \lambda)) = 1$ for all $\lambda \in \mathrm{Ev}(\delta)$, and the additive monoid $\mathrm{Ev}(\delta)$ of eigenvalues of the derivation $\delta$ is a submonoid of the additive monoid $\mathbb{N}\rho$ for some $\rho \in K^*$ such that $\mathbb{Z}\mathrm{Ev}(\delta) = \mathbb{Z}\rho$. Moreover, one of the following three cases occurs (the cases (ii) and (iii) are not mutually exclusive):*

   *(i) if $C = K[H]$ where $H := YX$ then $\delta = \rho H \frac{d}{dH}$ and $\mathrm{Ev}(\delta) = \rho\mathbb{N}$;*

   *(ii) if $0 \subsetneqq \mathrm{supp}(C) \subseteq \mathbb{N}$ then $\mathrm{Ev}(\delta) = \rho' \mathrm{supp}(C)$ for some $\rho' \in K^*$. Moreover, $\mathrm{Ev}(\delta) = \frac{\lambda}{v(c_\lambda)} \mathrm{supp}(C)$ for all $0 \neq \lambda \in \mathrm{Ev}(\delta)$ and $0 \neq c_\lambda \in \ker(\delta - \lambda)$; and $\delta$ is the unique extension of the derivation $\lambda c_\lambda \frac{d}{dc_\lambda}$ of the polynomial algebra $K[c_\lambda]$ to $C$;*

   *(iii) If $0 \subsetneqq \mathrm{supp}_-(C) \subseteq -\mathbb{N}$ then $\mathrm{Ev}(\delta) = \rho' \mathrm{supp}_-(C)$ for some $\rho' \in K^*$. Moreover, $\mathrm{Ev}(\delta) = \frac{\lambda}{v_-(c_\lambda)} \mathrm{supp}_-(C)$ for all $0 \neq \lambda \in \mathrm{Ev}(\delta)$ and $0 \neq c_\lambda \in \ker(\delta - \lambda)$; and $\delta$ is the unique extension of the derivation $\lambda c_\lambda \frac{d}{dc_\lambda}$ of the polynomial algebra $K[c_\lambda]$ to $C$;*

   *iff there exists a nonzero additive submonoid $E$ of $(\mathbb{N}, +)$ such that the algebra $C$ is isomorphic to the monoid subalgebra $\bigoplus_{i \in E} Kt^i$ of the polynomial algebra $K[t]$ in a variable $t$, $\delta = \rho t \frac{d}{dt}$ for some $\rho \in K^*$ and $\mathrm{Ev}(\delta) = \rho E$.*
3. $\ker(\delta) = K$.

*Remarks.* 1. In general, it is not true that in the cases 2(ii),(iii) the algebra $C$ is isomorphic to a polynomial algebra or is a non-singular algebra (see Example 4, Section 2).

2. Theorem 1.1 is a particular case Theorem 1.2, [10]. Here a shorter and different proof is given.

As a corollary two (short) proofs are given to the following theorem of J. A. Guccione, J. J. Guccione and C. Valqui.

**Theorem 1.5** *[9] If $[y, x] = 1$ for some elements $x$ and $y$ of the Weyl algebra $A_1$ then $C(x) = K[x]$.*

If the Problem/Conjecture of Dixmier is true then Theorems 1.1, 1.2, 1.3 and 1.5 would be its easy corollaries.

**Proof I of Theorem 1.5**. The inner derivations $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ of the Weyl algebra $A_1$ commute: $0 = \mathrm{ad}(1) = \mathrm{ad}([y, x]) = [\mathrm{ad}(y), \mathrm{ad}(x)]$. So, $\mathrm{ad}(y)$ is a non-zero derivation of the algebra $C(x) = \ker(\mathrm{ad}(x))$ such that $\mathrm{ad}(y)^2(x) = [y, [y, x]] = [y, 1] = 0$ and $x \in C(x) \backslash \ker(\mathrm{ad}(y))$ since $\mathrm{ad}(y)(x) = 1$. By Theorem 1.4.(1), $C(x) = K[t]$ for some element $t \in C(x)$ such that $[y, t] = 1$. Then $[y, t - x] = 0$, and so $t - x \in K$, by Theorem 1.4.(3) or by Theorem 1.6.(2) (since $t - x \in C(x) \cap C(y) = K$), and so $C(x) = K[x]$. $\square$

**Proof II of Theorem 1.5**. The proof follows from the following two facts:

**Theorem 1.6** 1. *(Lemma 2.1, [5]) Let $R$ be a ring $R$ and $\delta$ be a locally nilpotent derivation such that $\delta(x) = 1$ for some element $x \in R$. Then the ring $R$ is the skew polynomial ring $\ker(\delta)[x; \mathrm{ad}(x)]$.*

2. *[8] Let $a$ and $b$ be non-scalar elements of the Weyl algebra $A_1$ such that $ab \neq ba$. Then $C(a)\bigcap C(b) = K$.*

Since $[y,x] = 1 \neq 0$, $C(x)\bigcap C(y) = K$ (by the second fact) and then, by the first fact where $R = C(x)$ and $\delta = \mathrm{ad}(y)$ ($\delta$ is a locally nilpotent derivation of $R$ by Theorem 1.4 since $\delta^2(x) = 0$ and $\delta(x) = 1 \neq 0$):

$$C(x) = C(x) \cap C(y)[x; \mathrm{ad}(x) = 0] = K[x]. \quad \square$$

All the results of Sections 2 and 3 can be seen as intermediates steps in the proofs of Theorems 1.1, 1.2, 1.3 and 1.4. One of the key ideas in the proofs is to show that certain maps are locally nilpotent. The concept of the *drop* of a linear map and the linear maps of *constant drop* are an important tool in proving that certain linear maps are locally nilpotent (see Theorem 2.2). It turns out that the drop of a linear map coincides up to a negative rational multiple with the index of the map (Theorem 2.2.(2)). Theorems 1.1 and 1.5 beg to ask the following question (see also Questions 2 and 3 at the end of the paper).

*Question 1. Is it true that if $ab = ba$ for some elements $a \in A_1'$ and $b \in A_1$ then $b \in A_1'$, i.e. $C_{A_1'}(a) = C_{A_1}(a)$?*

## 2 Linear maps of constant drop, proof of Theorem 1.1

In this section proofs of Theorems 1.1 and 1.4 are given. Many results of this section are used in the proof of Theorem 1.2 which is given in Section 3.

Let a $K$-algebra $A$ be a domain (not necessarily commutative). A function $v : A\backslash\{0\} \to \mathbb{Z}$ is called a *degree function* on $A$, if for all elements $a, b \in A\backslash\{0\}$,

1. $v(ab) = v(a) + v(b)$,
2. $v(a + b) \leq \max\{v(a), v(b)\}$ where $v(0) := -\infty$, and
3. $v(\lambda) = 0$ for all $\lambda \in K^* := K\backslash\{0\}$.

The degree function $v$ on $A$ determines the ascending algebra filtration $A = \bigcup_{i\in\mathbb{Z}} A_{\leq i}$ which is called the *$v$-filtration* ($A_{\leq i}A_{\leq j} \subseteq A_{\leq i+j}$ for all $i, j \in \mathbb{Z}$) where $A_{\leq i} := \{a \in A \,|\, v(a) \leq i\}$, $\bigcap_{i\in\mathbb{Z}} A_{\leq i} = 0$ and the associated graded algebra $\mathrm{gr}_v(A) := \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}_{v,i}(A)$ is a domain where $\mathrm{gr}_{v,i}(A) := A_{\leq i}/A_{\leq i-1}$. If $v(a) \neq v(b)$ then $v(a + b) = \max\{v(a), v(b)\}$.

*Example 1.* The usual degree $\deg_x$ of a polynomial is a degree function on the polynomial algebra $K[x]$ or $K[x = x_1, x_2, \ldots, x_n]$. The *total degree* deg on $K[x_1, x_2, \ldots, x_n]$ is another example of a degree function. More generally, for each nonzero vector $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, the function $v_d : K[x_1, x_2, \ldots, x_n] \to \mathbb{Z}$,

$$v_d(\sum \lambda_{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}) := \max\{d_1\alpha_1 + \cdots + d_n\alpha_n \,|\, \lambda_{\alpha_1,\ldots,\alpha_n} \neq 0\}$$

is a degree function where $\lambda_{\alpha_1,\ldots,\alpha_n} \in K$.

*Example 2.* Let $\nu$ be a *discrete valuation* on the $K$-algebra $A$ which is a domain, i.e. $\nu : A\backslash\{0\} \to \mathbb{Z}$ is a function such that for all elements $a, b \in A\backslash\{0\}$,

1. $\nu(ab) = \nu(a) + \nu(b)$,
2. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ where $\nu(0) := \infty$, and
3. $\nu(\lambda) = 0$ for all $\lambda \in K^*$.

Then $v := -\nu$ is a degree function on $A$, and vice versa.

*Example 3.* Let a $K$-algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded domain with $K \subseteq A_0$. Then the *$\mathbb{Z}$-graded degrees* $v, v_- : A \to \mathbb{Z}$,

$$v(a) := \min\{d \in \mathbb{Z} \,|\, a \in \bigoplus_{i\leq d} A_i\}, \quad v_-(a) := -\max\{d \in \mathbb{Z} \,|\, a \in \bigoplus_{i\geq d} A_i\} \tag{1}$$

are degree functions on $A$. In particular, we have the $\mathbb{Z}$-graded degree $v$ on the Weyl algebra $A_1$ (see below). Let $C$ be a subalgebra of the algebra $A$. Then $\mathrm{supp}(C) := v(C\backslash\{0\})$ and $\mathrm{supp}_-(C) := -v_-(C\backslash\{0\})$ are additive submonoids of $(\mathbb{Z}, +)$.

