UNIVERSITY of York

This is a repository copy of *On irrationality measure of Thue-Morse constant*.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/127418/

Version: Accepted Version

Article:

Zorin, Evgeniy orcid.org/0000-0002-3092-340X and Badziahin, Dzmitry (2018) On irrationality measure of Thue-Morse constant. Mathematical Proceedings of the Cambridge Philosophical Society. pp. 1-18. ISSN 1469-8064

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

White Rose university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

arXiv:1707.06677v1 [math.NT] 20 Jul 2017

On irrationality measure of Thue-Morse constant

Dzmitry Badziahin* and Evgeniy Zorin‡

July 24, 2017

Abstract

We provide a non-trivial measure of irrationality for a class of Mahler numbers defined with infinite products which cover the Thue-Morse constant. Among the other things, our results imply a generalization to [10].

1 Introduction

Let $\xi \in \mathbb{R}$ be an irrational number. Its irrationality exponent $\mu(\xi)$ is defined to be the supremum of all μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < q^{-\mu}$$

has infinitely many rational solutions p/q. This is an important property of a real number since it shows, how close the given real number can be approximated by rational numbers in terms of their denominators. The irrationality exponent can be further refined by the following notion. Let

*Univerity of Sydney, School of Mathematics and Statistics, NSW 2006
†University of York, Department of Mathematics, York, YO10 5DD, United Kingdom.
‡Evgeniy Zorin acknowledges the support of EPSRC Grant EP/M021858/1

$\psi(q) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function which tends to zero as $q \to \infty$. Any function ψ with these properties is referred to as the *approximation function*. We say that an irrational number ξ is *ψ-well approximable* if the inequality

$$\left|\xi - \frac{p}{q}\right| < \psi(q) \tag{1}$$

has infinitely many solutions $p/q \in \mathbb{Q}$. Conversely, we say that ξ is *ψ-badly approximable* if (1) has only finitely many solutions. Finally, we say that ξ is *badly approximable* if it is c/q-badly approximable for some positive costant $c > 0$.

If a number $\xi \in \mathbb{R}$ is ψ-badly approximable, we also say that ψ is a *measure of irrationality of ξ*.

The statement $\mu(\xi) = \mu$ is equivalent to saying that for any $\epsilon > 0$, ξ is both $q^{-\mu-\epsilon}$-well approximable and $q^{-\mu+\epsilon}$-badly approximable. On the other hand, $(q^2 \log q)^{-1}$-badly approximable numbers are in general worse approached by rationals when compared to $(q^2 \log^2 q)^{-1}$-badly approximable numbers, even though that both of them have irrationality exponent equal to 2.

Remark 1. It is quite easy to verify that, for any approximation function ψ, for any $\xi \in \mathbb{R}$ and any $c \in \mathbb{Q} \setminus \{0\}$, the numbers ξ and $c\xi$ simultaneously are or are not ψ-badly approximable. Similarly, they simultaneously are or are not ψ-well approximable.

A big progress has been made recently in determining Diophantine approximation properties of so called Mahler numbers. Their definition slightly varies in the literature. In the present paper we define Mahler functions and Mahler numbers as follows. An analytic function $F(z)$ is called *Mahler function* if it satisfies the functional equation

$$\sum_{i=0}^{n} P_i(z) F(z^{d^i}) = Q(z) \tag{2}$$

where n and d are positive integers with $d \geq 2$, $P_i(z), Q(z) \in \mathbb{Q}[z]$, $i = 0, \ldots, n$ and $P_0(z)P_n(z) \neq 0$. We will only consider those Mahler functions $F(z)$ which lie in the space $\mathbb{Q}((z^{-1}))$ of Laurent series. Then, for any $\alpha \in \overline{\mathbb{Q}}$ inside the disc of convergence of $F(z)$, a real number $F(\alpha)$ is called a *Mahler number*.

One of the classical examples of Mahler numbers is the so called Thue-Morse constant which is defined as follows. Let $\mathbf{t} = (t_0, t_1, \dots) = (0, 1, 1, 0, 1, 0, 0, \dots)$ be the Thue-Morse sequence, that is the sequence $(t_n)_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, defined by the recurrence relations $t_0 = 0$ and for all $n \in \mathbb{N}_0$

$$t_{2n} = t_n,$$
$$t_{2n+1} = 1 - t_n.$$

Then, the Thue-Morse constant τ_{TM} is a real number which binary expansion is the Thue-Morse word. In other words,

$$\tau_{TM} := \sum_{k=0}^{\infty} \frac{t_k}{2^{k+1}}. \tag{3}$$

It is well known that τ_{TM} is a Mahler number. Indeed, one can check that τ_{TM} is related with the generating function

$$f_{TM}(z) := \sum_{i=0}^{\infty} (-1)^{t_i} z^{-i} \tag{4}$$

by the formula $\tau_{TM} = \frac{1}{2}(1 - \frac{1}{2} f_{TM}(2))$. At the same time, the function $f_{TM}(z)$, defined by (4), admits the following presentation [4, §13.4]:

$$f_{TM}(z) = \prod_{k=0}^{\infty} \left(1 - z^{-2^k}\right),$$

and the following functional equation holds:

$$f_{TM}(z^2) = \frac{z}{z-1} f_{TM}(z). \tag{5}$$

So it is indeed a Mahler function.

Approximation of Mahler numbers by algebraic numbers has been studied within a broad research direction on transcendence and algebraic independence of these numbers. We refer the reader to the monograph [18] for more details on this topic.

It has to be mentioned that, though some results on approximation by algebraic numbers can be specialized to results on rational approximations, most often they become rather weak. This happens because the results on approximations by algebraic numbers necessarily involve complicated constructions, which results in some loss of precision. More fundamental reason is that rational numbers enjoy significantly more regular (and much better understood) distribution in the real line when compared to the algebraic numbers.

The history of the research of approximation properties of Mahler numbers by rational numbers probably started in the beginning of 1990th with the work of Shallit and van der Poorten [19], where they considered a class of numbers that contains some Mahler numbers, including Fredholm constant $\sum_{n=0}^{\infty} 10^{-2^n}$, and they proved that all numbers from that class are badly approximable.

The next result on the subject, the authors are aware of, is due to Adamczewski and Cassaigne. In 2006, they proved [1] that every automatic number (which, according to [8, Theorem 1], is a subset of Mahler numbers) has finite irrationality exponent, or, equivalently, every automatic number is not a Liouville number. Later, this result was extended to all Mahler numbers [9]. We also mention here the result by Adamczewski and Rivoal [2], where they showed that some classes of Mahler numbers are ψ-well approximable, for various functions ψ depending on a class under consideration.

The Thue-Morse constant is one of the first Mahler numbers which irrationality exponent was computed precisely, it has been done by Bugeaud in 2011 [10]. This result served as a foundation for several other works, establishing precise values of irrationality exponents for wider and wider classes of Mahler numbers, see for example [12, 14, 21].