Let $C$ be a $K$-subalgebra of the algebra $A$. For a $K$-linear map $\delta : C \to C$, define the map $\Delta_{\delta,v}$, which is called the *drop* of $\delta$ with respect to the degree function $v$ on the algebra $A$, by the rule

$$\Delta = \Delta_{\delta,v} : C \to \mathbb{Z}\bigcup\{-\infty\}, \;\; a \mapsto v(\delta(a)) - v(a), \;\; \Delta(0) := -\infty, \tag{2}$$

i.e. $v(\delta(a)) = v(a) + \Delta(a)$ for all elements $a \in C$ where $-\infty + z = -\infty$ for all elements $z \in \mathbb{Z}$. For all elements $a \in C$ and $\lambda \in K^*$, $\Delta(\lambda a) = \Delta(a)$. For all elements $a, b \in C$ such that either $v(a) \neq v(b)$ or $v(a) = v(b) = v(a + b)$,

$$\Delta(a + b) \leq \max\{\Delta(a), \Delta(b)\}. \tag{3}$$

For linear maps $\delta, \delta' : C \to C$ and for all elements $a \in C$,

$$\Delta_{\delta\delta',v}(a) = \Delta_{\delta,v}(\delta'(a)) + \Delta_{\delta',v}(a). \tag{4}$$

**Lemma 2.1** *Let $A$, $C$, $v$, $\delta$ and $\Delta = \Delta_{\delta,v}$ be as above. Suppose that $\delta$ is a derivation of the algebra $C$. Then*

1. *$\Delta(ab) \leq \max\{\Delta(a), \Delta(b)\}$ for all elements $a, b \in A$.*
2. *If, in addition, $C$ is a commutative algebra and* $\mathrm{char}(K) = 0$ *then $\Delta(a^n) = \Delta(a)$ for all elements $a \in C$ and $n \geq 1$.*

*Proof.* 1.

$$\begin{aligned}\delta(ab) &= v(\delta(ab)) - v(ab) = v(\delta(a)b + a\delta(b)) - v(a) - v(b) \leq \max\{v(\delta(a)b), v(a\delta(b))\} - v(a) - v(b)\\ &= \max\{v(\delta(a)) + v(b) - v(a) - v(b), v(a) + v(\delta(b)) - v(a) - v(b)\}\\ &= \max\{\Delta(a), \Delta(b)\}.\end{aligned}$$

2. $\delta(a^n) = v(\delta(a^n)) - v(a^n) = v(na^{n-1}\delta(a)) - nv(a) = (n-1)v(a) + v(\delta(a)) - nv(a) = \Delta(a)$. □

*Definition.* The linear map $\delta : C \to C$ is called a *linear map of constant drop* if $\Delta(a) = \Delta(b)$ for all elements $a, b \in C\backslash\ker(\delta)$, and the common value of $\Delta(a)$ where $a \in C\backslash\ker(\delta)$ is called the *drop* of the linear map $\delta$ denoted by $\Delta = \Delta_{\delta,v}$.

Let $V$ be a vector space over the field $K$, a linear map $\varphi : V \to V$ is called a *Fredholm map/operator* if it has finite dimensional kernel and cokernel, and

$$\mathrm{ind}(\varphi) := \dim_K(\ker(\varphi)) - \dim_K(\mathrm{coker}(\varphi))$$

is called the *index* of the map $\varphi$. Let $\mathcal{F}(V)$ be the set of all Fredholm linear maps in $V$, it is, in fact, a monoid since

$$\mathrm{ind}(\varphi\psi) = \mathrm{ind}(\varphi) + \mathrm{ind}(\psi) \;\text{ for all }\; \varphi, \psi \in \mathcal{F}(V). \tag{5}$$

The next theorem provides examples of Fredholm linear maps of constant drop and of locally nilpotent maps.

**Theorem 2.2** *Let $A$, $C$, $v$, $\delta$ and $\Delta_{\delta,v}$ be as above and $C' := C\backslash K$. Suppose that the following conditions hold.*

1. $\Delta(ab) \leq \max\{\Delta(a), \Delta(b)\}$ *for all elements* $a, b \in C$.
2. $\Delta(a^n) = \Delta(a)$ *for all elements* $a \in C$ *and* $n \geq 1$.
3. $v(c) \geq 0$ *for all elements* $c \in C$*, and* $v(C') \neq 0$.
4. *For all* $i \geq 0$, $\dim_K(C_{\leq i}/C_{\leq i-1}) \leq 1$ *where* $C_{\leq i} := C \cap A_{\leq i} = \{c \in C \,|\, v(c) \leq i\}$.
5. $\ker(\delta) = K$.

*Then*

1. $\Delta(a) = \Delta(b)$ *for all elements* $a, b \in C'$*, i.e. the map* $\delta$ *has constant drop* $\Delta = \mathrm{im}(C')$.
2. *The map* $\delta \in \mathcal{F}(C)$ *is a Fredholm map with index* $\mathrm{ind}(\delta) = -\frac{\Delta}{g}$ *and* $\dim_K(\mathrm{coker}(\delta)) = \frac{\Delta}{g} + 1$ *where* $g$ *is the unique positive integer such that* $\mathbb{Z}v(C') = \mathbb{Z}g$*. Moreover, there exists an explicit natural number* $\gamma$ *(see (11)) and a vector subspace* $V$ *of* $C_{\leq \gamma}$ *such that* $C = V \bigoplus \mathrm{im}(\delta)$*, i.e.* $V \simeq \mathrm{coker}(\delta)$.
3. *If* $\Delta(a) < 0$ *for some element* $a \in C$*, i.e.* $\Delta < 0$*, then* $\Delta = -g$ *and the map* $\delta$ *is a locally nilpotent map such that there exists a* $K$*-basis* $\{e_i\}_{i\in\mathbb{N}}$ *of the algebra* $C$ *such that, for all* $i \geq 0$, $\delta(e_i) = e_{i-1}$ *and* $C_{\leq ig} = \bigoplus_{j=0}^{i} Ke_j$ *where* $e_{-1} := 0$.
4. (Additivity of the drop) *Let* $\delta' : C \to C$ *be another linear map that satisfies conditions 1–5 then* $\Delta_{\delta\delta',v} = \Delta_{\delta,v} + \Delta_{\delta',v}$*, i.e. for all elements* $a \in C\backslash\ker(\delta\delta')$, $v(\delta\delta'(a)) = v(a) + \Delta_{\delta,v} + \Delta_{\delta',v}$.

*Proof.* 1. By condition 3, $C \cap A_{\leq -1} = 0$. Then, by condition 4, $C \cap A_{\leq 0} = K$, and so the map $v : C\backslash\{0\} \to \mathbb{N}$, $c \mapsto v(c)$, is a non-zero homomorphism of monoids (by condition 3). Let $H$ be its image. The $\mathbb{Z}$-submodule $\mathbb{Z}H$ of $\mathbb{Z}$ is equal to $\mathbb{Z}g$ where $g = \gcd\{i \,|\, i \in H\}$. It is a well-known fact (and easy to show) that

$$|g\mathbb{N}\backslash H| < \infty. \tag{6}$$

In more detail, let $g = s - t$ for some elements $s, t \in H$. Let $m$ be the least positive element of $H$. For each $k = 0, 1, \dots, mg^{-1} - 1$, the element $h_k := t(mg^{-1} - k) + ks \in H$. Since $h_k = tg^{-1}m + k(s - t) = tg^{-1}m + kg \equiv kg \mod m$ for all $k = 0, 1, \dots, mg^{-1} - 1$, it follows that $|g\mathbb{N}\backslash \bigcup_{k=0}^{mg^{-1}-1}(h_k + \mathbb{N}m)| < \infty$, and (6) follows since

$$H' := \bigcup_{k=0}^{mg^{-1}-1} (h_k + \mathbb{N}m) \subseteq H. \tag{7}$$

Claim: *there exists an integer* $l$ *such that* $\Delta(c) \leq l$ *for all* $c \in C$.

To prove this fact choose elements $a, b \in C$ such that $v(a) = m$ and $v(b) + m\mathbb{Z}$ is a generator for the finite group $\mathbb{Z}H/\mathbb{Z}m = \mathbb{Z}g/\mathbb{Z}m$. Let $g_1 = v(b)$. Then $|H\backslash \bigcup_{i=0}^{mg^{-1}-1}(ig_1 + \mathbb{N}m)| < \infty$, and so $\dim_K(C/\bigoplus_{j\in\mathbb{N}}\bigoplus_{i=0}^{mg^{-1}-1} Ka^jb^i) < \infty$, and so the algebra $C$ has a $K$-basis of the type $\{e_1, \dots, e_t, a^jb^i \,|\, j \in \mathbb{N}, i = 0, 1, \dots, mg^{-1} - 1\}$ where the degree function $v$ takes *distinct* values on the elements of the basis. By (3) and conditions 1 and 2, for all elements $c \in C'$,

$$\Delta(c) \leq l := \max\{\Delta(e_1), \dots, \Delta(e_t), \Delta(a), \Delta(b)\}.$$

Fix an element $c \in C'$ such that $\Delta(c)$ is the largest possible. Then $p := v(c) \geq 1$ since $C \cap A_{\leq 0} = K$. Suppose that there exists an element $d \in C'$ with $\Delta(d) < \Delta(c)$, we seek a contradiction. Let $q := v(d)$. Then $q \geq 1$ since $C \cap A_{\leq 0} = K$. Clearly, $v(c^q) = pq = v(d^p)$. Then, by condition

4, $c^q = \lambda d^p + e$ for some nonzero scalar $\lambda$ and an element $e \in C$ such that $v(e) < v(c^q)$. Using condition 2 we have the following strict inequalities:

$$\begin{aligned} v(\delta(c^q)) &= v(c^q) + \Delta(c^q) = v(\lambda d^p) + \Delta(c) > v(\lambda d^p) + \Delta(d) = v(\lambda d^p) + \Delta(d^p) \\ &= v(\lambda d^p) + \Delta(\lambda d^p) = v(\delta(\lambda d^p)), \\ v(\delta(c^q)) &= v(c^q) + \Delta(c^q) = v(c^q) + \Delta(c) > v(e) + \Delta(e) = v(\delta(e)). \end{aligned}$$

Now, $v(\delta(c^q)) > \max\{v(\delta(\lambda d^p)), v(\delta(e))\} \geq v(\delta(\lambda d^p) + \delta(e)) = v(\delta(\lambda d^p + e)) = v(\delta(c^q))$, a contradiction.

2. We keep the notation of the proof of statement 1. Let

$$\mu := \max\{h_k \mid k = 0, 1, \ldots, mg^{-1} - 1\}. \tag{8}$$

For all natural numbers $j \geq \mu g^{-1}$,

$$\dim_K(C_{\leq jg}) = j + 1 - \nu \text{ where } \nu := |g\mathbb{N}\backslash H|. \tag{9}$$

Notice that the number $g$ divides the drop $\Delta$ of the map $\delta$. For each natural number $j \geq \mu g^{-1}$, there is the short exact sequence of vector spaces

$$0 \to K \to C_{\leq jg} \stackrel{\delta}{\to} C_{\leq jg+\Delta} \to C_{\leq jg+\Delta}/\delta(C_{\leq jg}) \to 0,$$

and so, by (9),

$$\dim_K(C_{\leq jg+\Delta}/\delta(C_{\leq jg})) = \dim_K(C_{\leq jg+\Delta}) - \dim_K(C_{\leq jg}) + 1 = \frac{\Delta}{g} + 1. \tag{10}$$

Fix a subspace $U$ of $C$ such that $C = U \bigoplus \mathrm{im}(\delta)$, and so $U \simeq \mathrm{coker}(\delta)$. Let $U'$ be a finite dimensional subspace of $U$. Then $U' \subseteq C_{\leq jg+\Delta}$ for some $j$, hence $U' \bigoplus \delta(C_{\leq jg}) \subseteq C_{\leq jg+\Delta}$ and, by (10), $\dim_K(U') \leq \frac{\Delta}{g} + 1$. This means that $\dim_K(\mathrm{coker}(\delta)) \leq \frac{\Delta}{g} + 1$, i.e. the map $\delta$ is Fredholm since $\dim_K(\ker(\delta)) = 1$, by condition 5. Let

$$\gamma := \mu + \Delta. \tag{11}$$

Then $C_{\leq \gamma} + \delta(C_{\leq jg}) = C_{\leq jg+\Delta}$ for all $j \geq j_0 := \mu g^{-1}$ since $\delta$ is the map of constant drop $\Delta$. The ascending chain of vector spaces $\{W_j := C_{\leq \gamma} \cap \delta(C_{\leq jg})\}_{j\in\mathbb{N}}$ of the finite dimensional vector space $C_{\leq \gamma}$ stabilizers say at step $p$ and let $W = W_p$. Choose a complementary subspace, say $V$, to $W$ in $C_{\leq \gamma}$, i.e. $C_{\leq \gamma} = V \bigoplus W$. Then, for all $j \geq \max\{p, j_0\}$, $C_{\leq jg+\Delta} = V \bigoplus \delta(C_{\leq jg})$. Since $C = \bigcup_{j\in\mathbb{N}} C_{\leq jg}$, we must have $C = V \bigoplus \mathrm{im}(\delta)$, and so $\dim_K(\mathrm{coker}(\delta)) = \dim_K(V) = \dim_K(C_{jg+\Delta}/\delta(C_{\leq jg})) = \frac{\Delta}{g} + 1$, by (10). Then, $\mathrm{ind}(\delta) = -\frac{\Delta}{g}$.

3. If $\Delta := \Delta(a) < 0$ for some element $a \in C'$ then $v(\delta(c)) = v(c) + \Delta < v(c)$ for all elements $c \in C'$. By conditions 3 and 5, the linear map $\delta$ is locally nilpotent with kernel $K$. Since $g|\Delta$ and $\dim_K(\ker(\delta)) = 1$, we must have $\Delta = -g$ and $H = \mathbb{N}g$; moreover, $\delta(C_{\leq jg}) = C_{\leq (j-1)g}$ for all $j \in \mathbb{N}$ where $C_{\leq -g} := 0$. Then we can find a $K$-basis $\{e_i\}_{i\in\mathbb{N}}$ for the algebra $C$ such that $\delta(e_i) = e_{i-1}$ and $C_{\leq ig} = \bigoplus_{j=0}^{i} Ke_j$.

4. For all $a \in C\backslash\ker(\delta\delta')$, $\delta'(a) \notin \ker(\delta)$ and so $v(\delta\delta'(a)) = v(\delta'(a)) + \Delta_{\delta,v} = v(a) + \Delta_{\delta',v} + \Delta_{\delta,v}$. Statement 4 follows also at once from statement 2 and the additivity of the index.

The proof of the theorem is complete. □

Let $D$ be a ring with an automorphism $\sigma$ and a central element $a$. The **generalized Weyl algebra** $A = D(\sigma, a)$ of degree 1 is the ring generated by $D$ and two indeterminates $X$ an $Y$ subject to the defining relations [2], [3]:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, \ YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded algebra where $A_n = Dv_n = v_nD$, $v_n = X^n$ $(n > 0)$, $v_n = Y^{-n}$ $(n < 0)$, $v_0 = 1$.

Let $K[H]$ be a polynomial algebra in a variable $H$ over the field $K$, $\sigma : H \to H-1$ be the $K$-automorphism of the algebra $K[H]$ and $a = H$. The first Weyl algebra $A_1 = K < X, Y \mid YX - XY = 1 >$ is isomorphic to the generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, H), \ X \leftrightarrow X, \ Y \leftrightarrow Y, \ YX \leftrightarrow H.$$

We identify both these algebras via this isomorphism, that is $A_1 = K[H](\sigma, H) = \bigoplus_{i\in\mathbb{Z}} K[H]v_i$ and $H := YX$.

The Weyl algebra $A_1$ admits the following $K$-algebra automorphism $\theta$ and the anti-automorphism $\theta'$ (i.e. $\theta'(ab) = \theta'(b)\theta'(a)$ for all elements $a, b \in A_1$):

$$\theta : A_1 \to A_1, \quad X \mapsto Y, \quad Y \mapsto -X, \quad (H \mapsto -H+1); \tag{12}$$

$$\theta' : A_1 \to A_1, \quad X \mapsto Y, \quad Y \mapsto X, \quad (H \mapsto H). \tag{13}$$

They reverse the $\mathbb{Z}$-grading of the Weyl algebra $A_1$, i.e. $\theta(K[H]v_i) = K[H]v_{-i}$ and $\theta'(K[H]v_i) = K[H]v_{-i}$ for all elements $i \in \mathbb{Z}$.

**Proof of Theorem 1.4**. 3. Clearly, $K \subseteq \ker(\delta)$. Suppose that $\delta(z) = 0$ for some element $z \in C\backslash K$, we seek a contradiction. Recall that $C = C(z')$ for all elements $z' \in C\backslash K$ [8] and $C(z')$ is a finitely generated $K[z']$-module [1]. In particular, $C$ is a finitely generated $K[z]$-module, and so every non-zero element $c$ of $C$ is algebraic over $K[z]$, i.e. $f(c) = 0$ for some non-zero polynomial $f(t) \in K[z][t]$. We may assume that its degree in $t$ is the least possible, then $f'(c) \neq 0$ where $f' = \frac{df}{dt}$, and $0 = \delta(f(c)) = f'(c)\delta(c)$, and so $\delta(c) = 0$. This means that $\delta = 0$, a contradiction.

1. We have to prove that the three statements are equivalent: $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are trivial.

$(b) \Rightarrow (c)$: Suppose that the statement $(b)$ holds, i.e. there exists an element $c \in C\backslash\ker(\delta)$ such that $\delta^n(c) = 0$ for some $n \geq 2$. Then $0 \neq \delta^m(c) \in \ker(\delta) = K$ (statement 3) for some $m$ such that $1 \leq m < n$. Clearly, $t := \delta^{m-1}(c)/\delta^m(c) \in C$ and $\delta(t) = 1$.

*Claim: $\delta$ is a locally nilpotent derivation of the algebra $C$.*

Now, the implication $(b) \Rightarrow (c)$ follows from the Claim and the well-known fact (which is a particular case of Theorem 1.6.(1)): *Suppose that $A$ be a commutative algebra over a field $K$ of characteristic zero, $\delta \in \mathrm{Der}_K(A)$ and $\delta(t) = 1$ for some element $t \in A$. Then $A$ is a polynomial algebra* $\ker(\delta)[t]$ *in $t$ with coefficients in the kernel* $\ker(\delta)$ *of $\delta$*. In our situation, $\ker(\delta) = K$ (statement 3), hence $C = K[t]$. Then $\delta = \frac{d}{dt}$.

**Proof of the Claim**. Let $v$ be the $\mathbb{Z}$-graded degree on the Weyl algebra $A_1$. In view of existence of grading reversing automorphism $\theta$ of the Weyl algebra $A_1$ see (12), it suffices to consider only two cases: either $t \in K[H]$ or $v(t) > 0$.

Suppose that $t \in K[H]$. Since $K[H] = C(H)$, and $t \in C(H)\backslash K$, we have $C(H) = C(t)$. This means that $\delta \in \mathrm{Der}_K(K[H])$ and $1 = \delta(t) = \frac{dt}{dH}\delta(H)$. Therefore, $\frac{dt}{dH}\delta(H) \in K^*$, and so $t = \lambda H + \mu$ for some $\lambda \in K^*$ and $\mu \in K$. Now, it is obvious that $K[H] = K[t]$ and $\delta = \frac{d}{dt}$.

Let $v(t) > 0$ and $\Delta = \Delta_{\delta,v}$. Notice that $\Delta(t) = v(1) - v(t) = 0 - v(t) = -v(t) < 0$. Now, the Claim follows from the following lemma.

**Lemma 2.3** *Suppose that $v(t) > 0$. Then the conditions of Theorem 2.2 hold for $A = A_1$, $C$, $\delta$, $\Delta = \Delta_{\delta,v}$ with $\Delta(t) = -v(t) < 0$, and so $\delta$ is a locally nilpotent derivation of $C$.*

*Proof.* Condition 5 has been already established above. Since $\delta$ is a derivation of the commutative algebra $C$ over a field of characteristic zero, conditions 1 and 2 of Theorem 2.2 hold by Lemma 2.1. Notice that if non-zero elements of the Weyl algebra $A_1$ commute then so do their leading terms with respect to the $\mathbb{Z}$-grading of the Weyl algebra $A_1$. The centralizer of all the homogeneous elements of the Weyl algebra $A_1$ are found in Proposition 3.1, [4]. This description makes conditions 3 and 4 obvious. The proof of Lemma 2.3 is complete. $\square$

2. We have to prove that the four statements are equivalent: $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. The implications $(a) \Rightarrow (b)$, $(c) \Rightarrow (a)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ are trivial. It remains to show that the implication $(b) \Rightarrow (c)$ holds.