Bugeaud, Han, Wen and Yao [11] computed the estimates of $\mu(f(b))$ for a large class of Mahler functions $f(z)$, provided that the distribution of indices at which Hankel determinants of $f(z)$ do not vanish or, equivalently, the continued fraction of $f(z)$ is known. In many cases, these estimates lead to the precise value of $\mu(f(b))$. We will consider this result in more details in the next subsection. Later, Badziahin [5] provided a continued fraction expansion for the functions of the form

$$f(z) = \prod_{t=0}^{\infty} P(z^{-d^t})$$

where $d \in \mathbb{N}, d \geq 2$ and $P(z) \in \mathbb{Q}[z]$ with $\deg P < d$. This result, complimented with [11], allows to find sharp estimates for the values of these functions at integer points.

Despite rather extensive studies on irrationality exponents of Mahler numbers, very little is known about their sharper Diophantine approximation properties. In 2015, Badziahin and Zorin [6] proved that the Thue-Morse constant τ_{TM}, together with many other values of $f_{TM}(b)$, $b \in \mathbb{N}$, are not badly approximable. Moreover, they proved

Theorem BZ . *Thue-Morse constant* τ_{TM} *is* $\frac{C}{q^2 \log\log q}$*-well approximable, for some explicit constant* $C > 0$.

Later, in [7] they extended this result to the values $f_3(b)$, where b is from a ceratin subset of positive integers, and

$$f_3(z) := \prod_{t=0}^{\infty}(1 - z^{-3^t}).$$

Khintchine's Theorem implies that outside of a set of the Lebesgue measure zero, all real numbers are $\frac{1}{q^2 \log q}$-well approximable and $\frac{1}{q^2 \log^2 q}$-badly approximable. Of course, this metric result implies nothing for any particular real number, or countable family of real numbers. However, it sets some expectations on the Diophantine approximaiton properties of real numbers.

The result of Theorem BZ does not provide the well-approximability result for the Thue-Morse constant suggested by Khintchine's theorem, but it falls rather short to it. At the same time, the bad-approximability side, suggested by Khintchine theorem, seems to be hard to establish (or even to approach to it) in the case of Thue-Morse constant and related numbers. In this paper we prove that a subclass of Mahler numbers, containing, in particular, Thue-Morse constant, is $(q \exp(K\sqrt{\log q \log\log q}))^{-2}$-badly approximable for some constant $K > 0$, see Theorem 2 at the end of Subsection 1.1. This result is still pretty far from what is suggested by Khintchine's theorem, however it significantly improves the best result [10] available at this moment, namely, that the irrationality exponent of Thue-Morse constant equals 2.

1.1 Continued fractions of Laurent series

Consider the set $\mathbb{Q}((z^{-1}))$ of Laurent series equipped with the standard valuation which is defined as follows: for $f(z) = \sum_{k=-d}^{\infty} c_k z^{-k} \in \mathbb{Q}((z^{-1}))$, its valuation $\|f(z)\|$ is the biggest degree d of z having non-zero coefficient c_{-d}. For example, for polynomials $f(z)$ the valuation $\|f(z)\|$ coincides with their degree. It is well known that in this setting the notion of continued fraction is well defined. In other words, every $f(z) \in \mathbb{Q}((z^{-1}))$ can be written as

$$f(z) = [a_0(z), a_1(z), a_2(z), \ldots] = a_0(z) + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{1}{a_n(z)},$$

where $a_i(z)$, $i \in \mathbb{Z}_{\geq 0}$, are non-zero polynomials with rational coefficients of degree at least 1.

The continued fractions of Laurent series share most of the properties of classical ones [20]. Furthermore, in this setting we have even stronger version of Legendre theorem:

Theorem L . *Let $f(z) \in \mathbb{Q}((z^{-1}))$. Then $p(z)/q(z) \in \mathbb{Q}(z)$ in a reduced form is a convergent of $f(z)$ if and only if*

$$\left\| f(z) - \frac{p(z)}{q(z)} \right\| < -2\|q(z)\|.$$

Its proof can be found in [20]. Moreover, if $p_k(z)/q_k(z)$ is the kth convergent of $f(z)$ in its reduced form, then

$$\left\| f(z) - \frac{p_k(z)}{q_k(z)} \right\| = -\|q_k(z)\| - \|q_{k+1}(z)\|. \tag{6}$$

For a Laurent series $f(z) \in \mathbb{Q}((z^{-1}))$, consider its value $f(b)$, where $b \in \mathbb{N}$ lies within the disc of convergence of f. It is well known that the continued fraction of $f(b)$ (or indeed of any real number x) encodes, in a pretty straightforward way, approximational properties of this number. At the same time, it is a much subtler question how to read such properties of $f(b)$ from the continued fraction of $f(z)$. The problem comes from the fact that after specialization at $z = b$, partial quotients of $f(z)$ become rational, but often not integer numbers, or they may even vanish. Therefore the necessary recombination of partial quotients is often needed to construct the proper continued fraction of $f(b)$. The problem of this type has been studied in the beautiful article [19]. Despite this complication, in many cases some information on Diophantine approximaiton properties of $f(b)$ can be extracted. In particular, this is the case for Mahler numbers. Bugeaud, Han, Wen and Yao [11] provided the following result that links the continued fraction of $f(z)$ and the irrationality exponents of values $f(b), b \in \mathbb{N}$. In fact, they formulated it in terms of Hankel determinants. The present reformulation can be found in [5]:

Theorem BHWY . *Let $d \geq 2$ be an integer and $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converge inside the unit disk. Suppose that there exist integer polynomials $A(z), B(z), C(z), D(z)$ with $B(0)D(0) \neq 0$ such that*

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d). \tag{7}$$

Let $b \geq 2$ be an integer such that $B(b^{-d^n})C(b^{-d^n})D(b^{-d^n}) \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Define

$$\rho := \limsup_{k\to\infty} \frac{\deg q_{k+1}(z)}{\deg q_k(z)},$$

where $q_k(z)$ is the denominator of kth convergent to $z^{-1}f(z^{-1})$. Then $f(1/b)$ is transcendental and

$$\mu(f(1/b)) \leq (1+\rho)\min\{\rho^2, d\}.$$

The corollary of this theorem is that, as soon as

$$\limsup_{k\to\infty} \frac{\deg q_{k+1}(z)}{\deg q_k(z)} = 1, \tag{8}$$

the irrationality exponent of $f(1/b)$ equals two. Then the natural question arises: can we say anything better on the Diophantine approximation properties of $f(1/b)$ in the case if the continued fraction of $z^{-1}f(z^{-1})$ satisfies a stronger condition than (8)? In particular, what if the degrees of all partial quotients $a_k(z)$ are bounded by some absolute constant or even are all linear? Here we answer this question for a subclass of Mahler functions.

The main result of this paper is the following.

Theorem 2. *Let $d \geq 2$ be an integer and*

$$f(z) = \prod_{t=0}^{\infty} P(z^{-d^t}), \tag{9}$$

where $P(z) \in \mathbb{Z}[z]$ is a polynomial such that $P(1) = 1$ and $\deg P(z) < d$. Assume that the series $f(z)$ is badly approximable (i.e. the degrees of all partial quotients of $f(z)$ are bounded from above by an absolute constant). Then there exists a positive constant K such that for any $b \in \mathbb{Z}$, $|b| \geq 2$, we have either $f(b) = 0$ or $f(b)$ is $q^{-2}\exp(-K\sqrt{\log q \log\log q})$-badly approximable.