$(b) \Rightarrow (c)$ Suppose that $\delta(c) = \lambda c$ for some $0 \neq c \in C$ and $\lambda \in K^*$. In view of existence of the grading reversing automorphism $\theta$ of the Weyl algebra $A_1$ see (12), it suffices to consider only two cases: either $c \in K[H]$ or $v(c) > 0$.

Suppose that $c \in K[H]$. Since $\ker(\delta) = K$ (statement 3), we see that $c \in C \backslash K$, and so $C = C(c) = C(H) = K[H]$. The five conditions of Theorem 2.2 are satisfied for $A = C = K[H]$, $v = \deg_H$, $\delta$ and $\Delta = \Delta_{\delta,v}$ and so, for all polynomials $p \in K[H] \backslash K$, $\deg_H(\delta(p)) = \deg_H(p)$ since $\deg_H(\delta(c)) = \deg_H(\lambda c) = \deg_H(c)$. Then $\delta = (\mu H + \nu)\frac{d}{dH}$ for some scalars $\mu \in K^*$ and $\nu \in K$. Let $H' := \mu H + \nu$. Then $K[H] = K[H']$ and $\delta = H' \frac{dH'}{dH} \frac{d}{dH'} = \mu H' \frac{d}{dH'}$. Now, the statement $(c).(i)$ is obvious as $\delta(H'^i) = i\mu H'^i$ for all $i \in \mathbb{N}$.

Let $v(c) > 0$ and $\Delta = \Delta_{\delta,v}$. Notice that $\Delta(c) = v(\lambda c) - v(c) = v(c) - v(c) = 0$.

**Lemma 2.4** *Suppose that $v(c) > 0$. Then the conditions of Theorem 2.2 hold for $A = A_1$, $C$, $\delta$, $\Delta = \Delta_{\delta,v}$ with $\Delta(c) = 0$, and so the derivation $\delta$ respects the filtration $C = \bigcup_{i \in \mathbb{N}} C_{\leq i}$ where $C_{\leq i} := \{c \in C \,|\, v(c) \leq i\}$, i.e. $\delta(C_{\leq i}) \subseteq C_{\leq i}$ for all $i \in \mathbb{N}$.*

*Proof.* Repeat word for word the proof of Lemma 2.3 $\square$

By Lemma 2.4, $C_0 = K$ and $\mathrm{supp}(C) \subseteq \mathbb{N}$. By Lemma 2.4 and the fact that $\dim_K(C_{\leq i}/C_{\leq i-1}) \leq 1$ for all $i \in \mathbb{N}$, the algebra $C$ is the direct sum $C = \bigoplus_{\lambda \in \mathrm{Ev}(\delta)} C^\lambda$ where $C^\lambda := \bigcup_{n \geq 1} \ker((\delta - \lambda)^n)$ where $\mathrm{Ev}(\delta)$ is the set of eigenvalues of the derivation $\delta \in \mathrm{Der}_K(C)$ in the field $K$.

*Claim: $C^\lambda = \ker(\delta - \lambda)$ for all $\lambda \in \mathrm{Ev}(\delta)$.*

To prove the claim, for each nonzero element $c_\lambda \in C^\lambda$, we introduce the *nilpotency degree* $d_\lambda(c_\lambda)$ of $c_\lambda$ by the rule

$$d_\lambda(c_\lambda) := \min\{n \in \mathbb{N} \,|\, (\delta - \lambda)^{n+1}(c_\lambda) = 0\}.$$

For all $\lambda, \mu \in \mathrm{Ev}(\delta)$, $0 \neq c_\lambda \in C^\lambda$ and $0 \neq c_\mu \in C^\mu$,

$$d_{\lambda+\mu}(c_\lambda c_\mu) = d_\lambda(c_\lambda) + d_\mu(c_\mu). \tag{14}$$

In more detail, let $n = d_\lambda(c_\lambda)$ and $m = d_\mu(c_\mu)$. It follows from the equality: for all elements $a, b \in C$,

$$(\delta - \lambda - \mu)^i(ab) = \sum_{i=0}^{n} \binom{n}{i} (\delta - \lambda)^i(a)(\delta - \mu)^{n-i}(b) \tag{15}$$

that $(\delta - \lambda - \mu)^{n+m+1}(c_\lambda c_\mu) = 0$ and $(\delta - \lambda - \mu)^{n+m}(c_\lambda c_\mu) = \binom{n+m}{n}(\delta - \lambda)^n(c_\lambda) \cdot (\delta - \mu)^m(c_\mu) \neq 0$. This proves (14).

Recall the Theorem of Amitsur [1]: *for any element $c \in C\backslash K$, the algebra $C$ is a finitely generated free $K[c]$-module*. In particular, the Gelfand-Kirillov dimension of the algebra $C$ is 1. Suppose that $C^\lambda \neq \ker(\delta - \lambda)$ for some $\lambda \in \mathrm{Ev}(\delta)$, we seek a contradiction. By (14), we may assume that $\lambda \neq 0$. Fix nonzero elements, say $s, t \in C^\lambda$, such that $s \in \ker(\delta - \lambda)$, $t \notin \ker(\delta - \lambda)$ but $(\delta - \lambda)^2(t) = 0$. Then the elements $s$ and $t$ are algebraically independent: if $\sum \lambda_{ij} s^i t^j = 0$ for some scalars $\lambda_{ij} \in K$ not all of which are equal to zero then we may assume that, for all $i$ and $j$, $i + j = m = const$ since $s^k t^l \in C^{(k+l)\lambda}$ for all $k$ and $l$; let $p := \max\{j \,|\, \lambda_{m-j,j} \neq 0\}$; then, by (15),

$$\begin{aligned} 0 &= (\delta - m\lambda)^p(\sum_{j=0}^{p} \lambda_{m-j,j} s^{m-j} t^j) = \sum_{j=0}^{p} \lambda_{m-j,j} (\delta - (m-j)\lambda - j\lambda)^p (s^{m-j} t^j) \\ &= \sum_{j=0}^{p} \lambda_{m-j,j} \sum_{i=0}^{p} \binom{p}{i} (\delta - (m-j)\lambda)^{p-i}(s^{m-j})(\delta - j\lambda)^i(t^j) = \sum_{j=0}^{p} \lambda_{m-j,j} s^{m-j} (\delta - j\lambda)^p(t^j) \\ &= \lambda_{m-p,p} s^{m-p} (\delta - p\lambda)^p(t^p) = \lambda_{m-p,p} s^{m-p} \cdot p! \cdot ((\delta - \lambda)(t))^p \neq 0, \end{aligned}$$

a contradiction. Since the elements $s$ and $t$ of the commutative algebra $C$ are algebraically independent its Gelfand-Kirillov is at least 2, a contradiction. The proof of the Claim is complete.

Let $\lambda, \mu \in \mathrm{Ev}(\delta)\backslash\{0\}$, $0 \neq c_\lambda \in \ker(\delta - \lambda)$ and $0 \neq c_\mu \in \ker(\delta - \mu)$. By Lemma 2.4, $C_0 = K$, $\ker(\delta) = K$, and so $p := v(c_\lambda) \geq 1$, $q := v(c_\mu) \geq 1$, and there exists a scalar $\nu \in K^*$ such that $c_\mu^p = \nu c_\lambda^q + \cdots$ where the three dots denote smaller terms (i.e. of smaller $\mathbb{Z}$-graded degree than $v(\nu c_\lambda^q) = pq$). Applying the derivation $\delta$ to this equality and using Lemma 2.4, we see that $p\mu c_\mu^p = \nu q \lambda c_\lambda^q + \cdots$. Comparing the leading terms of the two equalities we obtain the equality

$$\mu = \frac{\lambda}{p} q = \frac{\lambda}{v(c_\lambda)} v(c_\mu).$$

Therefore, $\mathrm{Ev}(\delta) = \frac{\lambda}{v(c_\lambda)} \mathrm{supp}(C)$, by the Claim, and the case $(ii)$ follows. The proof of Theorem 1.4 is complete. $\square$

In general, it is not true that in cases $2(ii), (iii)$ of Theorem 1.4, the algebra $C$ is isomorphic to a polynomial algebra or is a non-singular algebra.

*Example 4.* Let $v = H(H-1)^{-1}(H-2)X \in \mathrm{Frac}(A_1)$. Then $v \notin A_1$ but $v^i \in A_1$ for all $i \geq 2$. By Proposition 3.1, [4], $C := C(v^2) = K[v] \cap A_1 = K \bigoplus \bigoplus_{i \geq 2} K v^i$. Therefore, $\mathrm{supp}(C) = \{0, 2, 3, \ldots\} \neq \mathbb{N}$. For all $i \geq 0$, $[H, v^i] = i v^i$. Therefore, the restriction $\delta$ of the inner derivation $\mathrm{ad}(H)$ to the $\mathrm{ad}(H)$-invariant algebra $C$ yields a semi-simple derivation with the set of eigenvalues $\{0, 2, 3, \ldots\}$, and the algebra $C$ is not isomorphic to a polynomial algebra. The algebra $C$ is isomorphic to the algebra $K[s,t]/(s^2 - t^3)$ of regular functions on the cusp $s^2 = t^3$. In particular, the algebra $C$ is a singular one.

For each element $a \in A_1$, the union $N(a) := N(a, A_1) := \bigcup_{i \geq 0} N(a, A_1, i)$, where $N(a, A_1, i) := \ker(\mathrm{ad}(a)^{i+1})$, is a filtered algebra ($N(a, A_1, i) N(a, A_1, j) \subseteq N(a, A_1, i+j)$ for all $i, j \geq 0$). By the very definition, the algebra $N(a)$ is the largest subalgebra of the Weyl algebra $A_1$ on which the inner derivation $\mathrm{ad}(a)$ acts locally nilpotently. Little is known about these algebras in the case when $N(a) \neq C(a)$. In particular, it is not known of whether these algebras are finitely generated or Noetherian. Though, a positive answer to Dixmier's Fourth Problem [8], which is still open, would imply that the algebras $N(a)$ are finitely generated and Noetherian. In case of *homogeneous* elements of the Weyl algebra $A_1$, a positive answer to Dixmier's Fourth Problem was given in [4]. In particular, for all homogeneous elements $a$ of $A_1$, the algebra $N(a)$ is a finitely generated and Noetherian.

**Proposition 2.5** *If* $[y, x] = 1$ *for some elements* $x, y \in A_1$ *then* $N(x, A_1) = N(y, A_1) = K\langle x, y \rangle$.

*Proof.* In view of existence of the $K$-algebra automorphism of the Weyl algebra $A_1' := K\langle x, y \rangle \to A_1'$, $x \mapsto y$, $y \mapsto -x$, it suffices to show that the algebra $N := N(y, A_1)$ is equal to $A_1'$. The inner derivation $\delta = \mathrm{ad}(y)$ is a locally nilpotent derivation of the algebra $N$ with $\delta(x) = 1$. By Theorem 1.6.(1) and Theorem 1.5, $N = \ker(\delta)[x; \mathrm{ad}(x)] = C(y)[x; \mathrm{ad}(x)] = K[y][x; \mathrm{ad}(x)] = A_1'$. $\square$

For each element $a \in A_1 \backslash K$, let $\mathrm{Ev}(a)$ be the set of eigenvalues in the field $K$ of the inner derivation $\mathrm{ad}(a)$ of the Weyl algebra $A_1$. Then $\mathrm{Ev}(a)$ is an additive submonoid of $(K, +)$ and

$$D(a) := \bigoplus_{\lambda \in \mathrm{Ev}(a)} D(a, \lambda), \quad \text{where} \quad D(a, \lambda) := \ker_{A_1}(\mathrm{ad}(a) - \lambda), \tag{16}$$

is an $\mathrm{Ev}(a)$-graded subalgebra of the Weyl algebra $A_1$, i.e.

$$D(a, \lambda) D(a, \mu) \subseteq D(a, \lambda + \mu) \quad \text{for all} \quad \lambda, \mu \in \mathrm{Ev}(a).$$

Little is known about the algebras $D(a)$ in general.

The next corollary is a first step in the proof of Theorem 1.2.

**Corollary 2.6** *Suppose that $[y,x]=1$ for some elements $x,y\in A_1$. Let $A_1':=K\langle x,y\rangle$, $\delta_x:=\mathrm{ad}(x)$, $\delta_y:=\mathrm{ad}(y)$, and $h:=yx$. Then $D(h)=A_1'$ iff $\delta_x$ is a locally nilpotent derivation of the algebra $D(h)$ iff $\delta_y$ is a locally nilpotent derivation of the algebra $D(h)$ iff $\delta_x\delta_y$ is a locally nilpotent map in $D(h)$.*

*Proof.* If $D(h)=A_1'$ then $\delta_x$ is a locally nilpotent derivation of the algebra $D(h)$. Conversely, if $\delta_x$ is a locally nilpotent derivation of the algebra $D(h)$ then $D(h)\subseteq N(x,A_1)=A_1'$, by Proposition 2.5. Therefore, $D(h)=A_1'$ since the inclusion $A_1'\subseteq D(h)$ is obvious. By symmetry, the second 'iff' is also true. The derivations $\delta_x$ and $\delta_y$ commute. So, if $D(h)=A_1'$ then the map $\delta:=\delta_x\delta_y$ is a locally nilpotent map on $D(h)=A_1'$ since $\delta_x$ and $\delta_y$ are commuting locally nilpotent derivations of the Weyl algebra $A_1'$. Conversely, if $\delta$ is a locally nilpotent map on $D(h)$ then for any element $a\in D(h)$, $0=\delta^n(a)=\delta_x^n\delta_y^n(a)$ for some natural number $n\geq 1$. Thus $\delta_y^n(a)\in N(x,A_1)=A_1'=N(y,A_1)$ (by Proposition 2.5), and so $a\in N(y,A_1)=A_1'$. This means that $D(h)=A_1'$. $\Box$

**Proposition 2.7** *Suppose that $[y,x]=1$ for some elements $x,y\in A_1$. Let $h:=yx$, $\delta_x:=\mathrm{ad}(x)$, $\delta_y:=\mathrm{ad}(y)$, $\delta:=\delta_x\delta_y$ and $A_1':=K\langle x,y\rangle$. Then*

1. *$N(\delta,A_1)=A_1'$.*
2. *$\ker_{A_1}(\delta)=K[x]+K[y]$.*
3. *$\ker_{C(h)}(\delta)=K$.*
4. *$C(h)\cap A_1'=K[h]$.*
5. *$C(h)=K[h]$ iff the map $\delta$ acts locally nilpotently on $C(h)$.*
6. *$K[h]=\bigoplus_{i\geq 0}K\frac{y^ix^i}{(i!)^2}=\bigoplus_{i\geq 0}K\frac{x^iy^i}{(i!)^2}$, $\delta((-1)^i\frac{y^ix^i}{(i!)^2})=(-1)^{i-1}\frac{y^{i-1}x^{i-1}}{((i-1)!)^2}$ and $\delta((-1)^i\frac{x^iy^i}{(i!)^2})=(-1)^{i-1}\frac{x^{i-1}y^{i-1}}{((i-1)!)^2}$ for all $i\geq 1$ where $\frac{y^ix^i}{(i!)^2}=\frac{h(h+1)\cdots(h+i-1)}{(i!)^2}$ and $\frac{x^iy^i}{(i!)^2}=\frac{(h-1)(h-2)\cdots(h-i)}{(i!)^2}$.*

*Proof.* 1. Since the derivations $\delta_x$ and $\delta_y$ commute and $N(x,A_1)=N(y,A_1)=A_1'$ (by Proposition 2.5), the inclusion $A_1'\subseteq N(\delta,A_1)$ follows. Let $a\in N(\delta,A_1)$. Then $0=\delta^n(a)=\delta_x^n\delta_y^n(a)$ for some $n\geq 1$, hence $\delta_y^n(a)\in N(\delta_x,A_1)=A_1'=N(y,A_1)$ (by Proposition 2.5), and so $a\in N(\delta_y,A_1)=A_1'$, i.e. $N(\delta,A_1)=A_1'$.

2. Clearly, $K[x]+K[y]\subseteq\ker_{A_1}(\delta)\subseteq N(\delta,A_1)=A_1'$, by statement 1. Therefore, $\ker_{A_1}(\delta)=\ker_{A_1'}(\delta)$. Since $\delta(x^iy^j)=-ijx^{i-1}y^{j-1}$ for all $i,j\geq 1$, we have the opposite inclusion $\ker_{A_1}(\delta)\subseteq K[x]+K[y]$.

3. $\ker_{C(h)}(\delta)=C(h)\cap\ker_{A_1}(\delta)=C(h)\cap(K[x]+K[y])=K$, by statement 2 and the fact that $[h,x^i]=ix^i$ and $[h,y^i]=-iy^i$ for all $i\geq 0$.

4. The Weyl algebra $A_1'=\bigoplus_{i\in\mathbb{Z}}K[h]v_i'$ is a $\mathbb{Z}$-graded algebra where

$$v_i:=\begin{cases}x^i & \text{if } i\geq 0,\\ y^{-i} & \text{if } i<0.\end{cases}$$

and $[h,u]=iu$ for all $u\in K[h]v_i'$. Therefore, $C(h)\cap A_1'=K[h]$.

5. ($\Rightarrow$) This follows from $\delta_x^2(h)=0$, $\delta_y^2(h)=0$ and $\delta_x\delta_y=\delta_y\delta_x$ (and so $\delta_x^{n+1}(h^n)=\delta_y^{n+1}(h^n)=0$ for all $n\geq 1$).

($\Leftarrow$) If $\delta$ acts locally nilpotently on $C(h)$ then $C(h)\subseteq N(\delta,A_1)=A_1'$, by statement 1, and so $C(h)=C(h)\cap A_1'=K[h]$, by statement 4.

6. Statement 6 follows at once from the following two facts: $\deg_h(y^ix^i)=\deg_h(x^iy^i)=i$ for all $i\geq 0$ and $\delta(y^ix^i)=-i^2y^{i-1}x^{i-1}$ and $\delta(x^iy^i)=-i^2x^{i-1}y^{i-1}$ for all $i\geq 1$. $\Box$

**The degree function $v_{\rho,\eta}$ on the Weyl algebra** $A_1$. The elements $\{Y^iX^i\,|\,(i,j)\in\mathbb{N}^2\}$ is a $K$-basis of the Weyl algebra $A_1$. Any pair $(\rho,\eta)$ of positive integers determines the degree function $v_{\rho,\eta}$ on the Weyl algebra $A_1$ by the rule

$$v_{\rho,\eta}(\sum_{i,j\in\mathbb{N}}a_{ij}Y^iX^j):=\max\{\rho i+\eta j\,|\,a_{ij}\in K^*\}. \quad (17)$$

Since $\text{im}(v_{\rho,\eta}) \subseteq \mathbb{N}$, the negative terms of the $v_{\rho,\eta}$-filtration are all equal to zero, i.e. $A_1 = \bigcup_{i\in\mathbb{N}} A_{1,\leq i}(\rho,\eta)$ is a positively filtered algebra where $A_{1,\leq i}(\rho,\eta) := \{a \in A_1 \,|\, v_{\rho,\eta}(a) \leq i\}$, and $A_{1,\leq 0}(\rho,\eta) = K$. It follows from the relation $[Y,X] = 1$ that, for all elements $a, b \in A_1$,

$$v_{\rho,\eta}([a,b]) \leq v_{\rho,\eta}(a) + v_{\rho,\eta}(b) - \rho - \eta, \quad (18)$$

and so the associated graded algebra $\text{gr}_{v_{\rho,\eta}}(A_1)$ is a polynomial algebra in two variables which are the images of the elements $X$ and $Y$ in $\text{gr}_{v_{\rho,\eta}}(A_1)$.