2 Preliminary information on series $f(z)$.

In the further discussion, we consider series $f(z)$ which satisfies all the conditions of Theorem 2. Most of these conditions are straightforward to verify, the only non-evident point is to check whether the product function $f(z)$, defined by (9), is badly approximable. To address this, one can find a nice criteria in [5, Proposition 1]: $f(z)$ is badly approximable if and only if all its partial quotients are linear. This in turn is equivalent to the claim that the degree of denominator of the kth convergent of $f(z)$ is precisely k, for all $k \in \mathbb{N}$.

As shown in [5], it is easier to compute the continued fraction of a slightly modified series

$$g(z) = z^{-1} f(z). \tag{10}$$

Since Diophantine approximtion properties of numbers $f(b)$ and $g(b) = f(b)/b$ essentially coincide, for any $b \in \mathbb{N}$, we will further focus on the work with the function $g(z)$. As we assume that $f(z)$ is a badly approximable function, the function $g(z)$ defined by (10) is also badly approximable. In what follows, we will denote by $p_k(z)/q_k(z)$ the kth convergent of $g(z)$, and then, by [5, Proposition 1], we infer that $\deg q_k(z) = k$.

Write down the polynomial $P(z)$ in the form

$$P(z) = 1 + u_1 z + \ldots + u_{d-1} z^{d-1}.$$

Then $P(z)$ is defined by the vector $\mathbf{u} = (u_1, \ldots, u_{d-1}) \in \mathbb{Z}^{d-1}$ and, via (9) and (10), so is $g(z)$. To emphasize this fact, we will often write $g(z)$ as $g_{\mathbf{u}}(z)$.

2.1 Coefficients of the series, convergents and Hankel determinants

We write the Laurent series $g_{\mathbf{u}}(z) \in \mathbb{Z}[[z^{-1}]]$ in the following form

$$g_{\mathbf{u}}(z) = \sum_{n=1}^{\infty} c_n z^{-n}. \tag{11}$$

We denote by $\mathbf{c}_n$ the vector $(c_1, c_2, \ldots, c_n)$. Naturally, the definition of $g_{\mathbf{u}}(z)$ via the infinite product (see (9) and (10)) imposes the upper bound on $|c_n|$, $n \in \mathbb{N}$.

Lemma 3. *The term c_n satisfies*

$$|c_n| \le \|\mathbf{u}\|_\infty^{\lceil \log_d n \rceil} \le \|\mathbf{u}\|_\infty^{\log_d n+1}. \tag{12}$$

Consequently,

$$\|\mathbf{c}_n\|_\infty \le \|\mathbf{u}\|_\infty^{\log_d n+1} \tag{13}$$

Proof. Look at two different formulae for $g_{\mathbf{u}}(z)$:

$$g_{\mathbf{u}}(z) = z^{-1}\prod_{t=0}^{\infty}(1 + u_1 z^{-d^t} + \ldots + u_{d-1} z^{-(d-1)d^t}) = \sum_{n=1}^{\infty} c_n z^{-n}.$$

By comparing the right and the left hand sides one can notice that c_n can be computed as follows:

$$c_n = \prod_{j=0}^{l(n)} u_{d_{n,j}} \tag{14}$$

where $d_{n,0} d_{n,1} \cdots d_{n,l(n)}$ is the d-ary expansion of the number $n-1$. Here we formally define $u_0 = 1$. Equation (14) readily implies that $|c_n| \le \|\mathbf{u}\|^{l(n)}$. Finally, $l(n)$ is estimated by $l(n) \le \lceil \log_d(n-1) \rceil \le \lceil \log_d n \rceil$. The last two inequalities clearly imply (12), hence (13). □

Let $p_k(z)/q_k(z)$ be a convergent of $g_{\mathbf{u}}(z)$ in its reduced form. Recall that throughout the text we assume that $f(z)$ is badly approximable, hence $g_{\mathbf{u}}(z)$ defined by (10) is badly approximable, and because of this (and employing [5, Proposition 1]) we have

$$\deg q_k = k. \tag{15}$$

Denote

$$q_k(z) = a_{k,0} + a_{k,1}z + \ldots + a_{k,k}z^k, \qquad \mathbf{a}_k := (a_{k,0}, \ldots, a_{k,k})$$
$$p_k(z) = b_{k,0} + b_{k,1}z + \ldots + b_{k,k-1}z^{k-1}, \quad \mathbf{b}_k := (b_{k,0}, \ldots, b_{k,k-1}). \tag{16}$$

Because of (15), we have

$$a_{k,k} \neq 0. \tag{17}$$

The Hankel matrix is defined as follows:

$$H_k = H_k(g_{\mathbf{u}}) = \begin{pmatrix} c_1 & c_2 & \ldots & c_k \\ c_2 & c_3 & \ldots & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & c_{k+l} & \ldots & c_{2k-1} \end{pmatrix}.$$

It is known (see, for example, [5, Section 3]) that the convergent in its reduced form with $\deg q_k(z) = k$ exists if and only if the Hankel matrix H_k is invertible. Thus in our case we necessarily have that $H_k(g_{\mathbf{u}})$ is invertible for any positive integer k.

From (6), we have that

$$\|q_k(z)g_{\mathbf{u}}(z) - p_k(z)\| = -k - 1. \tag{18}$$

In other words, the coefficients for $x^{-1}, \ldots, x^{-k}$ in $q_k(z)g_{\mathbf{u}}(z)$ are all zero and the coefficient for x^{-k-1} is not. This suggests a method for computing $q_k(x)$. One can check that the vector $\mathbf{a}_k = (a_{k,0}, a_{k,1}, \ldots, a_{k,k})$ is the solution of the matrix equation $H_{k+1}\mathbf{a}_k = c \cdot \mathbf{e}_{k+1}$, where c is a non-zero constant and

$$\mathbf{e}_{k+1} = (0, \ldots, 0, 1)^t.$$

This equation has the unique solution since the matrix H_{k+1} is invertible. So, we can write the solution vector $\mathbf{a}$ as

$$\mathbf{a}_k = c \cdot H_{k+1}^{-1}\mathbf{e}_{k+1}. \tag{19}$$

In what follows, we will use the norm of the matrix $\|H\|_\infty$, defined to be the maximum of the absolute values of all its entries. Given a polynomial $P(z)$ we define its height $h(P)$ as the maximum of absolute values of its coefficients. In particular, we have $h(p_k(z)) = \|\mathbf{b}_k\|_\infty$ and $h(q_k(z)) = \|\mathbf{a}_k\|_\infty$.