For each non-zero element $a = \sum_{i,j\in\mathbb{N}} a_{ij}Y^iX^j$ where $a_{ij} \in K$, define its *Newton polygon* $\text{NP}(a)$ as the the convex hull of the *support* $\text{supp}(a) := \{(i,j) \in \mathbb{N}^2 \,|\, a_{ij} \neq 0\}$ of the element $a$. The pair $(\rho,\eta)$ is called *$a$-generic* if

$$\#\{(i,j) \in \text{supp}(a) \,|\, \rho i + \eta j = v_{\rho,\eta}(a)\} = 1. \quad (19)$$

The set of $a$-generic pairs is a non-empty set (moreover, all but finitely many pairs $(\rho,\eta)$ are $a$-generic, eg, pairs such that the lines $\rho y + \eta x = 1$ are not parallel to the edges of the Newton polygon of the element $a$).

**Proof of Theorem 1.3**. Let $v = v_{\rho,\eta}$ where $\rho$ and $\eta$ are positive integers, $A_1'' := \text{Frac}(A_1') \cap A_1$. The algebras $A_1$, $A_1'$ and $A_1''$ are invariant under the action of the inner derivation $\delta_x := \text{ad}(x)$.

Clearly, $A_1' \subseteq A_1''$. Suppose that $A_1' \neq A_1''$, we seek a contradiction. Fix an element $a \in A_1''\backslash A_1'$ with the least possible $v(a)$. There exist nonzero elements $p, q \in A_1'$ such that $pa = q$. Let $\delta_x(\cdot) = (\cdot)'$. Then, by (18), $v(p') < v(p)$, $v(q') < v(q)$, and $q' = p'a + pa'$, and so

$$v(p)+v(a') = v(pa') = v(q'-p'a) \leq \max\{v(q'), v(p')+v(a)\} < \max\{v(q), v(p)+v(a)\} = v(p)+v(a).$$

Therefore, $v(a') < v(a)$ and $a' \in A_1''$. By the minimality of $v(a)$, $a' \in A_1' = N(\delta_x, A_1)$ (Proposition 2.5), hence $a \in N(\delta_x, A_1) = A_1'$, a contradiction. $\square$

Till the end of this section, we assume that $[y,x] = 1$ for some elements $x$ and $y$ of the Weyl algebra $A_1$ and $h := yx$. The results below are some of the key steps in the proof of Theorems 1.1 and 1.2.

Let $d := \delta_y(\cdot)x : A_1 \to A_1$, $a \mapsto [y,a]x$. Then, for all elements $a, b \in A_1$,

$$d(ab) = d(a)b + ad(b) + d(a)h^{-1}y[b,x]. \quad (20)$$

In more detail, using the fact that that $\delta_y$ is a derivation of the Weyl algebra $A_1$, we obtain the equality

$$\begin{aligned} d(ab) &= \delta_y(a)xy \cdot (xy)^{-1}bx + ad(b) = d(a)y\frac{1}{h-1}bx + ad(b) \\ &= d(a)\frac{1}{h}ybx + ad(b) = d(a)b + ad(b) + d(a)(h^{-1}ybx - b) \\ &= d(a)b + ad(b) + d(a)h^{-1}y[b,x]. \end{aligned}$$

Similarly, let $d' := \delta_x(\cdot)y : A_1 \to A_1$, $a \mapsto [x,a]y$. Then, for all elements $a, b \in A_1$,

$$d'(ab) = d'(a)b + ad'(b) + d'(a)(h-1)^{-1}x[b,y]. \quad (21)$$

In more detail, using the fact that $\delta_x$ is a derivation of the Weyl algebra $A_1$, we obtain the equality

$$\begin{aligned} d'(ab) &= \delta_x(a)yx \cdot (yx)^{-1}by + ad'(b) = d'(a)xh^{-1}by + ad'(b) \\ &= d'(a)\frac{1}{h-1}xby + ad'(b) = d'(a)b + ad'(b) + d'(a)((h-1)^{-1}xby - b) \\ &= d'(a)b + ad'(b) + d'(a)(h-1)^{-1}x[b,y]. \end{aligned}$$

Let $d \in \{\delta_y(\cdot)x, \delta_x(\cdot)y, x\delta_y(\cdot), y\delta_x(\cdot)\}$ . Then $d(C(h)) \subseteq C(h)$ since

$$d(C(h)) \subseteq D(h,1)D(h,0)D(h,-1) + D(h,-1)D(h,0)D(h,1) \subseteq D(h,1+0-1) = D(h,0) = C(h).$$

**Lemma 2.8** *Let $d := \delta_y(\cdot)x$, $v = v_{\rho,\eta}$ where $\rho$ and $\eta$ are positive integers. Then $v(d(a^n)) = v(a^{n-1}d(a))$ for all elements $a \in C(h)\backslash K$ and $n \geq 1$.*

*Proof.* Notice that $\ker_{C(h)}(d) = C(h) \cap C(y) = K$ since the elements $h$ and $y$ do not commute. Therefore, if $a \in C(h)\backslash K$ then $d(a^n) \neq 0$ for all $n \geq 1$. Let us prove by induction on $n$ that

$$d(a^n) = na^{n-1}d(a) + \cdots \quad \text{for all} \;\; a \in C(h)\backslash K, \tag{22}$$

where the three dots denote smaller terms, i.e. $v(\cdots) < v(na^{n-1}d(a))$. The claim is trivially true for $n = 1$. So, let $n > 1$. By (20),

$$d(a^n) = d(a \cdot a^{n-1}) = d(a)a^{n-1} + ad(a^{n-1}) + d(a)h^{-1}y[a^{n-1}, x].$$

We can extend the degree function $v$ from the Weyl algebra $A_1$ to its quotient ring $\mathrm{Frac}(A_1)$, the, so-called, *Weyl skew field* of fractions of $A_1$, by the rule $v(s^{-1}a) = v(a) - v(s)$. Then, by (18),

$$v(d(a)h^{-1}y[a^{n-1}, x]) \leq v(d(a)) - v(h) + v(y) + v(a^{n-1}) + v(x) - \rho - \eta = v(a^{n-1}d(a)) - \rho - \eta. \tag{23}$$

By induction on $n$, $a \cdot d(a^{n-1}) = (n-1)a \cdot a^{n-2}d(a) + \cdots = (n-1)a^{n-1}d(a) + \cdots$. Recall that the algebra $C(h)$ is commutative. In particular, $d(a)a^{n-1} = a^{n-1}d(a)$. Then

$$\begin{aligned} d(a^n) &= a^{n-1}d(a) + ((n-1)a^{n-1}d(a) + \cdots) + d(a)h^{-1}y[a^{n-1}, x] \\ &= (na^{n-1}d(a) + \cdots) + d(a)h^{-1}y[a^{n-1}, x] = na^{n-1}d(a) + \cdots \quad (\text{by } (23)). \; \square \end{aligned}$$

**Theorem 2.9** *Let $v = v_{\rho,\eta}$ where $(\rho, \eta)$ is an $h$-generic pair and let $d \in \{\delta_y(\cdot)x, \delta_x(\cdot)y, x\delta_y(\cdot), y\delta_x(\cdot)\}$ where $\rho$ and $\eta$ are positive integers. Then $\Delta_{d,v}(c) = 0$ for all elements $c \in C(h)\backslash K$. Moreover, $A = A_1$, $C = C(h)$, $v$, $\delta := d$ and $\Delta = \Delta_{\delta,v}$ satisfy the five conditions of Theorem 2.2.*

*Proof.* In view of existence of the $K$-algebra isomorphism of the Weyl algebra $A_1' := K\langle x, y\rangle \to A_1'$, $x \mapsto y$, $y \mapsto -x$ ($h = yx \mapsto -xy = -h+1$), and the $K$-algebra anti-isomorphism $A_1' \to A_1'$, $x \mapsto y$, $y \mapsto x$ ($h = yx \mapsto yx = h$), and the fact that $C(h) = C(-h+1)$, it suffices to prove the theorem only for the map $d = \delta_y(\cdot)x$. Notice that the fifth condition of Theorem 2.2 holds: $\ker_{C(h)}(d) = \ker_{C(h)}(\delta_y) = K$, by Proposition 2.7.(3) (since $K \subseteq \ker_{C(h)}(\delta_y) \subseteq \ker_{C(h)}(\delta_x\delta_y) = K$); and $d(h) = [y, h]x = yx = h$. Therefore, $\Delta(h) = 0$. Now, to finish the proof of the theorem it suffices to show that $A_1$, $C(h)$, $v$, $d$ and $\Delta = \Delta_{d,v}$ satisfy the first four conditions of Theorem 2.2.

The first condition of Theorem 2.2 follows from (18) and (20): for all elements $a, b \in C(h)$,

$$v(d(a)h^{-1}y[b, x]) \leq v(d(a)) - v(h) + v(y) + v(b) + v(x) - \rho - \eta = v(d(a)b) - \rho - \eta < v(d(a)b).$$

Then, by (20),

$$\begin{aligned} \Delta(ab) &= v(d(ab)) - v(ab) \leq \max\{v(d(a)b + d(a)h^{-1}y[b, x]), v(ad(b))\} - v(a) - v(b) \\ &= \max\{v(d(a)b) - v(a) - v(b), v(ad(b)) - v(a) - v(b) = \max\{\Delta(a), \Delta(b)\}. \end{aligned}$$

The second condition of Theorem 2.2 follows from Lemma 2.8: for all $a \in C(h)\backslash K$ and $n \geq 1$,

$$\begin{aligned} \Delta(a^n) &= v(d(a^n)) - v(a^n) = v(a^{n-1}d(a)) - nv(a) = (n-1)v(a) + v(d(a)) - nv(a) \\ &= v(d(a)) - v(a) = \Delta(a); \end{aligned}$$

for all $a \in K$ and $n \geq 1$, $\Delta(a^n) = -\infty = \Delta(a)$. The third condition of Theorem 2.2 is obvious.