Lemma 4. *For any $k \in \mathbb{N}$, the k-th convergent $p_k(z)/q_k(z)$ to $g_{\mathbf{u}}(z)$ can be represented by $p_k(z)/q_k(z) = \widetilde{p}_k(z)/\widetilde{q}_k(z)$, where $\widetilde{p}_k, \widetilde{q}_k \in \mathbb{Z}[z]$ and*

$$h(\widetilde{q}_k) \leq (\|\mathbf{c}_{2k+1}\|_\infty^2 \cdot k)^{k/2}. \tag{20}$$
$$h(\widetilde{p}_k) \leq \|\mathbf{c}_{2k+1}\|_\infty^{k+1} \cdot k^{(k+2)/2}. \tag{21}$$

Consecutively, the following upper bounds hold true:

$$h(\widetilde{q}_k) \leq \|\mathbf{u}\|_\infty^{k(\log_d(2k+1)+1)} \cdot k^{k/2}. \tag{22}$$
$$h(\widetilde{p}_k) \leq \|\mathbf{u}\|_\infty^{(k+1)(\log_d(2k+1)+1)} \cdot k^{(k+2)/2}. \tag{23}$$

Proof. By applying Cramer's rule to the equation $H_{k+1}\mathbf{a}_k = c \cdot \mathbf{e}_{k+1}$ we infer that

$$a_{k,i} = c \cdot \frac{\Delta_{k+1,i}}{\det H_{k+1}}, \quad i = 0, \ldots, k, \tag{24}$$

where $\Delta_{k+1,i}$ denotes the determinant of the matrix H_{k+1} with the i-th column replaced by $\mathbf{e}_{k+1}$, $i = 1, \ldots, k+1$. Then we use the Hadamard's determinant upper bound to derive

$$|\det H_{k+1}| \leq \|H_{k+1}\|_\infty^{k+1} \cdot (k+1)^{(k+1)/2} = (\|\mathbf{c}_{2k+1}\|_\infty^2 (k+1))^{(k+1)/2}. \tag{25}$$

Moreover, by expanding the matrix involved in $\Delta_{k+1,i}$ along the ith column and by using Hadamard's upper bound again we get

$$|\Delta_{k+1,i}| \leq \|H_{k+1}\|_\infty^k \cdot k^{k/2} = (\|\mathbf{c}_{2k+1}\|_\infty^2 \cdot k)^{k/2}, \quad i = 0, \ldots, k.$$

To define $\widetilde{q}_k(z)$, set $c = \det H_{k+1}$ in (24). Then we readily have $\widetilde{q}_k(z) = \sum_{i=0}^k \Delta_{k+1,i+1} z^i$. By construction, it has integer coefficients and $h(\widetilde{q})$ satisfies (20).

Next, from (18) we get that the coefficients of $\widetilde{p}_k(z)$ coincide with the coefficients for positive powers of z of $\widetilde{q}_k(z) g_{\mathbf{u}}(z)$. By expanding the latter product, we get

$$|b_{k,i}| = \left| \sum_{j=i+1}^{k} a_{k,j} c_{j-i-1} \right| \leq \|\mathbf{c}_{2k+1}\|_\infty^{k+1} \cdot k^{(k+2)/2}.$$

Hence (21) is also satisfied.

The upper bounds (22) and (23) follow from (20) and (21) respectively by applying Lemma 3. □

Notation 5. For the sake of convenience, further in this text we will assume that all the convergents to $g_{\mathbf{u}}(z)$ are in the form described in Lemma 4. That is, we will always assume that $p_k(z)$ and $q_k(z)$ have integer coefficients and verify the upper bounds (20) and (21), as well as (22) and (23).

For any $k \in \mathbb{N}$ we define a suite of coefficients $(\alpha_{k,i})_{i>k}$ by

$$q_k(z)g_{\mathbf{u}}(z) - p_k(z) =: \sum_{i=k+1}^{\infty} \alpha_{k,i} z^{-i}. \tag{26}$$

Note that from the equation $H_{k+1}\mathbf{a}_k = c \cdot \mathbf{e}_{k+1}$ we can get that $\alpha_{k,k+1} = c = \det H_{k+1}$. In particular, it is a non-zero integer.

Lemma 6. *For any $i, k \in \mathbb{N}$, $i > k \geq 1$, we have*

$$\begin{aligned} |\alpha_{k,i}| &\leq (k+1)\|\mathbf{c}_{k+i}\|_\infty (\|\mathbf{c}_{2k+1}\|_\infty^2 \cdot k)^{k/2} \\ &\leq (k+1)\|\mathbf{u}\|_\infty^{\log_d(k+i)+1} \|\mathbf{u}\|_\infty^{k(\log_d(2k+1)+1)} \cdot k^{k/2}. \end{aligned} \tag{27}$$

Proof. One can check that $\alpha_{k,i}$ is defined by the formula $\alpha_{k,i} = \sum_{j=0}^{k} a_{k,j} c_{j+i}$, which in view of (20) from Lemma 4 implies the first inequality in (27). Then, the second inequality in (27) follows by applying Lemma 3. □

2.2 Using functional equation to study Diophantine approximaiton properties.

From (9) one can easily get a functional equation for $g_{\mathbf{u}}(z) = z^{-1} f(z)$:

$$g_{\mathbf{u}}(z) = \frac{g_{\mathbf{u}}(z^d)}{P^*(z)}, \quad P^*(z) = z^{d-1}P(z^{-1}) = z^{d-1} + u_1 z^{d-2} + \ldots + u_{d-1}. \tag{28}$$

This equation allows us, starting from the convergent $p_k(z)/q_k(z)$ to $g_{\mathbf{u}}(z)$, to construct an infinite sequence of convergents $\left(p_{k,m}(z)/q_{k,m}(z)\right)_{m \in \mathbb{N}_0}$ to $g_{\mathbf{u}}(z)$ by

$$\begin{aligned} q_{k,m}(z) &:= q_k(z^{d^m}), \\ p_{k,m}(z) &:= \prod_{t=0}^{m-1} P^*(z^{d^t}) p_k(z^{d^m}). \end{aligned} \tag{29}$$

This fact can be checked by substituting the functional equation (28) into the condition of Theorem L. The reader can also compare with [5, Lemma 3].

By employing (28) and (26), we find

$$q_{k,m}(z)g_{\mathbf{u}}(z)-p_{k,m}(z)=\prod_{t=0}^{m-1}P^*(z^{d^t})\cdot\sum_{i=k+1}^{\infty}\alpha_{k,i}z^{-d^m\cdot i}. \tag{30}$$

Consider an integer value b which satisfies the conditions of Theorem 2. Define[1]

$$p_{k,m} := p_{k,m}(b), \tag{31}$$

$$q_{k,m} := q_{k,m}(b), \tag{32}$$

where $p_{k,m}(z)$ and $q_{k,m}(z)$ are polynomials defined by (29).

Clearly, for any $k\in\mathbb{N}$, $m\in\mathbb{N}_0$ we have $p_{k,m},q_{k,m}\in\mathbb{Z}$.

Lemma 7. *Let $b,k,m\in\mathbb{N}$, $b\geq 2$. Assume*

$$b^{d^m}>2^{1+\log_d\|\mathbf{u}\|_\infty}. \tag{33}$$

Then the integers $p_{k,m}$ and $q_{k,m}$ verify

$$\left|g_{\mathbf{u}}(b)-\frac{p_{k,m}}{q_{k,m}}\right|\leq\frac{2(k+1)k^{k/2}d^m\|\mathbf{u}\|_\infty^{m+(k+1)(\log_d(2k+1)+1)}}{q_{k,m}\cdot b^{d^m\cdot k+1}}. \tag{34}$$

Moreover, if we make in addition a stronger assumption

$$b^{d^m}\geq 4(k+1)k^{k/2}\|\mathbf{u}\|_\infty^{(k+1)(\log_d(2k+1)+1)}, \tag{35}$$

then

$$\frac{|g_{\mathbf{u}}(b)|}{4q_{k,m}\cdot b^{d^m\cdot k+1}}\leq\left|g_{\mathbf{u}}(b)-\frac{p_{k,m}}{q_{k,m}}\right|. \tag{36}$$

[1]There is a slight abuse of notation in using the same letters $p_{k,m}$ and $q_{k,m}$ both for polynomials from $\mathbb{Z}[z]$ and for their values at $z=b$. However, we believe that in this particular case such a notation constitutes the best choice. Indeed, the main reason to consider polynomials $p_{k,m}(z)$ and $q_{k,m}(z)$ is to define eventually $p_{k,m}=p_{k,m}(b)$ and $q_{k,m}=q_{k,m}(b)$, which will play the key role in the further proofs. At the same time, it is easy to distinguish the polynomials $p_{k,m}(z)$, $q_{k,m}(z)$ and the corresponding integers $p_{k,m}$ and $q_{k,m}$ by the context. Moreover, we will always specify which object we mean and always refer to the polynomials specifying explicitly the variable, that is $p_{k,m}(z)$, $q_{k,m}(z)$ and not $p_{k,m}$ and $q_{k,m}$.

Proof. Consider Equation (30) with substituted $z := b$:

$$q_{k,m}g_{\mathbf{u}}(b) - p_{k,m} = \prod_{t=0}^{m-1} P^*(b^{d^t}) \cdot \sum_{i=k+1}^{\infty} \alpha_{k,i} b^{-d^m \cdot i}. \tag{37}$$

Each of the factors in $\left|P^*(b^{d^t})\right|$ in the right hand side of (37) can be upper bounded by $d \cdot \|\mathbf{u}\|_\infty b^{d^t(d-1)}$. So, the product in the right hand side of (37) can be estimated by

$$\left|\prod_{t=0}^{m-1} P^*(b^{d^t})\right| \leq d^m \|\mathbf{u}\|_\infty^m \cdot b^{d^m-1}. \tag{38}$$

Further, we estimate the second term on the right hand side of (37) by employing Lemma 6:

$$\left|\sum_{i=k+1}^{\infty} \alpha_{k,i} b^{-d^m \cdot i}\right| \leq (k+1)\|\mathbf{u}\|_\infty^{k(\log_d(2k+1)+1)} \cdot k^{k/2} \sum_{i=k+1}^{\infty} \frac{\|\mathbf{u}\|_\infty^{\log_d(k+i)+1}}{b^{d^m \cdot i}}. \tag{39}$$

The last sum in the right hand side of (39) is bounded from above by

$$\sum_{i=k+1}^{\infty} \frac{\|\mathbf{u}\|_\infty^{\log_d(k+i)+1}}{b^{d^m \cdot i}} \leq \frac{\|\mathbf{u}\|_\infty}{b^{d^m(k+1)}} \cdot \sum_{i=0}^{\infty} \frac{\|\mathbf{u}\|_\infty^{\log_d(2k+1+i)}}{b^{d^m \cdot i}}$$

$$\leq \frac{\|\mathbf{u}\|_\infty^{1+\log_d(2k+1)}}{b^{d^m(k+1)}} \sum_{i=0}^{\infty} \frac{(i+1)^{\log_d \|\mathbf{u}\|_\infty}}{b^{d^m \cdot i}} \leq \frac{\|\mathbf{u}\|_\infty^{1+\log_d(2k+1)} \cdot C(b,d,m,\|\mathbf{u}\|_\infty)}{b^{d^m(k+1)}}, \tag{40}$$

where

$$C(b,d,m,\|\mathbf{u}\|_\infty) = \sum_{i=0}^{\infty} \frac{(i+1)^{\log_d \|\mathbf{u}\|_\infty}}{b^{d^m \cdot i}}.$$

Note that for any $i \in \mathbb{Z}$, we have $i+1 \leq 2^i$. Because of this, assumption (33) implies

$$C(b,d,m,\|\mathbf{u}\|_\infty) \leq 2. \tag{41}$$

Finally, by putting together, (37), (38), (39), (40) and (41) we get

$$|q_{k,m}g_{\mathbf{u}}(b) - p_{k,m}| \leq \frac{2(k+1)k^{k/2}d^m\|\mathbf{u}\|_\infty^{m+(k+1)(\log_d(2k+1)+1)}}{b^{d^m \cdot k+1}}.$$

Dividing both sides by $q_{k,m}$ gives (34).

To get the lower bound, we first estimate the product in (30).

$$\prod_{t=0}^{m-1} P^*(b^{d^t}) = b^{d^m-1} \prod_{t=0}^{m-1} P(b^{-d^t}) \geq b^{d^m-1} \frac{g_{\mathbf{u}}(b)}{\prod_{t=m}^{\infty} P(b^{-d^t})}.$$

By (35), the denominator can easily be estimated as

$$\prod_{t=m}^{\infty} P(b^{-d^t}) \leq \prod_{t=m}^{\infty} \left(1 + \frac{2\|\mathbf{u}\|_\infty}{b^{d^t}}\right) < 2.$$

Therefore,

$$\prod_{t=0}^{m-1} P^*(b^{d^t}) \geq \frac{1}{2} b^{d^m-1} g_{\mathbf{u}}(b).$$

For the series in the right hand side of (30), we show that the first term dominates this series. Indeed, we have $|\alpha_{k,k+1}| \geq 1$ since it is a non-zero integer. Then,

$$\begin{aligned} |q_{k,m} g_{\mathbf{u}}(b) - p_{k,m}| &= \left| \prod_{t=0}^{m-1} P^*(b^{d^t}) \cdot \sum_{i=k+1}^{\infty} \alpha_{k,i} b^{-d^m \cdot i} \right| \\ &\geq \frac{1}{2} b^{d^m-1} |g_{\mathbf{u}}(b)| \left(b^{-d^m(k+1)} - \sum_{i=k+2}^{\infty} |\alpha_{k,i}|\, b^{-d^m \cdot i} \right) \\ &\overset{(39),(40)}{\geq} \frac{1}{2} b^{d^m-1} |g_{\mathbf{u}}(b)| \left(b^{-d^m(k+1)} - \frac{C(b,d,m,\|\mathbf{u}\|_\infty)(k+1)k^{k/2} \|\mathbf{u}\|_\infty^{(k+1)(\log_d(2k+1)+1)}}{b^{d^m(k+2)}} \right) \end{aligned} \tag{42}$$