Let $(i, j)$ be the only pair satisfying (19) for the element $h$. Then $h = \lambda Y^i X^j + \cdots$ for some $\lambda \in K^*$ where the three dots mean smaller terms with respect to the $v$-filtration on the Weyl algebra $A_1$. The element $l(h) := \lambda Y^i X^j$ is called the *leading term* of the element $h$ with respect to the $v$-filtration. Since

$$Y^i X^j = \begin{cases} (H+i-1)(H+i-2)\cdots(H+1)HX^{j-i} & \text{if } i \leq j, \\ (H+i-1)(H+i-2)\cdots(H+i-j)Y^{i-j} & \text{if } i > j, \end{cases}$$

the leading element $l(h)$ is a *homogeneous* element of the $\mathbb{Z}$-graded algebra $A_1$. The centralizers of all homogeneous elements of $A_1$ are found explicitly Proposition 3.1, [4]. It follows easily from this result and the obvious fact that $[h, h'] = 0$ implies $[l(h), l(h')] = 0$ that the fourth condition of Theorem 2.2 holds. $\square$

**Corollary 2.10** *Let the degree function $v$ on the Weyl algebra $A_1$ be as in Theorem 2.9, $\delta := \delta_x\delta_y$ and $\Delta = \Delta_{\delta,v}$. Then $\Delta(c) = -v(h) < 0$ for all elements $c \in C(h)\backslash K$.*

*Proof.* Let $d := \delta_y(\cdot)x$ and $d' := \delta_x(\cdot)y$. Then

$$dd' = \delta_y(\delta_x(\cdot)y)x = \delta_y\delta_x(\cdot)yx = \delta(\cdot)h. \tag{24}$$

Recall that $\ker_{C(h)}(\delta) = K$ (Proposition 2.7.(3)), $\ker_{C(h)}(d) = C(h) \cap C(y) = K$ since $hy \neq yh$, and $\ker_{C(h)}(d') = C(h) \cap C(x) = K$ since $hx \neq xh$. Therefore, if $\delta(c) \neq 0$ for some element $c \in C(h)$ then $d'(c) \notin K$. By Theorem 2.2.(4) and Theorem 2.9, for all elements $c \in C(h)\backslash K$, $v(dd'(c)) = v(c) + \Delta_{d,v} + \Delta_{d',v} = v(c) + 0 + 0 = v(c)$, and so $\Delta_{dd',v}(c) = 0$. On the other hand, for all elements $c \in C(h)\backslash K$,

$$0 = \Delta_{dd',v}(c) = \Delta_{\delta(\cdot)h,v}(c) = v(\delta(c)h) - v(c) = \Delta(c) + v(h),$$

and so $\Delta(c) = -v(h) < 0$. $\square$

**Proof of Theorem 1.1**. By Corollary 2.10, the map $\delta$ acts locally nilpotently on the algebra $C(h)$ since, for all elements $c \in C(h)\backslash K$, $v(\delta(c)) = v(c) + \Delta(c) = v(c) - v(h) < v(c)$ and $v(c') \geq 0$ for all elements $c' \in C$ and $\delta(K) = 0$. Then, $C(h) = K[h]$, by Proposition 2.7.(5). $\square$.

# 3 Proof of Theorem 1.2

The aim of this section is to give the proof of Theorem 1.2. In this section, we assume that $[y, x] = 1$ for some elements $x$ and $y$ of the Weyl algebra $A_1$ and $h := yx$. Let $A_1' := K\langle x, y\rangle$, $\delta_x := \text{ad}(x)$, $\delta_y := \text{ad}(y)$ and $\delta := \delta_x\delta_y = \delta_y\delta_x$.

**Proof of Theorem 1.2**. Notice that the Weyl algebra $A_1' := K\langle x, y\rangle = \bigoplus_{i\in\mathbb{Z}} K[h]v_i'$ is a subalgebra of $D(h)$ since, for all elements $u \in K[h]v_i'$, $[h, u] = iu$; and the set of integers $\mathbb{Z}$ is a subset of the set $\text{Ev}(h)$ of eigenvalues of the inner derivation $\delta := \text{ad}(h)$ of the Weyl algebra $A_1$. By the Theorem of Joseph ([10], Note added in proof), $\text{Ev}(h) = \mathbb{Z}\theta$ for some nonzero scalar $\theta$ necessarily of the form $\pm\frac{1}{n}$ since $1 \in \mathbb{Z}\theta$. Without loss of generality we may assume that $\theta = \frac{1}{n}$. Then

$$D(h) = \bigoplus_{i\in\mathbb{Z}} D_{i\theta} \text{ where } D_{i\theta} := \{a \in A_1 \,|\, [h, a] = i\theta a\}.$$

For all $\lambda, \mu \in \text{Ev}(h)$, $D_\lambda D_\mu \subseteq D_{\lambda+\mu}$. By Theorem 1.1, $D_0 = C(h) = K[h]$.

*Claim 1: For each $\lambda \in \text{Ev}(h)$, there exists an element $u_\lambda \in D_\lambda$ such that $D_\lambda = K[h]u_\lambda = u_\lambda K[h]$.*

If $\lambda = 0$ then take $u_0 = 1$, by Theorem 1.1. If $\lambda \neq 0$ then take any nonzero element, say $w_\lambda$, of $D_\lambda$ then the map $\cdot w_{-\lambda} : D_\lambda \to D_0 = K[h]$, $d \mapsto dw_{-\lambda}$, is a monomorphism of left $K[h]$-modules. Therefore, $D_\lambda = K[h]u_\lambda$ for some element $u_\lambda \in D_\lambda$. Similarly, the map $w_{-\lambda}\cdot : D_\lambda \to D_0 = K[h]$, $d \mapsto w_{-\lambda}d$, is a monomorphism of right $K[h]$-modules. Therefore, $D_\lambda = u_\lambda' K[h]$ for some element $u_\lambda' \in D_\lambda$. Then $u_\lambda = u_\lambda' p$ for some nonzero element $p \in K[h]$, and so

$$u_\lambda' K[h] = D_\lambda = K[h]u_\lambda = K[h]u_\lambda' p \subseteq D_\lambda p = u_\lambda' K[h]p \subseteq u_\lambda' K[h].$$

This gives the equality $K[h] = K[h]p$, and so $p \in K^*$. This implies the equality $K[h]u_\lambda = u_\lambda H[h]$.

*Claim 2: $x = \mu u_1$ and $y = \nu u_{-1}$ for some $\mu, \nu \in K^*$.*

By Claim 1, $x = u_1\mu$ and $y = \nu u_{-1}$ for some non-zero polynomials $\mu, \nu \in K[h]$. Then $h = yx = \nu u_{-1}u_1\mu$. Since the polynomial $u_{-1}u_1 \in K[h]$ is not a scalar polynomial (as the product of non-scalar elements in the Weyl algebra $A_1$ is a non-scalar element), the polynomials $\nu$ and $\mu$ must be non-zero scalars since

$$1 = \deg_h(h) = \deg_h(\lambda) + \deg_h(\mu) + \deg_h(u_{-1}u_1).$$

For every eigenvalue $\lambda \in \mathrm{Ev}(h)$, the vector space $D_\lambda$ is invariant under the linear maps $\delta_y(\cdot)x$ and $\delta_x(\cdot)y$.

**Lemma 3.1** *Let $d \in \{\delta_y(\cdot)x, \delta_x(\cdot)y\}$, $v = v_{\rho,\eta}$ where $(\rho, \eta) \in \mathbb{Z}^2$ with $\rho + \eta > 0$. Then $v(d(a^n)) = v(a^{n-1}d(a))$ and $\Delta_{d,v}(a^n) = \Delta_{d,v}(a)$ for all elements $a \in A_1\backslash\ker(d)$ and $n \geq 1$.*

*Proof.* The second equality (i.e. $\Delta_{d,v}(a^n) = \Delta_{d,v}(a)$) follows at once from the first one. In view of existence of the automorphism of the Weyl algebra $A_1' := K\langle x, y\rangle \to A_1'$, $x \mapsto y$, $y \mapsto -x$, it suffices to prove the first equality only for $\delta = \delta_y(\cdot)x$. Let us prove by induction on $n$ that

$$d(a^n) = na^{n-1}d(a) + \cdots \quad \text{for all} \;\; a \in A_1\backslash\ker(\delta_y) \tag{25}$$

where the three dots denote smaller terms, i.e. $v(\cdots) < v(na^{n-1}d(a))$. The claim is trivially true for $n = 1$. So, let $n > 2$. By (20),

$$d(a^n) = d(a \cdot a^{n-1}) = d(a)a^{n-1} + ad(a^{n-1}) + d(a)h^{-1}y[a^{n-1}, x].$$

Then, by (18),

$$v(d(a)h^{-1}y[a^{n-1}, x]) \leq v(d(a)) - v(h) + v(y) + v(a^{n-1}) + v(x) - \rho - \eta = v(a^{n-1}d(a)) - \rho - \eta. \tag{26}$$

By induction on $n$, $a \cdot d(a^{n-1}) = (n-1)a \cdot a^{n-2}d(a) + \cdots = (n-1)a^{n-1}d(a) + \cdots$. Then

$$\begin{aligned} d(a^n) &= (a^{n-1}d(a) + \cdots) + ((n-1)a^{n-1}d(a) + \cdots) + d(a)h^{-1}y[a^{n-1}, x] \\ &= (na^{n-1}d(a) + \cdots) + d(a)h^{-1}y[a^{n-1}, x] = na^{n-1}d(a) + \cdots . \quad \square \end{aligned}$$

**Lemma 3.2** *1. Each element of the basis $\{y^n \cdot y^ix^i, y^ix^i, y^ix^i \cdot x^n \,|\, n \geq 1, i \geq 0\}$ of the Weyl algebra $A_1'$ is an eigenvector for the linear map $d = \delta_y(\cdot)x$:*

$$d(y^n \cdot y^ix^i) = iy^n \cdot y^ix^i, \;\; d(y^ix^i) = iy^ix^i, \;\; d(y^ix^i \cdot x^n) = (i+n)y^ix^i \cdot x^n.$$

*2. Each element of the basis $\{x^iy^i \cdot y^n, x^iy^i, x^n \cdot x^iy^i \,|\, n \geq 1, i \geq 0\}$ of the Weyl algebra $A_1'$ is an eigenvector for the linear map $d' = \delta_x(\cdot)y$:*

$$d'(x^iy^i \cdot y^n) = -(i+n)x^iy^i \cdot y^n, \;\; d'(x^iy^i) = -ix^iy^i, \;\; d'(x^n \cdot x^iy^i) = -ix^n \cdot x^iy^i.$$