Recall that by (41), we have $C(b,d,m,\|\mathbf{u}\|_\infty) \leq 2$. So, by using assumption (35), we finally get

$$|q_{k,m} g_{\mathbf{u}}(b) - p_{k,m}| \geq \frac{b^{d^m-1} |g_{\mathbf{u}}(b)|}{4 b^{d^m(k+1)}} = \frac{|g_{\mathbf{u}}(b)|}{4 b^{d^m \cdot k+1}}.$$

Finally, dividing both sides by $q_{k,m}$ leads to (36). □

Lemma 8. *Let $b, k, m \in \mathbb{N}$, $k \geq 1$ and let*

$$b^{d^m} > 3 \cdot \left(\|\mathbf{c}_{2k+1}\|_\infty^2 k \right)^{k/2}. \tag{43}$$

Recall the notations $a_{k,i}$, $i = 0, \ldots, k$, for the coefficients of q_k, $k \in \mathbb{N}$, is defined in (16)*. Then,*

$$\frac{1}{2}|a_{k,k}| \cdot b^{kd^m} \le q_{k,m} \le \frac{3}{2}|a_{k,k}| \cdot b^{kd^m}. \tag{44}$$

Proof. The leading term of $q_{k,m}(z)$ is $a_{k,k}z^{kd^m}$. We know that $\deg q_k(z) = k$, therefore $a_{k,k} \neq 0$ and $a_{k,k}$ is an integer. Recall also that by (20) the maximum of the coefficients $a_{k,i}$, $i = 0, \ldots, k$, does not exceed $(\|\mathbf{c}_{2k+1}\|_\infty^2 \cdot k)^{k/2}$. Thus we find, by using assumption (43),

$$\left|\sum_{n=0}^{k-1} a_{k,n} \cdot b^{nd^m}\right| \le b^{kd^m}\left|\sum_{n=1}^{k} 3^{-n}\right| \le \frac{1}{2}b^{kd^m}.$$

We readily infer, by taking into account $q_{k,m} = a_{k,0} + a_{k,1}b^{d^m} + \ldots + a_{k,k}b^{kd^m}$,

$$\frac{1}{2}|a_{k,k}|b^{kd^m} \le |a_{k,k}|b^{kd^m} - \frac{1}{2}b^{kd^m} \le |q_{k,m}| \le |a_{k,k}|q^{kd^m} + \frac{1}{2}q^{kd^m} = \frac{3}{2}|a_{k,k}|q^{kd^m}.$$

This completes the proof of the lemma. □

Proposition 1. *Let $k \ge 2$, $m \ge 1$ be integers and assume that* (35) *is satisfied. Then, the integers $p_{k,m} = p_{k,m}(b)$ and $q_{k,m} = q_{k,m}(b)$, defined by* (31) *and by* (32)*, satisfy*

$$\left|g_{\mathbf{u}}(b) - \frac{p_{k,m}}{q_{k,m}}\right| \le \frac{3(k+1)k^k d^m \|\mathbf{u}\|_\infty^{m+(2k+1)(\log_d(2k+1)+1)}}{b \cdot q_{k,m}^2}, \tag{45}$$

$$\frac{|g_{\mathbf{u}}(b)|}{8bq_{k,m}^2} \le \left|g_{\mathbf{u}}(b) - \frac{p_{k,m}}{q_{k,m}}\right|. \tag{46}$$

Moreover, if, additionally, k and m satisfy

$$k \cdot d^m \log_2 b - 1 \ge \frac{1}{3}m^2(\log\|\mathbf{u}\|_\infty)^2, \tag{47}$$

then

$$\left|g_{\mathbf{u}}(b) - \frac{p_{k,m}}{q_{k,m}}\right| \le \frac{3 \cdot 2^{C\sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}}}{q_{k,m}^2}, \tag{48}$$

where

$$C = 2\sqrt{2} + 2\sqrt{5 \cdot \log\|\mathbf{u}\|_\infty} + 2. \tag{49}$$

Proof. From Lemma 8 we have

$$b^{k\cdot d^m} \geq \frac{2q_{k,m}}{3|a_{k,k}|} \overset{(22)}{\geq} \frac{2q_{k,m}}{3\|\mathbf{u}\|_\infty^{k(\log_d(2k+1)+1)} k^{k/2}}.$$

Similarly, by using $|a_{k,k}| \geq 1$ together with Lemma 8, we get the lower bound

$$b^{k\cdot d^m} \leq 2q_{k,m}. \tag{50}$$

These two bounds on b^{kd^m} allow to infer the inequalities (45) and (46) straightforwardly from the corresponding bounds in Lemma 7.

We proceed with the proof of the estimate (48). We are going to deduce it as a corollary of (45). To this end, we are going to prove, under the assumptions of this proposition,

$$(k+1)k^k d^m \|\mathbf{u}\|_\infty^{m+(2k+1)(\log_d(2k+1)+1)} \leq 2^{C\sqrt{\log_2 q_{k,m} \log_2\log_2 q_{k,m}}}, \tag{51}$$

where the constant C is defined by (49). It is easy to verify that (45) and (51) indeed imply (48). Therefore in the remaining part of the proof we will focus on verifying (51).

The inequality (50) together with condition (35) imply

$$\begin{aligned} \log_2 q_{k,m} \geq \quad & (2k-1) + k\log_2(k+1) + \tfrac{k^2}{2}\log_2 k \\ & + k(k+1)(\log_d(2k+1)+1)\log_2\|\mathbf{u}\|_\infty. \end{aligned} \tag{52}$$

By taking logarithms again one can derive that $\log_2\log_2 q_{k,m} \geq \log_2 k$. Now we compute

$$\log_2 q_{k,m}\log_2\log_2 q_{k,m} \geq \frac{k^2}{2}(\log_2 k)^2 > \frac{1}{8}(k\log_2 k + \log_2(k+1))^2. \tag{53}$$

The last inequality in (53) holds true because $k\log_2 k > \log_2(k+1)$ for $k \geq 2$.

Another implication of (52) is

$$\log_2 q_{k,m}\log_2\log_2 q_{k,m} \geq k(k+1)(\log_d(2k+1)+1)\log_2 k\log_2\|\mathbf{u}\|_\infty. \tag{54}$$

Since for $d \geq 2$ and $k \geq 2$ we have $\log_2 k \geq \frac{1}{4}(\log_d(2k+1)+1)$ and $k(k+1) \geq \frac{1}{5}(2k+1)^2$, therefore we readily infer from (54)

$$\log_2 q_{k,m}\log_2\log_2 q_{k,m} \geq \frac{1}{20\log_2\|\mathbf{u}\|_\infty}(2k+1)^2(\log_d(2k+1)+1)^2(\log_2\|\mathbf{u}\|_\infty)^2. \tag{55}$$

Next, it follows from (50) that

$$\log_2 q_{k,m} \geq k \cdot d^m \log_2 b - 1. \tag{56}$$

Therefore assumption (47) implies that $\log_2 q_{k,m} \geq \frac{1}{3} m^2 (\log_2 \|\mathbf{u}\|_\infty)^2$. At the same time, the assumptions $k \geq 2$ joint with (35) readily imply $b^{k \cdot d^m} \geq 576$, hence, by adding (50), we find $\log_2 \log_2 q_{k,m} \geq \log_2 \log_2 288 > 3$. So,

$$\log_2 q_{k,m} \log_2 \log_2 q_{k,m} > m^2 (\log_2 \|\mathbf{u}\|_\infty)^2. \tag{57}$$

Also, by these considerations we deduce from (56)

$$\log_2 q_{k,m} \log_2 \log_2 q_{k,m} > 3d^m > (m \cdot \log_2 d)^2 . \tag{58}$$

Finally, by taking square root in the both sides of (53), (55), (57) and (58) and summing up the results we find

$$\begin{aligned} C\sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}} \geq{}& \log_2(k+1) + k \log_2 k + m \log_2 d \\ &+ (m + (2k+1)(\log_d(2k+1)+1)) \log_2 \|\mathbf{u}\|_\infty, \end{aligned} \tag{59}$$

where the constant C is defined by (49). Finally, by taking the exponents base two from both sides of (59), we find (51), hence derive (48). ☐

Remark 9. Note that the constant C in Proposition 1 is rather far from being optimal. The proof above can be significantly optimized to reduce its value. However that would result in more tedious computations. All one needs to show is the inequality (59).

3 Proof of Theorem 2

We will prove the folowing result.

Theorem 10. *Let $b \geq 2$. There exists an effectively computable constant γ, which only depends on d and $\mathbf{u}$, such that for any $p \in \mathbb{Z}$ and any sufficiently large $q \in \mathbb{N}$, we have*

$$\left| g_{\mathbf{u}}(b) - \frac{p}{q} \right| \geq \frac{|g_{\mathbf{u}}(b)|}{4b \cdot q^2 \cdot \exp\left(\gamma \sqrt{\log_2 q \log_2 \log_2 q}\right)}. \tag{60}$$

It is easy to see that Theorem 2 is a straight corollary of Theorem 10. Indeed, if $f(b)$ from Theorem 2 is not zero then so is $g_{\mathbf{u}}(b)$ and the lower bound (60) is satisfied for all large enough q, therefore the inequality

$$\left|g_{\mathbf{u}}(b) - \frac{p}{q}\right| < q^{-2}\exp(-\gamma\sqrt{\log q \log\log q})$$

has only finitely many solutions. By definition, this implies that $g_{\mathbf{u}}(b)$ and in turn $f(b)$ are both $q^{-2}\exp(-\gamma\sqrt{\log q\log\log q})$-badly approximable.

Proof of Theorem 10. In this proof, we will use the constant C defined by (49). Fix a couple of integers p and q. We start with some preliminary calculations and estimates.

Define $x > 2$ to be the unique solution of the following equation

$$q = \frac{1}{12} \cdot x \cdot 2^{-\frac{3}{2}C\sqrt{\log_2 x \log_2 \log_2 x}}, \tag{61}$$

where the constant C is defined by (49).

The condition $x > 2$ ensures that both $\log_2 x$ and the double logarithm $\log_2 \log_2 x$ exist and are positive, hence $2^{-C\sqrt{\log_2 x \log_2 \log_2 x}} < 1$ and thus

$$12q < x. \tag{62}$$

For large enough q we then have

$$\frac{81}{4}C^2 \log_2 \log_2 x < \log_2 x$$

and therefore

$$2^{\frac{3}{2}C\sqrt{\log_2 x \log_2 \log_2 x}} < x^{1/3}. \tag{63}$$

From (61) and (63) we readily infer

$$x < (12q)^{3/2}, \tag{64}$$

Rewrite (61) in the following form

$$x = 12q \cdot 2^{\frac{3}{2}C\sqrt{\log_2 x \log_2 \log_2 x}}. \tag{65}$$

Then, by applying (64) to it we find that, for large enough q,

$$x < 12q \cdot 2^{2C\sqrt{\log_2 q \log_2 \log_2 q}}. \tag{66}$$

Denote

$$t := \log_b x. \tag{67}$$

Fix an arbitrary value $\tau \geq \tau_0 > 1$, where $\tau_0 = \tau_0(\mathbf{u})$ is a parameter which only depends on $\mathbf{u}$ and which we will fix later (namely, it has to ensure inequality (72)). Assume that $t > 2$ is large enough (that is, assume q is large enough, then by (62) x is large enough, hence by (67) t is large enough), so that

$$d \leq \frac{1}{\tau}\sqrt{\frac{t}{\log_2 t}}. \tag{68}$$

As $t > 2$, we also have $t \log_2 t > 2$. Choose an integer n of the form $n := k \cdot d^m$ with $m \in \mathbb{N}$, $k \in \mathbb{Z}$ such that

$$t \leq \quad n \quad \leq t + d\tau\sqrt{t \log_2 t}, \tag{69}$$

$$\tau\sqrt{t \log_2 t} \leq \quad d^m \quad \leq d\tau\sqrt{t \log_2 t}. \tag{70}$$

One can easily check that such n always exists.

Inequalities (68), (69) and (70) imply

$$k = \frac{n}{d^m} \leq \frac{t + d\tau\sqrt{t \cdot \log_2 t}}{\tau\sqrt{t \cdot \log_2 t}} = \frac{1}{\tau}\sqrt{\frac{t}{\log_2 t}} + d \leq \frac{2}{\tau}\sqrt{\frac{t}{\log_2 t}}. \tag{71}$$

Then we deduce, for t large enough,

$$k \log_2 k \leq \frac{2}{\tau}\sqrt{\frac{t}{\log_2 t}}\left(\log_2(2/\tau) + \frac{1}{2}\log_2 t - \frac{1}{2}\log_2\log_2 t\right) < \frac{2}{\tau}\sqrt{t \log_2 t}.$$

Therefore, for any τ large enough, that is for any $\tau \geq \tau_0$, where τ_0 depends only on $\mathbf{u}$, we have

$$2 + \log_2(k+1) + \frac{k}{2}\log_2 k + (k+1)(\log_d(2k+1) + 1)\log_2 \|\mathbf{u}\|_\infty) < \tau\sqrt{t \log_2 t}. \tag{72}$$

By taking the exponent base two of the left hand side of (72) and the exponent base $b \geq 2$ of the right hand side of (72), and then using (70), we ensure that (35) is satisfied. We can also take q (and, consecutively, t) large enough so that m, bounded from below by (70), satisfies $d^m \geq m^2(\log_2 \|\mathbf{u}\|_\infty)^2$, and then necessarily (47) is verified. Also, (69) and (70) imply that, for t large enough, $k \geq 2$.

Hence we have checked all the conditions on k and m from Proposition 1. It implies that the integers $p_{k,m}$ and $q_{k,m}$, defined by (31) and (32), satisfy inequalities (46) and (48). Lemma 3 and inequality (35) imply the inequality (43), so we can use Lemma 8, i.e. we have

$$\frac{1}{2}|a_{k,k}|b^n \leq q_{k,m} \leq \frac{3}{2}|a_{k,k}|b^n. \tag{73}$$

In case if

$$\frac{p}{q} = \frac{p_{k,m}}{q_{k,m}},$$

the result (60) readily follows from the lower bound (46) in Proposition 1.