*Proof.* Straightforward. $\square$

**Corollary 3.3** *Let $d \in \{\delta_y(\cdot)x, \delta_x(\cdot)y\}$, $v = v_{\rho,\eta}$ where $(\rho, \eta) \in \mathbb{Z}^2$ with $\rho + \eta > 0$, and $\Delta = \Delta_{d,v}$. Then $v(d(a)) = v(a)$ for all $a \in A_1'\backslash\ker(d)$, i.e. $\Delta(a) = 0$ for all $a \in A_1'\backslash\ker(d)$.*

*Proof.* This follows at once from Lemma 3.2. $\square$

*Claim 3: $\theta = 1$, i.e. $n = 1$.*

Suppose that $n \neq 1$, we seek a contradiction. Fix a pair $(\rho, \eta) \in \mathbb{Z}^2$ with $\rho + \eta > 0$. Let $v = v_{\rho,\eta}$, $d \in \{\delta_y(\cdot)x, \delta_x(\cdot)y\}$, and $\Delta = \Delta_{d,v}$. Since $(u_{\frac{1}{n}})^n \in D_{n\frac{1}{n}} = D_1 = xK[h]$ (by Claims 1 and 2), there exists a nonzero polynomial $\alpha \in K[H]\backslash\{0\}$ such that $(u_{\frac{1}{n}})^n = x\alpha$. Clearly, $u_{\frac{1}{n}} \notin \ker(d)$ since $\ker(\delta_y(\cdot)x) = \ker(\delta_y) = K[y]$ and $\ker(\delta_x(\cdot)y) = \ker(\delta_x) = K[x]$, by Theorem 1.5, and $K[x] + K[y] \subseteq \bigoplus_{i\in\mathbb{Z}} D_i$. By Lemma 3.1 and Corollary 3.3, $\Delta(u_{\frac{1}{n}}) = \Delta((u_{\frac{1}{n}})^n) = \Delta(x\alpha) = 0$. Therefore, $d(u_{\frac{1}{n}}) = u_{\frac{1}{n}}\gamma_d$ for some $\gamma_d \in K^*$ since $d(D_{\frac{1}{n}}) \subseteq D_{\frac{1}{n}}$ and $D_{\frac{1}{n}} = u_{\frac{1}{n}}K[h]$ (Claim 1).

So, $d(u_{\frac{1}{n}}) = u_{\frac{1}{n}}\gamma$ and $d'(u_{\frac{1}{n}}) = u_{\frac{1}{n}}\gamma'$ for some $\gamma, \gamma' \in K^*$ where $d = \delta_y(\cdot)x$ and $d' = \delta_x(\cdot)y$. Let $\delta = \delta_x\delta_y$. Then $\delta(D_{\frac{1}{n}}) \subseteq D_{\frac{1}{n}}$, and so $\delta(u_{\frac{1}{n}}) = u_{\frac{1}{n}}\alpha$ for some $\alpha \in K[h]$. By (24),

$$u_{\frac{1}{n}}\gamma\gamma' = dd'(u_{\frac{1}{n}}) = \delta(u_{\frac{1}{n}})h = u_{\frac{1}{n}}\alpha h,$$

and so $\alpha \neq 0$ and $0 = \deg_h(\gamma\gamma') = \deg_h(\alpha h) \geq 1$, a contradiction.

*Claim 4: For all $n \geq 1$, $u_n = \lambda_n x^n$ and $u_{-n} = \lambda_{-n}y^n$ for some scalars $\lambda_n, \lambda_{-n} \in K^*$.*

In view of existence of the automorphism of the Weyl algebra $A_1' \to A_1'$, $x \mapsto y$, $y \mapsto -x$ ($h \mapsto -h+1$), it suffices to prove Claim 4 for positive $n$. By Claim 1, $x^n = u_n\alpha$ for some element $\alpha \in K[h]\backslash\{0\}$. Suppose that $\alpha \notin K^*$, we seek a contradiction. We keep the notation of the proof of Claim 3. In addition, we assume that $\rho > 0$ and $\eta > 0$. Then $d(u_n) = u_n\gamma$ and $d'(u_n) = u_n\gamma'$ for some elements $\gamma, \gamma' \in K[h]\backslash\{0\}$, by Claim 1 and since $\ker_{A_1}(d) = \ker_{A_1}(\delta_y) = K[y]$ and $\ker_{A_1}(d') = \ker_{A_1}(\delta_x) = K[x]$, by Theorem 1.5 (where $d := \delta_y(\cdot)x$ and $d' := \delta_x(\cdot)y$). Let $m := \eta n = v(x^n) = v(u_n\alpha)$. Applying the linear map $d$ to the equality $x^n = u_n\alpha$ and using (20) we obtain the equality

$$nx^n = d(x^n) = d(u_n\alpha) = d(u_n)\alpha + u_nd(\alpha) - d(u_n)h^{-1}yx(\alpha - \sigma^{-1}(\alpha)) = u_n\gamma\alpha + u_nd(\alpha) - u_n\gamma(\alpha - \sigma^{-1}(\alpha))$$

where $\sigma : h \mapsto h-1$ is an automorphism of the polynomial algebra $K[h]$. Notice that $\deg_h(\alpha - \sigma^{-1}(\alpha)) < \deg_h(\alpha)$, and so $v(u_n\gamma(\alpha - \sigma^{-1}(\alpha))) < v(u_n\gamma\alpha)$. By the assumption, $\alpha \in K[h]\backslash K$, hence $d(\alpha) \neq 0$ since $\ker(d) = \ker(\delta_y) = K[y]$ and $K[h] \cap K[y] = K$. Then, by Corollary 3.3, $v(d(\alpha)) = v(\alpha)$, and so $v(u_nd(\alpha)) = v(u_n) + v(d(\alpha)) = v(u_n) + v(\alpha) = v(u_n\alpha) = m$. Therefore,

$$m + (\rho + \eta)\deg_h(\gamma) = v(u_n\gamma\alpha) = v(u_n\gamma\alpha - u_n\gamma(\alpha - \sigma^{-1}(\alpha)) \leq \max\{v(nx^n), v(u_nd(\alpha))\} = m.$$

Therefore, $\deg_h(\gamma) = 0$ since $\rho + \eta > 0$, and so $\gamma \in K^*$.

Similarly, applying the linear map $d' := \delta_x(\cdot)y$ to the equality $x^n = u_n\alpha$ and using (21) we obtain the equality

$$\begin{aligned} 0 &= d'(u_n\alpha) = d'(u_n)\alpha + u_nd'(\alpha) + d'(u_n)(h-1)^{-1}x[\alpha, y] \\ &= u_n\gamma'\alpha + u_nd'(\alpha) - u_n\gamma'(xy)^{-1}xy(\alpha - \sigma(\alpha)) \\ &= u_n\gamma'\alpha + u_nd'(\alpha) - u_n\gamma'(\alpha - \sigma(\alpha)). \end{aligned}$$

Notice that $\deg_h(\alpha - \sigma(\alpha)) < \deg_h(\alpha)$, and so

$$v(\alpha - \sigma(\alpha)) = v(h)\deg_h(\alpha - \sigma(\alpha)) < v(h)\deg_h(\alpha) = v(\alpha),$$

hence $v(u_n\gamma'\alpha) > v(u_n\gamma'(\alpha - \sigma(\alpha)))$. By the assumption, $\alpha \in K[h]\backslash K$, hence $d'(\alpha) \neq 0$ since $\ker(d') = \ker(\delta_x) = K[x]$ and $K[h] \cap K[x] = K$. Then, by Corollary 3.3, $v(d'(\alpha)) = v(\alpha)$, and so

$$v(u_n) + v(\gamma') + v(\alpha) = v(u_n\gamma'\alpha) = v(u_nd'(\alpha)) = v(u_n) + v(d'(\alpha)) = v(u_n) + v(\alpha).$$

Therefore, $0 = v(\gamma') = v(h)\deg_h(\gamma')$, i.e. $\gamma' \in K^*$.

Recall that $\delta = \delta_x\delta_y$. Since $\delta(D_n) \subseteq D_n$ and $D_n = u_nK[h]$, there exists $\beta \in K[h]$ such that $\delta(u_n) = u_n\beta$. By (24),

$$u_n\gamma\gamma' = dd'(u_n) = \delta(u_n)h = u_n\beta h,$$

hence $\beta \neq 0$ and $0 = \deg_h(\gamma\gamma') = \deg_h(\beta h) \geq 1$, a contradiction. The proof of Claim 4 is complete. By Claims 1, 3 and 4, $D(h) = A_1'$. The proof of Theorem 1.2 is complete. $\Box$

**Proposition 3.4** *Let $d = \delta_y(\cdot)x$, $d' = \delta_x(\cdot)y$, and $a \in A_1$. If $(dd')^n(a) \in A_1'$ for some $n \geq 1$ then $a \in A_1'$.*

*Proof.* We use induction on $n$ to prove the result. Notice that $dd' = \delta(\cdot)h$ where $\delta = \delta_x\delta_y$, see (24). If $n = 1$, i.e. $\delta(a)h \in A_1'$ then $\delta(a) \in \mathrm{Frac}(A_1') \cap A_1 = A_1'$ (Theorem 1.3). Since $A_1' = N(\delta, A_1)$ (Proposition 2.7.(1)) and $\delta(a) \in A_1'$, we must have $a \in N(\delta, A_1) = A_1'$.

Suppose that $n > 1$ and the statement is true for all $n' < n$. Then $(dd')^n(a) = (dd')^{n-1}(dd'(a)) \in A_1'$, hence $dd'(a) \in A_1'$ (by the inductive hypothesis), and so $a \in A_1'$, by the case $n = 1$. $\square$

*Question 2. Let $a \in A_1'$. Is it true that $N(a, A_1') = N(a, A_1)$?*

*Question 3. Let $a \in A_1'$. Is it true that $D(a, A_1') = D(a, A_1)$?*

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