We proceed with the case

$$\frac{p}{q} \neq \frac{p_{k,m}}{q_{k,m}}. \tag{74}$$

By triangle inequality, and then by the upper bound (48), we have

$$\begin{aligned} \left|g_{\mathbf{u}}(b) - \frac{p}{q}\right| &\geq \left|\frac{p_{k,m}}{q_{k,m}} - \frac{p}{q}\right| - \left|g_{\mathbf{u}}(b) - \frac{p_{k,m}}{q_{k,m}}\right| \\ &\geq \frac{1}{q_{k,m}q} - \frac{3 \cdot 2^{C\sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}}}{q_{k,m}^2}. \end{aligned} \tag{75}$$

By applying the upper bound in (73) complimented with (22), we find

$$\log_2 q_{k,m} \leq \log_2 \frac{3}{2} + k/2 \log_2 k + k(\log_d(2k+1)+1) \log_2 \|\mathbf{u}\|_\infty + n \log_2 b$$

Upper bounds (71) on k and (69) on n ensure that for large enough q we have

$$2^{C\sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}} \leq 2^{\frac{3}{2}C\sqrt{\log_2 x \log_2 \log_2 x}}. \tag{76}$$

The formula (67) for t and the lower bound in (69) together give $b^n \geq x$. Then, by using the lower bound (73), we find

$$q_{k,m} \geq \frac{1}{2}b^n \geq \frac{1}{2}x. \tag{77}$$

By using the estimates (76) and (77) on the numerator and denominator respectively, and then by substituting the value of x given by (65), we find

$$\frac{3 \cdot 2^{C\sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}}}{q_{k,m}^2} \leq \frac{3 \cdot 2^{\frac{3}{2}C\sqrt{\log_2 x \log_2 \log_2 x}}}{\frac{1}{2}x \cdot q_{k,m}} \overset{(65)}{=} \frac{1}{2q_{k,m}q},$$

hence, recalling (75), we find

$$\left|g_{\mathbf{u}}(b) - \frac{p}{q}\right| \geq \frac{1}{2q_{k,m}q}. \tag{78}$$

By inequality (73) combined with the upper bound in (69) and then (67) and (66) we get that, for q large enough,

$$q_{k,m} \leq \frac{3}{2}|a_{k,k}|b^n \leq \frac{3}{2}|a_{k,k}|b^{t+d\tau\sqrt{t\log_2 t}} \leq 18|a_{k,k}|q \cdot 2^{(2d\tau \log_2 b+2C)\sqrt{\log_2 q \log_2 \log_2 q}}.$$

The bound (22) implies

$$\log_2 |a_{k,k}| \leq \frac{k}{2}\log_2 k + k(\log_d(2k+1)+1)\log_2 \|\mathbf{u}\|_\infty. \tag{79}$$

By comparing the right hand side of this inequality with the left hand side in (72) we find

$$|a_{k,k}| \leq 2^{2\tau\sqrt{\log_2 q \log_2 \log_2 q}}$$

and then

$$q_{k,m} \leq 18q \cdot 2^{(2d\tau \log_2 b+2\tau+2C)\sqrt{\log_2 q \log_2 \log_2 q}}$$

Finally, (78) implies

$$\left|g_{\mathbf{u}}(b) - \frac{p}{q}\right| \geq \frac{1}{36q^2 \cdot 2^{(2d\tau \log_2 b+2\tau+2C)\sqrt{\log_2 q \log_2 \log_2 q}}}.$$

This completes the proof of the theorem with $\gamma = \ln 2 \cdot (2d\tau \log_2 b + 2\tau + 2C)$. □

References

[1] B. Adamczewski, J. Cassaigne, "Diophantine properties of real numbers generated by finite automata", Comp. Math., 142, 2006, 1351 – 1372.

[2] B. Adamczewski, T. Rivoal, "Irrationality measures for some automatic real numbers", Math. Proc. Cambridge Phil. Soc., 147 (2009), 659–678.

[3] J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen, "Hankel determinants of the Thue-Morse sequence", Ann. Inst. Fourier (Grenoble), 48, 1998, 1-27.

[4] J.-P. Allouche and J. Shallit, "Automatic Sequences: Theory, Applications, Generalizations". Cambridge University Press, Cambridge, 2003.

[5] D. Badziahin, "Finding special factors of values of polynomials at integer points". Preprint.

[6] Badziahin, D. Zorin, E., "Thue-Morse constant is not badly approximable". International Mathematics Research Notices 2015(19): 9618-9637.

[7] Badziahin, D. Zorin, E., "On generalized Thue-Morse functions and their values.". Preprint, https://arxiv.org/abs/1509.00297.

[8] P. G. Becker, "k-regular power series and Mahler-type functional equations". J. Numb. Theor., 49, 1994, 269 – 286.

[9] J. P. Bell, Y. Bugeaud, M. Coons, "Diophantine approximation of Mahler numbers". Proc. Lond. Math. Soc., 110(5), 2015, 1157 – 1206.

[10] Y. Bugeaud, "On the rational approximation to the Thue-Morse-Mahler numbers". Ann. Inst. Fourier, 61, 2011, 2065 – 2076.

[11] Y. Bugeaud, G. N. Han, Z. Y. Wen, J. Y Yao, "Hankel determinants, Padé approximations and irrationality exponents". Int. Math. Res. Notes, 2016(5), 2016, 1467 – 1496.

[12] M. Coons, "On the rational approximation to the sum of reciprocals of the Fermat numbers". Ramanujan J., 30, 2013, 39 – 65.

[13] D. Duverney, Ke. Nishioka, Ku. Nishioka, I. Shiokawa. “Transcendence of Jacobi’s theta series”, Proc. Japan. Acad. Sci, Ser A, 72, (1996), 202-203.

[14] Y. J. Guo, Z. X. Wen, W. Wu, “On the irrationality exponent of the regular paperfolding numbers”. Lin. Alg. Appl., 446, 2014, 237 – 264.

[15] J. Hartmanis and R. E. Stearns, “On the Computational Complexity of Algorithms”, Transactions of the American Mathematical Society, Vol. 117 (May, 1965), pp. 285-306.

[16] J. Liouville, “Sur des classes très-étendues de quantités dont la valeur nest ni algébrique, ni même réductible à des irrationnelles algébriques”, J. de Math. Pures et Appl.,1851.

[17] K. Mahler, “Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen”, Math. Z. 32 (1930), 545-585.

[18] K. Nishioka, “Mahler Functions and Transcendence”, Lecture Notes in Math. 1631, Springer, 1996.

[19] A. J. van der Poorten, J. O. Shallit, “Folded Continued Fractions”, J. Number Theory, 40, 1992, no. 2, 237–250.

[20] A. J. van der Poorten., “Formal power series and their continued fraction expansion”. Algorithmic Number Theory. Lecture notes in Computer Science, 1423, pp 358 – 371, Springer, Berlin, 1998.

[21] Z. X. Wen, W. Wu, “Hankel determinant of the Cantor sequence” (in Chinese), Scienta Sinica Math., 44, 2014, 1059 – 1072.